1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $T > 0$ be fixed. For $(t, x) \in [0, T] \times \mathbb{R}$ let

$$f(t,x) = \ln\left(e^{2xe^{T-t}} + e^{-2xe^{T-t}}\right), \quad g(x) = x^2\,.$$

Let $A = \{(a^1, a^2) \in \mathbb{R}^2 : a^1 \geq 0, a^2 \geq 0, a^1 + a^2 = 1\}$. Let W be a 1-dimensional Wiener process, let $\mathcal{F}_t = \sigma(W_s; s \leq t)$ and let $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}$. Let $\mathcal{A}$ denote the class of processes $\alpha_t = (\alpha_t^1, \alpha_t^2)$ that are $\mathbf{F}$-adapted and taking values in A. We will call elements of $\mathcal{A}$ admissible controls.

Consider the controlled SDE

$$dX_s^{t,x,\alpha} = (\alpha_s^1 - \alpha_s^2)\,ds + X_s^{t,x,\alpha}\,dW_s\,, \quad s \in [t,T]\,, \quad X_t^{t,x,\alpha} = x \in \mathbb{R}\,.$$

The optimization objective is to *minimize*

$$J(t,x,\alpha) = \mathbb{E}\Big[\int_t^T \left(f(X_s^{t,x,\alpha}) + \alpha_s^1 \ln \alpha_s^1 + \alpha_s^2 \ln \alpha_s^2\right) ds + g(X_T^{t,x,\alpha})\Big]$$

over admissible controls α.

(a) Let the value function be

$$w(t,x) = \inf_\alpha J(t,x,\alpha)\,.$$

Write down the HJB equation for the control problem. [5 marks]

Solution

The HJB is

$$\partial_t v + \tfrac{1}{2}x^2\partial_{xx}v + f + \inf_{a\in A}(a^1 \ln a^1 + a^2 \ln a^2 + (a^1 - a^2)\partial_x v) = 0 \ \text{ in } \ [0,T]\times\mathbb{R}\,,$$
$$v(T,\cdot) = g \ \text{ on } \ \mathbb{R}\,.$$

(b) Show that minimizers of the infimum (candidate optimal controls) are

$$a^1 = \frac{e^{-\partial_x v - 1}}{e^{-\partial_x v-1} + e^{\partial_x v - 1}} = \frac{e^{-\partial_x v}}{e^{-\partial_x v} + e^{\partial_x v}}\,, \quad a^2 = \frac{e^{\partial_x v - 1}}{e^{-\partial_x v-1} + e^{\partial_x v - 1}} = \frac{e^{\partial_x v}}{e^{-\partial_x v} + e^{\partial_x v}}$$

and that

$$\inf_{a\in A}(a^1 \ln a^1 + a^2 \ln a^2 + (a^1 - a^2)\partial_x v) = -\ln(e^{\partial_x v} + e^{-\partial_x v})\,.$$

[15 marks]

Solution

Since $a^1 + a^2 = 1$ we can see that

$$\begin{aligned}
&\inf_{(a^1,a^2)\in A}(a^1 \ln a^1 + a^2 \ln a^2 + (a^1 - a^2)\partial_x v)\\
&= \inf_{a\in[0,1]}(a\ln a + (1-a)\ln(1-a) + (a - (1-a))\partial_x v)\\
&= \inf_{a\in[0,1]}(a\ln a + (1-a)\ln(1-a) - \partial_x v + 2a\partial_x v)\,.
\end{aligned}$$

Moreover

$$\begin{aligned}
&\tfrac{d}{da}(a\ln a + (1-a)\ln(1-a) - \partial_x v + 2a\partial_x v)\\
&= 1 + \ln a - 1 - \ln(1-a) + 2\partial_x v = \ln\tfrac{a}{1-a} + 2\partial_x v\,.
\end{aligned}$$

Further

$$\tfrac{d}{da}\left(\ln \tfrac{a}{1-a} + 2\partial_x v\right) = \frac{1}{a-a^2} > 0 \text{ for } a \in (0,1)\,.$$

Thus any $a \in (0,1)$ which satisfies the first order condition will be a minimizer. The first order condition then leads to $\ln \frac{a}{1-a} = -2\partial_x v$ which can be solved:

$$a^1 = a = \frac{e^{-2\partial_x v}}{1+e^{-2\partial_x v}} = \frac{e^{-\partial_x v}e^{-\partial_x v}}{e^{-\partial_x v}e^{\partial_x v} + e^{-\partial_x v}e^{-\partial_x v}} = \frac{e^{-\partial_x v}}{e^{\partial_x v}+e^{-\partial_x v}} \in (0,1)\,.$$

Moreover

$$a^2 = (1-a) = \frac{e^{\partial_x v}+e^{-\partial_x v}}{e^{\partial_x v}+e^{-\partial_x v}} - \frac{e^{-\partial_x v}}{e^{\partial_x v}+e^{-\partial_x v}} = \frac{e^{\partial_x v}}{e^{\partial_x v}+e^{-\partial_x v}}\,.$$

Substituting the minimizer back into the expression to be minimized we obtain

$$\begin{aligned}
&\frac{e^{-\partial_x v}}{e^{-\partial_x v}+e^{\partial_x v}}\Big(-\partial_x v - \ln(e^{-\partial_x v}+e^{\partial_x v})\Big) + \frac{e^{\partial_x v}}{e^{-\partial_x v}+e^{\partial_x v}}\Big(\partial_x v - \ln(e^{-\partial_x v}+e^{\partial_x v})\Big)\\
&+ \frac{e^{-\partial_x v}}{e^{-\partial_x v}+e^{\partial_x v}}\partial_x v - \frac{e^{\partial_x v}}{e^{-\partial_x v}+e^{\partial_x v}}\partial_x v\\
&= -\frac{e^{-\partial_x v}}{e^{-\partial_x v}+e^{\partial_x v}}\ln(e^{-\partial_x v}+e^{\partial_x v}) - \frac{e^{\partial_x v}}{e^{-\partial_x v}+e^{\partial_x v}}\ln(e^{-\partial_x v}+e^{\partial_x v})\\
&= -\ln(e^{-\partial_x v}+e^{\partial_x v})\,.
\end{aligned}$$

This completes one version of solution to part (b).
An alternative approach is to observe that

$$\partial_a\Big(a^1\ln a^1 + a^2 \ln a^2 + (a^1-a^2)\partial_x v\Big) = \begin{pmatrix} 1+\ln a^1 + \partial_x v \\ 1 + \ln a^2 - \partial_x v\end{pmatrix}\,.$$

Moreover

$$\partial_{aa}\Big(a^1\ln a^1 + a^2 \ln a^2 + (a^1-a^2)\partial_x v\Big) = \begin{pmatrix} \frac{1}{a^1} & 0 \\ 0 & \frac{1}{a^2}\end{pmatrix}\,.$$

This is positive definite (e.g. because it's diagonal, so the eigenvalues are the entries on the diagonal and these are positive). So the function $A \ni a \mapsto \Big(a^1\ln a^1 + a^2 \ln a^2 + (a^1-a^2)\partial_x v\Big)$ strictly convex and it will have only one minimum characterised by the first order condition

$$1+\ln a^1 + \partial_x v = 0 \quad 1+\ln a^2 - \partial_x v = 0\,.$$

Solving (and scaling to ensure the resulting point is in A) we get

$$a^1 = \frac{e^{-\partial_x v - 1}}{e^{-\partial_x v-1}+e^{\partial_x v-1}} = \frac{e^{-\partial_x v}}{e^{-\partial_x v}+e^{\partial_x v}}\,,\quad a^2 = \frac{e^{\partial_x v-1}}{e^{-\partial_x v-1}+e^{\partial_x v-1}} = \frac{e^{\partial_x v}}{e^{-\partial_x v}+e^{\partial_x v}}\,.$$

(c) Use the ansatz $v(t,x) = \psi(t)x^2$ with $\psi \in C^1([0,T])$, $\psi(T) = 1$, to solve the HJB equation. [10 marks]

Solution

From parts (a) and (b) we have the HJB

$$\partial_t v + \tfrac{1}{2}x^2\partial_{xx}v + f - \ln(e^{\partial_x v} + e^{-\partial_x v}) = 0 \ \text{ in } \ [0,T]\times\mathbb{R}\,,$$
$$v(T,\cdot) = g \ \text{ on } \ \mathbb{R}\,.$$

With the ansatz this becomes

$$\psi'(t)x^2 + x^2\psi(t) + \ln\left(e^{2xe^{T-t}} + e^{-2xe^{T-t}}\right) - \ln(e^{2x\psi(t)} + e^{-2x\psi(t)}) = 0\,.$$

This will hold provided that $\psi'(t)+\psi(t)=0$ and $\psi(t)=e^{T-t}$ which is happily true. Thus the solution is $v(t,x) = x^2e^{T-t}$.

(d) Carry out the verification argument (either directly or appealing to a theorem in the course) to show that the solution of the HJB equation is equal to the value function w and that the feedback controls found in (b) are the optimal ones. [20 marks]

Solution

We will appeal to a theorem from the course whereby we need to check that: 1) the infimum in the HJB is attainable with measurable minimisers, 2) the resulting SDE has unique solution X^* and 3) $t' \mapsto \int_t^{t'} \partial_x v(s,X_s^*)X_s^*\,dW_s$ is a martingale.
Condition 1) has already been checked by (b). For 2), let us write the SDE for X^*:

$$dX_s^* = (a^1(t,X_s^*) - a^2(t,X_s^*)\,ds + X_s^*\,dW_s\,,\quad s\in[t,T]\,,\quad X_t^* = x\in\mathbb{R}\,.$$

This SDE will have a unique solution as long as the drift and diffusion coefficients are Lipschitz continuous in x. The diffusion coefficient is linear in x and thus clearly Lipschitz continuous in x. The drift is a difference of two functions which will be Lipschitz continuous in x as long as both functions in the difference are. Let us show that this is the case. Note that by the mean-value theorem, for any $t\in[0,T]$ and $x,x'\in\mathbb{R}$ we have

$$|a^2(t,x) - a^2(t,x')| \le |\partial_x a^1(t,\xi)||x-x'|$$

for some $\xi\in[x,x']$. Thus, we only need to show that $\partial_x a^2$ is uniformly bounded. A calculation using the quotient rule of calculus shows that

$$\partial_x a^2(t,\xi) = -\frac{4\psi(t)}{2+e^{-4\xi\psi(t)}+e^{4\xi\psi(t)}}\,.$$

Hence $|\partial_x a^2| \le 2e^T$. Similarly $|\partial_x a^1| \le 2e^T$ and thus the drift is Lipschitz continuous in x and the SDE for X^* has a unique solution. Moreover, as an SDE with Lipschitz-in-x coefficients that are uniformly bounded in t we have that $\sup_{s\in[t,T]}\mathbb{E}|X_s^*|^4 < \infty$ for initial $x\in\mathbb{R}$.
Condition 3) will hold as long as we can show that

$$I := \mathbb{E}\int_0^T |\partial_x v(t,X_t^*)X_t^*|^2\,dt < \infty\,.$$

We see that

$$I = \mathbb{E}\int_0^T |2\psi(t)(X_t^*)^2|^2\,dt \le 2e^T\int_0^T \mathbb{E}(X_t^*)^4\,dt < \infty\,.$$

This completes the verification.

2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Consider the following model for optimal market making (same as in lectures): Fix real constants $\sigma > 0$, $\kappa > 0$, $\lambda^b > 0$, $\lambda^a > 0$ and $T > 0$.

- Let $dS_r^{t,S} = \sigma dW_r$ for $r \in [t, T]$ with initial value $S_t = S \in \mathbb{R}$ given (mid price process).
- Let $M^b = (M_t^b)_{t \geq 0}$ and $M^a = (M_t^a)_{t \geq 0}$ be two Poisson jump processes with intensities λ^b, λ^a respectively counting sell and buy market order arrivals.
- Let $(U_i^b)_{i \in \mathbb{N}}$, $(U_i^a)_{i \in \mathbb{N}}$ be iid r.v.s with uniform distribution on $[0, 1]$.
- Let $\mathcal{F}_t = \sigma(W_s : s \leq t) \vee \sigma(M_s^b : s \leq t) \vee \sigma(M_s^a : s \leq t) \vee \sigma(U_i^b : i \leq M_t^b) \vee \sigma(U_i^a : i \leq M_t^a)$. Let $\mathcal{A}$ denote $\mathbb{R}^+ \cup \{+\infty\}$-valued stochastic processes that are square integrable and progressively measurable w.r.t $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$. We shall call these admissible controls. Let $\delta^b, \delta^a \in \mathcal{A}$. Let $\delta = (\delta^b, \delta^a)$.
- Let N^{b,δ_t^b} and N^{a,δ_t^a} be stochastic processes satisfying

$$N_t^{b,\delta_t^b} = N_{t-}^{b,\delta_t^b} + (M_t^b - M_{t-}^b) \mathbf{1}_{U^b_{M_t^b} \geq e^{-\kappa \delta_t^b}} ,$$

$$N_t^{a,\delta_t^a} = N_{t-}^{a,\delta_t^a} + (M_t^a - M_{t-}^a) \mathbf{1}_{U^a_{M_t^a} \geq e^{-\kappa \delta_t^a}} .$$

  In other words these are doubly stochastic Poisson processes with stochastic intensities given by $\lambda^b e^{-\kappa \delta_t^b}$ and $\lambda^a e^{-\kappa \delta_t^a}$ counting when the market makers orders get picked up by incoming market orders.
- Let $Q_r^{t,q,\delta} = q + N_r^{b,\delta^b} - N_r^{a,\delta^a}$ for $r \in [t, T]$ with initial value $Q_t = q \in \{0\} \cup \mathbb{N}$ given.
- Let $dX_r^{t,q,S,x,\delta} = -(S_r - \delta_r^b) dN_r^{b,\delta^b} + (S_r + \delta_r^a) dN_r^{a,\delta^a}$ for $r \in [t, T]$ with initial value $X_t = x \in \mathbb{R}$ given.
- The market maker has inventory lower and upper bound $\underline{q}, \bar{q} \in \mathbb{Z}$.

So far this was exactly the market making model we used in lectures. We shall now change the optimization objective. Let $\gamma > 0$ be a fixed real constant and let $u(x) = -e^{-\gamma x}$ (exponential utility). Our aim is to *maximize*

$$J(t, q, S, x, \delta) = \mathbb{E}_{t,q,S,x,\delta} \Big[u(X_T + S_T Q_T - \alpha Q_T^2) \Big]$$

over $\mathcal{A}$. Here $\mathbb{E}_{t,S,q,x,\delta}[\cdot]$ denotes the conditional expectation given $S_t = S \in \mathbb{R}$, $X_t = x \in \mathbb{R}$, $Q_t = q \in \mathbb{Z} \cap [\underline{q}, \bar{q}]$ and given the process control δ is used.

(a) The value function is $w(t, S, q, x) = \sup_{\delta^b, \delta^a \in \mathcal{A}} J(t, S, q, x, (\delta^b, \delta^a))$. Write down the HJB satisfied by w. [10 marks]

Solution

The HJB is

$$\partial_t v + \tfrac{1}{2}\sigma^2 \partial_{SS} v + \sup_{\delta^b > 0} \lambda^b e^{-\kappa \delta^b} \Big(v(t, S, q+1, x - (S - \delta^b)) - v(t, S, q, x) \Big) \mathbf{1}_{q < \bar{q}}$$
$$+ \sup_{\delta^a > 0} \lambda^a e^{-\kappa \delta^a} \Big(v(t, S, q-1, x + (S + \delta^a)) - v(t, S, q, x) \Big) \mathbf{1}_{q > \underline{q}} = 0$$

on $[0, T] \times \mathbb{R} \times \mathbb{Z} \cap [\underline{q}, \bar{q}] \times \mathbb{R}$ with the terminal condition

$$v(T, S, q, x) = u(x + Sq - \alpha q^2) .$$

(b) Use the ansatz $w(t,S,q,x) = u(x + Sq - g(t,q)) = -e^{-\gamma(x+qS+g(t,q))}$ for some $g = g(t,q)$ to show that the offsets achieving the supremums in the HJB are

$$\delta^b(t,q) = \tfrac{1}{\gamma}\ln\tfrac{\kappa+\gamma}{\kappa} + g(t,q) - g(t,q+1)\,, \quad \delta^a(t,q) = \tfrac{1}{\gamma}\ln\tfrac{\kappa+\gamma}{\kappa} + g(t,q) - g(t,q-1)\,.$$

You do not need to justify why points satisfying the first order condition are maximizers. [15 marks]

Solution

We start by noting that the ansatz implies that $g(T,q) = -\alpha q^2$ and moreover

$$\partial_t w = (-\gamma \partial_t g) w\,, \quad \partial_{SS} w = \gamma^2 q^2 w\,.$$

Moreover, writing $g^+ = g(t,q+1)$, we now wish to see what happens with the first supremum. We have

$$\begin{aligned} &e^{-\kappa\delta}(w(t,S,q+1,x-(s-\delta)) - w)\\ &= e^{-\kappa\delta}\Big[e^{-\gamma(x+qS+g)} - e^{-\gamma(x-(S-\delta)+(q+1)S+g^+)}\Big]\\ &= e^{-\kappa\delta}\Big[e^{-\gamma(x+qS+g)} - e^{-\gamma(x+\delta+qS+g^+)}\Big]\\ &= e^{-\gamma(x+qS)}\Big[e^{-\kappa\delta}e^{-\gamma g} - e^{-(\kappa+\gamma)\delta}e^{-\gamma g^+}\Big]\,. \end{aligned}$$

To maximize this over δ we can note that

$$\frac{d}{d\delta}\Big[e^{-\kappa\delta}e^{-\gamma g} - e^{-(\kappa+\gamma)\delta}e^{-\gamma g^+}\Big] = -\kappa e^{-\kappa\delta}e^{-\gamma g} + (\kappa+\gamma)e^{-\kappa\delta}e^{-\gamma\delta}e^{-\gamma g^+}\,.$$

To find when this is 0 we solve

$$0 = -\kappa e^{-\gamma g} + (\kappa+\gamma)e^{-\gamma\delta}e^{-\gamma g^+}$$

which yields that

$$\delta^b = \tfrac{1}{\gamma}\ln\tfrac{\kappa+\gamma}{\kappa} + g - g^+\,.$$

We still need to justify that this is indeed the maximum. The function $\delta \mapsto e^{-\kappa\delta}e^{-\gamma g} - e^{-(\kappa+\gamma)\delta}e^{-\gamma g^+}$ is a weighted sum of convex functions $\delta \mapsto e^{-\gamma g}$. Let's see about the second supremum:

$$\begin{aligned} &e^{-\kappa\delta}(w(t,S,q-1,x+(s+\delta)) - w)\\ &= e^{-\kappa\delta}\Big[e^{-\gamma(x+qS+g)} - e^{-\gamma(x+\delta+qS+g^-)}\Big]\,. \end{aligned}$$

This is the same as before except with g^- instead of g^+ and so

$$\delta^a = \tfrac{1}{\gamma}\ln\tfrac{\kappa+\gamma}{\kappa} + g - g^-\,.$$

(c) Hence show that g satisfies the nonlinear ODE

$$\partial_t g - \tfrac{1}{2}\sigma^2\gamma q^2 + \hat\lambda^b e^{-\kappa(g-g^+)}\mathbf{1}_{q<\bar q} + \hat\lambda^a e^{-\kappa(g-g^-)}\mathbf{1}_{q>\underline{q}} = 0\,.$$

for some constants $\hat\lambda^b$, $\hat\lambda^a$. Write down what these are in terms of the original problem constants. [20 marks]

Solution

Plugging in the maximizing δ^b into the expression for the supremum we get

$$\begin{aligned}
&e^{-\kappa\delta^b}(w(t,S,q+1,x-(S-\delta^b))-w) \\
&= e^{-\kappa\delta^b}\Big[e^{-\gamma(x+qS+g)} - e^{-\gamma(x+\frac{1}{\gamma}\ln\frac{\kappa+\gamma}{\kappa}+g-g^{+}+qS+g^{+})}\Big] \\
&= e^{-\kappa\delta^b}\Big[e^{-\gamma(x+qS+g)} - e^{-\gamma(x+\frac{1}{\gamma}\ln\frac{\kappa+\gamma}{\kappa}+g+qS)}\Big] \\
&= e^{-\kappa\delta^b}e^{-\gamma(x+qS+g)}\Big[1-\tfrac{\kappa}{\kappa+\gamma}\Big] = -e^{-\kappa\delta^b}w\tfrac{\gamma}{\kappa+\gamma}\,.
\end{aligned}$$

For the supremum over δ^a we similarly get

$$e^{-\kappa\delta^a}(w(t,S,q-1,x+(S+\delta^b))-w) = -e^{-\kappa\delta^a}w\tfrac{\gamma}{\kappa+\gamma}\,.$$

Thus the HJB equation reads

$$(-\gamma\partial_t g)w + \tfrac{1}{2}\sigma^2\gamma^2q^2w - \lambda^b\tfrac{\gamma}{\kappa+\gamma}e^{-\kappa\delta^b}w\mathbf{1}_{q<\bar{q}} - \lambda^a\tfrac{\gamma}{\kappa+\gamma}e^{-\kappa\delta^a}w\mathbf{1}_{q>\underline{q}} = 0\,.$$

Dividing by $-\gamma w$ we thus get

$$\partial_t g - \tfrac{1}{2}\sigma^2\gamma q^2 + \lambda^b\tfrac{1}{\kappa+\gamma}e^{-\kappa\left(\frac{1}{\gamma}\ln\frac{\kappa+\gamma}{\kappa}+g-g^{+}\right)}\mathbf{1}_{q<\bar{q}} + \lambda^a\tfrac{1}{\kappa+\gamma}e^{-\kappa\left(\frac{1}{\gamma}\ln\frac{\kappa+\gamma}{\kappa}+g-g^{-}\right)}\mathbf{1}_{q>\underline{q}} = 0\,.$$

Which, letting $\hat{\lambda}^b := \lambda^b\frac{1}{\kappa+\gamma}(\frac{\kappa}{\kappa+\gamma})^{-\kappa/\gamma}$ and $\hat{\lambda}^a := \lambda^a\frac{1}{\kappa+\gamma}(\frac{\kappa}{\kappa+\gamma})^{-\kappa/\gamma}$, is

$$\partial_t g - \tfrac{1}{2}\sigma^2\gamma q^2 + \hat{\lambda}^b e^{-\kappa(g-g^{+})}\mathbf{1}_{q<\bar{q}} + \hat{\lambda}^a e^{-\kappa(g-g^{-})}\mathbf{1}_{q>\underline{q}} = 0\,.$$

(d) Show that after a transformation g can be expressed in terms of a solution of a linear ODE. [5 marks]

Solution

The final transformation to get a linear ODE is $g = \frac{1}{\kappa}\ln z$ for some z leads to

$$\tfrac{1}{\kappa}\tfrac{\partial_t z}{z} - \tfrac{1}{2}\sigma^2\gamma q^2 + \hat{\lambda}^b\tfrac{z^{+}}{z}\mathbf{1}_{q<\bar{q}} + \hat{\lambda}^a\tfrac{z^{-}}{z}\mathbf{1}_{q>\underline{q}} = 0\,.$$

Writing $Z(t)_q = z(t,q) \in \mathbb{Z}\cap[\underline{q},\bar{q}]$ we thus get

$$Z'(t)_q - \tfrac{1}{2}\sigma^2\kappa\gamma q^2 Z(t)_q + \hat{\lambda}^b Z(t)_{q+1} + \hat{\lambda}^a Z(t)_{q-1} = 0\,,\ \ t\in[0,T]\,,\ \ q\in\mathbb{Z}\cap(\underline{q},\bar{q})$$

which is the desired linear ODE.