Localisable monads

Nuiok Dicaire

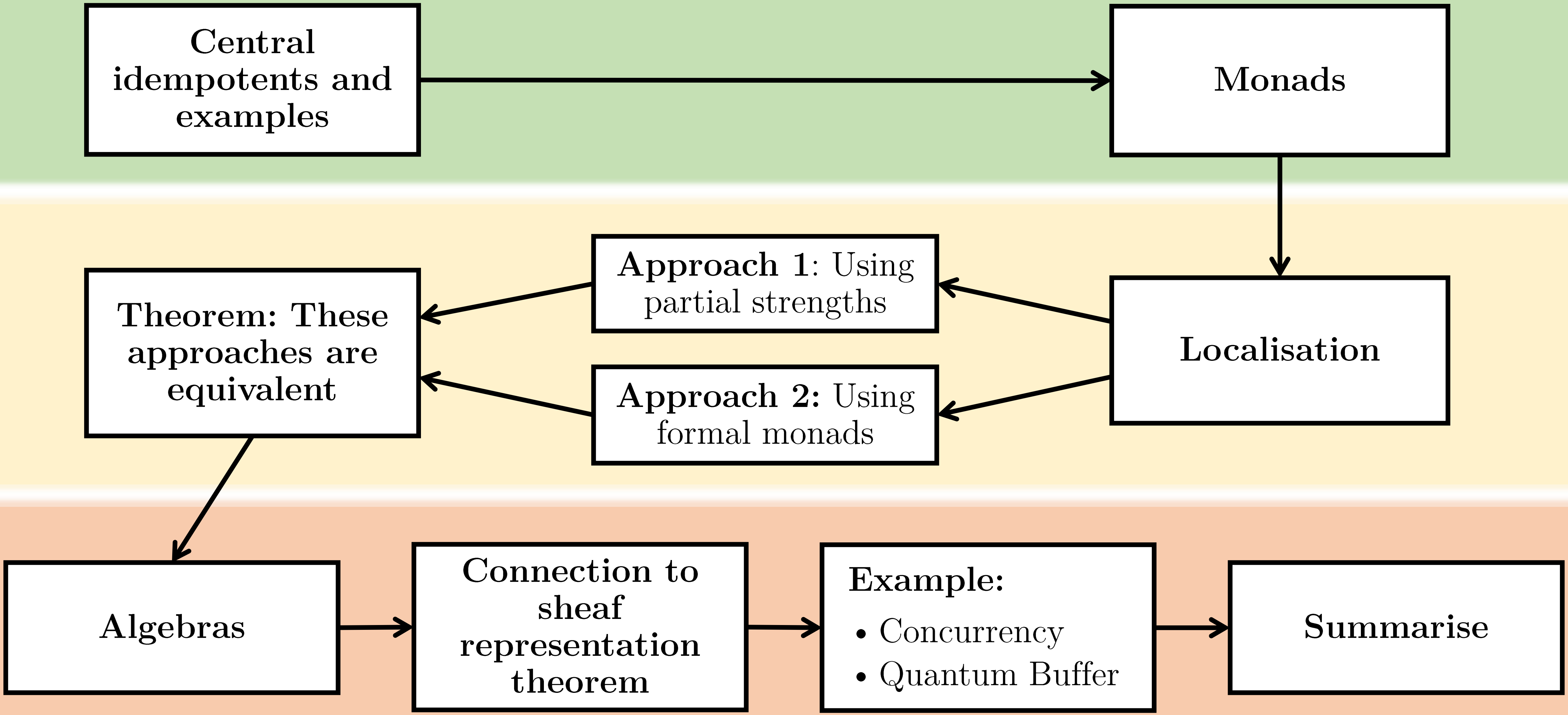
Joint work with:
Carmen Constantin
Chris Heunen

Categorical Late Lunch – August 2021

Objective

Localising monads using central idempotents

This Talk!


Central idempotents and examples
Monads
Localisation
Approach 1: Using partial strengths
Approach 2: Using formal monads
Theorem: These approaches are equivalent
Algebras
Connection to sheaf representation theorem
Example:
• Concurrency
• Quantum Buffer
Summarise

Central idempotents

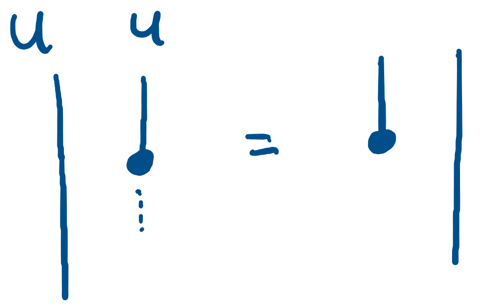
(in a symmetric monoidal category)

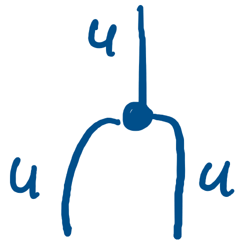
<u>**Definition**</u>

- Morphism $u : U \to I$
- Such that

$$\rho_U \circ (U \otimes u) = \lambda_U \circ (u \otimes U) : U \otimes U \to U$$

is invertible.





<u>**Equivalence Class**</u>

- Identify $u : U \to I$ and $v : V \to I$ when there is an isomorphism $m : U \to V$ such that $u = v \circ m$.

Examples

(of central idempotents)

1) $(\mathbf{Set}, \times, \{*\})$

Central idempotents: $\emptyset, \{*\}$

2) $(L, \wedge, 1)$ meet semilattice as a category

Central idempotents: all elements of L

3) (In a cartesian category, central idempotents are exactly subterminal objects.)

Sheaf $F : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$

Monoidal category of sheaves over X

$$\chi_u : \mathcal{O}^{\mathrm{op}} \to \mathbf{Set}$$

$$V \mapsto \begin{cases} \{*\} & \text{if } V \subset U \\ \emptyset & \text{if } V \not\subset U \end{cases}$$

Central idempotents: opens of X

4) R commutative unital ring

Category of R-modules (and R linear morphisms)

($\otimes$: $\otimes$ of R-modules and $I = R$)

Central idempotents: idempotent ideals

Monads and examples

- A monad is a triple $(\underset{\text{Endofunctor}}{T}, \underbrace{\mu, \eta}_{\text{Natural transformations}})$.

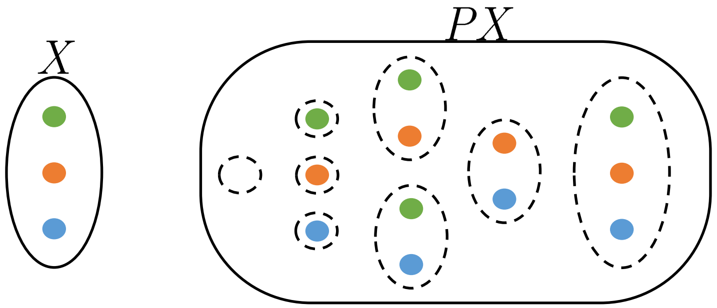
Examples:

- List/free monoid monad
- Maybe/option monad
- Writer/action monad
- etc.
- State monad
- Continuation monad
- Input-output monad

- Way of embedding objects and morphisms into additional context.

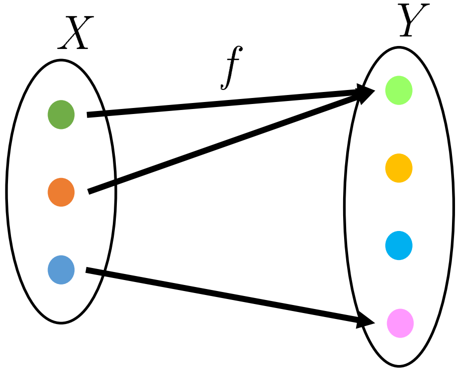
- E.g. Power Set Monad

$$\begin{aligned} P : \mathbf{Set} &\longrightarrow \mathbf{Set} \\ X &\longrightarrow PX \\ f_{X \to Y} &\longrightarrow P(f) \end{aligned}$$



η: maps to singletons

μ: union



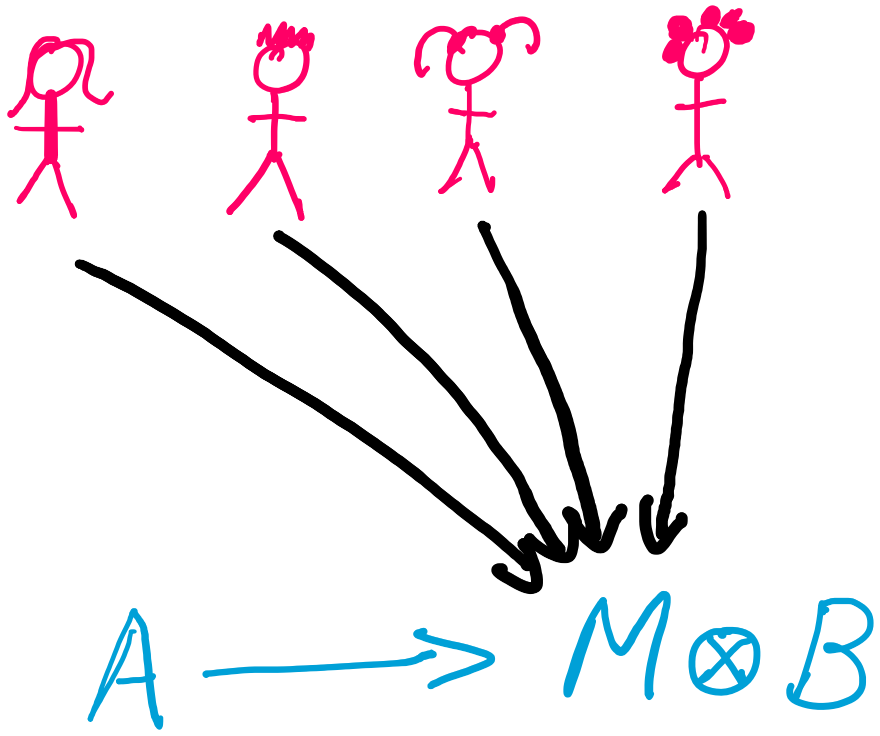
Localisation

<u>Writer monad:</u>

$$T : \mathbf{Set} \longrightarrow \mathbf{Set}$$
$$A \longmapsto M \times A$$

<u>Kleisli maps:</u>

$$f : A \longrightarrow M \times B$$



Approach 1: Partial strengths

Idea: Restrict $\mathbf{C}$ to central idempotents u.

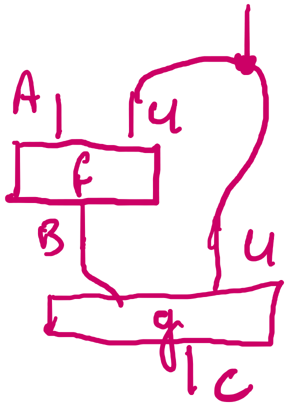
- Category $\mathbf{C}||_u$

Objects: Ob of $\mathbb{C}$

Morphisms: $A \longrightarrow B$ in $\mathbb{C}||_u$

$A \otimes u \longrightarrow B$ in $\mathbb{C}$

Composition:



Identity: $A \otimes u$

Definition: A monad T is localisable if it has partial strengths

$$\mathrm{st}_{A,U} : T(A) \otimes U \to T(A \otimes U)$$

(satisfying some compatibility axioms)

Example: Strong monads.

Example: a monad T on a cartesian closed category if

$$T(A \times B) \simeq T(A) \times T(B)$$

Approach 1: Partial strengths

Idea: Restrict $\mathbf{C}$ to central idempotents u.

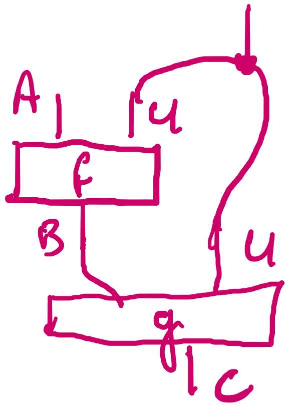
- Category $\mathbf{C}||_u$

Objects: Ob of $\mathbb{C}$

Morphisms: $A \longrightarrow B$ in $\mathbb{C}||_u$

$A \otimes u \longrightarrow B$ in $\mathbb{C}$

Composition:



Identity: $A \otimes u$

Definition: A monad T is localisable if it has partial strengths

$$\mathrm{st}_{A,U} : T(A) \otimes U \to T(A \otimes U)$$

(satisfying some compatibility axioms)

Proposition: If T is localisable, we can define "small monads" on $\mathbf{C}||_u$

$$T_u(A) = T(A)$$

$$T_u(f) = T(A) \otimes u \xrightarrow{st} T(A \otimes u) \xrightarrow{T(f)} T(B)$$

$$\eta^u_A = \eta_A \otimes u$$

$$\mu^u_A = \mu_A \otimes u$$

Approach 2: Formal monads

Street (1972): Formal theory of monads $\Longleftrightarrow$ Theory of monads in arbitrary 2-categories

$$K = [\mathrm{ZI}(\mathbf{C})^{\mathrm{op}}, \mathbf{Cat}]$$

$$\begin{aligned} \overline{\mathbf{C}} : \mathrm{ZI}(\mathbf{C})^{\mathrm{op}} &\longrightarrow \mathbf{Cat} \\ u &\longmapsto \mathbf{C}||_u \end{aligned}$$

$$T : \overline{\mathbf{C}} \longrightarrow \overline{\mathbf{C}}$$

$$T_u : \mathbf{C}||_u \longrightarrow \mathbf{C}||_u$$

$$\mu : TT \longrightarrow T$$

$$\mu_u : T_u T_u \longrightarrow T_u$$

$$\eta : 1_{\overline{\mathbf{C}}} \longrightarrow T$$

$$\eta_u : 1_{\mathbf{C}||_u} \longrightarrow T_u$$

Equivalence Theorem

$$\begin{tikzcd}[column sep=huge] \text{Localisable monads} \arrow[r, bend left, "\text{induces (using small monads)}"] & \text{Formal monads in } K = [\mathrm{ZI}(\mathbf{C})^{\mathrm{op}}, \mathbf{Cat}] \arrow[l, bend left, "\text{induces (by defining a strength)}"] \end{tikzcd}$$

<u>**Theorem:**</u> The above are equivalent.

Extra details

$$st : T_1(A) \otimes U \longrightarrow T_1(A \otimes U)$$

$$\begin{tikzcd}
\mathbb{C} \arrow[loop left, "\substack{FG \\ \| \\ -\otimes U}"] \arrow[rr, bend left, "G"] \arrow[rr, phantom, "T"] & & \mathbb{C}/\!\!/_U \arrow[ll, bend left, "F"]
\end{tikzcd}$$

$\uparrow$ coKleisli of comonad $-\otimes U$

$$G : \mathbb{C} \longrightarrow \mathbb{C}/\!\!/_U$$

$$A \longmapsto A$$

$$A \xrightarrow{f} B \longmapsto \frac{A \longrightarrow B}{A \otimes U \longrightarrow B}$$

$$A \otimes U \xrightarrow{A \otimes u} A \xrightarrow{f} B$$

$$F : \mathbb{C}/\!\!/_U \longrightarrow \mathbb{C}$$

$$A \longmapsto A \otimes U$$

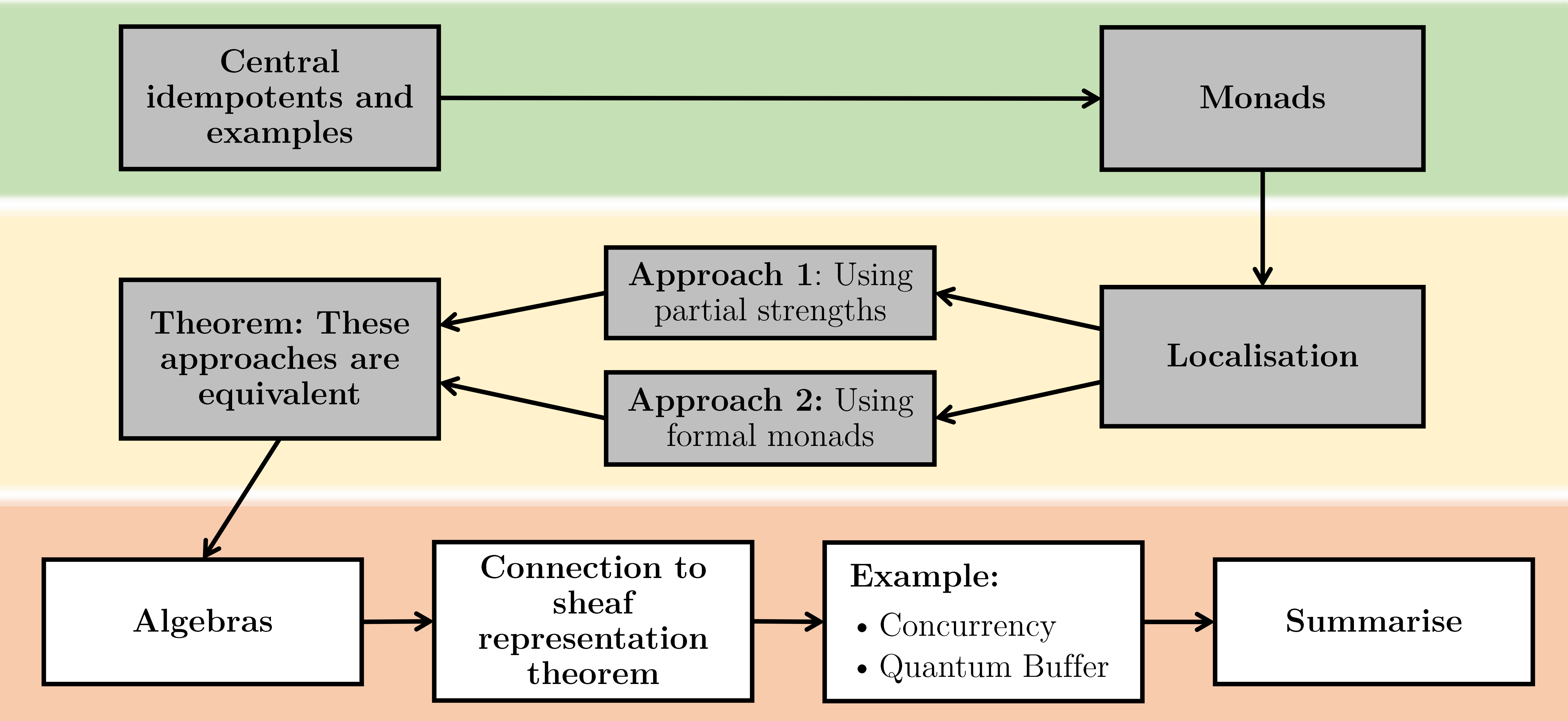
$$A \xrightarrow{f} B \longmapsto A \otimes U \to B \otimes U$$

$$(A \otimes U \to B)$$

$$A \otimes U \to A \otimes U \otimes U \xrightarrow{f \otimes U} B \otimes U$$

$$\begin{tikzcd}
st : T_1(A) \otimes U \arrow[rrr] \arrow[d, equal] & & & T_1(A \otimes U) \arrow[d, equal] \\
FGT_1A \arrow[d, equal] & & & T_1FGA \\
FT_UGA \arrow[r, "FT_U\eta_{GA}"'] & FT_UGFGA & = & FGT_1FGA \arrow[u, "\varepsilon_{T_1FGA}"']
\end{tikzcd}$$

This Talk!



Central idempotents and examples
Monads
Localisation
Approach 1: Using partial strengths
Approach 2: Using formal monads
Theorem: These approaches are equivalent
Algebras
Connection to sheaf representation theorem
Example:
• Concurrency
• Quantum Buffer
Summarise

Algebras

- R. Street. The formal theory of monads. *Journal of Pure and Applied Algebra*, 1972.
- S. Lack and R. Street. The formal theory of monads II. *Journal of Pure and Applied Algebra*, 2002.

Formal algebra category:

An object of $\mathbf{K}$ such that, for any object $X \in \mathbf{K}$:

$$T\|_{-} \circlearrowright \mathbf{C}\|_{-} \quad \rightsquigarrow \quad \mathbf{K}(X, T\|_{-}) \circlearrowright \mathbf{K}(X, \mathbf{C}\|_{-})$$

$$\beta : X \Rightarrow \mathbf{C}\|_{-} \quad \longmapsto \quad T\|_u \circ \beta_u : X_u \to \mathbf{C}\|_u$$

- This monad has a **concrete Eilenberg-Moore category** of algebras.
- Ob: pairs β, θ of type

$$\begin{tikzcd}
 & \mathbf{C}\|_u \arrow[dr, bend left, "T\|_u"] \arrow[d, phantom, "\theta_u \Downarrow"] & \\
X_u \arrow[ur, bend left, "\beta_u"] \arrow[rr, "\beta_u"'] & {} & \mathbf{C}\|_u
\end{tikzcd}$$

- Mor: modifications $\varphi : \beta \Rrightarrow \beta'$

$$\begin{tikzcd}
 & X_u \arrow[dl, bend right, "\beta_u"'] \arrow[dd, "\beta_u" description] \arrow[dd, bend left=80, "\beta'_u"] & \\
\mathbf{C}\|_u \arrow[dr, bend right, "T\|_u"'] \arrow[r, phantom, "\overset{\theta_u}{\Rightarrow}"] & {} \arrow[r, phantom, "\overset{\varphi_u}{\Rrightarrow}"] & {} \\
 & \mathbf{C}\|_u &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & X_u \arrow[dl, bend right, "\beta_u"'] \arrow[dl, bend left, "\beta'_u" description] \arrow[dd, bend left=80, "\beta'_u"] \arrow[dl, phantom, "\varphi_u \Rrightarrow"] & \\
\mathbf{C}\|_u \arrow[dr, bend right, "T\|_u"'] \arrow[r, phantom, "\underset{\theta'_u}{\Rightarrow}"] & {} & {} \\
 & \mathbf{C}\|_u &
\end{tikzcd}$$

Connections to sheaf representation theorem

Theorem[1]: Any monoidal category $\mathbf{C}$ with universal finite joins of central idempotents is monoidally equivalent to a category of global sections of a sheaf $u \mapsto \mathbf{C}||_u$ of local monoidal categories over $\mathrm{ZI}(\mathbf{C})$.

$$\mathbf{MonCat} \rightleftarrows \mathbf{MonScheme}^{op}$$

$$\begin{tikzcd} \mathbb{C} \arrow[d, "F"'] \\ \mathbb{D} \end{tikzcd} \qquad \begin{tikzcd} (\mathrm{ZI}(\mathbb{C}), \mathbb{C}||\_) \\ (\mathrm{ZI}(\mathbb{D}), \mathbb{D}||\_) \arrow[u, "\ell"'] \end{tikzcd}$$

$$F_u : \mathbb{C}||_u \to \mathbb{D}_{\ell^{-1}(u)}$$

$$T \circlearrowright \mathbb{C} \quad \longmapsto \quad \begin{tikzcd} \mathrm{ZI}(\mathbb{C}) \\ \mathrm{ZI}(\mathbb{C}) \arrow[u] \end{tikzcd}$$

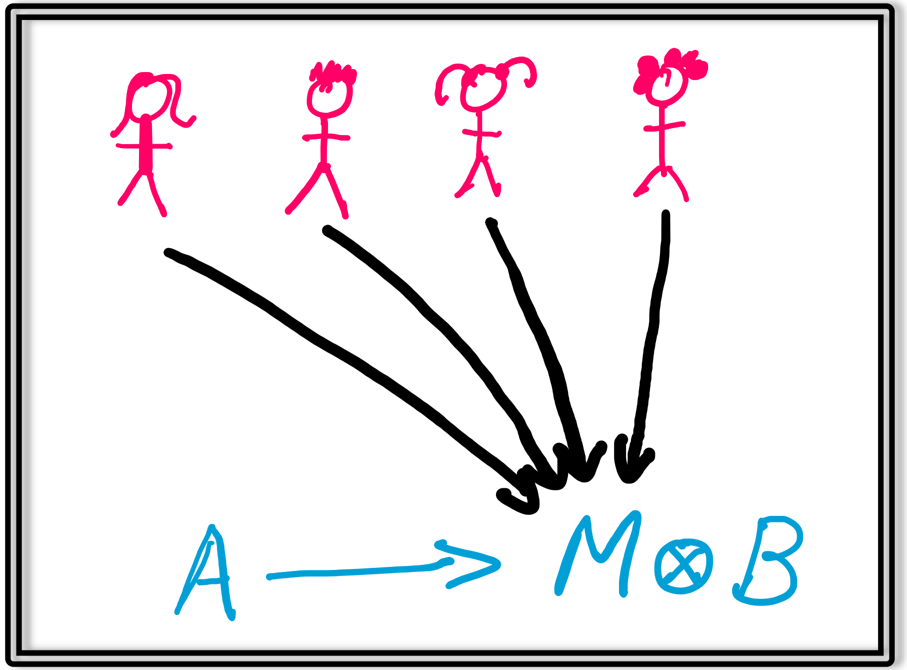
localisable

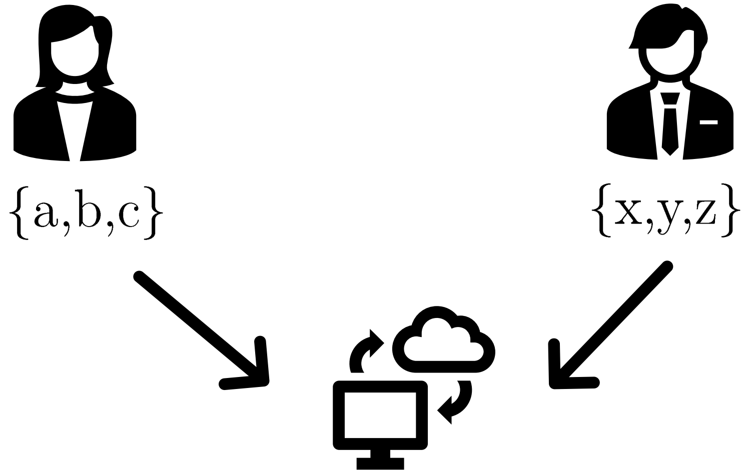
$$T_u : \mathbb{C}||_u \to \mathbb{C}||_u$$

monads

[1] R. Soares Barbosa and C. Heunen. Sheaf representation of monoidal categories. arxiv:2106.08896, 2021.

Example: Concurrency





Basic idea: M_1 and M_2 monoids

Coproduct $M_1 + M_2$

Suppose $ab = ba$ for $a \in M_1$ and $b \in M_2$

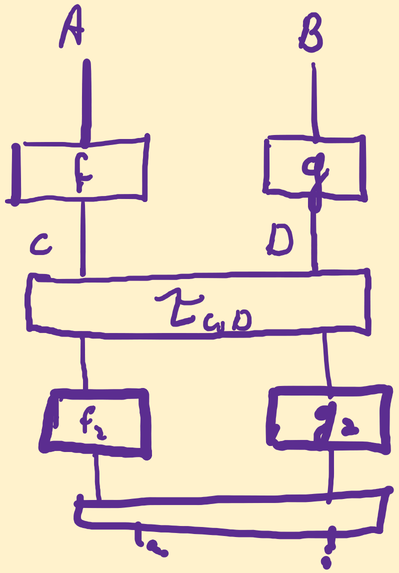
Quotient out to obtain trace monoid M

Let M_1 and M_2 be monoids in symmetric monoidal categories $\mathbf{C}_1$ and $\mathbf{C}_2$.

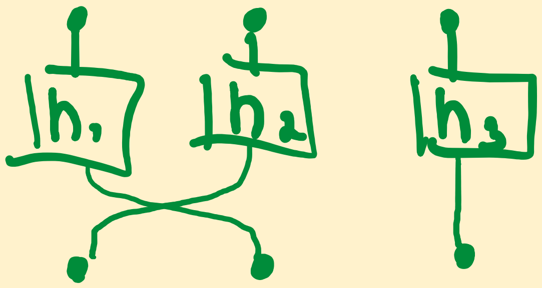
Want: Category $\mathbf{C}$ with monoid M and central idempotents u_1 and u_2, that restricts to $\mathbf{C}_1$ and $\mathbf{C}_2$.

Building C:

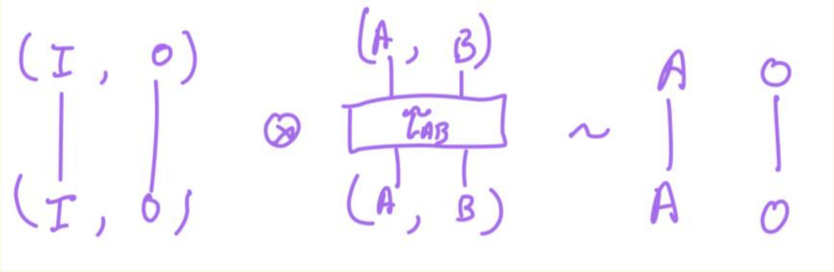
- Start with $\mathbf{C}'$:
  - Ob: pairs (A, B) of $A \in \mathbf{C}_1$ and $B \in \mathbf{C}_2$.
  - Mor:

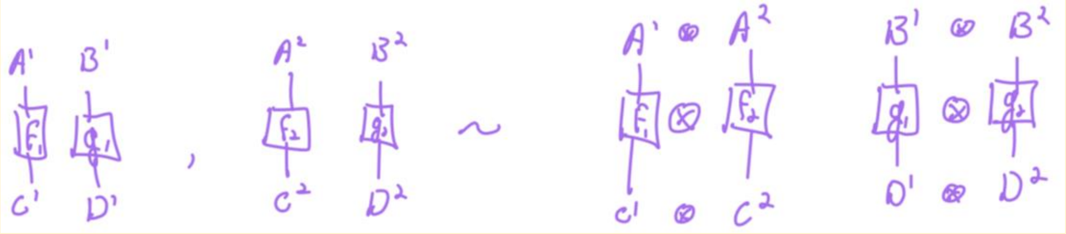


- Take the free SMC $\mathbf{C}''$ on $\mathbf{C}'$
  - Ob: lists of objects of $\mathbf{C}'$ $(A^1, B^1), (A^2, B^2), \ldots (A^n, B^n)$
  - Mor: lists of mor. with permutations



- Consider the equivalence relation $\sim$ generated by:





- Then $\mathbf{C} = \mathbf{C}''/\sim$

$$u_1 = (I, 0),\ u_2 = (0, I)$$

$$M = (M_1, M_2)$$

- For $\mathbf{C}_i = \mathbf{Set}$ this gives us a writer monad on $\mathbf{C}$.

Example: Quantum Buffer

- State monad on **Set**: $T(-) = S \multimap (- \times S)$ E.g. $S = \{0, 1\}$
- Central idempotents: $\emptyset, 1$
- This is (trivially) a localisable monad!

- State monad on $\mathbf{Set}^n$:

$$T(A_1, \ldots, A_n) = (S_1, \ldots, S_n) \multimap \big((A_1, \ldots, A_n) \times (S_1, \ldots, S_n)\big) \qquad \mathrm{ZI}(\mathbf{Set}^n) \simeq 2^n$$

- Strength: curry of

$$T(A_1, \ldots, A_n) \times (U_1, \ldots, U_n) \times (S_1, \ldots, S_n) \longrightarrow (S_1, \ldots, S_n) \times (A_1, \ldots, A_n) \times (U_1, \ldots, U_n)$$

Example: Quantum Buffer

$$\begin{tikzcd}
T(A) \otimes U \otimes V \arrow[r, "\mathrm{st}_{A,U} \otimes V"] \arrow[d, "T(A)\otimes\sigma_{U,V}"'] & T(A \otimes U) \otimes V \arrow[r, "\mathrm{st}_{A\otimes U,V}"] & T(A \otimes U \otimes V) \\
T(A) \otimes V \otimes U \arrow[r, "\mathrm{st}_{A,V} \otimes U"'] & T(A \otimes V) \otimes U \arrow[r, "\mathrm{st}_{A\otimes V,U}"'] & T(A \otimes V \otimes U) \arrow[u, "T(A\otimes\sigma_{V,U})"']
\end{tikzcd}$$

- State monad on $\mathbf{Set}^n$:

$$T(A_1, \ldots, A_n) = (S_1, \ldots, S_n) \multimap \big((A_1, \ldots, A_n) \times (S_1, \ldots, S_n)\big)$$

$$\mathrm{ZI}(\mathbf{Set}^n) \simeq 2^n$$

- Strength: curry of

$$T(A_1, \ldots, A_n) \times (U_1, \ldots, U_n) \times (S_1, \ldots, S_n) \longrightarrow (S_1, \ldots, S_n) \times (A_1, \ldots, A_n) \times (U_1, \ldots, U_n)$$

Example: Quantum Buffer

- This also works for exponentiable objects in **Hilb**

$$T(-) = S^* \otimes - \otimes S \qquad \text{where } S^* = \mathbf{Hilb}(S, \mathbb{C}) \qquad S = \mathbb{C}^2 \text{ (one qubit)} \qquad \mathrm{ZI}(\mathbf{Hilb}) = \{0, \mathbb{C}\}$$

- Similar situation for $\mathbf{Hilb}^n$ as for $\mathbf{Set}^n$.

- We can also replace the finite n in $\mathbf{Set}^n$ with an arbitrary topological space X.
- Category $\mathrm{Sh}(X)$ of sheaves on X and S the constant sheaf $S(U) = \{0, 1\}$
  - Central idempotents: open subsets
  - Monad $T(-) = S \multimap (- \otimes S)$ is localisable
  - Stalks are the (global) state monads on **Set** (storing one bit each)
- Can now model a quantum buffer over an arbitrary locally compact Hausdorff topological space.

Summary

2 ways to define local monads: equivalent

Idea was to localise on central idempotents (think opens)

Concrete examples: could be explored further

Future work: Commutativity

Thank you!

Paper: https://arxiv.org/abs/2108.01756