Sheaf representation of monoidal categories

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Categories should be nice and easy

Category **Vect** of vector spaces is monoidal. So is **Vect** $\times$ **Vect**.
Clearly **Vect** is easier: does not decompose as product.

Any monoidal category embeds into a nice one, and
any nice monoidal category is dependent product of easy ones.

Nice and easy

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Can reconstruct opens of $\{0, 1\}$ as *subunits* of **Vect** $\times$ **Vect**

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- stiff: subunits form semilattice
- universal joins of subunits: subunits form complete lattice

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Category is easy if subunits are like singletons:

- (sub)local: any (finite) cover contains the open that is covered
  every net converges to a single *focal point*

Sheaves are continuously parametrised objects

Write $\mathcal{O}(X)$ for open sets of space X.

Presheaf on X is functor $F\colon \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$
Elements of $F(U)$ are called *local sections*.
Elements of $F(X)$ are called *global sections*.
Map $F(U \subseteq V)\colon F(V) \to F(U)$ is called *restriction*.

Sheaf condition

Sheaf is continuous presheaf: $F(\text{colim } U_i) = \lim F(U_i)$

- Elements of $F(U)$ are *global sections* over $U = \text{colim } U_i = \bigcup U_i$
- Elements of $\lim F(U_i)$ are *compatible local sections*:

$$\lim F(U_i) = \big\{ (s_i) \mid F(U_i \cap U_j \subseteq U_i)(s_i) = F(U_i \cap U_j \subseteq U_j)(s_j) \big\}$$

Compatible local sections must glue together to unique global section

Example: F(U) $= \{$ continuous functions $U \to \mathbb{R}$ $\}$

Sheaves of categories

What if F takes values not in **Set** but in **V**?

Then sheaf condition becomes equaliser in **V**:

$$F(\bigcup_i U_i) \xrightarrow{\langle F(U_i \subseteq \bigcup U_i)\rangle_i} \prod_i F(U_i) \underset{\langle F(U_i \cap U_j \subseteq U_j)\circ \pi_j\rangle_{i,j}}{\overset{\langle F(U_i \cap U_j \subseteq U_i)\circ \pi_i\rangle_{i,j}}{\rightrightarrows}} \prod_{i,j} F(U_i \cap U_j)$$

Stalk

of sheaf F at point x is $\operatorname{colim}\{F(U) \mid x \in U\}$

Say F is a "sheaf of ..." when its stalks are "..."
E.g. sheaves of local rings

Sheaf representation

Literature:

- Boolean algebra is global sections of sheaf of spaces $\{0, 1\}$
- ring is ring of global sections of sheaf of local rings
- topos is category of global sections of sheaf of local toposes

Will generalise all three into:

- monoidal category with universal join of subunits is
  category of global sections of sheaf of local monoidal categories

Corollary:

- stiff monoidal category embeds into
  category of global sections of sheaf of local monoidal categories

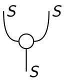
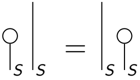
Subunits

How to recover $\mathcal{O}(X)$ from $\mathrm{Sh}(X)$?
Look at subobjects of terminal object $s \colon S \rightarrowtail 1$.

What if we want sheaves with values not in **Set**?
A subunit in a monoidal category $\mathbf{C}$ is a subobject $s \colon S \rightarrowtail I$ such that $S \otimes s \colon S \otimes S \to S \otimes I$ is invertible. They form set $\mathsf{ISub}(\mathbf{C})$.

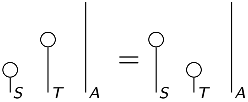
- $\mathsf{ISub}(\mathrm{Sh}(X)) = \mathcal{O}(X)$
- $\mathsf{ISub}(L) = L$ for semilattice L
- $\mathsf{ISub}(\mathbf{Mod}_R) = \{I \subseteq R \text{ ideal} \mid I^2 = I\}$ for commutative ring R
- $\mathsf{ISub}(\mathbf{Hilb}_{C(X)}) = \mathcal{O}(X)$

Nice subunits

Draw subunit as , and draw  for inverse of 

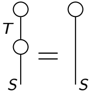
ISub($\mathbf{C}$) semilattice $\Longleftarrow$ $\mathbf{C}$ is stiff $\Longleftrightarrow$

$$\begin{tikzcd}
S \otimes T \otimes A \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & T \otimes A \arrow[d, tail] \\
S \otimes A \arrow[r, tail] & A
\end{tikzcd}$$



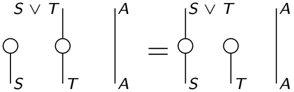
Nicer subunits

$s \leq t$ if there is unique $m\colon S \to T$ with $s = t \circ m$:



$$
\begin{array}{ll}
 & \mathrm{ISub}(\mathbf{C}) \text{ distributive lattice} \\
\Longleftarrow & \mathbf{C} \text{ has universal finite joins of subunits} \\
\Longleftrightarrow & \mathrm{ISub}(\mathbf{C}) \text{ has finite joins, } 0 \simeq 0 \otimes A \text{ is initial, and}
\end{array}
$$

$$
\begin{tikzcd}
S \otimes T \otimes A \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "\ulcorner", very near end] & T \otimes A \arrow[d, tail] \\
S \otimes A \arrow[r, tail] & (S \vee T) \otimes A
\end{tikzcd}
$$



Embedding

Stiff $\mathbf{C}$ embeds into category with universal finite joins of subunits
embeds into category with universal joins of subunits

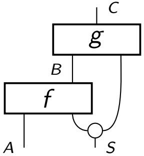
Universally, faithfully, preserving subunits and tensor products

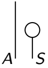
Base space

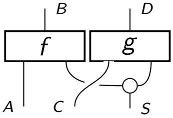
$\mathbf{C}$ has universal (finite) joins of subunits
$\implies$ $\mathsf{ISub}(\mathbf{C})$ is a (distributive lattice) frame
$\implies$ *Zariski spectrum* $X = \mathsf{Spec}(\mathsf{ISub}(\mathbf{C}))$ is topological space

points x are (completely) prime filters in $\mathsf{ISub}(\mathbf{C})$

Local sections $F(s)$

- Objects: as in **C**
- Morphisms: $A \otimes S \to B$ in **C**
- Composition: 

- Identity: 

- Tensor product: 

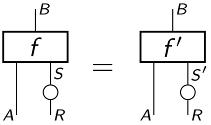
Sheaf condition

To specify a sheaf $F\colon \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{MonCat}$,
it's enough to give a presheaf $F\colon \mathrm{ISub}(\mathbf{C})^{\mathrm{op}} \to \mathbf{MonCat}$,
such that $F(0)$ is terminal and the following is an equaliser:

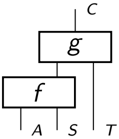
$$F(s \vee t) \xrightarrow{\langle F(s \leq s \vee t),\, F(t \leq s \vee t)\rangle} F(s) \times F(t) \underset{F(s \wedge t \leq t) \circ \pi_2}{\overset{F(s \wedge t \leq s) \circ \pi_1}{\rightrightarrows}} F(s \wedge t)$$

Stalks $F(x)$ are (sub)local

- Objects: as in $\mathbf{C}$
- Morphisms: $A \otimes S \to B$ in $\mathbf{C}$ for $s \in x$, identified when



- Composition of (s, f) and (t, g) is



Theorem

Any small stiff category with universal (finite) joins of subunits
is monoidally equivalent to category of global sections of sheaf
of (sub)local categories.

Any small stiff category
embeds into a category of global sections of a sheaf
of local categories.

Preservation

category	local sections	stalks
stiff	monoidal	stiff
closed	closed	closed
traced	traced	traced
compact	compact	compact
Boolean		two-valued
limits	limits	limits
projective colimits	colimits	colimits

Conclusion

- Cleanly separate 'spatial' from 'temporal' directions
- Does for multiplicative linear logic
  what was known for intuitionistic logic
- Directly capture more examples
- Concrete proof

- Completeness theorem?
- Coherence theorem?
- Restriction categories?
- Applications in computer science? Probability? Quantum theory?

References

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