Axioms for the category of Hilbert spaces

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Chris Heunen
<chris.heunen@ed.ac.uk>

Andre Kornell
<akornell@tulane.edu>

THE UNIVERSITY OF EDINBURGH

Tulane University

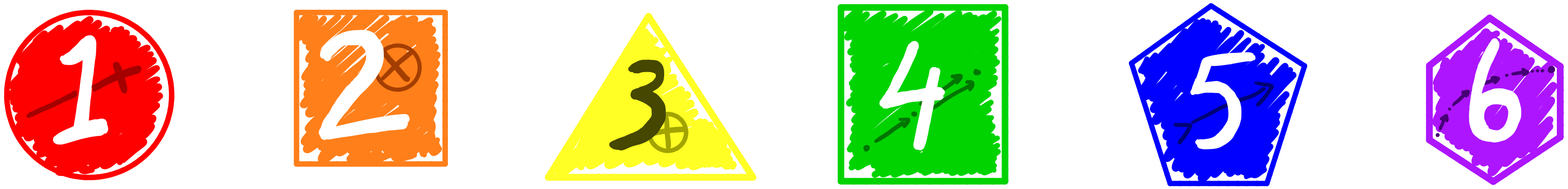
$\mathbf{Hilb}$

objects : Hilbert spaces (over fixed field $\mathbb{R}$ or $\mathbb{C}$)
(of arbitrary dimension)

arrows : continuous linear functions
$(= \text{bounded: } \exists \|f\| < \infty \, \forall x: \|f(x)\| \leq \|f\| \cdot \|x\|)$

Theorem:

If a (locally small) category $\mathbf{C}$ satisfies axioms



then $\mathbf{C} \simeq \mathbf{Hilb}$

(over the real or complex numbers)

1

Dagger

$\dagger : \mathcal{C}^{op} \longrightarrow \mathcal{C}$

$$H \xrightarrow{\ f = f^{\dagger\dagger}\ } K$$

$$H \xleftarrow{\ f^{\dagger}\ } K$$

(adjoint)

$\langle f(x) | y \rangle = \langle x | f^{\dagger}(y) \rangle$

"Way of the dagger"

dagger monomorphism

$$\begin{tikzcd} H \arrow[dr, "f"] \arrow[dd, equal] & \\ & K \arrow[dl, "f^{\dagger}"] \\ H & \end{tikzcd}$$

(isometry)

dagger isomorphism

$$\begin{tikzcd} H \arrow[dr, "f"] \arrow[dd, equal] & \\ & K \arrow[dl, "f^{\dagger}"] \arrow[dd, equal] \\ H \arrow[dr, "f"'] & \\ & K \end{tikzcd}$$

(unitary)

2⊗ Tensor

$\mathbb{C}$ is dagger symmetric monoidal (tensor product)

$$(H\otimes K)\otimes L \simeq H\otimes(K\otimes L)$$

$$I\otimes H \simeq H \simeq H\otimes I$$ dagger isomorphisms

$$H\otimes K \simeq K\otimes H$$

I is monoidal separator

$$f=g: H\otimes K \longrightarrow L \Longleftarrow \forall \begin{matrix} I \xrightarrow{h} H \\ I \longrightarrow K \end{matrix} : \; I \simeq I\otimes I \xrightarrow{h\otimes k} H\otimes K \underset{g}{\overset{f}{\rightrightarrows}} L$$

I is simple (ℝ or ℂ is 1-dim'l)

$$S \overset{s}{\rightarrowtail} I \quad \Longrightarrow \quad \begin{tikzcd} I \arrow[dr, equal] \arrow[dd, dashed] & \\ & I \\ S \arrow[ur, "s"'] & \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} S \arrow[dr, "s"] \arrow[dd, dashed] & \\ & I \\ 0 \arrow[ur, "0"'] & \end{tikzcd}$$

3

Biproducts

zero object

$$\begin{tikzcd} H \arrow[r, dashed, "\exists!"] \arrow[rr, bend right, "0_{H,K}"'] & 0 \arrow[r, dashed, "\exists!"] & K \end{tikzcd}$$

(0-dim'l space)

(binary) dagger (co)products

(direct sum)

$$\begin{tikzcd} H \arrow[r, "i"] \arrow[rr, bend left, "0_{H,K}"] \arrow[dr] & H \oplus K \arrow[r, shift left, "j^\dagger"] \arrow[d, dashed] & K \arrow[l, shift left, "j"] \arrow[dl] \\ & L & \end{tikzcd}$$

4 Equalisers

dagger equaliser = equaliser that is dagger monomorphism

$$\begin{tikzcd}
E \arrow[loop left, "id_E"] \arrow[r, dashed, "e"', shift right] & H \arrow[l, "e^\dagger"', shift right] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & K \\
E' \arrow[u, dashed] \arrow[ur] & &
\end{tikzcd}$$

$\left(\{x \mid f(x) = g(x)\} \text{ is closed subspace} \right)$

5 Kernels

any dagger monomorphism is a kernel

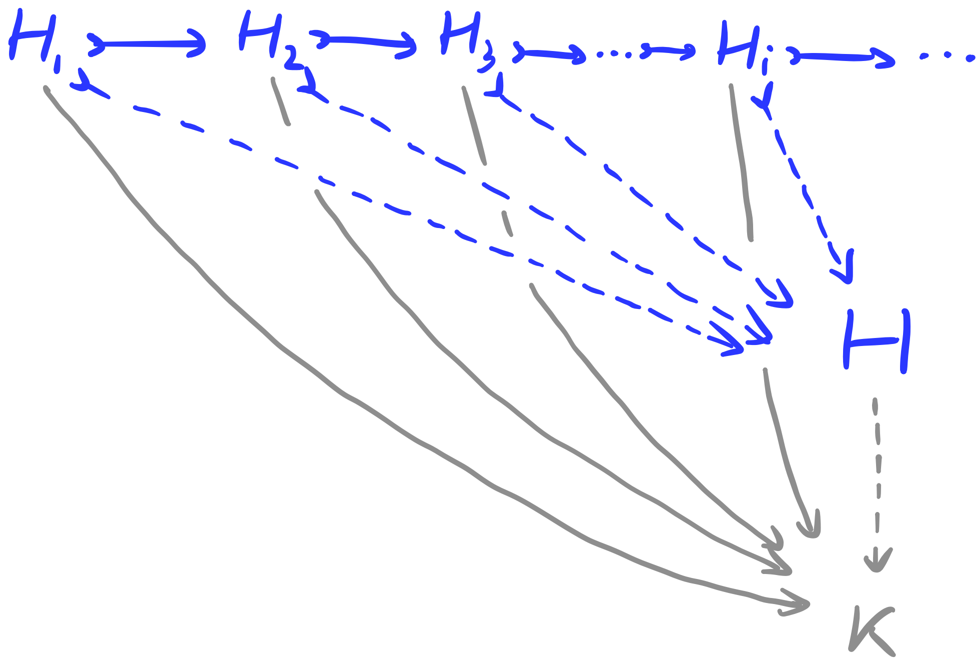
$$
\begin{tikzcd}
N \arrow[loop left, "id_N"] \arrow[r, shift right, "f"'] & H \arrow[l, shift right, "f^{\dagger}"'] \arrow[r, dashed, shift left] \arrow[r, shift right, "0_{H,K}"'] & K \\
N' \arrow[u, dashed] \arrow[ur] & &
\end{tikzcd}
$$

(closed subspace $N \subseteq H$
is kernel of $H \simeq N \oplus N^{\perp} \to N^{\perp}$)

6 Directed colimits*

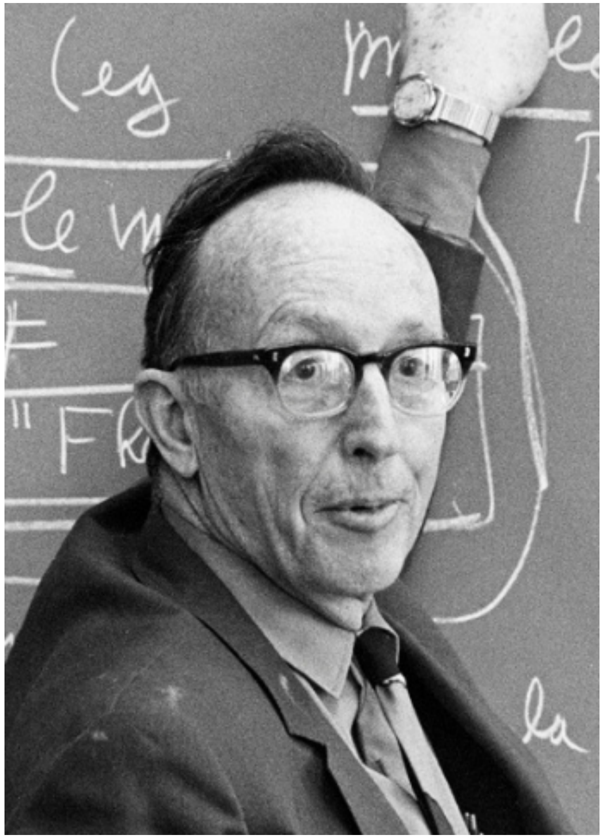
(* of dagger monomorphisms)

Wide subcat of dagger monomorphisms has directed colimits



(completion of union)

Saunders Mac Lane



Peter Selinger

Jamie Vicary

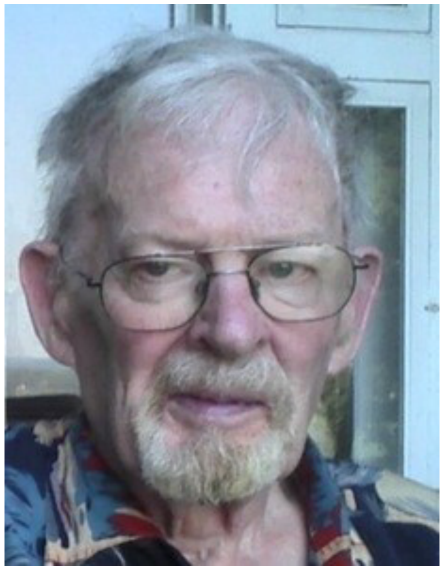
Martti Karvonen

Samson Abramsky

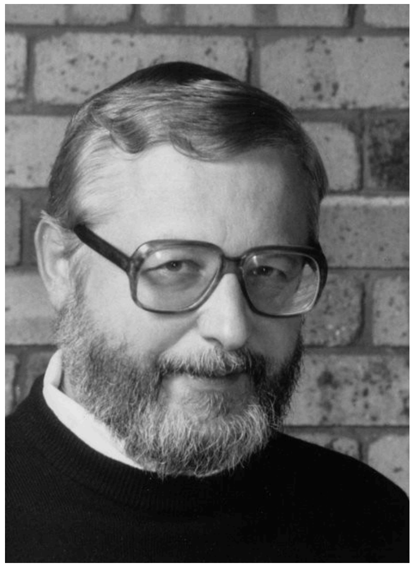
Bob Coecke

Peter Freyd

Barry Mitchell

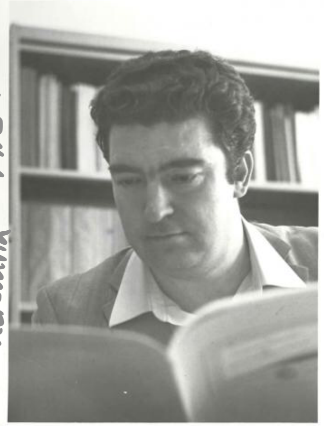


Bill Lawvere



John von Neumann

Constantin Piron



Josef-Maria Jauch

Huzihiro Araki

Samuel Holland Jr.

John Wilbur

Garrett Birkhoff

Shuichiro Maeda

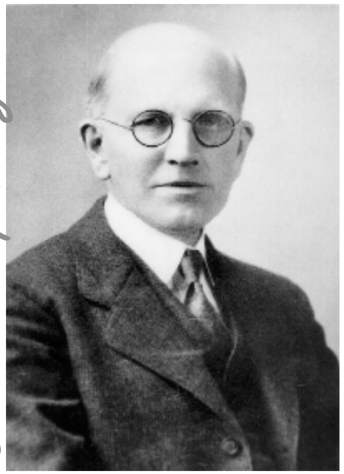
Ichiro Amemiya

Maria Pia Solèr

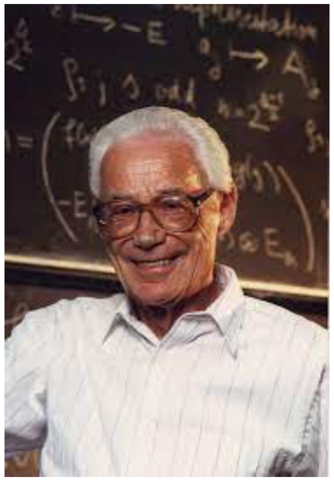
Hans Keller

Oswald Veblen

John Wesley Young



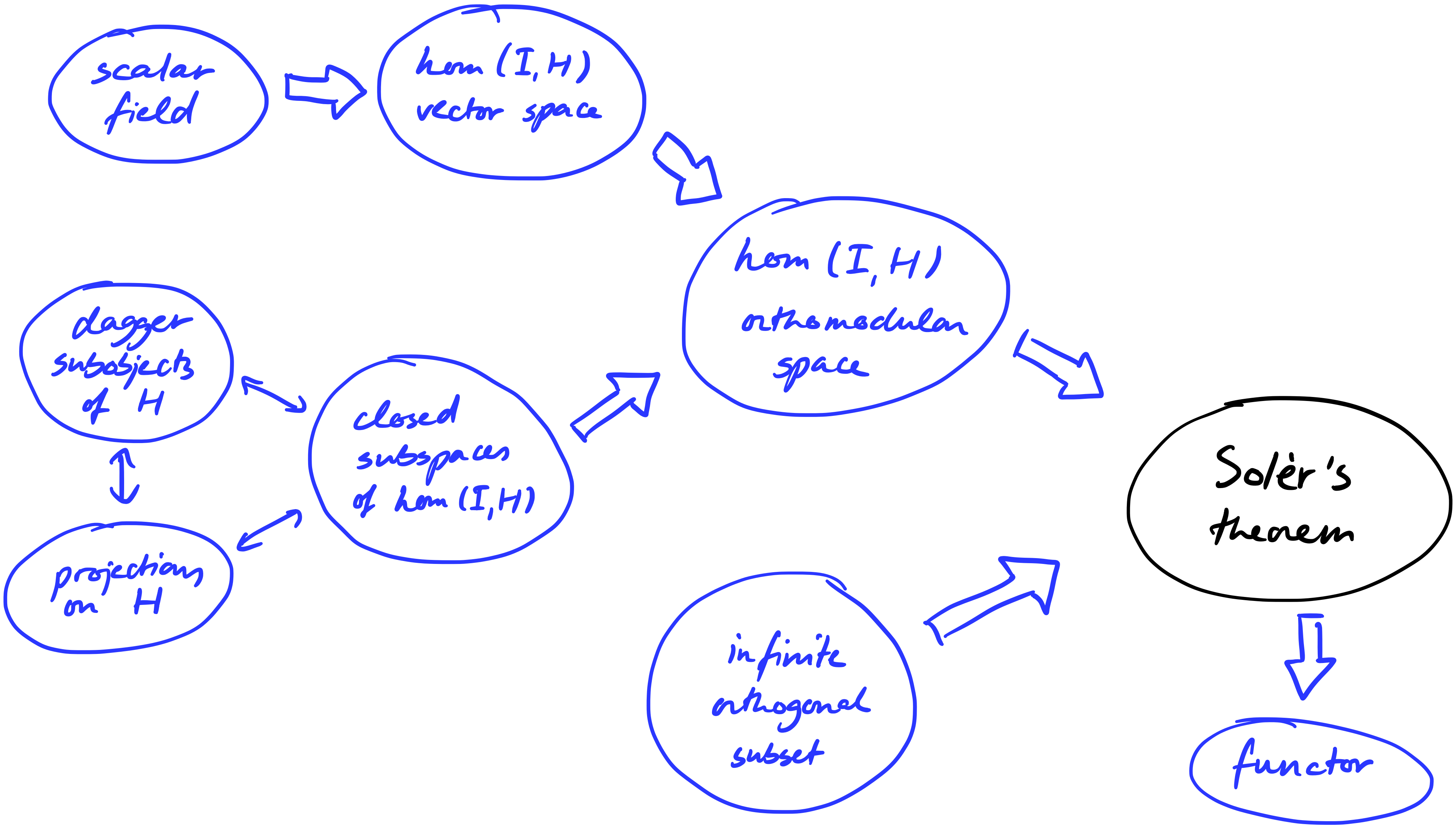
Beno Eckmann



Peter Hilton

Bart Jacobs

Proof


scalar field
hom (I, H) vector space
hom (I, H) orthomodular space
dagger subobjects of H
closed subspaces of hom (I, H)
projections on H
infinite orthogonal subset
Solèr's theorem
functor

Scalars $I \xrightarrow{z} I$ form:

- monoid

$$\begin{tikzcd} & I \arrow[dr, "w"] & \\ I \arrow[ur, "z"] \arrow[rr, dashed, "w \cdot z"'] & & I \end{tikzcd} \qquad I \xrightarrow{id_I} I$$

- commutative monoid

$$\begin{tikzcd}
I \arrow[rr, "w"] \arrow[dr, "z"] \arrow[dd, equal] & & I \arrow[dr, "z"] \arrow[dd, equal] & \\
& I \arrow[rr, "w"] \arrow[dd, equal] & & I \arrow[dd, equal] \\
I\otimes I \arrow[rr, "w\otimes id"] \arrow[dr, "id\otimes z"'] & & I\otimes I \arrow[dr, "id\otimes z"] & \\
& I\otimes I \arrow[rr, "w\otimes id"'] & & I\otimes I
\end{tikzcd}$$

- commutative semiring

$$\begin{tikzcd} I \arrow[r, dashed, "w+z"] \arrow[d] & I \\ I\oplus I \arrow[r, "w\oplus z"'] & I\oplus I \arrow[u] \end{tikzcd} \qquad I \xrightarrow{0_{I,I}} I$$

- involutive commutative semiring $\qquad I \xrightarrow{z^\dagger} I$

Scalars $I \xrightarrow{z} I$ form involutive field:

$$
\begin{tikzcd}
I \arrow[rr, "z"] \arrow[dr, dashed, two heads] & & I \\
& \bullet \arrow[ur, hook, "f"'] &
\end{tikzcd}
$$

$f = \ker(\underbrace{\mathrm{coker}(z)}_{=\ker(z^\dagger)^\dagger})$

$f = 0$ or f iso

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

$z = 0$ ✓ $\qquad$ z epi $\Longrightarrow$ $z^\dagger = 0$ or $z^\dagger$ iso ✓

$\Downarrow$

$z = 0$ ✓

Semiring theory: zerosum-free or field ✓

$(w + z = 0 \Longrightarrow w = z = 0)$

$\Downarrow$

$\ker(I \oplus I \longrightarrow I) = 0$ ✗

Projections $\left\{ H \xrightarrow{p = p^\dagger \circ p} H \right\} \simeq$ dagger subobjects $\left\{ S \overset{s}{\rightarrowtail} H \mid s^\dagger \circ s = id_S \right\} / \simeq$

$$s \circ s^\dagger \longleftarrow\!\shortmid \; s$$

$$\begin{tikzcd} H \arrow[rr, "p"] \arrow[dr, dashed, two heads] & & H \\ & \bullet \arrow[ur, tail] & \end{tikzcd} \longmapsto \ker(\mathrm{coker}(p))$$

$$p \leq q \iff s \leq t$$

$(p \circ q = p)$

$$\left(\begin{tikzcd} T \arrow[r, tail, "t"] & H \\ S \arrow[u, dashed] \arrow[ur, tail, "s"'] & \end{tikzcd} \right)$$

Dagger subobjects $\{ S \xrightarrow{s} H \mid s^\dagger \circ s = id_S \}$

- have directed least upper bounds
- have binary greatest lower bounds (pullbacks)
- have orthocomplement

$$s^\perp = \ker(s^\dagger)$$

$$s \leq t \iff t^\perp \leq s^\perp$$

$$s^{\perp\perp} = s$$

$\Longrightarrow$ form complete lattice

So: projections $\{ H \xrightarrow{p = p^\dagger \circ p} H \}$ are also complete lattice with orthocomplement $p^\perp = id_H - p$

<u>Hermitean</u> <u>form</u> on vector space $\mathcal{H} = \mathrm{Hom}(I, H)$

$$\langle f | g \rangle = g^\dagger \circ f \quad \text{is nondegenerate}$$

subspace $V \subseteq \mathcal{H}$ has orthocomplement $V^\perp = \{ f \in \mathcal{H} \mid \forall v \in V : \langle f | v \rangle = 0 \}$

is <u>closed</u> when $V^{\perp\perp} = V$

$$\{ V \subseteq \mathcal{H} \text{ closed subspace} \} \simeq \{ H \xrightarrow{p} H \text{ projection} \}$$

$$p \circ \mathcal{H} = \{ p \circ h \mid h \in \mathcal{H} \} \longmapsfrom p$$

Hermitean space $\mathcal{H} = \hom(I, H)$

- is <u>orthomodular</u> ($\mathcal{H} = V \oplus V^{\perp}$ for all closed $V \subseteq \mathcal{H}$)

  (Pf: $V = p \circ \mathcal{H}$ for some $H \xrightarrow{p} H$

  $p^{\perp} \circ v \in V^{\perp} \Leftarrow w^{\dagger} \circ (p^{\perp} \circ v) = ((id_H - p) \circ w)^{\dagger} \circ v = (w - (p \circ w))^{\dagger} \circ v = 0$

  $p \circ v \in V$

  $v = (p \circ v) + (p^{\perp} \circ v) \in V + V^{\perp}$)

- has orthogonal subset of size n iff $H = \underbrace{I \oplus \cdots \oplus I}_{n}$

- has infinite orthogonal subset if $H = I \oplus I \oplus \cdots$

Solèr's theorem $\Longrightarrow$

- field is $\mathbb{R}$ or $\mathbb{C}$ ~~or $\mathbb{H}$~~

- $\hom(I, I \oplus I \oplus \cdots)$ is Hilbert space

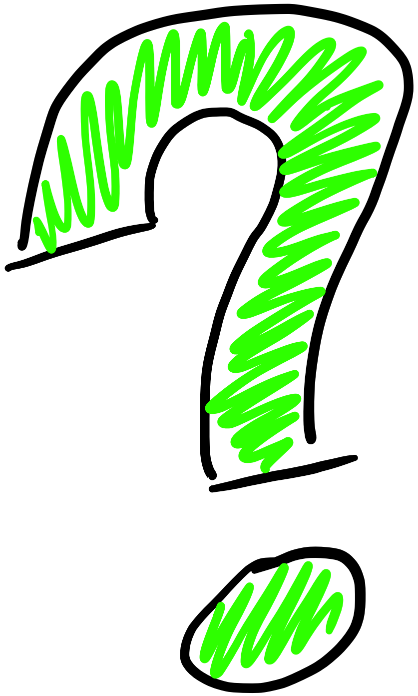
$\Longrightarrow$ $\hom(I, H \oplus I \oplus I \oplus \cdots)$ is Hilbert space

$\Longrightarrow$ $\hom(I, H)$ is Hilbert space

Functor $\hom(I, -): \mathbf{C} \longrightarrow \mathbf{Hilb}$

- is functorial 1 2 3
  (use Hellinger-Toeplitz)
- is faithful 2
- is full 3 6
  (continuous linear map = linear combination of isometries)
- is essentially surjective (Hilbert space $\mathcal{H} = \underbrace{I \oplus I \oplus \cdots}_{\dim(\mathcal{H})}$)
- preserves $\dagger$ and $\otimes$

Q.E.D.

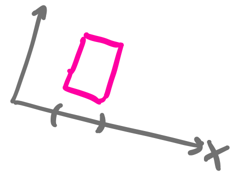


- Variations?

  contractions, isometries, unitaries $\|f\| \le 1$

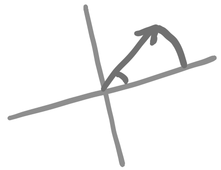
  finite-dimensional Hilbert spaces

  Hilbert modules



- Structure directly from axioms?

  polar decomposition



  Stone's theorem

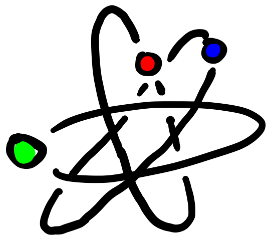
  Gleason's theorem

  Wigner's theorem

- Physical interpretation of axioms?

  reformulation of axioms

  reconstruction of quantum mechanics



- Logic?

  toposes

  abelian categories

  effectuses