Automated Market Makers Designs beyond Constant Functions

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1. Literature on constant function markets (CFMs) & market making (MM)
2. Automated market makers (AMMs) using constant function markets (CFMs)
3. Automated market makers (AMMs) using stochastic control
   i) Recap of Avellanda and Stoikov model for market making
   ii) Arithmetic liquidity pool (ALP) design
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   iv) ALP backtesting performance
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Literature on CFMs & MM

Literature on CFMs

- [Angeris and Chitra, 2020] show that the convexity of the trading function is key in CFMs,
- [Lehar and Parlour, 2021] discuss the competition between CFMs and LOBs,
- [Angeris et al., 2022] study the returns of LPs in simple setups
- [Neuder et al., 2021] and [Cartea et al., 2022a] study strategic liquidity provision in CFMs with concentrated liquidity,
- [Li et al., 2023] study strategic liquidity provision in different types of AMMs,
- [Cartea et al., 2023] derive the predictable losses of LPs in CFMs and in concentrated liquidity AMMs,
- [Milionis et al., 2022] study the arbitrage gains of LTs in CFMs, and [Fukasawa et al., 2023] study the hedging of the impermanent losses of LPs,
- A strand of the literature studies liquidity taking strategies in AMMs; see [Cartea et al., 2022b] and [Jaimungal et al., 2023].
- [Goyal et al., 2023] study an AMM with a dynamic trading function that incorporates the beliefs of LPs about future asset prices,
- [Sabate-Vidales and Šiška, 2022] study variable fees in CPMs, and [Cohen et al., 2023] derive no-arbitrage relationship between fee revenue and the perpetual option premium of CFM LP.

Literature on MM

Liquidity provision in OTC and LOB markets:

- [Ho and Stoll, 1983]
- [Glosten and Milgrom, 1985]
- **[Avellaneda and Stoikov, 2008]**
- extended in many directions [Guéant et al., 2012], [Guéant et al., 2013], [Cartea et al., 2015], [Guéant, 2016].
- [Bergault et al., 2022] design an AMM where LPs set quotes around an exogenous oracle.

In contrast to all the above, we **avoid need for exogenous price input**.

AMMs based on CFMs

CFMs: an overview

A constant function market (CFM) is characterised by

i) The reserves $(x^{(1)}, x^{(2)}) \in \mathbb{R}_+^2$ describing amounts of assets in the pool.

ii) A "trade" function $\Psi : \mathbb{R}_+^2 \to \mathbb{R}$ which determines valid states of the pool after each trade:

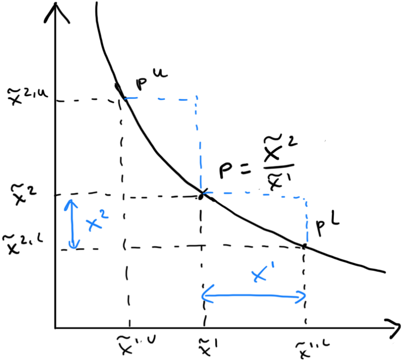
$$\left\{ (x^{(1)}, x^{(2)}) \in \mathbb{R}_+^2 \; : \; \Psi(x^{(1)}, x^{(2)}) = \text{constant} \right\} . \tag{1}$$

iii) A trading fee $(1 - \gamma)$, for $\gamma \in (0, 1]$.

CFMs: an overview

To buy $\Delta x^{(1)}$ of asset $x^{(1)}$:

1. Deposit (i.e. sell) a quantity $\Delta x^{(2)}$ of asset $x^{(2)}$ into the pool s.t.
$$\Psi(x^{(1)} - \Delta x^{(1)}, x^{(2)} + \Delta x^{(2)}) = \Psi(x^{(1)}, x^{(2)}) . \tag{2}$$
2. Pay a fee $(1-\gamma)\Delta x^{(2)}$.
3. Reserves get updated
$$x^{(1)} \leftarrow x^{(1)} - \Delta x^{(1)} \quad \text{and} \quad x^{(2)} \leftarrow x^{(2)} + \Delta x^{(2)} . \tag{3}$$



CFMs: an overview

The relative price of trading $\Delta x^{(1)}$ for $\Delta x^{(2)}$ is defined as

$$\frac{P^{1,CFM}(\Delta x^{(1)})}{P^{2,CFM}(\Delta x^{(2)})} := \frac{\Delta x^{(2)}}{\Delta x^{(1)}} \quad \text{s.t.} \quad \Psi(x^{(1)} - \Delta x^{(1)}, x^{(2)} + \Delta x^{(2)}) = \Psi(x^{(1)}, x^{(2)}) \,.$$

Observe that

$$\begin{aligned} 0 &= \Psi(x^{(1)} - \Delta x^{(1)}, x^{(2)} + \Delta x^{(2)}) - \Psi(x^{(1)}, x^{(2)}) \\ &= -\partial_{x^{(1)}}\Psi(x^{(1)}, x^{(2)})\Delta x^{(1)} + \partial_{x^{(2)}}\Psi(x^{(1)}, x^{(2)})\Delta x^{(2)} + \mathcal{O}((\Delta x^{(1)})^2) + \mathcal{O}((\Delta x^{(2)})^2) \,. \end{aligned}$$

Hence relative “price” is given by

$$\frac{P^{1,CFM}}{P^{2,CFM}} := \lim_{\Delta x^{(1)} \to 0} \frac{P^{1,CFM}(\Delta x^{(1)})}{P^{2,CFM}(\Delta x^{(2)})} = \frac{\partial_{x^{(1)}}\Psi(x^{(1)}, x^{(2)})}{\partial_{x^{(2)}}\Psi(x^{(1)}, x^{(2)})} \,. \tag{4}$$

Assume frictionless external market with $S = (S^{(1)}, S^{(2)})$. No-arbitrage condition in the case of no fees ($\gamma = 1$) implies that

$$\frac{P_t^{1,CFM}}{P_t^{2,CFM}} = \frac{S_t^1}{S_t^2}\,. \tag{5}$$

Example 1 (GMM)

Let the trading function be

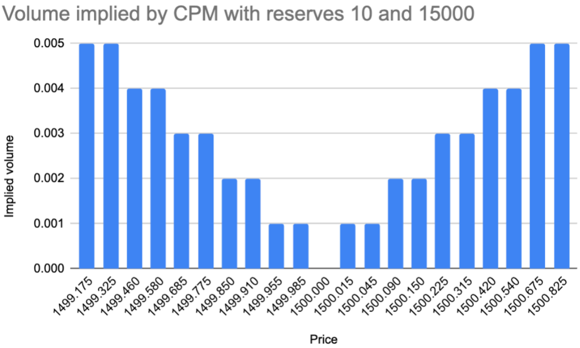
$$\Psi(x^{(1)}, x^{(2)}) = (x^{(1)})^{\theta}(x^{(2)})^{1-\theta} \tag{6}$$

for $\theta \in (0, 1)$. The no arbitrage relationship (5), in GMM is given by

$$\frac{P^{1,CFM}}{P^{2,CFM}} = \frac{\theta x^{(2)}}{(1-\theta)x^{(1)}} = \frac{S^{(1)}}{S^{(2)}} \,. \tag{7}$$

Example 2 (GMM with $\theta = 1/2$ LOB)

- $x^{(1)} = 10$ (e.g. ETH), $x^{(2)} = 15\,000$ (e.g. USDT)
- $$\frac{P^{1,CFM}}{P^{2,CFM}} = \frac{x^{(2)}}{x^{(1)}} = \frac{15\,000}{10} = 1\,500\,.$$
- Fix tick size e.g. $0.015 = 1.5 \cdot 10^{-2}$.



AMMs using stochastic control

Avellanda–Stoikov market making model

- **Mid-price process** $\mathrm{d}S_t = \sigma\,\mathrm{d}W_t$**.**
- MM quotes prices at $S_t + \delta_t^a$ (MM sells) and $S_t - \delta_t^b$ (MM buys); $\delta = (\delta_t)_{t\in[0,T]} = (\delta_t^a, \delta_t^b)_{t\in[0,T]}$ is the strategy.
- N_t^b counts the number of times the MM bought ζ units.
- N_t^a counts the number of times the MM sold ζ units.
- Trade intensity depends on MM quotes:
  - $\lambda_t^b(\delta_t^b)$ is the arrival intensity for N_t^b and
  - $\lambda_t^a(\delta_t^a)$ is the arrival intensity for N_t^a.
  - E.g. $\lambda_t^a(\delta_t^a) = \exp(-\kappa\delta_t^a)$, $\lambda_t^b(\delta_t^b) = \exp(-\kappa\delta_t^b)$, $\kappa > 0$.
- MM has inventory
$$\mathrm{d}y_t = \zeta \mathrm{d}N_t^b - \zeta \mathrm{d}N_t^a$$
- and cash
$$\mathrm{d}x_t = \zeta(S_t + \delta_t^a)\,\mathrm{d}N_t^a + \zeta(S_t - \delta_t^b)\,\mathrm{d}N_t^b$$
- and objective[5]
$$v^\delta(t,x,y,S) = \mathbb{E}_{t,x,y,S}\left[x_T + y_T S_T - \alpha(y_T - \hat{y})^2 - \phi\int_t^T (y_s - \hat{y})^2\,\mathrm{d}s\right].$$

One can write down the HJB, solve, perform verification.

[5] In [Avellaneda and Stoikov, 2008] there is exponential utility.

Price formation

In Avellanda–Stoikov:

- We rely on **some exogenous** price formation process summarized by the mid price $dS_t = \sigma\, dW_t$.
- Prices at which the MM trades i.e. $S_t \pm \delta_t^{b;a}$ have **no impact** on S_t.

In contrast in a CFM-based AMM:

- Price forms as a result of incoming trades e.g.

$$\frac{P^{1,CFM}}{P^{2,CFM}} = \frac{x^{(2)}}{x^{(1)}} = \frac{15\,000}{10} = 1\,500\,.$$

- Can be purely "toxic" flow:

Arithmetic Liquidity Pool (ALP)

Arithmetic Liquidity Pool (ALP): The model

- **Impact functions** $y \mapsto \eta^a(y)$, $y \mapsto \eta^b(y)$ determine the pool's marginal rate response to incoming trades as a function of the LP's position.
- **Reference price process**

$$\mathrm{d}Z_t = -\,\eta^b(y_{t-})\,\mathrm{d}N_t^b + \eta^a(y_{t-})\,\mathrm{d}N_t^a\,. \tag{8}$$

- N_t^b counts the number of times the ALP bought ζ units.
- N_t^a counts the number of times the ALP sold ζ units.
- Trade intensity depends on MM quotes:
  - $\lambda_t^b(\delta_t^b)$ is the arrival intensity for N_t^b and
  - $\lambda_t^a(\delta_t^a)$ is the arrival intensity for N_t^a.
  -

$$\begin{cases} \lambda_t^b\left(\delta_t^b\right) = c^b\, e^{-\kappa\,\delta_t^b}\,\mathbf{1}^b\left(y_{t-}\right)\,, \\ \lambda_t^a\left(\delta_t^a\right) = c^a\, e^{-\kappa\,\delta_t^a}\,\mathbf{1}^a\left(y_{t-}\right)\,, \end{cases} \tag{9}$$

$$\mathbf{1}^b(y) = \mathbf{1}_{\{y+\zeta\le\overline{y}\}} \quad \text{and} \quad \mathbf{1}^a(y) = \mathbf{1}_{\{y-\zeta\ge\underline{y}\}}\,. \tag{10}$$

  - Inventory risk constraint $y_t \in \mathcal{Y} := \{\underline{y},\, \underline{y}+\zeta, \dots, \overline{y}-\zeta,\, \overline{y}\}$.
- MM has inventory

$$\mathrm{d}y_t = \zeta\mathrm{d}N_t^b - \zeta\mathrm{d}N_t^a$$

- and cash

$$\mathrm{d}x_t = -\zeta\,\left(Z_{t-} - \delta_t^b\right)\,\mathrm{d}N_t^b + \zeta\,\left(Z_{t-} + \delta_t^a\right)\mathrm{d}N_t^a\,.$$

ALP: Objective

For $t \in [0, T]$, we define the set $\mathcal{A}_t$ of admissible shifts

$$\mathcal{A}_t = \Big\{ \delta_s = (\delta_s^b, \delta_s^a)_{s\in[t,T]},\ \mathbb{R}^2\text{-valued},\ \mathbb{F}\text{-adapted},$$
$$\text{square-integrable, and bounded from below by } \underline{\delta} \Big\},$$

where $\underline{\delta} \in \mathbb{R}$ is given and write $\mathcal{A} := \mathcal{A}_0$.

The objective is to **maximize** $w^\delta \colon [0, T] \times \mathbb{R} \times \mathcal{Y} \times \mathbb{R} \to \mathbb{R}$, given by

$$w^\delta(t, x, y, z) = \mathbb{E}_{t,x,y,z} \left[x_T + y_T\, Z_T - \alpha\, (y_T - \hat{y})^2 - \phi \int_t^T (y_s - \hat{y})^2 \, \mathrm{d}s \right]$$

over $\delta = \left(\delta^b, \delta^a\right) \in \mathcal{A}$.

ALP: Value function

The value function $w \colon [0, T] \times \mathbb{R} \times \mathcal{Y} \times \mathbb{R} \to \mathbb{R}$ of the LP is

$$w(t, x, y, z) = \sup_{\delta \in \mathcal{A}_t} w^{\delta}(t, x, y, z) \,. \tag{11}$$

Proposition 1

There is $C \in \mathbb{R}$ such that for all $(\delta_s)_{s \in [t,T]} \in \mathcal{A}_t$, the performance criterion of the LP satisfies

$$w^{\delta}(t, x, y, z) \leq C < \infty \,,$$

so the value function w in (11) is well defined.

ALP: HJB

The HJB equation associated with problem (11) is given by

$$
\begin{aligned}
0 &= \partial_t \omega - \phi\,(y-\hat{y})^2 \\
&+ \sup_{\delta^b} \lambda^b(\delta^b)\Big\{\omega(t, x-\zeta\,(z-\delta^b), y+\zeta, z-\eta^b(y)) - \omega\,(t,x,y,z)\Big\} \\
&+ \sup_{\delta^a} \lambda^a(\delta^a)\Big\{\omega\,(t, x+\zeta\,(z+\delta^a)\,, y-\zeta, z+\eta^a(y)) - \omega\,(t,x,y,z)\Big\}
\end{aligned}
\tag{12}
$$

on $[0, T) \times \mathbb{R} \times \mathcal{Y} \times \mathbb{R}$ with the terminal condition

$$
\omega(T,x,y,z) = x + y\,z - \alpha\,(y-\hat{y})^2\,. \tag{13}
$$

ALP: HJB solution

Proposition 2 (Candidate closed-form solution: ALP)

Let $\underline{N} = \underline{y}/\zeta$, $\overline{N} = \overline{y}/\zeta$, *and* $N = \overline{N} - \underline{N} + 1$. *Define the matrix* $\mathbf{K} \in \mathbb{R}^{N\times N}$ *by*

$$\mathbf{K}_{mn} = \begin{cases} c^a\, e^{-1}\, e^{\kappa\,(m-1)\,\eta^a(m\,\zeta)} & \text{if } n = m-1 \text{ and } m > \underline{N}\,, \\ -\kappa\,\phi\,(m\,\zeta - \hat{y})^2\,/\zeta & \text{if } n = m\,, \\ c^b\, e^{-1}\, e^{-\kappa\,(m+1)\,\eta^b(m\,\zeta)} & \text{if } n = m+1 \text{ and } m < \overline{N}\,, \end{cases}$$

for $m, n \in \{\underline{N}, \underline{N}+1, \dots, \overline{N}\}$. *Let* $\mathbf{U} \in C^1([0,T], \mathbb{R}^N)$ *be*

$$\mathbf{U}(t) = \exp\left(\mathbf{K}\, t\right)\, \mathbf{U}(0)\,, \quad t \in [0,T]\,,$$

where

$$\mathbf{U}(0)_m = e^{-\alpha\ \frac{\kappa}{\zeta}\,(\zeta\, m - \hat{y})^2}\,, \quad m \in [\underline{N}, \bar{N}] \cap \mathbb{Z}\,.$$

For $m \in [\underline{N}, \bar{N}] \cap \mathbb{Z}$ *let*

$$u(t, m\,\zeta) = \mathbf{U}(T-t)_m\,, \tag{14}$$

and define

$$\theta(t,y) = \frac{\zeta}{\kappa}\, \log u(t,y)\,. \tag{15}$$

Then, the function $\omega\colon [0,T] \times \mathbb{R} \times \mathcal{Y} \times \mathbb{R} \to \mathbb{R}$ *given by*

$$\omega(t,x,y,z) = x + y\,z + \theta(t,y) \tag{16}$$

solves the HJB equation (12).

ALP: Verification and strategy

Theorem 3 (Verification: ALP)

Let ω be defined as in Proposition 2. Then the function ω in (16) *satisfies that for all* $(t, x, y, z) \in [0, T] \times \mathbf{R} \times \mathcal{Y} \times \mathbf{R}$ *and* $\delta = (\delta_s)_{s\in[t,T]} \in \mathcal{A}_t$,

$$w^{\delta}(t, x, y, z) \leq \omega\,(t, x, y, z)\ . \tag{17}$$

Moreover, equality is obtained in (17) *with the admissible optimal Markovian control* $(\delta^{\star}_s)_{s\in[t,T]} = \left(\delta^{b\star}_s, \delta^{a\star}_s\right)_{s\in[t,T]} \in \mathcal{A}_t$ *given by the feedback formulae*

$$\delta^{b\star}(t, y_{t-}) = \frac{1}{\kappa} - \frac{\theta\,(t, y_{t-} + \zeta) - \theta\,(t, y_{t-})}{\zeta} - \frac{(y_{t-} + \zeta)\,\eta^{b}(y_{t-})}{\zeta}\,, \tag{18}$$

$$\delta^{a\star}(t, y_{t-}) = \frac{1}{\kappa} - \frac{\theta\,(t, y_{t-} - \zeta) - \theta\,(t, y_{t-})}{\zeta} + \frac{(y_{t-} - \zeta)\,\eta^{a}(y_{t-})}{\zeta}\,, \tag{19}$$

where θ is in (15). *In particular,* $\omega = w$ *on* $[0, T] \times \mathbb{R} \times \mathcal{Y} \times \mathbb{R}$.

ALP: impact functions and arbitrage

ALP: impact functions and arbitrage

Poorly chosen impact functions may lead to arbitrage against the pool:

Definition 4 (Arbitrage)

Arbitrage is any (roundtrip) sequence of trades $\{\epsilon_1, \ldots, \epsilon_{\mathfrak{m}}\}$, where $\epsilon_k = \pm 1$ (buy/sell) for $k \in \{1, \ldots, \mathfrak{m}\}$ and $\sum_{k=1}^{\mathfrak{m}} \epsilon_k = 0$, such that the terminal cash of the liquidity taker (LT) is positive.

ALP: Roundtrip arb

P&L of the LT after the roundtrip trade as

$$\begin{aligned} \text{case (i)} \qquad & \text{P\&L} = \zeta\left(\eta^a(y_0) - \mathfrak{d}^a(y_0, Z_0) - \mathfrak{d}^b(y_0 - \zeta, Z + \eta^a(y_0)\right), \\ \text{case (ii)} \qquad & \text{P\&L} = \zeta\left(\eta^b(y_0) - \mathfrak{d}^b(y_0, Z_0) - \mathfrak{d}^a(y_0 + \zeta, Z_0 - \eta^b(y_0)\right). \end{aligned} \tag{20}$$

Clearly, the profits in (20) are non-positive if the bid quote

$$\underbrace{Z_0 + \eta^a(y_0) - \mathfrak{d}^b(y_0 - \zeta, Z_0 + \eta^a(y_0))}_{\text{the bid quote after a buy trade}} \leq \underbrace{Z_0 + \mathfrak{d}^a(y_0, Z_0)}_{\text{ask quote before the trade}} \tag{21}$$

because it guarantees

$$\eta^a(y_0) \leq \mathfrak{d}^b(y_0 - \zeta, Z_0 + \eta^a(y_0)) + \mathfrak{d}^a(y_0, Z_0)\,,$$

and conversely for a sell trade.

ALP: Marginal rate manipulation arb

The condition (21) doesn't guarantee that

$$Z + \eta^a(y) - \eta^b(y - \zeta) = Z$$

and that

$$Z - \eta^b(y) + \eta^a(y + \zeta) = Z$$

at the end of the arbitrage sequence of length $\mathfrak{m} = 2$.

Condition for Z to take values on a grid only: let $\mathfrak{y}_1 = \underline{y}$, $\mathfrak{y}_2 = \underline{y} + \zeta$, ..., and $\mathfrak{y}_N = \overline{y}$.

Proposition 3

The marginal rate Z takes only the ordered finitely many values $\mathcal{Z} = \{\mathfrak{z}_1, \dots, \mathfrak{z}_N\}$, with the property that $Z_0 \in \mathcal{Z}$ and for $i \in \{1, \dots, N-1\}$

$$\mathfrak{z}_{i+1} - \eta^b(\mathfrak{y}_{N-i}) = \mathfrak{z}_i \qquad \text{and} \qquad \mathfrak{z}_i + \eta^a(\mathfrak{y}_{N-i} + \zeta) = \mathfrak{z}_{i+1}\,, \tag{22}$$

if and only if $\eta^a(\,\cdot\,)$ and $\eta^b(\,\cdot\,)$ are such that

$$\eta^b(\mathfrak{y}_i) = \eta^a(\mathfrak{y}_i + \zeta)\,, \tag{23}$$

for $i \in \{1, \dots, N-1\}$.

ALP: no-arbitrage impact functions

Theorem 5

Let $\eta^a(\cdot)$ and $\eta^b(\cdot)$ satisfy (23) for $i \in \{1, \ldots, N-1\}$. For any liquidity provision strategy of the form $\left(\delta^b, \delta^a\right) = \left(\mathfrak{d}^b(y, Z), \mathfrak{d}^a(y, Z)\right)$, if for all $i \in \{1, \ldots, N-1\}$,

$$\eta^a(\mathfrak{y}_{i+1}) \leq \mathfrak{d}^a\left(\mathfrak{y}_{i+1}, \mathfrak{z}_{N-i}\right) + \mathfrak{d}^b\left(\mathfrak{y}_{i+1} - \zeta, \mathfrak{z}_{N-i} + \eta^a\left(\mathfrak{y}_{i+1}\right)\right) \tag{24}$$

$$\text{and} \quad \eta^b(\mathfrak{y}_i) \leq \mathfrak{d}^b\left(\mathfrak{y}_i, \mathfrak{z}_{N-i+1}\right) + \mathfrak{d}^a\left(\mathfrak{y}_i + \zeta, \mathfrak{z}_{N-i+1} - \eta^b\left(\mathfrak{y}_i\right)\right), \tag{25}$$

or equivalently

$$\eta^a(\mathfrak{y}_{i+1}) \leq \mathfrak{d}^a\left(\mathfrak{y}_{i+1}, \mathfrak{z}_{N-i}\right) + \mathfrak{d}^b\left(\mathfrak{y}_i, \mathfrak{z}_{N-i+1}\right) \quad \text{and}$$
$$\eta^b(\mathfrak{y}_i) \leq \mathfrak{d}^b\left(\mathfrak{y}_i, \mathfrak{z}_{N-i+1}\right) + \mathfrak{d}^a\left(\mathfrak{y}_{i+1}, \mathfrak{z}_{N-i}\right),$$

then there is no roundtrip sequence of trades that a liquidity taker can execute to arbitrage the ALP. For the liquidity provision strategy in (18), the condition simplifies to

$$\eta^a(\mathfrak{y}_i) \leq \frac{1}{\kappa}, \quad \text{and} \quad \eta^b(\mathfrak{y}_i) \leq \frac{1}{\kappa}, \tag{26}$$

for all $i \in \{1, \ldots, N\}$.

ALP: examples of no-arbitrage impact functions

1. $\eta^a(y) = \eta^b(y) = \eta \leq \frac{1}{\kappa}$ with $\eta \in \mathbb{R}^+$ a constant.
2. Fix $\underline{y} \geq \zeta$ and recall $y \in \mathcal{Y} = \{\underline{y}, \ldots, \overline{y}\}$. Fix $L < \frac{1}{\kappa}$ and let

$$\eta^b(y) = \frac{\zeta}{\frac{1}{2}\, y + \zeta}\, L \quad \text{and} \quad \eta^a(y) = \frac{\zeta}{\frac{1}{2}\, y - \zeta}\, L\,, \tag{27}$$

3. Impact functions built using a CFM trade function.

CFM as a special case of ALPs if LT trade size is ζ

CFMs are special case of ALPs if LT trade size is ζ: marginal price

Recall CFM is given by a convex differentiable trade function Ψ and the two pool balances satisfies:

$$\Psi(x_t, y_t) = \text{constant}\,.$$

Due to convexity of Ψ we know that $\exists$ a *level function* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$x_t = \varphi(y_t)\,.$$

So

$$\Psi(\varphi(y), y) = \text{constant}$$

so taking derivative in y we get

$$\partial_x \Psi(\varphi(y), y)\varphi'(y) + \partial_y \Psi(\varphi(y), y) = 0$$

and so, recalling (4)

$$\varphi'(y) = -\frac{\partial_y \Psi(\varphi(y), y)}{\partial_x \Psi(\varphi(y), y)\varphi'(y)} = -\text{marginal price in CFM}\,.$$

CFMs are special case of ALPs if LT trade size is ζ: CFM dynamics

The dynamics of the amounts of asset X and asset Y and the marginal rate Z^{CFM} in the CFM pool are given by

$$
\begin{aligned}
\mathrm{d}y_t^{\text{CFM}} &= \zeta\,\mathrm{d}N_t^b - \zeta\,\mathrm{d}N_t^a\,,\\
\mathrm{d}x_t^{\text{CFM}} &= \left(\varphi\left(y_{t-}^{\text{CFM}}+\zeta\right) - \varphi\left(y_{t-}^{\text{CFM}}\right) + \mathfrak{f}\,\zeta\left(-\varphi'(y_{t-})\right)\right)\mathrm{d}N_t^b\\
&\quad + \left(\varphi\left(y_{t-}^{\text{CFM}}-\zeta\right) - \varphi\left(y_{t-}^{\text{CFM}}\right) + \mathfrak{f}\,\zeta\left(-\varphi'(y_{t-})\right)\right)\mathrm{d}N_t^a\,.\\
\mathrm{d}Z_t^{\text{CFM}} &= \left(-\varphi'\left(y_{t-}^{\text{CFM}}+\zeta\right) + \varphi'\left(y_{t-}^{\text{CFM}}\right)\right)\mathrm{d}N_t^b\\
&\quad + \left(-\varphi'\left(y_{t-}^{\text{CFM}}-\zeta\right) + \varphi'\left(y_{t-}^{\text{CFM}}\right)\right)\mathrm{d}N_t^a\,,
\end{aligned}
$$

where $\mathfrak{f} \in [0,1)$ is a given CFM fee.

CFMs are special case of ALPs if LT trade size is ζ: impact fns and strategy

Theorem 6

Let $\varphi(\cdot)$ be the level function of a CFM. Assume the LP in the ALP chooses the impact functions

$$\eta^a(y) = \varphi'(y) - \varphi'(y-\zeta)\,, \qquad \eta^b(y) = -\varphi'(y) + \varphi'(y+\zeta)\,, \tag{28}$$

and chooses the offsets

$$\begin{aligned} \delta_t^a &= \frac{\varphi(y_{t-} - \zeta) - \varphi(y_{t-})}{\zeta} + \varphi'(y_{t-}) + \underbrace{\mathfrak{f}\,\zeta\,(-\varphi'(y_{t-}))}_{\text{if we include fees}}\,, \\ \delta_t^b &= \frac{\varphi(y_{t-} + \zeta) - \varphi(y_{t-})}{\zeta} - \varphi'(y_{t-}) + \underbrace{\mathfrak{f}\,\zeta\,(-\varphi'(y_{t-}))}_{\text{if we include fees}}\,. \end{aligned} \tag{29}$$

Then, the marginal rate dynamics, inventory dynamics, and execution costs in the ALP are the same as those in the CFM with level function $\varphi(\cdot)$.

CFMs are special case of ALPs if LT trade size is ζ: CFMs are suboptimal

Proposition 4

Let $\varphi(\cdot)$ be the level function of a CFM. Consider a CFM LP whose performance criterion is

$$J^{CFM} = \mathbb{E}\left[x_T^{CFM} + y_T^{CFM}\, Z_T^{CFM} - \alpha\,(y_T^{CFM} - \hat{y})^2 - \phi \int_0^T (y_s^{CFM} - \hat{y})^2\, \mathrm{d}s\right], \quad (30)$$

with $J^{CFM} \in \mathbb{R}$. Consider an ALP LP with impact functions given by (28). Let $\delta_t^{CFM} = \left(\delta_t^{a,CFM}, \delta_t^{b,CFM}\right)$ be given by (29). Consider the performance criterion $J : \mathcal{A}_0 \to \mathbb{R}$

$$J(\delta) = \mathbb{E}\left[x_T + y_T\, Z_T - \alpha\,(y_T - \hat{y})^2 - \phi \int_0^T (y_s - \hat{y})^2\, \mathrm{d}s\right]. \quad (31)$$

Then,

$$J^{CFM} = J\left(\delta^{CFM}\right) \qquad \text{and} \qquad J^{CFM} \leq J\left(\delta^{\star}\right), \quad (32)$$

where $\delta^{\star} = (\delta^{a,\star}, \delta^{b,\star})$ is given by (18).

Backtesting ALP

ALP evaluation: quotes

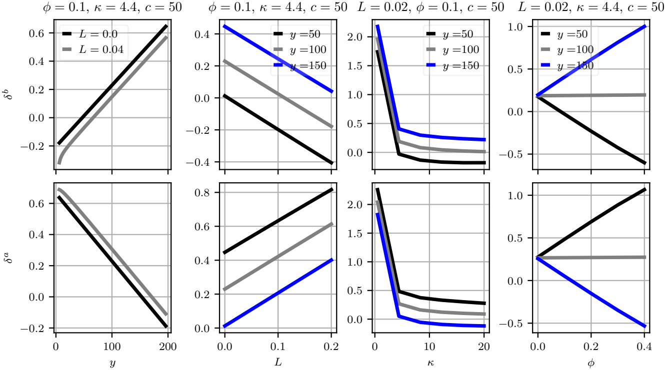
Fix $\underline{y} \geq \zeta$ and recall $y \in \mathcal{Y} = \{\underline{y}, \ldots, \overline{y}\}$. Fix $L < \frac{1}{\kappa}$ and let

$$\eta^b(y) = \frac{\zeta}{\frac{1}{2}\, y + \zeta}\, L \quad \text{and} \quad \eta^a(y) = \frac{\zeta}{\frac{1}{2}\, y - \zeta}\, L\,.$$

Then

$$\delta^{b\star}(t,y) = \frac{1}{\kappa} - \frac{\theta\,(t, y+\zeta) - \theta\,(t,y)}{\zeta} - L\,, \tag{33}$$

$$\delta^{a\star}(t,y) = \frac{1}{\kappa} - \frac{\theta\,(t, y-\zeta) - \theta\,(t,y)}{\zeta} + L\,. \tag{34}$$



ALP evaluation: Binance and Uniswap v3 data

	ETH/USDC 0.05%		Binance
	LT	LP	
Number of transactions	216,739	42,022	12,341,854
Average transaction size	\$ 109,037	\$ 2,765,499	\$ 1,735
Gross USD volume	$\approx$ \$ 185.57 $\times 10^9$	$\approx$ \$ 116.2 $\times 10^9$	$\approx$ \$ 21.42 $\times 10^9$
Average trading frequency	18.27 seconds	12.3 minutes	2.56 seconds
Median LP holding time	86 minutes		n.a.
Average pool depth	19,788,327 $\sqrt{\text{ETH} \cdot \text{USDC}}$		n.a.

Table: LT and LP activity statistics in the Uniswap v3 pool ETH/USDC 0.05% and in Binance between 5 May 2021 (Uniswap inception) and 30 April 2022; see [Drissi, 2023] for more details.

ALP evaluation: base case

ALP for ETH/USDC between 1 August 2021 09:00 and 09:30. The LP's strategy parameters are $\zeta = 1$ ETH, $\kappa = 1\,\mathrm{ETH}^{-1}$, $c = 100$, $L = 0.3$ ETH, $\underline{y} = -500$ ETH, $\overline{y} = 500$ ETH. Moreover, we set $T = 30$ minutes, $\phi = \alpha = 10^{-4}\,\mathrm{USDC} \cdot \mathrm{ETH}^{-2}$, and $y_0 = \hat{y} = 100$.

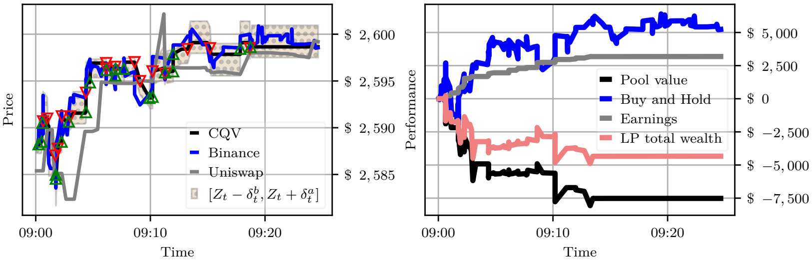


Figure: LP wealth when arbitrageurs trade in the ALP and Binance. **Left**: Exchange rates from ALP, Binance, and Uniswap v3. **Right**: *Pool value* is computed as $x_t + y_t\, Z_t$, *Buy and Hold* is computed as the wealth from holding the LP's inventory outside the ALP, i.e., $y_t\, Z_t$, *Earnings* are the revenue from the quotes, and *LP total wealth* is the total LP's wealth.

ALP evaluation: higher inventory penalty

As before but $\phi = \alpha = 10^{-4}\,\mathrm{USDC} \cdot \mathrm{ETH}^{-2}$.

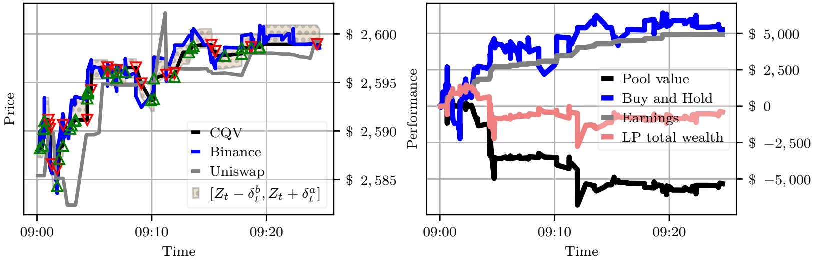


Figure: LP wealth when only an arbitrageur interacts in the ALP.

ALP evaluation: toxic flow impact

Scenario I: toxic flow only.

Scenario II: 1/2 volume is toxic, 1/2 volume is noise traders.

	Average	Standard deviation
ALP (scenario I)	−0.004%	0.719%
ALP (scenario II)	0.717%	2.584%
Buy and Hold	0.001%	0.741%
Uniswap v3	−1.485%	7.812%

Table: Average and standard deviation of 30-minutes performance of LPs in the ALP for both simulation scenarios, LPs in Uniswap v3 pool ETH/USDC 0.05% ., and buy-and-hold.

Discussion and References

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