A 2-dimensional bifunctor theorem

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(Joint work with Peter Faul and José Siqueira)

29 October 2020

The (classical) bifunctor theorem

Theorem

Suppose we have functors $L_C\colon \mathcal{B} \to \mathcal{D}$ and $M_B\colon \mathcal{C} \to \mathcal{D}$ for each $B \in \mathcal{B}$ and $C \in \mathcal{C}$ and that $L_C(B) = M_B(C)$.

There is a bifunctor $P\colon \mathcal{B} \times \mathcal{C} \to \mathcal{D}$ such that $L_C = P(-, C)$ and $M_B = P(B, -)$ if and only if for $f\colon B_1 \to B_2$ and $g\colon C_1 \to C_2$ we have

$$L_{C_2}(f)M_{B_1}(g) = M_{B_2}(g)L_{C_1}(f).$$

In this case, $P(B, C) = L_C(B) = M_B(C)$ and $P(f, g) = M_{B_2}(g)L_{C_1}(f)$.

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Consider pseudofunctors $L_C \colon \mathcal{B} \to \mathcal{D}$ *and* $M_B \colon \mathcal{C} \to \mathcal{D}$ *for each* $B \in \mathcal{B}$ *and* $C \in \mathcal{C}$ *and that* $L_C(B) = M_B(C)$. *Suppose we have an invertible 2-cell* $\sigma_{f,g} \colon L_{C_2}(f)M_{B_1}(g) \to M_{B_2}(g)L_{C_1}(f)$ *for each* $f \colon B_1 \to B_2$ *and* $g \colon C_1 \to C_2$ *and suppose these satisfy certain coherence conditions.*

Then there is a pseudo-bifunctor $P \colon \mathcal{B} \times \mathcal{C} \to \mathcal{D}$ *and canonical isomorphisms* $M_B \cong P(B, -)$ *and* $L_C \cong P(-, C)$.

Furthermore, every pseudofunctor $P' \colon \mathcal{B} \times \mathcal{C} \to \mathcal{D}$ *is pseudonaturally isomorphic to one of this form (for an essentially unique choice of* L*'s,* M*'s and* σ*'s).*

In fact, there is also a similar construction for lax functors.

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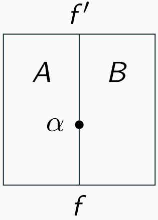
Crash course on string diagrams: basics

In string diagrams for 2-categories:

- 2-morphisms are represented by vertices,
- 1-morphisms are represented by wires,
- objects are represented by regions.

Our diagrams are read *from bottom to top* and *from left to right*.

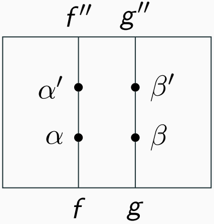
For instance, if $f, f' \colon A \to B$ then $\alpha \colon f \to f'$ would be represented as follows.



Crash course on string diagrams: composition

Placing 2-morphisms on top of each other denotes vertical composition and putting them side by side denotes horizontal composition.

For example, let $f, f', f'' \colon A \to B$, $\alpha \colon f \to f'$ and $\alpha' \colon f' \to f''$. And let $g, g', g'' \colon B \to C$, $\beta \colon g \to g'$ and $\beta' \colon g' \to g''$.
The following string diagram depicts the composite $(\alpha'\alpha) * (\beta'\beta)$.



Unit 1-morphisms and 2-morphisms are omitted. Perturbing the dots up and down leaves the meaning unchanged due to the interchange law.

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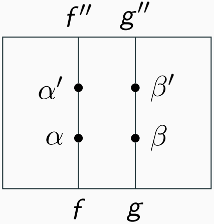

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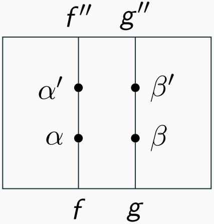

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An example: monads

Recall that (T, μ, η) is a monad in a 2-category if T is an endomorphism on an object X and $\mu\colon T^2 \to T$ and $\eta\colon \mathrm{Id}_X \to T$ satisfy associativity and unit axioms.

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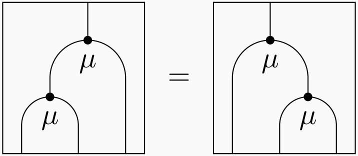

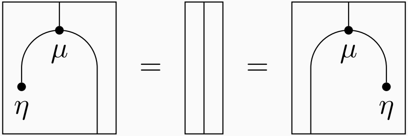

An example: distributive laws

Given two monads (T, μ^T, η^T) and (S, μ^S, η^S) on the same object we might ask for a monad structure on the composite TS. We will use a map $\sigma\colon ST \to TS$ (called a distributive law) satisfying four axioms.

It is easiest to understand these using string diagrams, where we write the 2-morphism σ as a crossing of the S wire over the T wire.

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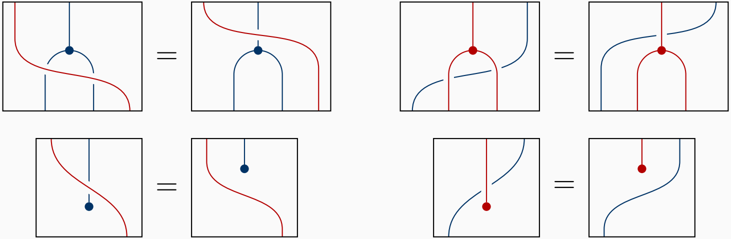

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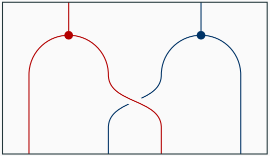
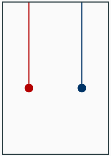
Below are the multiplication and unit of the composite monad TS.





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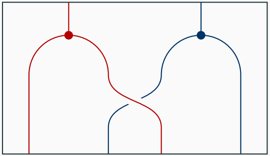

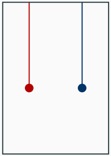

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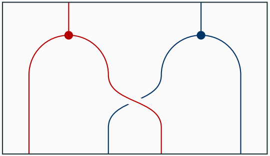

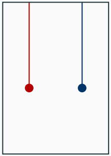

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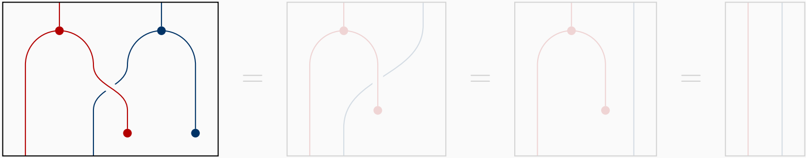

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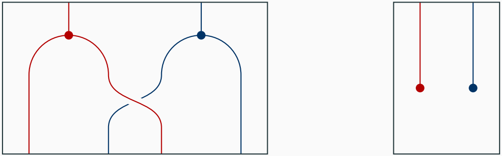

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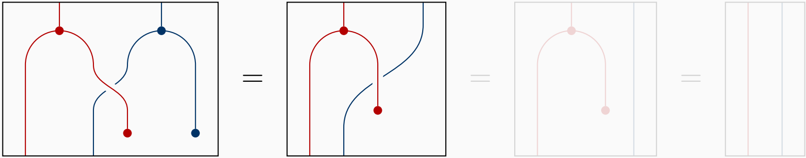

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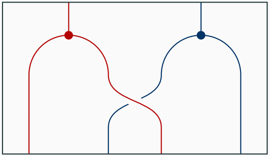

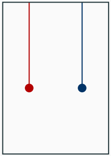

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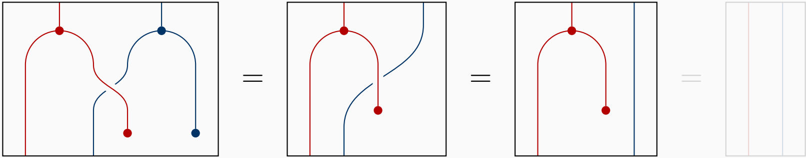

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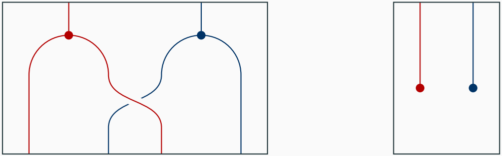

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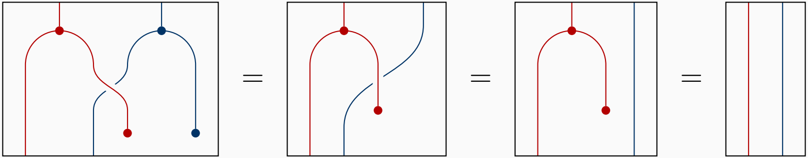

Lax functors

Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. A lax functor $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ consists of

- a function $\mathcal{F}$ sending objects in $\mathcal{C}$ to objects in $\mathcal{D}$,
- a functor $\mathcal{F}\colon \operatorname{Hom}(C_1, C_2) \to \operatorname{Hom}(\mathcal{F}(C_1), \mathcal{F}(C_2))$ for each pair of objects $C_1, C_2 \in \mathcal{C}$,
- a 2-morphism $\gamma_{g,f}: \mathcal{F}(g) \circ \mathcal{F}(f) \to \mathcal{F}(g \circ f)$ called the compositor for each each pair of composable 1-morphisms (f, g) in $\mathcal{C}$,
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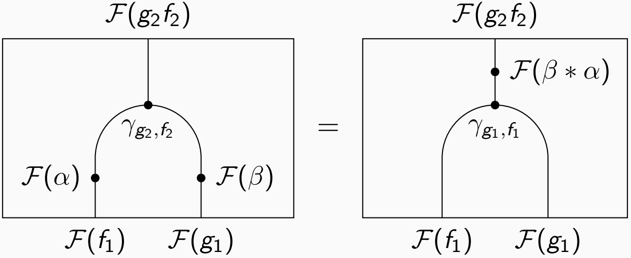

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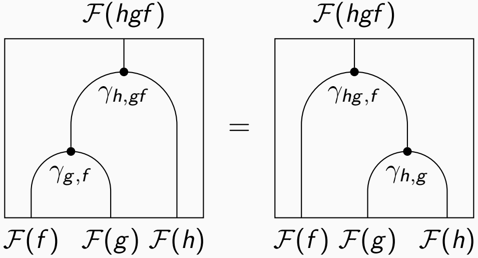

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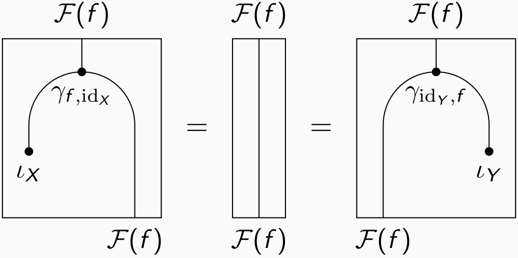

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Constructing lax bifunctors from families

Suppose we have lax functors $L_C \colon \mathcal{B} \to \mathcal{D}$ and $M_B \colon \mathcal{C} \to \mathcal{D}$ for each $B \in \mathcal{B}$ and $C \in \mathcal{C}$ and that $L_C(B) = M_B(C)$. Furthermore, suppose we have a 2-morphism $\sigma_{f,g} \colon L_{C_2}(f)M_{B_1}(g) \to M_{B_2}(g)L_{C_1}(f)$ for each $f \colon B_1 \to B_2$ and $g \colon C_1 \to C_2$.

Under certain assumptions on this data, we will construct a lax bifunctor $P \colon \mathcal{B} \times \mathcal{C} \to \mathcal{D}$ as follows.

- $P(B, C) := L_C(B) = M_B(C)$ on objects,
- $P(f, g) := M_{B_2}(g)L_{C_1}(f)$ on 1-morphisms,
- $P(\alpha, \beta) := M_{B_2}(\beta) * L_{C_1}(\alpha)$ on 2-morphisms,
- the unitor $\iota_{B,C}$ is $\iota^B_C * \iota^C_B$,
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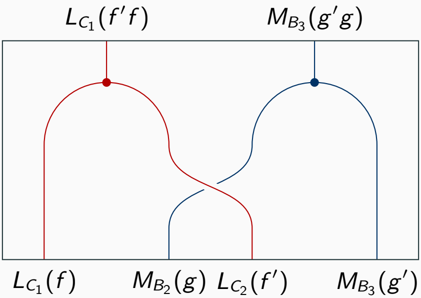
Constructing lax bifunctors from families: the compositor

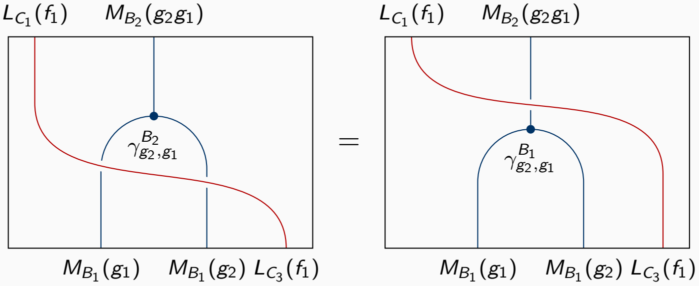
The compositor can be better expressed using string diagrams.

To exhibit the structure of the diagram we use

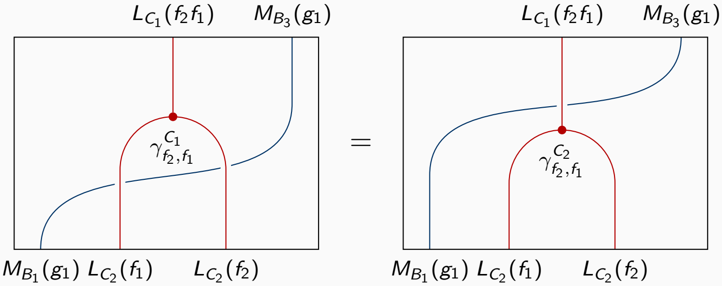
- red wires for morphisms an 'L' lax functor has been applied to,
- blue wires for morphisms an 'M' lax functor has been applied to,
- crossings of a red wire over a blue wire for 'σ' 2-morphisms.



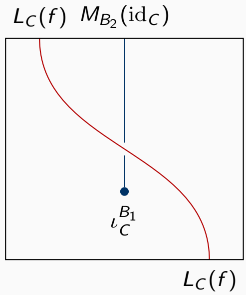
The coherence conditions



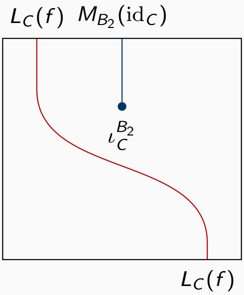
(D1)



(D2)


$L_C(f)$
$M_{B_2}(\mathrm{id}_C)$
$\iota_C^{B_1}$
$L_C(f)$

$=$


$L_C(f)$
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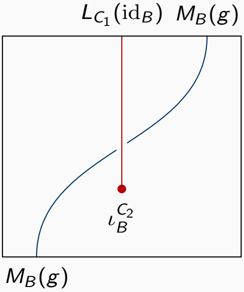
(D3)


$L_{C_1}(\mathrm{id}_B)$
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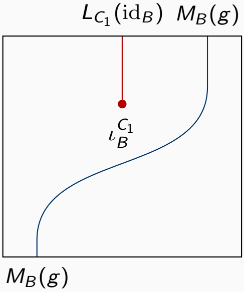
$L_{C_1}(\mathrm{id}_B)$
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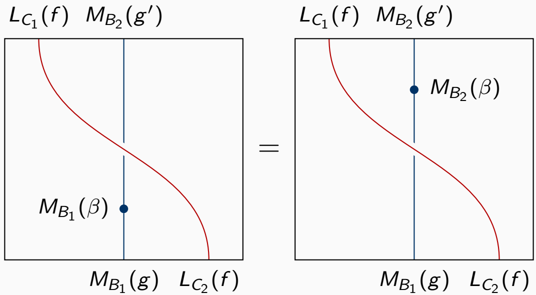
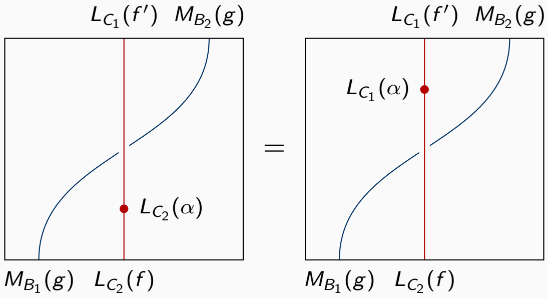
(D4)

The coherence conditions



(D5)



(D6)

Pseudo-bifunctor theorem

Theorem

Consider pseudofunctors $L_C \colon \mathcal{B} \to \mathcal{D}$ *and* $M_B \colon \mathcal{C} \to \mathcal{D}$ *for each* $B \in \mathcal{B}$ *and* $C \in \mathcal{C}$ *and that* $L_C(B) = M_B(C)$*. Suppose we have an invertible 2-cell* $\sigma_{f,g} \colon L_{C_2}(f)M_{B_1}(g) \to M_{B_2}(g)L_{C_1}(f)$ *for each* $f \colon B_1 \to B_2$ *and* $g \colon C_1 \to C_2$ *and these satisfy the aforementioned coherence conditions.*

Then there is a pseudo-bifunctor $P \colon \mathcal{B} \times \mathcal{C} \to \mathcal{D}$ *constructed as above and canonical isomorphisms* $M_B \cong P(B, -)$ *and* $L_C \cong P(-, C)$*.*

Furthermore, every pseudofunctor $P' \colon \mathcal{B} \times \mathcal{C} \to \mathcal{D}$ *is pseudonaturally isomorphic to one of this form (for an essentially unique choice of* L*'s,* M*'s and* σ*'s).*

Unit conditions are superfluous for pseudofunctor result

In fact, in the context of the pseudo-bifunctor theorem we only need to assume conditions D1, D2, D5, D6.

This is because when σ is invertible, D3 and D4 follow from D1 and D2.

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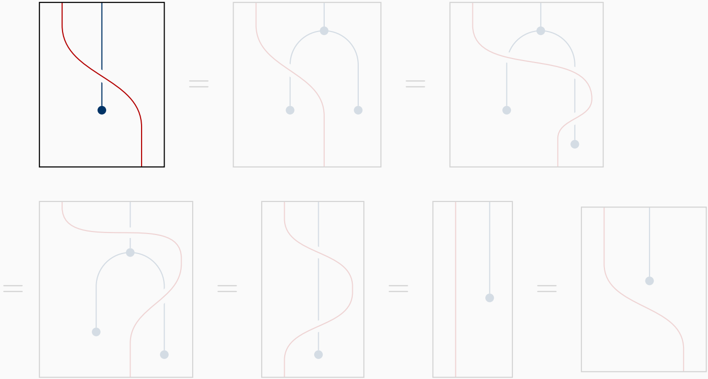

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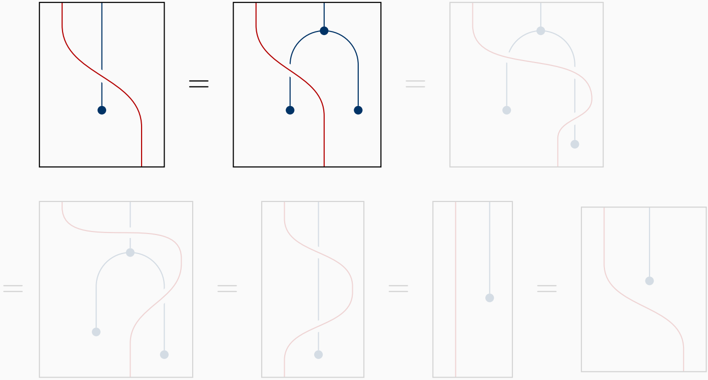

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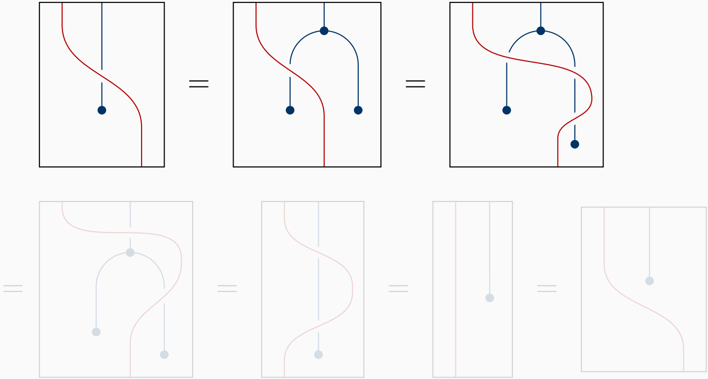

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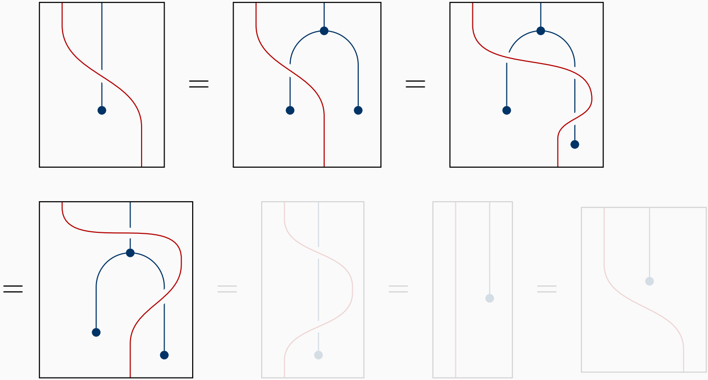

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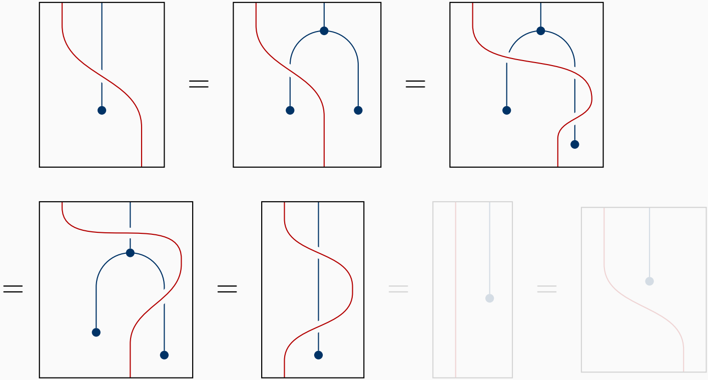

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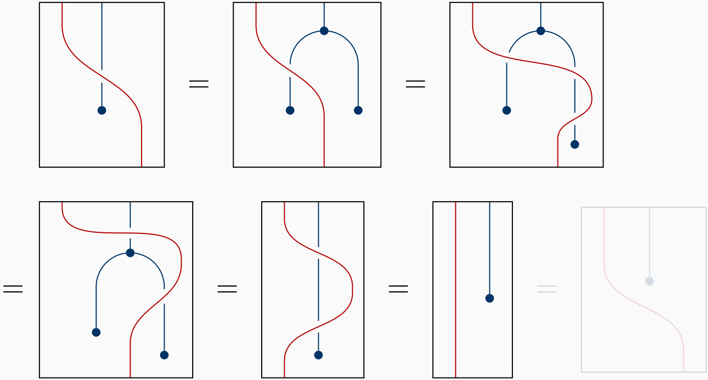

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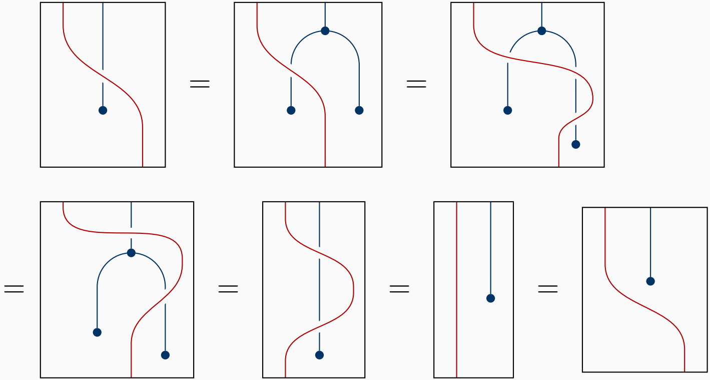

Application: distributive laws of monads

Monads in $\mathcal{D}$ are lax functors from the terminal 2-category into $\mathcal{D}$.

In this case our data reduces to two monads $S = L(\mathrm{id})$ and $T = M(\mathrm{id})$ on the same object $X = L(*) = M(*)$ and a single 2-cell $\sigma \colon ST \to TS$.

The conditions D1–D4 are precisely the axioms of a distributive law. Conditions D5 and D6 are automatic, since the only 2-cell in the terminal 2-category is the identity.

Now since $1 \times 1 \cong 1$, the construction of the bifunctor yields another monad, which is precisely the monad structure on TS obtained from the distributive law.

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A monoidal category $\mathcal{A}$ can be thought of as a 1-object bicategory.

Consider the pseudo-bifunctor result for $\mathcal{B} = \mathcal{C} = \mathcal{D} = \mathcal{A}$ and $L = M = \mathrm{Id}_{\mathcal{A}}$. Then $\sigma_{X,Y}$ is an isomorphism from $X \otimes Y$ to $Y \otimes X$.

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The resulting bifunctor is then a strong monoidal functor from $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ whose underlying functor is $\otimes$.

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Functoriality and further results

It is also possible to relate families of (op)lax transformations between the families of lax functors to (op)lax transformations between the resulting lax bifunctors, and similarly for modifications.

We obtain a 2-category of 'generalised distributive laws of lax functors' $\mathrm{Dist}(\mathcal{B}, \mathcal{C}, \mathcal{D})$ and a 2-functor $\mathrm{Dist}(\mathcal{B}, \mathcal{C}, \mathcal{D}) \to \mathrm{Lax}(\mathcal{B} \times \mathcal{C}, \mathcal{D})$. This restricts to an equivalence between the distributive laws of pseudofunctors and $\mathrm{Hom}(\mathcal{B} \times \mathcal{C}, \mathcal{D})$.

Moreover, we have an equivalence $\mathrm{Dist}(\mathcal{B}, \mathcal{C}, \mathcal{D}) \cong \mathrm{Lax}(\mathcal{B}, \mathrm{Lax}(\mathcal{C}, \mathcal{D}))$. This generalises the result that distributive laws of monads in $\mathcal{D}$ are monads in the 2-category of monads in $\mathcal{D}$.

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Thanks for listening!

More details can be found in our paper on the arXiv: `https://arxiv.org/abs/2010.07926`.