Lax algebras and topology

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Motivation

Definition

A concrete category $(\mathcal{C}, U : \mathcal{C} \to \textbf{Set})$ is called **algebraic** if U is monadic.

Examples:

- **Grp**, **CH** are algebraic.
- **Top** is not algebraic: the monad induced by the adjunction

$$\textbf{Set} \underset{U}{\overset{D}{\rightleftarrows}} \textbf{Top}$$

  is the identity monad.

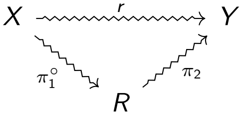
Question: Can we find a weaker notion of monads and algebras, such that **Top** can be expressed as a category of algebras?

Overview

1. Lax extensions of monads
   1. Barr extension
   2. Canonical extension
   3. Extensions to V-**Rel**
2. Lax algebras

Barr extension

Let (T, μ, η) be a monad in **Set**. We can factorize every relation $r : X \rightsquigarrow Y$ as



Define for every set X and every relation r:

$$T_B X := TX \quad \text{and} \quad T_B(r) := T\pi_2 \circ (T\pi_1)^\circ.$$

This defines a lax functor $T_B : \mathbf{Rel} \to \mathbf{Rel}$ such that the following diagram commute:

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{T} & \mathbf{Set} \\ \downarrow & & \downarrow \\ \mathbf{Rel} & \xrightarrow{T_B} & \mathbf{Rel} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{Set}^{\mathrm{op}} & \xrightarrow{T^{\mathrm{op}}} & \mathbf{Set}^{\mathrm{op}} \\ \downarrow & & \downarrow \\ \mathbf{Rel} & \xrightarrow{T_B} & \mathbf{Rel} \end{array}$$

Barr extension

Natural transformations $\eta : 1_{\textbf{Set}} \to T$ and $\mu : TT \to T$, become oplax natural transformations $\eta : 1_{\textbf{Rel}} \to T_B$ and $\mu : T_B T_B \to T_B$, i.e. for all relations $r : X \rightsquigarrow Y$ we have

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & T_B X \\ r \downarrow & \leq & \downarrow T_B r \\ Y & \xrightarrow{\eta_Y} & T_B Y \end{array} \quad \text{and} \quad \begin{array}{ccc} T_B T_B X & \xrightarrow{\mu_X} & T_B X \\ T_B T_B r \downarrow & \leq & \downarrow T_B r \\ T_B T_B Y & \xrightarrow{\mu_Y} & T_B Y \end{array}$$

Examples:

- Idenity monad: $(1_{\textbf{Set}})_B = 1_{\textbf{Rel}}$
- Powerset monad (P, μ, η): for a relation $R \subseteq X \times Y$, we find

$$A(P_B R)B \Leftrightarrow \forall a \in A\ \exists b \in B : aRb \quad \text{and} \quad \forall b \in B\ \exists a \in A : aRb$$

- Ultrafilter monad (U, μ, η): for a relation $R \subseteq X \times Y$, we find

$$u_1(U_B R)u_2 \Leftrightarrow \forall A \in u_1\ \forall B \in u_2\ \exists a \in A\ \exists b \in B : aRb$$

Flat lax extension of a monad

Definition

A **flat lax extension** of a monad (T, μ, η) on **Set** is a lax functor $\hat{T} : \mathbf{Rel} \to \mathbf{Rel}$ such that:

- $$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{T} & \mathbf{Set} \\ \downarrow & & \downarrow \\ \mathbf{Rel} & \xrightarrow{\hat{T}} & \mathbf{Rel} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{Set}^{\mathrm{op}} & \xrightarrow{T^{\mathrm{op}}} & \mathbf{Set}^{\mathrm{op}} \\ \downarrow & & \downarrow \\ \mathbf{Rel} & \xrightarrow{\hat{T}} & \mathbf{Rel} \end{array}$$

  commute.
- The natural transformations η and μ, become oplax natural transformations $\eta : 1_{\mathbf{Rel}} \to \hat{T}$ and $\mu : \hat{T}\hat{T} \to \hat{T}$.

Example: Barr extension

Remark: For all the monads in our examples, T_B is in fact a 2-functor.

Lax extension of a monad

Definition

A **lax extension** of a monad (T, μ, η) on **Set** is a lax functor $\hat{T} : \mathbf{Rel} \to \mathbf{Rel}$ such that:

- $$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{T} & \mathbf{Set} \\ \downarrow & & \downarrow \\ \mathbf{Rel} & \xrightarrow{\hat{T}} & \mathbf{Rel} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{Set}^{\mathrm{op}} & \xrightarrow{T^{\mathrm{op}}} & \mathbf{Set}^{\mathrm{op}} \\ \downarrow & & \downarrow \\ \mathbf{Rel} & \xrightarrow{\hat{T}} & \mathbf{Rel} \end{array}$$

  commute laxly, i.e.

  $$\hat{T}X = TX; \quad Tf \leq \hat{T}f \quad \text{and} \quad (Tf)^\circ \leq \hat{T}f^\circ$$

- The natural transformations η and μ, become oplax natural transformations $\eta : 1_{\mathbf{Rel}} \to \hat{T}$ and $\mu : \hat{T}\hat{T} \to \hat{T}$.

Remark: Every lax extension induces a dual lax extension via $\check{T}(r) := (\hat{T}(r^\circ))^\circ$.

Canonical extension

Let (T, μ, η) be a monad in **Set** and suppose that T preserves inverse images.
For a subset A of a set X, we can identify TA as a subset of TX.
For a relation $R \subseteq X \times Y$ and a subset $A \subseteq X$ define

$$r[A] := \{y \in Y \mid \exists a \in A : aRy\}$$

We define $p(T_{\text{can}}R)q :\Leftrightarrow \forall A \subseteq X : (p \in TA \Rightarrow q \in Tr[A])$.

Proposition ([4])

If (T, μ, η) is a taut monad, i.e.

- T preserves inverse images,
- $\eta_X(x) \in TA \Leftrightarrow x \in A$ and $\mu_X(\chi) \in TA \Leftrightarrow \chi \in TTA$ for all $A \subseteq X$.

Then T_{can} is a lax extensions of (T, μ, η).

We call this extension the *canonical extension* and its dual extension the *op-canonical extension*.

Canonical extension

Examples: The powerset monad, identity monad and (ultra)filter monad are all taut.

- Identity monad: $(1_{\mathbf{Set}})_{\text{can}} = (1_{\mathbf{Set}})_{\text{op-can}} = (1_{\mathbf{Set}})_B = 1_{\mathbf{Rel}}$
- Powerset monad: for a relation $R \subseteq X \times Y$, we find

$$A(P_{\text{can}}R)B \Leftrightarrow \forall b \in B\ \exists a \in A : aRb$$

and

$$A(P_{\text{op-can}}R)B \Leftrightarrow \forall a \in A\ \exists b \in B : aRb.$$

- Filter monad: for a relation $R \subseteq X \times Y$, we find

$$f_1(F_{\text{can}}R)f_2 \Leftrightarrow \forall A \in f_1\ \exists B \in f_2\ \forall b \in B\ \exists a \in A : aRb$$

and

$$f_1(F_{\text{can}}R)f_2 \Leftrightarrow \forall B \in f_2\ \exists A \in f_1\ \forall a \in A\ \exists b \in B : aRb.$$

- Ultrafilter monad: $U_{\text{can}} = U_{\text{op-can}} = U_B$

V-**Rel**

- Let $V = (L, \otimes, k)$ be a unital quantale, i.e. a complete lattice L with binary suprema preserving operation $\otimes$ and neutral element k.
  Examples: $2 = (\{0,1\}, \wedge, 1)$ and $P_+ = ([0,\infty]^{\mathrm{op}}, +, 0)$
- The 2-category V-**Rel**:
  - *Objects*: Sets
  - *Morphisms*: $X \rightsquigarrow Y$ is a map $X \times Y \to V$ (a V-relation). Composition of $r_1 : X \rightsquigarrow Y$ with $r_2 : Y \rightsquigarrow Z$ is given by
    $$r_2 \circ r_1(x, y) := \bigvee_{y \in Y} r_1(x, y) \otimes r_2(y, z).$$
  - *2-morphisms*: The order on V, induces an order on V-relations.
- We consider **Set** as a subcategory of V-**Rel**.
- For a V-relation $r : X \rightsquigarrow Y$, we denote r° for the opposite V-relation $Y \rightsquigarrow X$.

Lax extension of a monad to V-**Rel**

Definition

A **lax extension to** V-**Rel** of a monad (T, μ, η) on **Set** is a lax functor $\hat{T} : V\text{-}\mathbf{Rel} \to V\text{-}\mathbf{Rel}$ such that:

- $$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{T} & \mathbf{Set} \\ \downarrow & & \downarrow \\ V\text{-}\mathbf{Rel} & \xrightarrow{\hat{T}} & V\text{-}\mathbf{Rel} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{Set}^{\mathrm{op}} & \xrightarrow{T^{\mathrm{op}}} & \mathbf{Set}^{\mathrm{op}} \\ \downarrow & & \downarrow \\ V\text{-}\mathbf{Rel} & \xrightarrow{\hat{T}} & V\text{-}\mathbf{Rel} \end{array}$$

  commute laxly, i.e.

  $$\hat{T}X = TX; \quad Tf \leq \hat{T}f \quad \text{and} \quad (Tf)^\circ \leq \hat{T}f^\circ$$

- The natural transformations η and μ, become oplax natural transformations $\eta : 1_{V\text{-}\mathbf{Rel}} \to \hat{T}$ and $\mu : \hat{T}\hat{T} \to \hat{T}$.

Remark: Every lax extension induces a dual lax extension via $\check{T}(r) := (\hat{T}(r^\circ))^\circ$.

Extension to V-**Rel**

A V-relation $r : X \rightsquigarrow Y$ induces collection of relations

$$(x r_v y :\Leftrightarrow v \leq r(x,y))_{v \in V}$$

For a lax extension $\hat{T} : \mathbf{Rel} \to \mathbf{Rel}$ of a monad (T, μ, η), we can define

$$\hat{T}^V r(p,q) := \bigvee \{ v \in V \mid p(\hat{T} r_v) q \}$$

Proposition ([2])

Let $\hat{T} : \mathbf{Rel} \to \mathbf{Rel}$ be a lax extension of a monad (T, μ, η). If V is completely distributive and $k = 1$, then $\hat{T}^V : V\text{-}\mathbf{Rel} \to V\text{-}\mathbf{Rel}$ is a lax extension to V-**Rel** of the monad (T, μ, η).

Remark: The extension $\hat{T}^2$ is equal to $\hat{T}$.

Extension to V-**Rel**

Examples: Note that $P_+ = ([0,\infty]^{\text{op}}, +, 0)$ is completely distributive and 0 is the top element of the lattice $[0,\infty]^{\text{op}}$.

- Identity monad: $1^V_{\textbf{Rel}} = 1_{V\text{-}\textbf{Rel}}$.
- Powerset monad: for a P_+-relation $d : X \rightsquigarrow Y$, we find

$$P_B^{P_+} d(A, B) = \inf\{\epsilon \geq 0 \mid A \subseteq B^\epsilon \text{ and } B \subseteq A^\epsilon\},$$

  where $B^\epsilon = \{x \in X \mid \exists b \in B : d(x, b) \leq \epsilon\}$. This is the Hausdorff distance.
- Ultrafilter monad: for a P_+-relation $d : X \rightsquigarrow Y$, we find

$$U_B^{P_+}(u_1, u_2) = \sup_{A \in u_1, B \in u_2} \inf_{a \in A, b \in B} d(a, b).$$

Overview

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2. Lax algebras

Lax algebras

Let (T, μ, η) be a monad in **Set** and let V be a unital quantale. Let $\hat{T} : V\text{-}\mathbf{Rel} \to V\text{-}\mathbf{Rel}$ be a lax extension to V-**Rel**.

- The category of lax algebras $\hat{T}$-**Alg**:
  - *Objects*: (X, α), where X is a set and $\alpha : \hat{T}X \rightsquigarrow X$ is a V-relation such that:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \hat{T}X \\ & \searrow_{1_X} \quad \leq & \Big\downarrow\!\!\wr\, \alpha \\ & & X \end{array} \qquad \text{and} \qquad \begin{array}{ccc} TTX & \overset{\hat{T}\alpha}{\rightsquigarrow} & TX \\ \Big\downarrow \mu_X & \geq & \Big\downarrow\!\!\wr\, \alpha \\ TX & \overset{\alpha}{\rightsquigarrow} & X \end{array}$$

  - *Morphisms*: A morphism $f : (X, \alpha) \to (Y, \beta)$ is a map $f : X \to Y$ such that

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ \Big\downarrow\!\!\wr\, \alpha & \leq & \Big\downarrow\!\!\wr\, \beta \\ X & \xrightarrow{f} & Y \end{array}$$

- If $\hat{T}$ is flat, then $\mathbf{Set}^T$ is a full subcategory of $\hat{T}$-**Alg**.

Lax algebras

Examples:

- Identity monad:
  - $1_{\mathbf{Rel}}$-**Alg** $\simeq$ **Ord**
  - $1_{P_+\text{-}\mathbf{Rel}}$-**Alg** $\simeq$ **Met**: for a P_+-relation $d : X \rightsquigarrow X$, we find

$$1_X \geq d \Leftrightarrow d(x,x) = 0 \quad \forall x \in X$$

and

$$d \leq d \circ d \Leftrightarrow d(x,y) \leq \inf_y \{d(x,y) + d(y,z)\} \quad \forall x, z \in X$$

- Powerset monad:
  - P_{can}-**Alg** $\simeq$ **Clos**: For a closure space $(X, c : PX \to PX)$, define the relation

$$ARx :\Leftrightarrow x \in c(A).$$

For an object (X, R) in P_+-**Alg**, define $c : PX \to PX$ by

$$c(A) := \{x \in X \mid ARx\}.$$

  - $P_{\text{op-can}}$-**Alg** $\simeq$ **Ord**: $ARx :\Leftrightarrow \forall a \in A : a \leq x$ and $x \leq y :\Leftrightarrow \{x\}Ry$.

Lax algebras

- Powerset monad:
  - $P^{P_+}_{\text{can}}$-**Alg** $\simeq$ **Clsn**: $(X, c : PX \times X \to [0,\infty])$ metric counterpart of closure space.
  - $P^{P_+}_{\text{op-can}}$-**Alg** $\simeq$ **Met**: $\overline{d}(A, x) := \inf_{a \in A} d(a, x)$ and $d(x, y) := \overline{d}(\{x\}, y)$.
- Ultrafilter monad:
  - U_B-**Alg** $\simeq$ **Top**: For a topological space X, define the relation $UX \rightsquigarrow X$

    $$uRx \Leftrightarrow u \to x$$

    Then (X, R) is an object in U_B-**Alg**.
    For an object (X, R) in U_B-**Alg**, define a map $c : PX \to PX$ by

    $$c(A) := \{x \in X \mid \exists u \in UX : A \in u \text{ and } uRx\}$$

    Then c is a topological closure operator.
  - $U^{P_+}_B$-**Alg** $\simeq$ **App**: $(X, c : PX \times X \to [0,\infty])$ metric counterpart of topological space/topological closure space.
- Filter monad:
  - F_{can}-**Alg** $\simeq$ **Clos** and $F_{\text{op-can}}$-**Alg** $\simeq$ **Top**
  - $F^{P_+}_{\text{can}}$-**Alg** $\simeq$ **Clsn** and $F^{P_+}_{\text{op-can}}$-**Alg** $\simeq$ **App**

Results about lax algebras

Theorem ([3])

1. The category $\hat{T}$-**Alg** is (co)complete.
2. The forgetful functor $\hat{T}$-**Alg** $\to$ **Set** has a left and right adjoint.
3. $\hat{T}$-**Alg** is well-(co)powered and has a (co)generator.

Results about lax algebras

Let $\hat{T}_1$ be a lax extension to V-**Rel** and $\hat{T}_2$ a lax extension to W-**Rel** of a monad (T, μ, η). Every quantale morphism $\varphi : V \to W$ *compatible* with the extensions, induces a functor

$$B_\varphi : \hat{T}_1\text{-}\mathbf{Alg} \to \hat{T}_2\text{-}\mathbf{Alg}.$$

Example: For $\varphi : 2 \to P_+$, this gives us functors

$$\begin{array}{ccc} \mathbf{Ord} & & \mathbf{Top} \\ \downarrow & & \downarrow \\ \mathbf{Met} & & \mathbf{App} \end{array}$$

Proposition ([3])

If φ has a right adjoint ψ that is compatible with the extensions. Then B_ψ is right adjoint to B_φ.

Example: The quantale morphism $\varphi : 2 \to P_+$ has a right adjoint that is compatible with the extensions. Therefore the above functors have right adjoints.

Results about lax algebras

Let $\hat{T}$ be a lax extension to V-**Rel** of a monad $\mathbb{T}$ and let S be a lax extension to V-**Rel** of a monad $\mathbb{S}$. Every monad morphism $\alpha : \mathbb{T} \to \mathbb{S}$, such that $\alpha : T \to S$ becomes an oplax natural transformation, induces a functor

$$A_\alpha : S\text{-}\mathbf{Alg} \to T\text{-}\mathbf{Alg}$$

Proposition ([3])

The unit $\eta : \mathbf{I} \to (T, \eta, \mu)$ becomes an oplax natural transformation for all extensions of $\mathbf{I}$ and (T, η, μ) to V-**Rel**. The induced functor A_η has a left adjoint.

Example: If we apply this to the ultrafilter monad. We find functors that have a right adjoint.

$$\begin{array}{ccc} \mathbf{Ord} & \longrightarrow & \mathbf{Top} \\ \downarrow & & \downarrow \\ \mathbf{Met} & \longrightarrow & \mathbf{App} \end{array}$$

Compact Hausdorff lax algebras

Definition

Let (X, α) be an object in $\hat{T}$-**Alg**.

1. We call (X, α) **compact** if $1_{TX} \leq \alpha^\circ \circ \alpha$.
2. We call (X, α) **Hausdorff** if $1_X \geq \alpha \circ \alpha^\circ$.

Theorem ([3])

If V is *lean* with $k = 1$ and $\hat{T}$ is a flat lax extension to V-**Rel** of a monad (T, μ, η), then

1. $\hat{T}$-$\mathbf{Alg}_{\mathrm{CH}} \simeq \mathbf{Set}^T$;
2. $\hat{T}$-$\mathbf{Alg}_{CH} \to \hat{T}$-**Alg** has a left adjoint.

Example: The Barr extension of the ultrafilter monad satisfies the conditions. Therefore

$$\mathbf{CH} \simeq \mathbf{Set}^U \to \mathbf{Top}$$

has a left adjoint. This is the Čech-Stone compactification.

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