# What is a hole in a locally metric category?

Magnitude homology and iterated enrichment

Emily Roff
The Categorical Late Lunch
11th August 2021

# Plan

1. What is a hole?

2. What is a locally metric category?

3. Magnitude homology: our hole-detecting technology
   - Categorifying magnitude
   - Iterating magnitude homology
   - Deloopings, suspensions and "spheres"

4. What is a hole in a locally metric category?

# Part I

## What is a hole?

# Holes

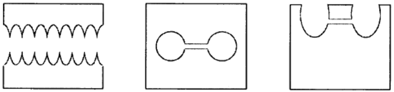


**Figure 7.13**
An "accordion" tunnel and two "dumbbell" holes.

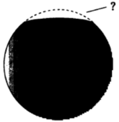



**Figure 2.5**
A hole in a sphere?

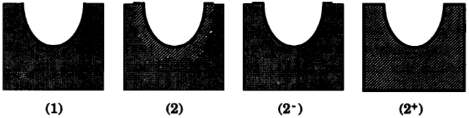



**Figure 3.7**
Four different ways of construing holes.

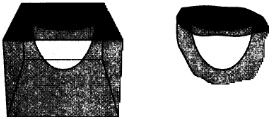


**Figure 3.1**
A hole (left); one of its hole-linings (right). i.e., a hole *tout court* according to the Lewises: a material body.

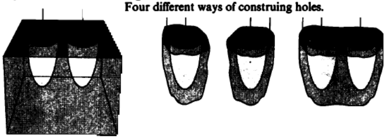


**Figure 3.6**
If being the same hole is having a common part that is a hole, then $l_1$ is not the same hole as $l_2$, though they are both the same hole as $l_3$.

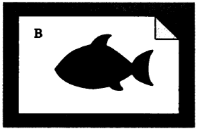



**Figure 3.3**
A fishy hole.

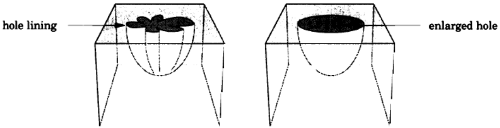



**Figure 3.4**
An enlarged hole with a smaller hole-lining.

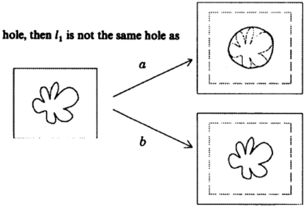
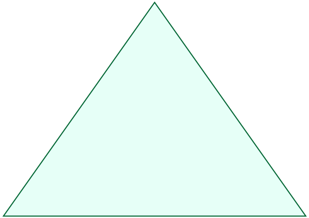


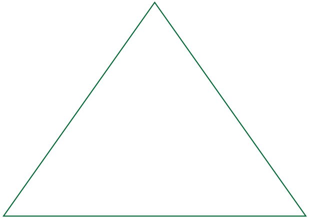

**Figure 3.5**
Enlarging a Ludovician hole—but how?

Casati and Varzi (1994)

# The archetypal hole



not a hole



a hole

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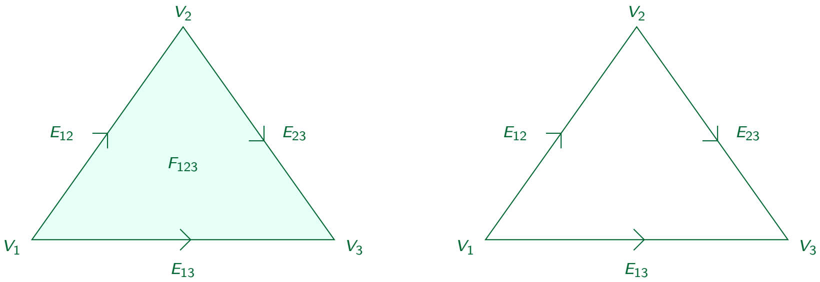


A hole is a cycle of edges that is <u>not</u> the boundary of a face.

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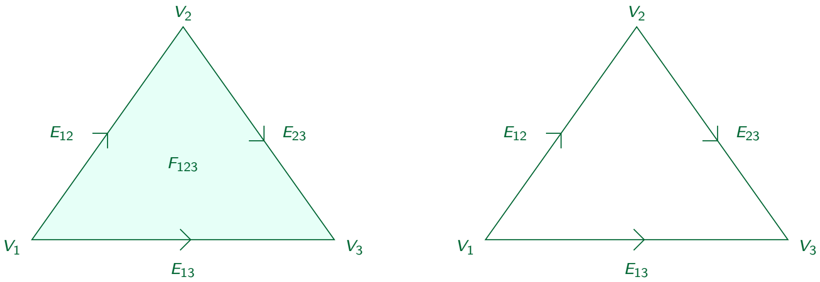


$$\mathbb{Z} \cdot \{\text{edges}\}$$

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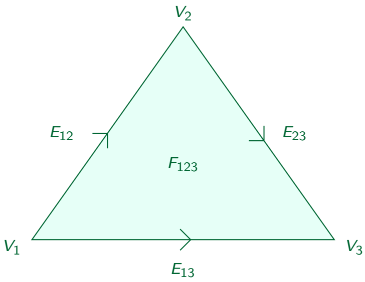


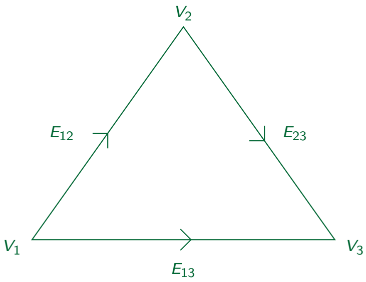


$$\mathbb{Z} \cdot \{\text{vertices}\} \xleftarrow[\partial_1]{} \mathbb{Z} \cdot \{\text{edges}\} \xleftarrow[\partial_2]{} \mathbb{Z} \cdot \{\text{faces}\}$$

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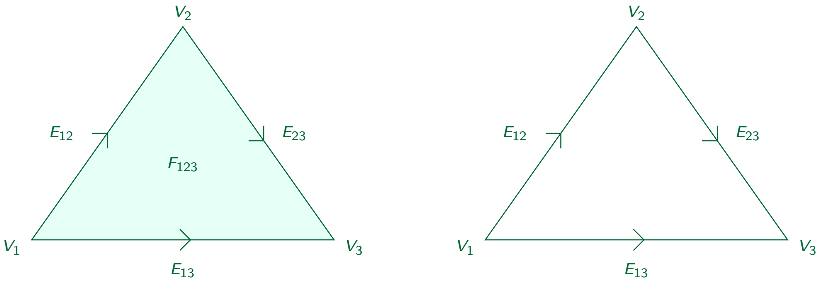


$$\mathbb{Z}\cdot\{\text{vertices}\} \xleftarrow[\partial_1]{} \mathbb{Z}\cdot\{\text{edges}\} \xleftarrow[\partial_2]{} \mathbb{Z}\cdot\{\text{faces}\}$$

$$\ker(\partial_1) = \langle\{\text{cycles}\}\rangle, \quad \operatorname{im}(\partial_2) = \langle\{\text{boundaries}\}\rangle$$

$$\ker(\partial_1)/\operatorname{im}(\partial_2) \sim \text{“cycles that are not boundaries”} = \text{“holes”}$$

# Chain complexes and homology

A **chain complex** $A$ is a sequence of objects and morphisms

$$A_0 \xleftarrow{\partial_1} A_1 \xleftarrow{\partial_2} A_2 \xleftarrow{\partial_3} A_3 \xleftarrow{\partial_4} \cdots$$

in an abelian category $\mathbb{A}$, such that $\operatorname{im}(\partial_{i+1}) \hookrightarrow \ker(\partial_i)$ for all $i$.

We refer to elements of $\ker(\partial_i)$ as **cycles** and elements of $\operatorname{im}(\partial_{i+1})$ as **boundaries**.

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The **homology** of $A$ is a sequence $H_\bullet(A)$ of objects in $\mathbb{A}$ defined by

$$H_i(A) = \frac{\ker(\partial_i)}{\mathrm{im}(\partial_{i+1})}.$$

There is a category $\mathrm{Ch}(\mathbb{A})$ of chain complexes in $\mathbb{A}$, and $H_\bullet : \mathrm{Ch}(\mathbb{A}) \to \mathbb{A}^{\mathbb{N}}$ is a functor.

# Homology theories

Roughly, a **homology theory** on a category **C** is a composite of functors*

$$\mathbf{C} \xrightarrow{A(-)} \mathrm{Ch}(\mathbb{A}) \xrightarrow{H_\bullet} \mathbb{A}^{\mathbb{N}}$$

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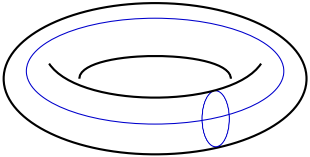
<u>Very</u> roughly, $H_i A(X)$ tells us about "the holes of dimension $i$" in an object $X$ of $\mathbf{C}$.
But $H_\bullet A(X)$ as a whole often tells us much more.



one connected component

two unshrinkable loops

$$SH_k(T) = \begin{cases} \mathbb{Q} & k = 0 \\ \mathbb{Q}^2 & k = 1 \\ \mathbb{Q} & k = 2 \\ 0 & k > 2 \end{cases}$$

$T$ is two-dimensional

and $1 - 2 + 1 = 0 = \chi(T)$

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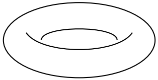
# Examples



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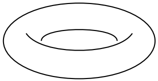
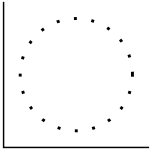


Persistent homology

$PH_\bullet : \mathbf{Data} \to \mathbf{Vect}^{\mathbb{R}\times\mathbb{N}}$



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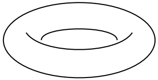

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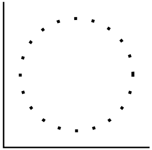

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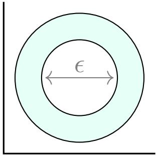

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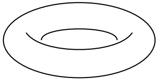

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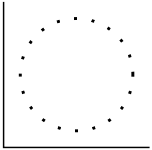

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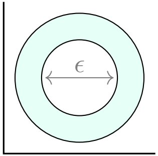


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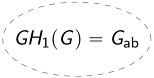

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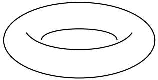

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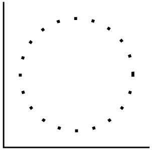

"I know what holes are; how can I detect them?"

Magnitude homology

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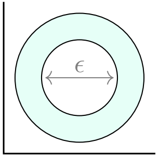


Group homology

$GH_\bullet : \mathbf{Gp} \to \mathbf{Ab}^{\mathbb{N}}$

$GH_1(G) = G_{ab}$

"I have a well-motivated homology theory; what does it think a hole is? What info does it carry?"

# Part II

## What is a locally metric category?

# Locally metric categories

A **(generalized) metric space** is a category enriched in $([0, \infty], +)$.

In the category **Met** of generalized metric spaces

- morphisms are distance-decreasing functions;
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A **locally metric category** is a category enriched in $(\mathbf{Met}, \otimes_1)$:

- its hom sets are generalized metric spaces;
- its composition satisfies $d(g \circ f, g' \circ f') \leqslant d(g, g') + d(f, f')$.

**Examples** include **Met** itself and $\mathbf{Ban}_1$: Banach spaces and operators of norm $\leqslant 1$.

Every metric space gives rise to a **Met**-category in (at least) two ways.

# Approximate category theory

Sometimes we might want to think about categories “with fuzz”. For example:

If our morphisms are processes with probabilistic outcomes.

If we want to infer categorical structure from noisy data.

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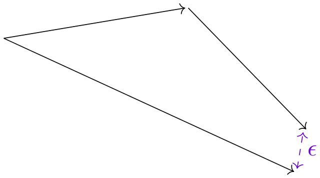
—see e.g. Tholen & Rosický (2018).



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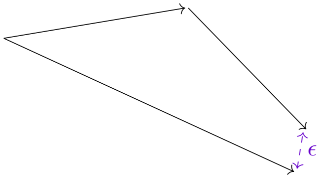


To deal with an "up-to-$\epsilon$" composition rule requires something more general, e.g. an $\epsilon$-approximate categorical structure—see Aliouche & Simpson (2014).

# Part III

## Magnitude homology: our hole-detecting technology

# Size and magnitude

notion of 'size' for objects in $\mathcal{V}$ $\longrightarrow$ $\boxed{\textbf{magnitude}}$ $\longrightarrow$ notion of 'size' for $\mathcal{V}$-categories

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Examples (Leinster, Willerton, Meckes, Carbery, Gimperlein, . . . )

Finite categories: magnitude is a generalized Euler characteristic

Finite metric spaces: magnitude is "effective number of points"

Compact metric spaces: magnitude knows volume, surface area, Euler characteristic. . .

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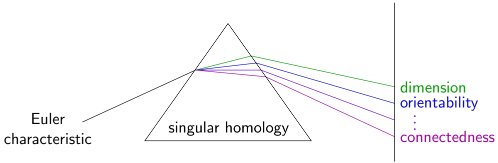

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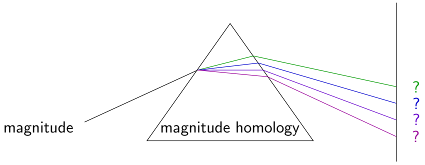
Idea Magnitude homology should be a functor

$MH_\bullet : \mathcal{V}\mathbf{Cat} \to \mathbb{A}^{\mathbb{N}}$ such that

$$\chi(MH_\bullet(\mathbf{X})) = \sum_i (-1)^i \mathrm{rk}(MH_i(\mathbf{X})) = \mathrm{Mag}(\mathbf{X}).$$



# Categorifying size

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{magnitude}\\ \mathrm{Mag}_{\#} : \mathcal{V}\mathbf{Cat} \to R\end{array}} & \\
\boxed{\begin{array}{c}\text{size homomorphism}\\ \# : (\mathcal{V}, \otimes) \to (R, \cdot)\end{array}} \arrow[u, "\text{enrich}"] \arrow[r, dashed, "\text{categorify}"'] & \boxed{\begin{array}{c}\text{size functor}\\ \Sigma : (\mathcal{V}, \otimes) \to (\mathbb{A}, \otimes)\end{array}}
\end{tikzcd}$$

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Suppose $R$ is a ring and

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$$\begin{tikzcd} \mathcal{V} \arrow[rr, "\#"] \arrow[dr, bend right, "\Sigma"'] & \arrow[d, phantom, "||"] & R \\ & \mathbb{A} \arrow[ur, bend right, "\mathrm{rk}"'] & \end{tikzcd}$$

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Then we say $\Sigma$ is a **size functor** categorifying $\#$.

# Categorifying magnitude

$$
\begin{tikzcd}
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# Magnitude homology

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**Definition (Leinster & Shulman, 2017)**

The **magnitude nerve** of a $\mathcal{V}$-category $\mathbf{X}$ is given for $n \in \mathbb{N}$ by

$$MB_n^{\Sigma}(\mathbf{X}) = \bigoplus_{x_0,\ldots,x_n \in \mathbf{X}} \Sigma\mathbf{X}(x_0, x_1) \otimes \cdots \otimes \Sigma\mathbf{X}(x_{n-1}, x_n)$$

with face maps $\delta^i$ induced by composition in $\mathbf{X}$ and degeneracies $\sigma^i$ by identities.

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Example If $\mathcal{V} = \mathbf{Set}$ and $\Sigma : \mathbf{Set} \to \mathbf{Ab}$ is the free abelian group functor, then

$$MB_n^{\Sigma}(\mathbf{X}) = \mathbb{Z} \cdot \{x_0 \to x_1 \to \cdots \to x_n \text{ in } \mathbf{X}\}.$$

# Magnitude homology

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\mathcal{V}\mathbf{Cat} \arrow[r, "MB^{\Sigma}"] \arrow[rr, bend right, "MC^{\Sigma}"'] & {[\triangle^{\mathsf{op}}, \mathbb{A}]} \arrow[r, "C"] & \mathrm{Ch}(\mathbb{A}) \arrow[r, "H_\bullet"] & \mathbb{A}^{\mathbb{N}}
\end{tikzcd}
$$

## Definition (Leinster & Shulman, 2017)

The **magnitude complex** of $\mathbf{X}$ has $MC_n^{\Sigma}(\mathbf{X}) = MB_n^{\Sigma}(\mathbf{X})$, with boundary maps

$$\partial_n : MC_n^{\Sigma}(\mathbf{X}) \to MC_{n-1}^{\Sigma}(\mathbf{X})$$

given by $\partial_n = \sum_{i=0}^{n} (-1)^i \delta^i$.

# Magnitude homology

$$\begin{tikzcd}
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given by $\partial_n = \sum_{i=0}^{n}(-1)^i \delta^i$.

The **magnitude homology** of $\mathbf{X}$ is the homology of $MC^{\Sigma}(\mathbf{X})$.

# Magnitude homology categorifies magnitude

**Theorem (Leinster & Shulman, 2017)**

*Under finiteness conditions, $MH^{\Sigma}$ categorifies magnitude:*

$$\chi(MH^{\Sigma}(\mathbf{X})) = \mathrm{Mag}_{\#}(\mathbf{X}).$$

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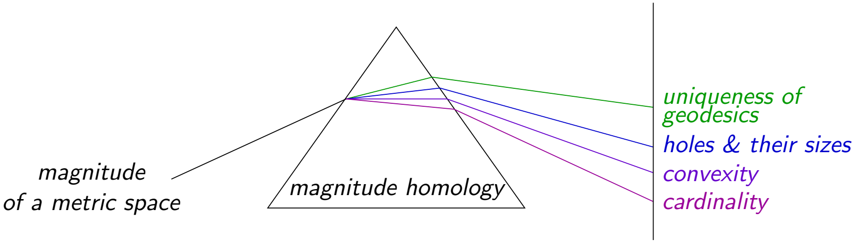

# Part IV

## Iterating magnitude homology

# Iterating magnitude

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\text{magnitude}\\ \mathrm{Mag}_{\#} : \mathcal{V}\mathbf{Cat} \to R\end{array}} \arrow[r, "\text{categorify}"'] & \boxed{\begin{array}{c}\text{magnitude homology}\\ MH^{\Sigma} : \mathcal{V}\mathbf{Cat} \to \mathbb{A}^{\mathbb{N}}\end{array}} \\
\boxed{\begin{array}{c}\text{size homomorphism}\\ \# : (\mathcal{V}, \otimes) \to (R, \cdot)\end{array}} \arrow[u, "\text{enrich}"] \arrow[r, "\text{categorify}"'] & \boxed{\begin{array}{c}\text{size functor}\\ \Sigma : (\mathcal{V}, \otimes) \to (\mathbb{A}, \otimes)\end{array}} \arrow[u, "\text{enrich}"']
\end{tikzcd}
$$

# Iterating magnitude

For bicategories, see Tanaka (2014)

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{magnitude}\\ \mathrm{Mag}_{\mathrm{Mag}} : 2\mathcal{V}\mathbf{Cat} \to R\end{array}} & \\
\boxed{\begin{array}{c}\text{magnitude}\\ \mathrm{Mag}_{\#} : \mathcal{V}\mathbf{Cat} \to R\end{array}} \arrow[u, "\text{enrich}"] \arrow[r, "\text{categorify}"'] & \boxed{\begin{array}{c}\text{magnitude homology}\\ MH^{\Sigma} : \mathcal{V}\mathbf{Cat} \to \mathbb{A}^{\mathbb{N}}\end{array}} \\
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$$
\begin{tikzcd}[column sep=huge, row sep=large]
\boxed{\begin{array}{c}\text{magnitude}\\ \mathrm{Mag}_{\mathrm{Mag}} : 2\mathcal{V}\mathbf{Cat} \to R\end{array}} \arrow[r, dashed] & MH^{MH}\,? \\
\boxed{\begin{array}{c}\text{magnitude}\\ \mathrm{Mag}_{\#} : \mathcal{V}\mathbf{Cat} \to R\end{array}} \arrow[u, "\text{enrich}"] \arrow[r, "\text{categorify}"'] & \boxed{\begin{array}{c}\text{magnitude homology}\\ MH^{\Sigma} : \mathcal{V}\mathbf{Cat} \to \mathbb{A}^{\mathbb{N}}\end{array}} \arrow[u, dashed] \\
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\end{tikzcd}
$$

# The magnitude nerve as a size functor

## Proposition

*The magnitude nerve is a strong symmetric monoidal functor*

$$MB^{\Sigma} : (\mathcal{V}\mathbf{Cat}, \otimes_{\mathcal{V}}) \to ([\triangle^{\mathsf{op}}, \mathbb{A}], \otimes_{pw}).$$

## Proof.

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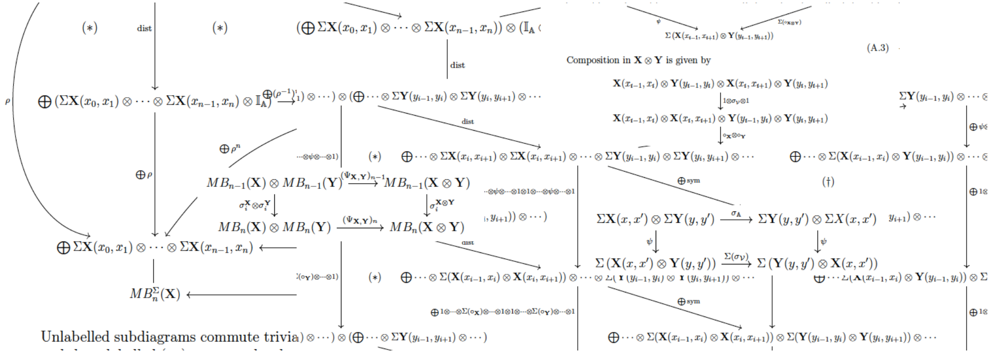


□

# Iterating magnitude homology

$$
\begin{tikzcd}
2\mathcal{V}\mathbf{Cat} \arrow[r, "MB^{MB^{\Sigma}}"] \arrow[rr, bend right, "MB^2"'] & {[\triangle^{op} \times \triangle^{op}, \mathbb{A}]} \arrow[r, "\mathrm{diag}"] & {[\triangle^{op}, \mathbb{A}]} \arrow[r, "C"] & \mathrm{Ch}(\mathbb{A}) \arrow[r, "H_\bullet"] & \mathbb{A}^{\mathbb{N}}
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$$

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The **(iterated) magnitude nerve** of a $\mathcal{V}\mathbf{Cat}$-category $\mathbf{X}$ is

$$MB^2(\mathbf{X}) = \mathrm{diag}\left(MB^{MB^{\Sigma}}(\mathbf{X})\right).$$

# Iterating magnitude homology

$$\begin{tikzcd}
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The **(iterated) magnitude homology** of $\mathbf{X}$ is

$$MH^2_\bullet(\mathbf{X}) = H_\bullet C(MB^2(\mathbf{X})).$$

## $MH^2$ categorifies iterated magnitude

**Lemma**
*For any 2-category $\mathbf{X}$, $MH^2(\mathbf{X})$ is the homology of the classifying space $B\mathbf{X}$.*

**Proof** is via the description of the Duskin nerve in Bullejos & Cegarra (2003). $\square$

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**Theorem**
*For any finite enough locally metric category* $\mathbf{X}$ *we have* $\chi(MH^2(\mathbf{X})) = \mathrm{Mag}_{\mathrm{Mag}}(\mathbf{X})$.

**Proof** uses facts about spectral sequences, plus simple linear algebra. □

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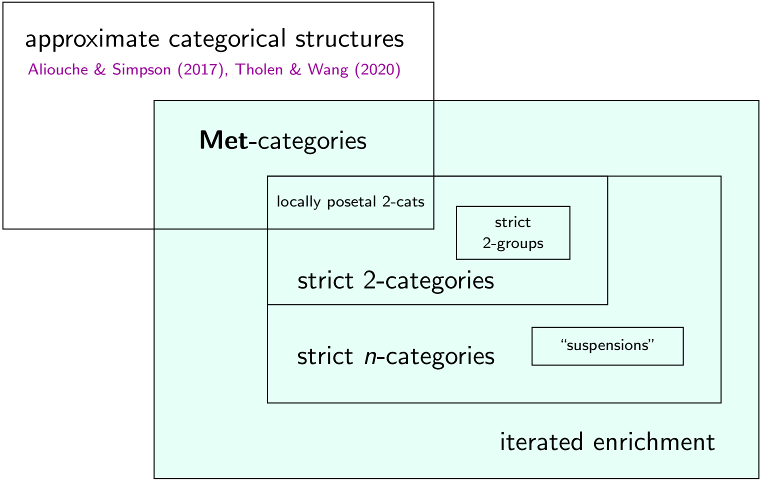
\end{tikzcd}$$

# Some classes of examples



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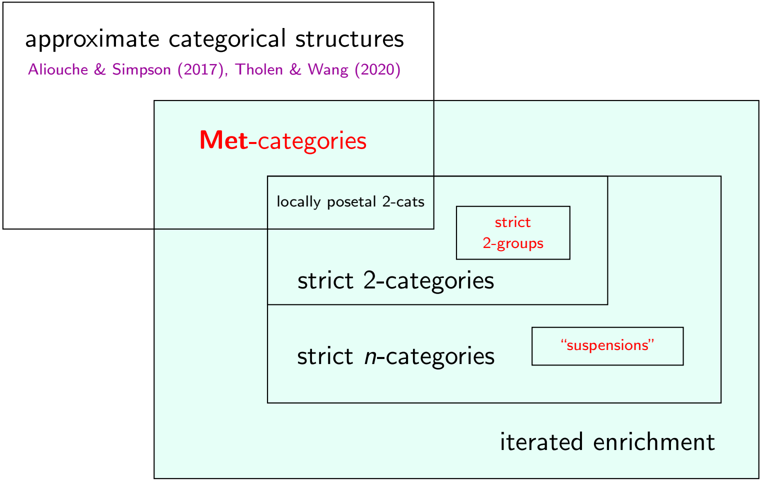

# Part V

## Deloopings, suspensions and “spheres”

# Deloopings

Let $G$ be a group and $N$ a normal subgroup of $G$.

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- 2-cells are elements of $N \times G$

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, "g"] \arrow[r, "hg"'] \arrow[r, phantom, shift left=2, "\Downarrow (h,g)"] & \bullet \end{tikzcd}$$

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$$\begin{tikzcd}
\bullet \arrow[r, bend left=60, "g"] \arrow[r] \arrow[r, bend right=60, "k(hg)"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow (h{,}g)"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow (k{,}hg)"] & \bullet
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$$
\begin{tikzcd}
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\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[r, bend left=60, "g"] \arrow[r, phantom, "\Downarrow (kh,g)"] \arrow[r, bend right=60, "(kh)g"'] & \bullet
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$$

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$$
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\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[r, bend left=60, "g'g"] \arrow[r, bend right=60, "hgh'g'"'] \arrow[r, phantom, "\Downarrow (hgh'g^{-1},\, gg')"] & \bullet
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Recall that group homology says

$$H_0(G) = \mathbb{Z} \text{ and } H_1(G) = G_{\text{ab}}.$$

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### Proposition

*Let $\Sigma : \mathbf{Set} \to \mathbf{Ab}$ be the free abelian group functor. Then*

$$MH_0^2(\mathbb{B}(N \lhd G)) = \mathbb{Z}$$

*and*

$$MH_1^2(\mathbb{B}(N \lhd G)) = (G/N)_{\mathrm{ab}}\,.$$

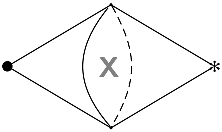
# Suspensions and "spheres"

Let $\mathbf{X}$ be any strict $(n-1)$-category.

## Definition

The **suspension** of $\mathbf{X}$ is a strict $n$-category $\Gamma\mathbf{X}$ with

- $\mathrm{ob}(\Gamma\mathbf{X}) = \{\bullet, *\}$
- $\Gamma\mathbf{X}(\bullet, *) \simeq \mathbf{X}$
- $\Gamma\mathbf{X}(*, \bullet) \simeq \varnothing$
- $\Gamma\mathbf{X}(\bullet, \bullet) \simeq \mathbf{1} \simeq \Gamma\mathbf{X}(*, *)$.



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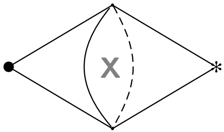
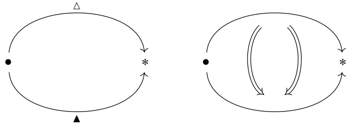



## Example

Set $\mathbb{S}^0 = \{\triangle, \blacktriangle\}$. For each $n > 0$ let $\mathbb{S}^n = \Gamma\mathbb{S}^{n-1}$. Then $\mathbb{S}^n$ is the strict $n$-category with two parallel $k$-cells in every dimension $k \leqslant n$.



$\mathbb{S}^1$ $\qquad$ $\mathbb{S}^2$

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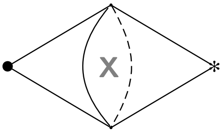


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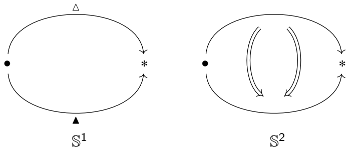
## Proposition

*$MH^n_\bullet$ behaves with respect to suspension of $(n-1)$-cats as singular homology behaves with respect to topological suspension.*

## Corollary

$$MH^n_k(\mathbb{S}^n) = \begin{cases} \mathbb{Z} & \text{if } k = 0, n \\ 0 & \text{otherwise.} \end{cases}$$



# Part VI

## What is a hole in a locally metric category?

# The magnitude nerve of a metric space

$$\begin{tikzcd} \mathbf{Met} \arrow[r, "MB^{\Sigma}"] \arrow[rrr, bend left, "MH^{\Sigma}"] & {[\triangle^{\mathrm{op}}, \mathbf{Ab}^{\mathbb{R}_+}]} \arrow[r, "C"] & \mathrm{Ch}(\mathbf{Ab}^{\mathbb{R}_+}) \arrow[r, "H_\bullet"] & \mathbf{Ab}^{\mathbb{R}_+ \times \mathbb{N}} \end{tikzcd}$$

For a metric space $X$ the magnitude nerve is given in degrees $n \in \mathbb{N}$ and $\ell \in \mathbb{R}_+$ by

$$\mathbb{Z} \cdot \left\{ (p_0, \ldots, p_n) \mid \text{each } p_i \in X \text{ and } \sum_{i=0}^{n-1} d(p_i, p_{i+1}) = \ell \right\}.$$

The face maps are given on generators by

$$\delta_n^i(p_0, \ldots, p_n) = (p_0, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n)$$

if $d(p_{i-1}, p_i) + d(p_i, p_{i+1}) = d(p_{i-1}, p_{i+1})$, and $\delta_n^i(p_0, \ldots, p_n) = 0$ otherwise.

# The magnitude nerve of a locally metric category

$$\begin{tikzcd}[column sep=large]
\mathbf{MetCat} \arrow[r, "MB^{MB^{\Sigma}}"] \arrow[rrrr, bend left, "MH^2"] \arrow[rr, bend right, red, "MB^2"'] & {[\triangle^{\mathrm{op}} \times \triangle^{\mathrm{op}}, \mathbf{Ab}^{\mathbb{R}_+}]} \arrow[r, "\mathrm{diag}"] & {[\triangle^{\mathrm{op}}, \mathbf{Ab}^{\mathbb{R}_+}]} \arrow[r, "C"] & \mathrm{Ch}(\mathbf{Ab}^{\mathbb{R}_+}) \arrow[r, "H_\bullet"] & \mathbf{Ab}^{\mathbb{R}_+ \times \mathbb{N}}
\end{tikzcd}$$

For a **Met**-category **X** the magnitude nerve is given in degrees $n \in \mathbb{N}$ and $\ell \in \mathbb{R}_+$ by

$$\mathbb{Z} \cdot \left\{ \begin{tikzcd}
x_0 \arrow[r, bend left=50, "f_{00}"] \arrow[r, "f_{01}"] \arrow[r, bend right=50, "f_{0n}"'] \arrow[r, phantom, shift right=3, "\vdots"] & x_1 \arrow[r, bend left=50, "f_{10}"] \arrow[r, "f_{11}"] \arrow[r, bend right=50, "f_{1n}"'] \arrow[r, phantom, shift right=3, "\vdots"] & x_2 & \cdots & x_{n-1} \arrow[r, bend left=50, "f_{n-1,0}"] \arrow[r, "f_{n-1,1}"] \arrow[r, bend right=50, "f_{n-1,n}"'] \arrow[r, phantom, shift right=3, "\vdots"] & x_n
\end{tikzcd} \;\middle|\; \sum_{p=0}^{j-1} \sum_{q=0}^{k-1} d(f_{pq}, f_{p,q+1}) = \ell \right\}.$$

# Gaps

A **gap of width $\ell$** in $\mathbf{X}$ is an equivalence class of irreducible pairs of arrows:

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\updownarrow \ell>0"] & y \end{tikzcd}$$

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$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\ell"] & y \arrow[r, "h"] & z \end{tikzcd}$$

A pair is reducible if

- we can tighten it by composing with another arrow
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$$\begin{tikzcd} x \arrow[r, bend left, "h\circ f"] \arrow[r, bend right, "h\circ g"'] \arrow[r, phantom, "<\ell"] & z \end{tikzcd}$$

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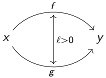
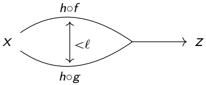
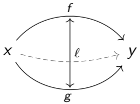
# Gaps

A **gap of width $\ell$** in $\mathbf{X}$ is an equivalence class of irreducible pairs of arrows:




A pair is reducible if

- we can tighten it by composing with another arrow
  - if $(f, g)$ can't be tightened, call it tight
- or we can bridge it with an arrow strictly between $f$ and $g$
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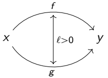


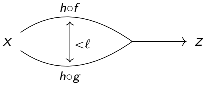


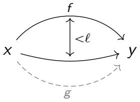


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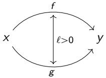


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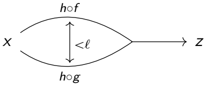


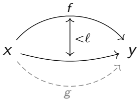


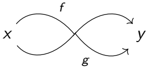

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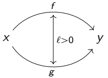


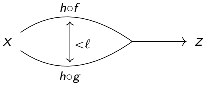


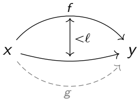


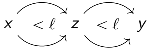


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# Gaps

A **gap of width $\ell$** in $\mathbf{X}$ is an equivalence class of irreducible pairs of arrows: $\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\ell>0"] & y \end{tikzcd}$

$$\begin{tikzcd} x \arrow[r, bend left, "h\circ f"] \arrow[r, bend right, "h\circ g"'] \arrow[r, phantom, "<\ell"] & z \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[r, bend left=50, "f"] \arrow[r, bend left=10] \arrow[r, phantom, bend left=30, "<\ell"] \arrow[r, dashed, bend right=40, "g"'] & y \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "<\ell"] & z \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "<\ell"] & y \end{tikzcd}$$

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A gap is a class of simple, tight, adjacent pairs under the equivalence relation gen'd by

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\ell"] & y \end{tikzcd} \sim \begin{tikzcd} x \arrow[r, bend left, "h\circ f"] \arrow[r, bend right, "h\circ g"'] \arrow[r, phantom, "\ell"] & \bullet \arrow[r] & z \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} w \arrow[r, no head] & \bullet \arrow[r, bend left, "f\circ k"] \arrow[r, bend right, "g\circ k"'] \arrow[r, phantom, "\ell"] & y \end{tikzcd} \sim \begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\ell"] & y \end{tikzcd}$$

# The magnitude homology of a locally metric category

**Theorem**

*Let* $\mathbf{X}$ *be a locally metric category in which all the hom-spaces are separated.*

*In real grading* $0$*, the magnitude homology of* $\mathbf{X}$ *is the homology of its underlying ordinary category* $\underline{\mathbf{X}}$*:*

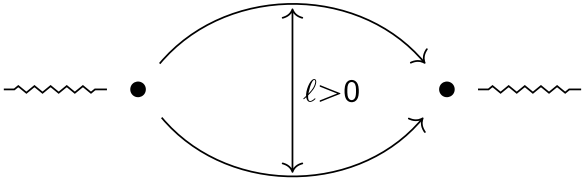
$$MH^0_\bullet(\mathbf{X}) \cong H_\bullet(\underline{\mathbf{X}}).$$

*In real gradings* $\ell > 0$*, the first three magnitude homology groups are given by*

$$MH^\ell_k(\mathbf{X}) \cong \begin{cases} 0 & k = 0, 1 \\ \mathbb{Z} \cdot \{\textit{gaps of width } \ell \textit{ in } \mathbf{X}\} & k = 2. \end{cases}$$

# Conclusion

A hole in a locally metric category is a **gap**: a class of pairs of parallel arrows that cannot be pulled tighter; cannot be bridged; and cannot be split into smaller gaps.



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