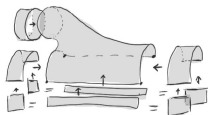


Relative Topological Quantum Field Theories and the Cobordism Hypothesis

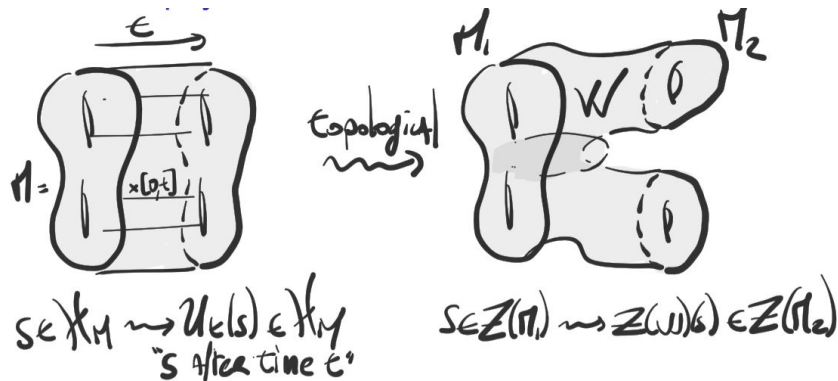
Benjamin Haïoun

Université Toulouse 3 Paul Sabatier, France
University of Edinburgh (sometimes)

February 8, 2023



Topological Quantum Field Theories



Definition (Atiyah, Segal):

A **TQFT** is a symmetric monoidal functor $\mathcal{Z} : Cob_n \rightarrow Vect$.

A **fully extended** (or local) **TQFT** is a symmetric monoidal n -functor

$$\mathcal{Z} : Bord_n \rightarrow nVect.$$

Examples

1. The trivial theory $Triv$:
$$\left\{ \begin{array}{lll} Bord_n & \rightarrow & \mathcal{C} \\ \text{any object} & \mapsto & \mathbf{1} \\ \text{any morphism} & \mapsto & Id_1 \end{array} \right. .$$

2. Witten–Reshetikhin–Turaev theories:

Input: semisimple modular tensor category $\mathcal{V}$.

$$\begin{array}{ll} \tilde{Cob}_3 & \rightarrow Vect \\ \Sigma & \mapsto k\langle \text{links in a handlebody} \rangle / \sim \\ M & \mapsto \text{some 3-manifold invariant} \end{array}$$

3. Non-semisimple variants $\tilde{Cob}_3^{adm} \rightarrow Vect$ (partially defined).

Conclusion:

Constructing TQFTs is hard!

Results and conjectures

Theorem (Brochier–Jordan–Safronov–Snyder):

Using the Cobordism Hypothesis, one gets a (framed) 4-TQFT $\mathcal{Z}_{\mathcal{V}} : \mathit{Bord}_4^{\text{fr}} \rightarrow 4\mathit{Vect}$ for any modular tensor category $\mathcal{V}$.

Theorem (H.):

Using the Cobordism Hypothesis, one gets a (framed) 3-TQFT relative to $\mathcal{Z}_{\mathcal{V}}$, $\mathcal{R}_{\mathcal{V}} : \mathit{Bord}_3^{\text{fr}} \rightarrow 4\mathit{Vect}^{\rightarrow}$ for semisimple modular tensor categories $\mathcal{V}$.
+ partially defined one in the non-semisimple case.

Conjecture (H–J–S, Freed–Teleman, Walker)

One can reconstruct the Witten–Reshetikhin–Turaev anomalous TQFTs $\mathit{WRT}_{\mathcal{V}} : \tilde{\mathit{Cob}}_3 \rightarrow \mathit{Vect}$ from the pair $(\mathcal{Z}_{\mathcal{V}}, \mathcal{R}_{\mathcal{V}})$.

Conjecture (H–J–S):

Also recovers the non-semisimple variants.

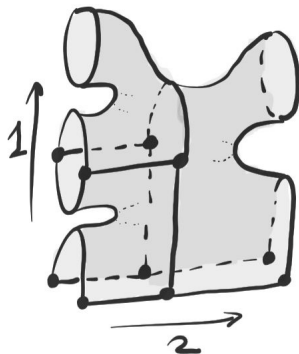
The source category $Bord_n$

objects: $\{\emptyset, \bullet, \bullet\bullet, \dots\}$

1-morphisms: $\{ \text{I I}, \text{X}, \dots \}$
($\bullet\bullet \rightarrow \bullet\bullet$)

2-morphisms: $\{ \text{I I} \rightarrow \text{X} \}$
($\text{X} \rightarrow \text{I I}$)

composition(s):



Theorem (Lurie, Calaque–Scheimbauer):

We can build this!

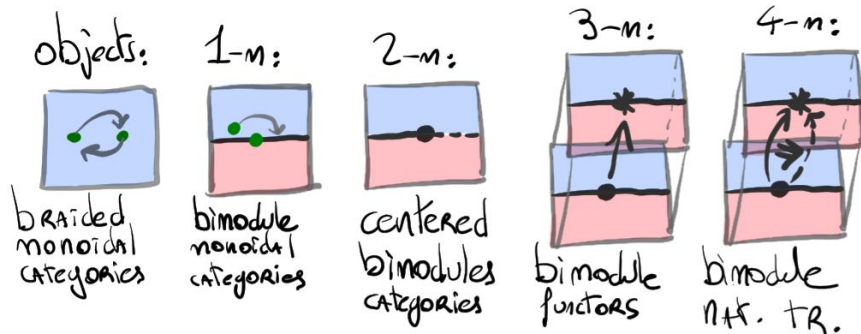
The target category $n\text{Vect}$

Definitions of 2Vect :

Idea 1: $Pr = 2\text{-category of presentable linear categories and cocontinuous functors.}$

Idea 2: $Alg_1(\text{Vect}) = 2\text{-category of algebras and bimodules.}$

$4\text{Vect} \stackrel{\text{today}}{=} Alg_2(Pr) :$



Theorem (Haugseeng + Johnson-Freyd-Scheimbauer):

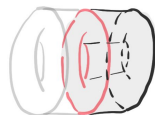
We can build this!

Relative TQFTs

Idea of definition (Freed-Teleman):

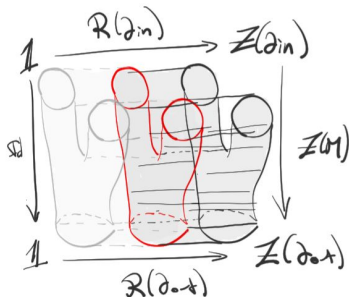
A **relative theory** to $\mathcal{Z}$ is a transformation $\mathcal{R} : \text{Triv} \rightarrow \mathcal{Z}$.

$$\text{---} \bullet \text{---} : \emptyset \rightarrow pt \quad \mapsto \quad \mathcal{R}(\text{---} \bullet \text{---}) : \mathbf{1} \rightarrow \mathcal{Z}(pt)$$



$$\mapsto \mathcal{R}(\text{---} \bullet \text{---} \times \Sigma) : \text{Triv}(\Sigma) \rightarrow \mathcal{Z}(\Sigma)$$

$$M : \partial_{in} \rightarrow \partial_{out} \quad \mathcal{R}(\text{---} \bullet \text{---} \times -) \quad \mapsto$$



All $n - 1$ -dimensional!

Arrow category

Definition (Johnson-Freyd–Scheimbauer):

$$\mathcal{C}^{\rightarrow} = \left\{ \begin{array}{ll} \text{objects} & : (s_f \in \mathcal{C}, t_f \in \mathcal{C}, f^{\#} : s_f \rightarrow t_f) \\ \\ \text{1-morphisms} & : h = (s_h, t_h, h^{\#}), \quad \begin{array}{ccc} s_{f_1} & \xrightarrow{f_1^{\#}} & t_{f_1} \\ s_h \downarrow & \nearrow h^{\#} & \downarrow t_h \\ s_{f_2} & \xrightarrow{f_2^{\#}} & t_{f_2} \end{array} \\ \\ \text{2-morphisms} & : \quad \begin{array}{ccc} s_{f_1} & \xrightarrow{f_1^{\#}} & t_{f_1} \\ \scriptstyle \left(\begin{array}{c} \scriptstyle \xrightarrow{\scriptstyle \alpha} \end{array} \right) \scriptstyle \downarrow \scriptstyle \left(\begin{array}{c} \scriptstyle \xrightarrow{\scriptstyle \alpha} \end{array} \right) \\ s_{f_2} & \xrightarrow{f_2^{\#}} & t_{f_2} \end{array} \quad \begin{array}{c} \scriptstyle \xrightarrow{\scriptstyle \alpha^{\#}} \\ \scriptstyle \xrightarrow{\scriptstyle \alpha^{\#}} \end{array} \quad \begin{array}{c} \scriptstyle \xrightarrow{\scriptstyle \alpha} \\ \scriptstyle \xrightarrow{\scriptstyle \alpha} \end{array} \end{array} \\ \\ \text{n-1-morphisms} & : \quad \dots \end{array} \right.$$

- symmetric monoidal
- source and target functors $s, t : \mathcal{C}^{\rightarrow} \rightarrow \mathcal{C}$

Defintion (J-S):

Let $\mathcal{Z} : \text{Bord}_{n-1} \rightarrow \mathcal{C}$ be a 'categorified' TQFT.

A **TQFT relative to $\mathcal{Z}$** is a s.m. functor $\mathcal{R} : \text{Bord}_{n-1} \rightarrow \mathcal{C}^\rightarrow$ such that $s \circ \mathcal{R} = \text{Triv}$ and $t \circ \mathcal{R} = \mathcal{Z}$.

$$\mathcal{R}(\Sigma) : \text{Triv}(\Sigma) \rightarrow \mathcal{Z}(\Sigma) \text{ and } \begin{array}{ccc} \text{Triv}(\partial_{in}) & \xrightarrow{\mathcal{R}(\partial_{in})} & \mathcal{Z}(\partial_{in}) \\ \text{Triv}(M) \downarrow & \nearrow \mathcal{R}(M) & \downarrow \mathcal{Z}(M) \\ \text{Triv}(\partial_{out}) & \xrightarrow{\mathcal{R}(\partial_{out})} & \mathcal{Z}(\partial_{out}) \end{array} .$$

Example (categorical number = 1):

$\mathcal{Z}_1, \mathcal{Z}_2 : \mathcal{B} \rightarrow \mathcal{C}$, $\mathcal{R} : \mathcal{B} \rightarrow \mathcal{C}^\rightarrow$, $s \circ \mathcal{R} = \mathcal{Z}_1$ and $t \circ \mathcal{R} = \mathcal{Z}_2$ means:

$$\mathcal{R}(X \text{ object}) : \mathcal{Z}_1(X) \rightarrow \mathcal{Z}_2(X), \text{ and } \begin{array}{ccc} \mathcal{Z}_1(X) & \xrightarrow{\mathcal{R}(X)} & \mathcal{Z}_2(X) \\ \mathcal{Z}_1(f) \downarrow & \circlearrowleft & \downarrow \mathcal{Z}_2(f) \\ \mathcal{Z}_1(Y) & \xrightarrow{\mathcal{R}(Y)} & \mathcal{Z}_2(Y) \end{array} .$$

Anomalous TQFTs

Construction (Bulk+Relative=Anomalous):

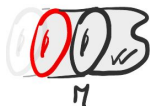
$$\text{Bulk } n\text{-TQFT } \mathcal{Z} : \text{Bord}_n \rightarrow \mathcal{C} \rightsquigarrow \mathcal{Z}^{\rightarrow 1} : \begin{cases} \text{Bord}_n^{\rightarrow \emptyset} & \rightarrow & \mathcal{C}^{\rightarrow 1} \\ M \xrightarrow{W} \emptyset & \mapsto & \mathcal{Z}(M) \xrightarrow{\mathcal{Z}(W)} \mathbf{1} \end{cases}$$

+ $\mathcal{R} : \text{Bord}_{n-1} \rightarrow \mathcal{C}^{1 \rightarrow}$ relative to $\mathcal{Z}$

$$\rightsquigarrow \mathcal{A} : \begin{array}{ccc} \text{Bord}_n^{\rightarrow \emptyset} & \xrightarrow{\mathcal{Z}^{\rightarrow 1}} & \mathcal{C}^{\rightarrow 1} \\ & \searrow s & \downarrow \mathcal{R} \\ & \text{Bord}_{n-1} & \rightarrow \mathcal{C}^{1 \rightarrow} \end{array} \begin{array}{c} \\ \\ \end{array} \xrightarrow{\text{compose}} \text{End}_{\mathcal{C}}(\mathbf{1})$$

In our example (conjectural WRT):

$$\mathcal{A}(M \xrightarrow{W} \emptyset) : k \xrightarrow{\mathcal{R}(M)} \mathcal{Z}(M) \xrightarrow{\mathcal{Z}(W)} k \quad \text{is a scalar.}$$



$$\mathcal{A}(\Sigma \xrightarrow{H} \emptyset) : \text{Vect} \xrightarrow{\mathcal{R}(\Sigma)} \mathcal{Z}(\Sigma) \xrightarrow{\mathcal{Z}(H)} \text{Vect} \quad \text{is vector space.}$$

How to construct TQFTs easily

Cobordism Hypothesis (Baez–Dolan, Lurie, Grady–Pavlov, Ayala–Francis, Schommer-Pries):

TQFTs are classified by their value at the point, which has to be a fully dualizable object:

$$ev_{pt} : Fun^{\otimes}(Bord_n^{fr}, \mathcal{C}) \xrightarrow{\sim} (\mathcal{C}^{fd})^{\sim}$$

is an equivalence of ∞ -groupoids.

Idea ($n=1$):

The point $\bullet^+$ has a dual in $Bord_n$, $\bullet^-$:

$$\begin{aligned} ev_{pt} &= \text{cap} : \bullet^+ \otimes \bullet^- \rightarrow \emptyset \\ coev_{pt} &= \text{cup} : \emptyset \rightarrow \bullet^- \otimes \bullet^+ \end{aligned} \quad \text{s.t.} \quad \text{cap}^+ = \text{cap}^-, \quad \text{cup}^+ = \text{cup}^-$$

And s.m. functors preserve dualizability data

Rigidity

Definition:

A **dual** of $X \in \mathcal{C}^\otimes$ is $(X^*, \text{ev}_X : X \otimes X^* \rightarrow \mathbf{1}_{\mathcal{C}}, \text{coev}_X : \mathbf{1} \rightarrow X^* \otimes X)$ s.t.

$$\smile = (\text{ev}_X \otimes \text{Id}_X) \circ (\text{Id}_X \otimes \text{coev}_X) \simeq \text{Id}_X = | \quad \text{and} \quad \frown \simeq |.$$

Proposition:

Duals are unique up to essentially unique isomorphism.

Definition:

An **adjoint** of a k -morphism $F : X \rightarrow Y$ is

$$(F^R : Y \rightarrow X, \text{Rco}(F) : F \circ F^R \Rightarrow \text{Id}_Y, \text{Ru}(f) : \text{Id}_X \Rightarrow F^R \circ F) \text{ s.t. } \smile \simeq | \text{ and } \frown \simeq |.$$

Proposition:

Adjoints are unique up to essentially unique isomorphism.

Higher dualizability

Observation:

For $0 \leq k < n$, every k -morphism $M: \partial_{in} \rightarrow \partial_{out}$ in $Bord_n$ has adjoint $(\overline{M}, M \times I / \partial_{in} \times I, M \times I / \partial_{out} \times I)$.

$$\eta = ev_{pt} = \cap \rightsquigarrow (\cup, \text{cylinder}, \text{box})$$

$$\text{snake: } \text{box} \cong \text{cylinder}$$

$$\eta = coev_{pt} = \cup \rightsquigarrow (\cap, \text{box}, \text{cylinder})$$

Definition:

A s.m. category $\mathcal{C}$ **has duals** if for $0 \leq k < n$, all k -morphisms have adjoints.
An object $X \in \mathcal{C}$ is **fully dualizable** if X lies in a sub- n -category with duals.

Proof of the Cobordism Hypothesis

Injectivity ($\mathcal{Z}$ is determined by its value on the point):

Higher dualizability data of the point generates $Bord_n$

+ s.m. functors preserve higher dualizability data:

$$[pt \mapsto X] \Rightarrow [\cap \mapsto ev_X \text{ and } \cup \mapsto coev_X]$$

$$\Rightarrow [\text{disk with hole} \mapsto Ru(coev_X) \text{ and } \text{pair of pants} \mapsto Rco(coev_X)]$$

And so on...

Surjectivity (a fully dualizable object induces a TQFT):

Set $\cap \mapsto ev_X$, $\cup \mapsto coev_X$, $\text{disk with hole} \mapsto Ru(coev_X)$ and so on...

Check that every relation in $Bord_n^{fr}$ holds in general for any fully dualizable object.

$\cap$ and $\cup$ biadjoints $\Rightarrow coev_X$ and ev_X biadjoints.

$$\text{pair of pants with cap} = \text{pair of pants} = \text{pair of pants with cup} \Rightarrow Rco(coev_X) \sim \sim Ru(ev_X).$$

Recap

Theorem (B-J-S-S):

$\mathcal{V}$ modular tensor category $\Rightarrow \mathcal{V}$ is fully dualizable in $Alg_2(Pr)$.

Theorem (H.):

Unit inclusion $\eta : Vect \rightarrow \mathcal{V} \rightsquigarrow \mathcal{A}_\eta \in Alg_2(Pr)^{\rightarrow}$

$\mathcal{A}_\eta$ fully dualizable $\Leftrightarrow \mathcal{V} \simeq \text{Free}(\text{rigid braided monoidal category})$.

Non-semisimple case: 2-dualizable and partially 3-dualizable.

Conclusion:

$$\begin{aligned} \mathcal{V} &\overset{C-H}{\rightsquigarrow} \mathcal{Z}_\mathcal{V} : Bord_4^{fr} \rightarrow Alg_2(Pr) \\ &\quad + \\ \eta &\overset{C-H}{\rightsquigarrow} \mathcal{R}_\mathcal{V} : Bord_3^{fr} \rightarrow Alg_2(Pr)^{\rightarrow} \\ &\quad = \\ &\quad \mathcal{A}_\mathcal{V} : Bord_4^{fr, \rightarrow \emptyset} \rightarrow Alg_1(Pr) \overset{?}{\simeq} WRT_\mathcal{V} \end{aligned}$$

Non-semisimple case: $\mathcal{R}_\mathcal{V}$ partially defined, $\mathcal{A}_\mathcal{V} \overset{?}{\simeq}$ non-semisimple variants

Any questions?

Thank you!