PART B Please complete in BLOCK CAPITALS

Student Number ________________ Name ________________________________ Signature ________________________________

This information will not be visible to the marker.

Stochastic Control and Dynamic Asset Allocation
MATH11150

Thursday 2 May 2024
13:00–15:00

Number of questions: 2
Total number of marks: 100

PART A Please complete clearly

Exam Number
as shown on your university card

IMPORTANT PLEASE READ CAREFULLY

Before the examination

1. Put your university card face up on the desk.
2. Complete PART A and PART B above. By completing PART B you are accepting the University Regulations on student conduct in an examination (see back cover).
3. Complete the attendance slip and leave it on the desk.
4. In this examination, candidates are allowed to have three sheets (six sides) of A4 paper with whatever notes they desire written or printed on one or both sides of the paper. Magnifying devices to enlarge the contents of the sheets for viewing are not permitted. No further notes, printed matter or books are allowed.
5. A scientific calculator is permitted in this examination. It must not be a graphic calculator. It must not be able to communicate with any other device.

During the examination

1. Write clearly, in ink, in the space provided after each question. If you need more space then please use the extra pages at the end of the examination script or ask an invigilator for additional paper.
2. Attempt all questions.
3. If you have rough work to do, simply include it within your overall answer – put brackets at the start and end of it if you want to highlight that it is rough work.

At the end of the examination

1. This examination script must not be removed from the examination venue.
2. There are extra pages for working at the end of this examination script. If used, you should clearly label your working with the question to which it relates.
3. Additional paper and graph paper, if used, should be attached to the back of this examination script. Write your examination number on the top of each additional sheet.

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1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $T > 0$ be fixed. For $(t, x) \in [0, T] \times \mathbb{R}$ let

$$f(t,x) = \ln\left(e^{2xe^{T-t}} + e^{-2xe^{T-t}}\right), \quad g(x) = x^2 .$$

Let $A = \{(a^1, a^2) \in \mathbb{R}^2 : a^1 \geq 0, a^2 \geq 0, a^1 + a^2 = 1\}$. Let W be a 1-dimensional Wiener process, let $\mathcal{F}_t = \sigma(W_s; s \leq t)$ and let $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}$. Let $\mathcal{A}$ denote the class of processes $\alpha_t = (\alpha_t^1, \alpha_t^2)$ that are $\mathbf{F}$-adapted and taking values in A. We will call elements of $\mathcal{A}$ admissible controls.

Consider the controlled SDE

$$dX_s^{t,x,\alpha} = (\alpha_s^1 - \alpha_s^2)\, ds + X_s^{t,x,\alpha}\, dW_s , \quad s \in [t, T] , \quad X_t^{t,x,\alpha} = x \in \mathbb{R} .$$

The optimization objective is to *minimize*

$$J(t, x, \alpha) = \mathbb{E}\Big[\int_t^T \Big(f(X_s^{t,x,\alpha}) + \alpha_s^1 \ln \alpha_s^1 + \alpha_s^2 \ln \alpha_s^2\Big)\, ds + g(X_T^{t,x,\alpha})\Big]$$

over admissible controls α.

(a) Let the value function be

$$w(t, x) = \inf_{\alpha} J(t, x, \alpha) .$$

Write down the HJB equation for the control problem. [5 marks]

(b) Show that minimizers of the infimum (candidate optimal controls) are

$$a^1 = \frac{e^{-\partial_x v - 1}}{e^{-\partial_x v-1} + e^{\partial_x v-1}} = \frac{e^{-\partial_x v}}{e^{-\partial_x v} + e^{\partial_x v}} , \quad a^2 = \frac{e^{\partial_x v-1}}{e^{-\partial_x v-1} + e^{\partial_x v-1}} = \frac{e^{\partial_x v}}{e^{-\partial_x v} + e^{\partial_x v}}$$

and that

$$\inf_{a\in A}(a^1 \ln a^1 + a^2 \ln a^2 + (a^1 - a^2)\partial_x v) = -\ln(e^{\partial_x v} + e^{-\partial_x v}) .$$

[15 marks]

(c) Use the ansatz $v(t, x) = \psi(t)x^2$ with $\psi \in C^1([0, T])$, $\psi(T) = 1$, to solve the HJB equation. [10 marks]

(d) Carry out the verification argument (either directly or appealing to a theorem in the course) to show that the solution of the HJB equation is equal to the value function w and that the feedback controls found in (b) are the optimal ones. [20 marks]

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2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Consider the following model for optimal market making (same as in lectures): Fix real constants $\sigma > 0$, $\kappa > 0$, $\lambda^b > 0$, $\lambda^a > 0$ and $T > 0$.

- Let $dS_r^{t,S} = \sigma dW_r$ for $r \in [t,T]$ with initial value $S_t = S \in \mathbb{R}$ given (mid price process).
- Let $M^b = (M_t^b)_{t\geq 0}$ and $M^a = (M_t^a)_{t\geq 0}$ be two Poisson jump processes with intensities λ^b, λ^a respectively counting sell and buy market order arrivals.
- Let $(U_i^b)_{i\in\mathbb{N}}$, $(U_i^a)_{i\in\mathbb{N}}$ be iid r.v.s with uniform distribution on $[0,1]$.
- Let $\mathcal{F}_t = \sigma(W_s : s \leq t) \vee \sigma(M_s^b : s \leq t) \vee \sigma(M_s^a : s \leq t) \vee \sigma(U_i^b : i \leq M_t^b) \vee \sigma(U_i^a : i \leq M_t^a)$. Let $\mathcal{A}$ denote $\mathbb{R}^+ \cup \{+\infty\}$-valued stochastic processes that are square integrable and progressively measurable w.r.t $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. We shall call these admissible controls. Let $\delta^b, \delta^a \in \mathcal{A}$. Let $\delta = (\delta^b, \delta^a)$.
- Let N^{b,δ_t^b} and N^{a,δ_t^a} be stochastic processes satisfying

$$N_t^{b,\delta_t^b} = N_{t-}^{b,\delta_t^b} + (M_t^b - M_{t-}^b)\mathbf{1}_{U^b_{M_t^b} \geq e^{-\kappa\delta_t^b}},$$

$$N_t^{a,\delta_t^a} = N_{t-}^{a,\delta_t^a} + (M_t^a - M_{t-}^a)\mathbf{1}_{U^a_{M_t^a} \geq e^{-\kappa\delta_t^a}}.$$

  In other words these are doubly stochastic Poisson processes with stochastic intensities given by $\lambda^b e^{-\kappa\delta_t^b}$ and $\lambda^a e^{-\kappa\delta_t^a}$ counting when the market makers orders get picked up by incoming market orders.
- Let $Q_r^{t,q,\delta} = q + N_r^{b,\delta^b} - N_r^{a,\delta^a}$ for $r \in [t,T]$ with initial value $Q_t = q \in \{0\} \cup \mathbb{N}$ given.
- Let $dX_r^{t,q,S,x,\delta} = -(S_r - \delta_r^b)dN_r^{b,\delta^b} + (S_r + \delta_r^a)dN_r^{a,\delta^a}$ for $r \in [t,T]$ with initial value $X_t = x \in \mathbb{R}$ given.
- The market maker has inventory lower and upper bound $\underline{q}, \bar{q} \in \mathbb{Z}$.

So far this was exactly the market making model we used in lectures. We shall now change the optimization objective. Let $\gamma > 0$ be a fixed real constant and let $u(x) = -e^{-\gamma x}$ (exponential utility). Our aim is to *maximize*

$$J(t,q,S,x,\delta) = \mathbb{E}_{t,q,S,x,\delta}\Big[u(X_T + S_T Q_T - \alpha Q_T^2)\Big]$$

over $\mathcal{A}$. Here $\mathbb{E}_{t,S,q,x,\delta}[\cdot]$ denotes the conditional expectation given $S_t = S \in \mathbb{R}$, $X_t = x \in \mathbb{R}$, $Q_t = q \in \mathbb{Z} \cap [\underline{q}, \bar{q}]$ and given the process control δ is used.

(a) The value function is $w(t,S,q,x) = \sup_{\delta^b,\delta^a\in\mathcal{A}} J(t,S,q,x,(\delta^b,\delta^a))$. Write down the HJB satisfied by w. [10 marks]

(b) Use the ansatz $w(t,S,q,x) = u(x + Sq - g(t,q)) = -e^{-\gamma(x+qS+g(t,q))}$ for some $g = g(t,q)$ to show that the offsets achieving the supremums in the HJB are

$$\delta^b(t,q) = \tfrac{1}{\gamma}\ln\tfrac{\kappa+\gamma}{\kappa} + g(t,q) - g(t,q+1), \quad \delta^a(t,q) = \tfrac{1}{\gamma}\ln\tfrac{\kappa+\gamma}{\kappa} + g(t,q) - g(t,q-1).$$

You do not need to justify why points satisfying the first order condition are maximizers. [15 marks]

(c) Hence show that g satisfies the nonlinear ODE

$$\partial_t g - \tfrac{1}{2}\sigma^2\gamma q^2 + \hat{\lambda}^b e^{-\kappa(g-g^+)}\mathbf{1}_{q<\bar{q}} + \hat{\lambda}^a e^{-\kappa(g-g^-)}\mathbf{1}_{q>\underline{q}} = 0.$$

for some constants $\hat{\lambda}^b$, $\hat{\lambda}^a$. Write down what these are in terms of the original problem constants. [20 marks]

(d) Show that after a transformation g can be expressed in terms of a solution of a linear ODE. [5 marks]

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Exam Hall Regulations

The following is a copy of a Notice which is displayed in Edinburgh University Examination Halls for the information of students and staff.

The University of Edinburgh Exam Hall Regulations

1. An examination attendance sheet is laid on the desk for each student to complete upon arrival. These are collected by an invigilator after thirty minutes have elapsed from the start of the examination. Students are not normally allowed to enter the examination hall more than thirty minutes after the start of the examination.
2. Students arriving after the start of the examination are required to complete a "Late arrival form" which requires them to sign a statement that they understand that they are not entitled to any additional time. Students are not allowed to leave the examination hall less than thirty minutes after the commencement of the examination or within the last fifteen minutes of the examination.
3. Books, papers, briefcases and cases must be left at the back or sides of the examination room. It is an offence against University discipline for a student to have in their possession in the examination any material relevant to the work being examined unless this has been authorised by the examiners.
4. Students must take their seats within the block of desks allocated to them and must not communicate with other students either by word or sign, nor let their papers be seen by any other student.
5. Students are prohibited from deliberately doing anything that might distract other students. Students wishing to attract the attention of an invigilator shall do so without causing a disturbance. Any student who causes a disturbance in an examination room may be required to leave the room, and shall be reported to the University Secretary.
6. Personal handbags must be placed on the floor at the student's feet; they should be opened only in full view of an invigilator.
7. An announcement will be made to students that they may start the examination. Students must stop writing immediately when the end of the examination is announced.
8. Answers should be written in the script book provided. Rough work, if any, should be completed within the script book and subsequently crossed out. Script books must be left in the examination hall.
9. During an examination, students will be permitted to use only such dictionaries, other reference books, computers, calculators and other electronic technology as have been issued or specifically authorised by the examiners. Such authorisation must be confirmed by the Registry.
10. The use of mobile telephones is not permitted and mobile telephones must be switched off during an examination.
11. It is an offence against University discipline for any student knowingly
    - to make use of unfair means in any University examination
    - to assist a student to make use of such unfair means
    - to do anything prejudicial to the good conduct of the examination, or
    - to impersonate another student or allow another student to impersonate them
12. Students will be required to display their University Card on the desk throughout all written degree examinations and certain other examinations. If a card is not produced, the student will be required to make alternative arrangements to allow their identity to be verified before the examination is marked.
13. Smoking and eating are not allowed inside the examination hall.
14. If an invigilator suspects a student of cheating, they shall impound any prohibited material and shall inform the Examinations Office as soon as possible.

15. Cheating is an extremely serious offence, and any student found by the Discipline Committee to have cheated or attempted to cheat in an examination may be deemed to have failed that examination or the entire diet of examinations, or be subject to such penalty as the Discipline Committee considers appropriate.