2024/25 Semester 2 Stochastic Control and Dynamic Asset Allocation Problem Sheet 5, for Week 10 - Last updated 26th March 2025

Recall the optimal liquidation problem: trader's inventory, an \mathbb{R} -valued process:

 $dQ_u = -\alpha_u du$ with $Q_t = q > 0$ initial inventory.

Here α will typically be mostly positive as the trader should sell all the assets. We will denote this process $Q_u = Q_u^{t,q,\alpha}$ because clearly it depends on the starting point q at time t and on the trading strategy α . Asset price, an \mathbb{R} -valued process:

$$dS_u = \lambda \, \alpha_u \, du + \sigma \, dW_u \,, \ S_t = S \,.$$

We will denote this process $S_u = S_u^{t,S,\alpha}$ because clearly it depends on the starting point S at time t and on the trading strategy. Here $\lambda \leq 0$ controls how much permanent impact the trader's own trades have on its price. Trader's execution price (for $\kappa > 0$):

$$\hat{S}_t = S_t - \kappa \alpha_t \,.$$

This means that there is a temporary price impact of the trader's trading: she doesn't receive the full price S_t but less, in proportion to her selling intensity.

Quite reasonably we wish to maximize (over trading strategies α that are adapted to the filtration generated by the Wiener process and such that $\mathbb{E} \int_0^T \alpha_u^2 du < \infty$), up to to some finite time T > 0, the expected amount gained in sales, whilst penalising the terminal inventory (with $\theta > 0$):

$$J(t,q,S,\alpha) := \mathbb{E}\bigg[\underbrace{\int_{t}^{T} \hat{S}_{u}^{t,S,\alpha} \,\alpha_{u} \,du}_{\text{gains from sale}} + \underbrace{Q_{T}^{t,q,\alpha} \, S_{T}^{t,S,\alpha}}_{\text{val. of inventory}} - \underbrace{\theta \, |Q_{T}^{t,q,\alpha}|^{2}}_{\text{penalty for unsold}}\bigg].$$

The goal is to find

$$V(t,q,S) := \sup_{\alpha} J(t,q,S,\alpha) \,.$$

Exercise 5.1 (Optimal liquidation with no permanent market impact). Solve the optimal liquidation problem above in the case $\lambda = 0$ (i.e. there is no permanent price impact of our trading on the market price).

Hint 1: By calculating $d(Q_tS_t) = \ldots$ reduce the dimensionality of the problem by showing that the process $(S_u)_{u \in [t,T]}$, as a martingale, only plays a role in terms of its starting value.

Hint 2: Next, follow the linear-quadratic example in the lecture notes but in this case the Ricatti ODE arising here has an explicit solution.