1. Let $T > 0$, H, M, $C \leq 0$, $D < 0$, $R \leq 0$, κ, $\lambda > 0$ be constants with values in $\mathbb{R}$. Consider the controlled jump-diffusion

$$dX_s = (HX_s^\alpha + M\alpha_s)\,ds + \kappa(dN_s^u - dN_s^d)\,, \quad s \in [t,T]\,, \ \ X_t = x \in \mathbb{R}\,,$$

where $N^u = (N_t^u)_{t\in[0,T]}$ and $N^d = (N_t^d)_{t\in[0,T]}$ are Poisson processes with arrival intensity λ. The optimization objective which is to maximize

$$J(t,x,\alpha) := \mathbb{E}_{t,x,\alpha}\left[\int_t^T (CX_s^2 + D\alpha_s^2)\,ds + RX_T^2\right]\,,$$

over all square integrable predictable processes α. Here $\mathbb{E}_{t,x,\alpha}[\cdot]$ denotes the conditional expectation where the process X starts from x at time t and the control process α is used.

(a) Write down the Bellman equation for $v(t,x) = \sup_\alpha J(t,x,\alpha)$. **[7 marks]**

(b) Assume that S and b are in $C^1([0,T])$. Use the ansatz $v(t,x) = S(t)x^2 + b(t)$ to derive a solution to the Bellman equation i.e. derive the ordinary differential equations that S and b satisfy. Write down an expression for $v(t,x)$ in terms of t, x, R and S only (i.e. solve the equation for b). **[23 marks]**

(c) Take $H = 0$, $M = 1$, $C = 0$, $D = -1$, $R = -1$. Solve the ODE for S and hence get an exact expression the optimal Markov control. **[5 marks]**

Hint: If you wish you can take $\kappa = 0$ throughout and still collect up to 80% of the marks.

Solution

(a) The Bellman equation is

$$\partial_t v + \sup_a \left[L^a v + Cx^2 + Da^2\right] = 0 \ \text{ on } \ [0,T) \times \mathbb{R}\,,$$
$$v(T,x) = Rx^2 \ \ \forall x \in \mathbb{R}\,,$$

where

$$L^a v(t,x) = \lambda\Big(v(t,x+\kappa) - v(t,x)\Big) + \lambda\Big(v(t,x-\kappa) - v(t,x)\Big) + (Hx + Ma)\partial_x v\,.$$

(b) First we note that with $v(t,x) = S(t)x^2 + b(t)$ we have

$$v(t,x+\kappa) - 2v(t,x) + v(t,x-\kappa) = S(t)\Big((x+\kappa)^2 - 2x^2 + (x-\kappa)^2\Big) = 2S(t)\kappa^2\,.$$

Next we note that $\partial_t v = S'(t)x^2 + b'(t)$ and $\partial_x v = 2S(t)x$ and so plugging into the Bellman equation we get

$$S'(t)x^2 + b'(t) + \sup_a\Big(2\kappa^2\lambda S(t) + 2S(t)Hx^2 + 2S(t)Max + Cx^2 + Da^2\Big) = 0 \ \text{ on } \ [0,T) \times \mathbb{R}\,.$$

The expression being maximized is concave in a since $D < 0$. From the first order condition we thus get that

$$a^*(t,x) = -D^{-1}MS(t)x\,.$$

Substituting this back in and collecting terms leads to

$$\Big(S'(t) + 2HS(t) - D^{-1}M^2S^2(t) + C\Big)x^2 + b'(t) + 2\kappa^2\lambda S(t) = 0 \text{ on } [0,T) \times \mathbb{R}\,.$$

Collecting the terms involving x^2 and those without we see that the Ricatti ODE for S is:

$$S'(t) + 2HS(t) - D^{-1}M^2S^2(t) + C = 0\,,\ \ t \in [0,T]\,,\ \ S(T) = R$$

and for b we have

$$b'(t) + 2\kappa^2\lambda S(t) = 0\,,\ \ t \in [0,T]\,,\ \ b(T) = 0\,.$$

Solving the equation for b is easier as that just involves integration:

$$b(t) = -\int_t^T 2\kappa^2\lambda S(r)\,dr\,.$$

Thus

$$v(t,x) = S(t)x^2 - \int_t^T 2\kappa^2\lambda S(r)\,dr\,.$$

(c) In this case the Ricatti ODE simplifies to $S' = -S^2$ with $S(T) = -1$. Separating the variables and integrating

$$\tfrac{1}{S^2}dS = -dt$$

leads to

$$-\tfrac{1}{S} = -t + K\,,\ \ t \in [0,T], S(T) = -1$$

for some constant K. From the terminal condition we then get $K = 1 + T$ and hence $S(t) = -\frac{1}{1+(T-t)}$. Hence

$$a^*(t,x) = -D^{-1}MS(t)x = S(t)x = \frac{-x}{1+(T-t)}\,.$$

Comment

The students have seen a solution the LQR problem for controlled diffusions but not the jump case. So in that sense it is “unseen”. However they’ve seen what infinitesimal generators jump diffusions lead to so the question should be tractable.

Note also that I am explicitly giving the option of treating the deterministic case to those who don’t wish to engage with the jump process.

2. Let W be a real-valued Wiener process, $x \in \mathbb{R}$, $b : \mathbb{R} \times A \to \mathbb{R}$ and $\sigma : \mathbb{R} \times A \to \mathbb{R}$ be Lipschitz continuous in $(x, a) \in \mathbb{R} \times A$ with $A \subseteq \mathbb{R}$. Let $\mathcal{A}$ denote A-valued processes that are square integrable and adapted to the filtration generated by W. For $\alpha \in \mathcal{A}$ consider the controlled SDE

$$dX_s^{t,x,\alpha} = b(X_s^{t,x,\alpha}, \alpha_s)\, ds + \sigma(X_s^{t,x,\alpha}, \alpha_s)\, dW_s, \,,\; t \in [t, T]\,, \;\; X_t = x\,.$$

(a) State the tower property of conditional expectation. [2 marks]

(b) State the flow property that solutions to the above controlled SDE satisfy. [2 marks]

(c) State the Markov property that solutions to the above controlled SDE satisfy. [3 marks]

(d) Let $g : \mathbb{R} \to \mathbb{R}$ be continuous and of polynomial growth. Let $f : \mathbb{R} \times A \to \mathbb{R}$ be continuous, of polynomial growth in x and at most square growth in a. Let

$$J(t, x, \alpha) := \mathbb{E}\left[\int_t^T f(X_s^{\alpha,t,x}, \alpha_s)\, ds + g(X_T^{\alpha,t,\xi})\right] \quad \text{and} \quad v(t, x) := \sup_{\alpha \in \mathcal{A}} J(t, x, \alpha)\,.$$

Prove that for $t \le \hat{t} \le T$ we have

$$v(t, x) \le \sup_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_t^{\hat{t}} f(X_s^{\alpha,t,x}, \alpha_s)\; ds + v(\hat{t}, X_{\hat{t}}^{\alpha,t,x}) \Big| X_t^{\alpha,t,x} = x\right].$$

Justify each step, in particular by stating when you're applying a), b) or c) above. [18 marks]

Solution

(a) Let $\mathcal{G} \subseteq H$ be σ-algebras. Let X be a r.v. such that $\mathbb{E}|X| < \infty$. Then

$$\mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}] = \mathbb{E}[X|\mathcal{G}]\,.$$

(b) Let $t \le \hat{t} \le T$. Then

$$X_s^{t,x,\alpha} = X_s^{\hat{t}, X_{\hat{t}}^{t,x,\alpha}} \qquad \forall s \in [\hat{t}, T]\,.$$

(c) Let $t \le \hat{t} \le T$. Let $F : C([\hat{t}, T]; \mathbb{R}) \to \mathbb{R}$. Then

$$\mathbb{E}[F((X_s^{t,x,\alpha})_{s \in [\hat{t},T]})|\mathcal{F}_{\hat{t}}] = \mathbb{E}[F((X_s^{t,x,\alpha})_{s \in [\hat{t},T]})|X_{\hat{t}}^{t,x,\alpha}]\,,$$

where $\mathcal{F}_s := \sigma(X_r^{t,x,\alpha} : r \le s)$.

(d) Take an arbitrary $\alpha \in \mathcal{A}$. We have

$$J(t, x, \alpha) = \mathbb{E}\left[\int_t^{\hat{t}} f(s, X_s^\alpha, \alpha_s) ds + \int_{\hat{t}}^T f(s, X_s^\alpha, \alpha_s) ds + g(X_T^\alpha) \Big| X_t^\alpha = x\right].$$

We will use the tower property of conditional expectation and use the Markov property of the process. Let $\mathcal{F}_{\hat{t}}^X := \sigma(X_s^\alpha : 0 \le s \le \hat{t})$. Then

$$\begin{aligned} &J(t, x, \alpha) \\ &= \mathbb{E}\left[\int_t^{\hat{t}} f(s, X_s^\alpha, \alpha_s) ds + \mathbb{E}\left[\int_{\hat{t}}^T f(s, X_s^\alpha, \alpha_s) ds + g(X_T^\alpha) \Big| \mathcal{F}_{\hat{t}}^X\right] \Big| X_t^\alpha = x\right] \\ &= \mathbb{E}\left[\int_t^{\hat{t}} f(s, X_s^\alpha, \alpha_s) ds + \mathbb{E}\left[\int_{\hat{t}}^T f(s, X_s^\alpha, \alpha_s) ds + g(X_T^\alpha) \Big| X_{\hat{t}}^{\alpha,t,x}\right] \Big| X_t^\alpha = x\right]. \end{aligned}$$

Now, because of the flow property of SDEs,

$$\mathbb{E}\left[\int_{\hat{t}}^{T} f(s, X_s^{\alpha,t,x}, \alpha_s)ds + g(X_T^{\alpha,t,x})\Big| X_{\hat{t}}^{\alpha,t,x}\right] = J\left(\hat{t}, X_{\hat{t}}^{\alpha,t,x}, \alpha\right) \leq v\left(\hat{t}, X_{\hat{t}}^{\alpha,t,x}\right).$$

Hence

$$J(t,x,\alpha) \leq \sup_{\alpha\in\mathcal{A}} \mathbb{E}\left[\int_{t}^{\hat{t}} f(s, X_s^{\alpha}, \alpha_s)ds + v\left(\hat{t}, X_{\hat{t}}^{\alpha,t,x}\right) \Big| X_t^{\alpha} = x\right].$$

Taking supremum over control processes α on the left provides the desired result.

Comment

This is the first part of proof of Bellman's principle seen in the lectures and provided in the lecture notes.

3. Consider the following control problem for a jump diffusion:

- Let $dS_r^{t,S} = \sigma dW_r$ for $r \in [t,T]$ with initial value $S_t = S \in \mathbb{R}$ given.
- Let $M = (M_t)_{t\geq 0}$ be a Poisson jump process with intensity λ.
- Let $(U_i)_{i\in\mathbb{N}}$ be iid r.v.s with uniform distribution on $[0,1]$.
- Let $\mathcal{F}_t = \sigma(W_s : s \leq t) \vee \sigma(M_s : s \leq t) \vee \sigma(U_i : i \leq M_t)$. Let $\mathcal{A}$ denote $\mathbb{R}^+$-valued stochastic processes that are square integrable and progressively measurable w.r.t $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. We shall call these admissible controls. Let $\delta \in \mathcal{A}$.
- Let N_t^δ be a stochastic process satisfying $N_t^\delta = N_{t-}^\delta + (M_t - M_{t-})\mathbf{1}_{U_{M_t} \geq e^{-\kappa\delta_t}}$. In other words it is a doubly stochastic Poisson process with stochastic intensity given by $\lambda e^{-\kappa\delta_t}$.
- Let $Q_r^{t,q,\delta} = q - N_r^\delta$ for $r \in [t,T]$ with initial value $Q_t = q \in \{0\} \cup \mathbb{N}$ given.
- Let $dX_r^{t,q,S,x,\delta} = (S_r + \delta_r)dN_r^\delta$ for $r \in [t,T]$ with initial value $X_t = x \in \mathbb{R}$ given.
- Let $\tau^{t,q,\delta} := T \wedge \inf\{r \geq t : Q_r^{t,q,\delta} \leq 0\}$.

Our aim is to maximize

$$J(t,q,S,x,\delta) = \mathbb{E}_{t,q,S,x,\delta}\Big[X_\tau + S_\tau Q_\tau - \alpha Q_\tau^2\Big]$$

over $\mathcal{A}$. Here $\mathbb{E}_{t,q,S,x,\delta}[\cdot]$ denotes the conditional expectation given $S_t = S$, $X_t = x$, $Q_t = q$ and given the process control δ is used. For simplicity we will assume $q \in \{0,1,2\}$. Let

$$v(t,q,S,x) = \sup_{\delta\in\mathcal{A}} J(t,q,S,x,\delta)\,.$$

(a) Write down the Bellman equation for v, including all boundary and terminal conditions and carefully specifying the domain. [7 marks]

(b) Assume that $\theta \in C^1([0,T])$. Use the ansatz $v(t,q,S,x) = x + Sq + \theta(t,q)$ and the Bellman PDE to derive that

$$\begin{aligned}\partial_t\theta + \frac{1}{\kappa}\lambda e^{-1}\exp\Big(\kappa(\theta(t,q-1) - \theta)\Big) &= 0 \text{ on } [0,T) \times \{1,2\}\,,\\ \theta(t,0) &= 0 \ \ \forall t \in [0,T),\\ \theta(T,q) &= -\alpha q^2 \ \ \forall q \in \{0,1,2\}\,.\end{aligned}$$

[23 marks]

(c) Let $w(t,q) := e^{\kappa\theta(t,q)}$ and write down the ODE w satisfies. [5 marks]

(d) Solve the ODE for w with $q = 1$ and thus derive an explicit expression for $\theta(t,1)$. [5 marks]

Solution

(a) Let $\mathcal{D} = \mathbb{R} \times \mathbb{R}^+ \times \{1,2\}$. Then

$$\begin{aligned}&\partial_t v + \tfrac{1}{2}\sigma^2\partial_{SS}v + \sup_{\delta\geq 0}\Big[\lambda e^{-\kappa\delta}(v(t,S,x+(S+\delta),q-1) - v)\Big] = 0 \text{ on } [0,T) \times \mathcal{D}\,,\\ &v(t,x,S,0) = x \ \ \forall t \in [0,T)\,, x \in \mathbb{R}^+\,, S \in \mathbb{R}\,,\\ &v(T,x,S,q) = x + qS - \alpha q^2 \ \ \forall (x,S,q) \in \mathcal{D}\,.\end{aligned}$$

(b) The ansatz

$$v(t,S,x,q) = x + qS + \theta(t,q)$$

allows us to match boundary conditions as long as $\theta(T,q) = -\alpha q^2$. Let us now solve the Bellman PDE with this ansatz. We note that

$$x + (S+\delta) + S(q-1) + \theta(t,q-1) - x - qS - \theta = \delta + \theta(t,q-1) - \theta$$

and hence

$$\partial_t \theta + \sup_{\delta \geq 0} \left[\lambda e^{-\kappa\delta}(\delta + \theta(t,q-1) - \theta) \right] = 0 \;\text{ on }\; [0,T) \times \mathcal{D}\,,$$
$$\theta(t,0) = 0 \;\; \forall t \in [0,T),$$
$$\theta(T,q) = -\alpha q^2 \;\; \forall q \in \{0,1,2\}\,.$$

Letting $g(t,q) = \theta(t,q-1) - \theta$ we observe that we need to maximize the function

$$\delta \mapsto e^{-\kappa\delta}(\delta + g)\,.$$

Calculating derivatives we see this is concave on $[0,\infty)$ and from the first order condition we see that

$$e^{-\kappa\delta} - (\delta + g)\kappa e^{-\kappa\delta} = 0$$

i.e.

$$\delta^*(t,q) = \frac{1}{\kappa} - g(t,q)\,.$$

To substitute this back into the equation for θ we see that

$$\sup_{\delta \geq 0} \left[e^{-\kappa\delta}(\delta + g(t,q)) \right] = e^{-1} e^{\kappa g(t,q)} \tfrac{1}{\kappa}$$

and so

$$\partial_t \theta + \frac{1}{\kappa}\lambda e^{-1} \exp\Big(\kappa(\theta(t,q-1) - \theta)\Big) = 0 \;\text{ on }\; [0,T) \times \{1,2\}\,,$$
$$\theta(t,0) = 0 \;\; \forall t \in [0,T),$$
$$\theta(T,q) = -\alpha q^2 \;\; \forall q = \{0, \Delta, 2\Delta, \ldots, \mathcal{N}\}\,.$$

(c) If $w(t,q) := e^{\kappa\theta(t,q)}$ then $\kappa\theta(t,q) = \ln w(t,q)$. Then $\partial_t \theta = \frac{1}{\kappa}\frac{1}{w}\partial_t w$ and the equation for w is

$$\frac{1}{\kappa}\frac{1}{w(t,q)}\partial_t w(t,q) + \frac{1}{\kappa}\lambda e^{-1}\frac{w(t,q-1)}{w(t,q)} = 0 \;\text{ on }\; [0,T) \times \{1,2\}$$

which upon multiplication by $w(t,q) > 0$ becomes

$$\partial_t w(t,q) + \lambda e^{-1} w(t,q-1) = 0 \;\text{ on }\; [0,T) \times \{1,2\}\,.$$

The terminal condition is $w(T,q) = e^{-\kappa\alpha q^2}$ and $w(t,0) = 1$.

(d) We know that $w(t,0) = 1$ and so the equation for $w(t,1)$ is

$$\partial_t w(t,1) + \lambda e^{-1} = 0 \;\text{ on }\; [0,T)\,, \;\; w(T,1) = e^{-\kappa\alpha}\,.$$

Integrating we get $w(t,1) = \lambda e^{-1}(T-t) + e^{-\kappa\alpha}$ and so

$$\theta(t,1) = \tfrac{1}{\kappa}\ln w(t,1) = \tfrac{1}{\kappa}\ln\Big(\lambda e^{-1}(T-t) + e^{-\kappa\alpha}\Big)\,.$$

Comment

Without saying so, this is exactly the problem of optimal asset disposal when using limit orders only which has been solved in class.