Return of the Alley Cats

or

Ten Minutes of Magnitude

Emily Roff

The Categorical Late Lunch
October 2021

Further shady deals

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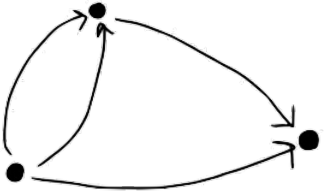

Fig 1. What we want

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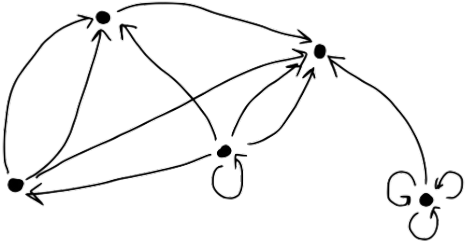

Fig 2. What we get

How to check for the absence of an adjunction

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Take two finite categories, **X** and **Y**.

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4. Apply this Lemma:

Lemma (Leinster, 2008)

Suppose $Z_{\mathbf{X}}$ and $Z_{\mathbf{Y}}$ are both invertible, and there exists an adjunction $\mathbf{X} \leftrightarrows \mathbf{Y}$. Then

$$\sum_{x,x'} Z_{\mathbf{X}}^{-1}(x, x') = \sum_{y,y'} Z_{\mathbf{Y}}^{-1}(y, y').$$

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Definition (Leinster, 2008)

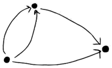
When $Z_{\mathbf{X}}$ is invertible, we call $\sum_{x,x'} Z_{\mathbf{X}}^{-1}(x, x') =: \mathrm{Mag}(\mathbf{X})$ the **magnitude** of $\mathbf{X}$.

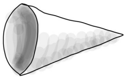
"Fancy" proof of the Lemma

The **classifying space** of a category

$$\mathbf{Cat} \longrightarrow [\triangle^{op}, \mathbf{Set}] \longrightarrow \mathbf{Top}$$

$$\mathbf{X} \longmapsto \mathrm{Nerve}(\mathbf{X}) \longmapsto \mathbb{B}\mathbf{X}$$





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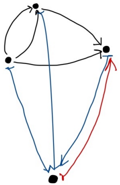

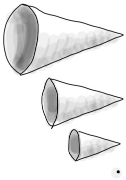

Combine these facts:

- Fact 1 $\mathbb{B}(-)$ is a 2-functor. Thus, any adjunction $\mathbf{X} \rightleftarrows \mathbf{Y}$ induces a homotopy equivalence $\mathbb{B}\mathbf{X} \simeq \mathbb{B}\mathbf{Y}$.

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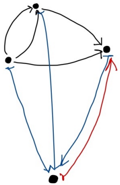

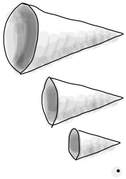

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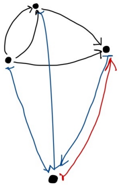

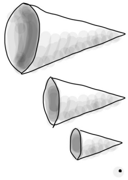

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- Fact 2 Homotopy equivalent spaces have isomorphic homology groups.
- Crucial Fact Under finiteness conditions,

$$\chi(H_\bullet(\mathbb{B}\mathbf{X})) = \mathrm{Mag}(\mathbf{X}).$$

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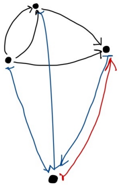

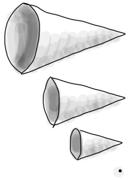

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$$\chi(H_\bullet(\mathbb{B}\mathbf{X})) = \mathrm{Mag}(\mathbf{X}).$$

Another Fact $H_\bullet(\mathbb{B}\mathbf{X}) \cong H_\bullet(C_\bullet)$ where $C_n = \mathbb{Q} \cdot \{x_0 \xrightarrow{f_1} x_1 \cdots \xrightarrow{f_n} x_n \mid f_i \neq \mathrm{Id} \text{ for all } i\}$.

Proof that $\chi(H_\bullet(\mathbb{B}\mathbf{X})) = \mathrm{Mag}(\mathbf{X})$*

$$\chi(H_\bullet(\mathbb{B}\mathbf{X})) = \sum_{n=0}^{\infty} (-1)^n \#\{x_0 \xrightarrow{f_1} x_1 \cdots \xrightarrow{f_n} x_n \mid f_i \neq \mathrm{Id} \text{ for all } i\}$$

Leinster & Shulman (2021)

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&= \sum_{n=0}^{\infty} (-1)^n \sum_{x_0, x_n \in \mathrm{ob}(\mathbf{X})} (Z_{\mathbf{X}} - I)^n (x_0, x_n)
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&= \sum_{x, y \in \mathrm{ob}(\mathbf{X})} Z_{\mathbf{X}}^{-1}(x, y) = \mathrm{Mag}(\mathbf{X}). \qquad \square
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Beyond ordinary categories

Magnitude makes sense for enriched categories, too! E.g. for a linear category $\mathbf{X}$, write

$$Z_{\mathbf{X}}(x, y) = \dim(\mathbf{X}(x, y))$$

and when $Z_{\mathbf{X}}$ is invertible, say the **magnitude** of $\mathbf{X}$ is $\sum_{x,y} Z_{\mathbf{X}}^{-1}(x, y)$.

But what's the enriched analogue of the topological story? Of $H_\bullet(\mathbb{B}\mathbf{X})$?

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What is the appropriate notion of **nerve** or **classifying space** for linear categories?

References

Leinster. The Euler characteristic of a category. *Documenta Mathematica* 13 (2008).

Leinster and Shulman. Magnitude homology of enriched categories and metric spaces. *Algebraic and Geometric Topology* 21 (2021).