Strict monoidal categories are monoids in what category?

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Introduction

- I was creating a presentation of a strict symmetric monoidal category (specifically a PRO), but was unhappy with my axiom schemas.
- While generalizing strict SMCs, I came across this characterization which happened to be exactly what I needed! The approach examined here is due to Soichiro Fujii.[1]
- There is another approach characterizing strict monoidal categories as monads in the bicategory of spans on the category of monoids.

[1] S. Fujii, 'A unified framework for notions of algebraic theory', Theory and Applications of Categories, vol. 34, pp. 1246–1316, Nov. 2019.

Key idea

- For a strict monoidal category $\mathcal{M}$, the homset $\mathcal{M}(-,=)\colon \mathsf{N}^{\mathsf{op}} \times \mathsf{N} \to \mathsf{Set}$ is a lax monoidal functor which is also a monoid with respect to profunctor composition $\diamond$ in a manner respecting the lax structure.
- Conversely, a lax endo-profunctor on N which is a monoid with respect to $\diamond$ is induces an identity-on-objects functor from N which strictly preserves the monoidal structure.

Lax monoidal functors

Definition

Let $(\mathcal{A}, \otimes_\mathcal{A}, I_\mathcal{A})$ and $(\mathcal{B}, \otimes_\mathcal{B}, I_\mathcal{B})$ be monoidal categories. A *lax monoidal functor* $F\colon \mathcal{A} \to \mathcal{B}$ is a functor from $\mathcal{A}$ to $\mathcal{B}$ (also denoted F) as well as

$$\eta_F\colon I_\mathcal{B} \to F(I_\mathcal{A}) \qquad \mu_F\colon F(X) \otimes_\mathcal{B} F(Y) \to F(X \otimes_\mathcal{A} Y)$$

such that the following diagrams commute:

$$\begin{array}{ccc}
(F(X) \otimes_\mathcal{B} F(Y)) \otimes_\mathcal{B} F(Z) & \longrightarrow & F(X) \otimes_\mathcal{B} (F(Y) \otimes_\mathcal{B} F(Z)) \\
{\scriptstyle \mu\otimes \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}\otimes\mu} \\
F(X \otimes_\mathcal{A} Y) \otimes_\mathcal{B} F(Z) & & F(X) \otimes_\mathcal{B} F(Y \otimes_\mathcal{A} Z) \\
{\scriptstyle \mu}\big\downarrow & & \big\downarrow{\scriptstyle \mu} \\
F((X \otimes_\mathcal{A} Y) \otimes_\mathcal{A} Z) & \longrightarrow & F(X \otimes_\mathcal{A} (Y \otimes_\mathcal{A} Z))
\end{array}$$

$$\begin{array}{ccc}
I_\mathcal{B} \otimes_\mathcal{B} F(X) & \xrightarrow{\eta\otimes\mathrm{id}} & F(I_\mathcal{A}) \otimes_\mathcal{B} F(X) \\
\big\downarrow & & \big\downarrow{\scriptstyle \mu} \\
F(X) & \longleftarrow & F(I_\mathcal{A} \otimes_\mathcal{A} X)
\end{array}
\qquad
\begin{array}{ccc}
F(X) \otimes_\mathcal{B} I_\mathcal{B} & \xrightarrow{\mathrm{id}\otimes\eta} & F(X) \otimes_\mathcal{B} F(I_\mathcal{A}) \\
\big\downarrow & & \big\downarrow{\scriptstyle \mu} \\
F(X) & \longleftarrow & F(X \otimes_\mathcal{A} I_\mathcal{A})
\end{array}$$

Monoidal natural transformation

Definition

Let $(\mathcal{A}, \otimes_{\mathcal{A}}, I_{\mathcal{A}})$ and $(\mathcal{B}, \otimes_{\mathcal{B}}, I_{\mathcal{B}})$ be monoidal categories and (F, η_F, μ_F) and (G, η_G, μ_G) be lax monoidal functors from $\mathcal{A}$ to $\mathcal{B}$. A *monoidal natural transformation* $\theta \colon F \Rightarrow G$ is a natural transformation from F to G (also denoted θ) such that the following diagrams commute:

$$\begin{tikzcd}
F(X) \otimes_{\mathcal{B}} F(Y) \arrow[r, "\theta \otimes \theta"] \arrow[d, "\mu_F"'] & G(X) \otimes_{\mathcal{B}} G(Y) \arrow[d, "\mu_G"] \\
F(X \otimes_{\mathcal{A}} Y) \arrow[r, "\theta"'] & G(X \otimes_{\mathcal{A}} Y)
\end{tikzcd}$$

$$\begin{tikzcd}
 & I_{\mathcal{B}} \arrow[dl, "\eta_F"'] \arrow[dr, "\eta_G"] & \\
F(I_{\mathcal{A}}) \arrow[rr, "\theta"'] & & G(I_{\mathcal{A}})
\end{tikzcd}$$

$\mathbf{MonCAT_{lax}}(\mathsf{N}^{\mathsf{op}} \times \mathsf{N}, \mathsf{Set})$

Definition

For monoidal categories $(\mathcal{A}, \otimes_{\mathcal{A}}, I_{\mathcal{A}})$ and $(\mathcal{B}, \otimes_{\mathcal{B}}, I_{\mathcal{B}})$, we write $\mathbf{MonCAT_{lax}}(\mathcal{A}, \mathcal{B})$ for the category which has lax monoidal functors from $\mathcal{A}$ to $\mathcal{B}$ as objects and monoidal natural transformations as morphisms.

Proposition

The functor $\hom_{\mathsf{N}} \colon \mathsf{N}^{\mathsf{op}} \times \mathsf{N} \to \mathsf{Set}$ *is lax monoidal with*

$$\eta \colon 1 \to \hom_{\mathsf{N}}(0,0),\ * \mapsto \mathrm{id}$$

$$\begin{aligned}\mu \colon \hom_{\mathsf{N}}(a_1, b_1) \times \hom_{\mathsf{N}}(a_2, b_2) &\to \hom_{\mathsf{N}}(a_1 + a_2, b_1 + b_2) \\ (f_1, f_2) &\mapsto f_1 + f_2\end{aligned}$$

Profunctor composition is lax compatible

Definition

Let $S, T\colon \mathsf{N}^{\mathsf{op}} \times \mathsf{N} \to \mathsf{Set}$, then we define the *profunctor composition* of S and T by

$$(S \diamond T)(a, c) := \int^{b} T(a, b) \times S(b, c)$$

Lemma

Suppose $S, T\colon \mathsf{N}^{\mathsf{op}} \times \mathsf{N} \to \mathsf{Set}$ *are lax monoidal. Then* $S \diamond T$ *is lax monoidal with*

$$\eta_{S \diamond T}\colon 1 \to 1 \times 1 \xrightarrow{\eta_S \times \eta_T} S(0,0) \times T(0,0) \to \int^{b} T(0,b) \times S(b,0) \cong (S \diamond T)(0,0)$$

$$
\begin{aligned}
\mu_{S\diamond T}\colon \left(\int^{b_1} T(a_1,b_1)\times S(b_1,c_1)\right)\times\left(\int^{b_2} T(a_2,b_2)\times S(b_2,c_2)\right)\\
&\cong \int^{b_1,b_2} T(a_1,b_1)\times S(b_1,c_1)\times T(a_2,b_2)\times S(b_2,c_2)\\
&\cong \int^{b_1,b_2} T(a_1,b_1)\times T(a_2,b_2)\times S(b_1,c_1)\times S(b_2,c_2)\\
&\xrightarrow{\int \mu_T\times\mu_S} \int^{b_1,b_2} T(a_1+a_2,b_1+b_2)\times S(b_1+b_2,c_1+c_2)\\
&\cong \int^{b_1,b_2,x,y} T(a_1+a_2,x)\times \mathrm{hom}_{\mathsf{N}}(x,b_1+b_2)\times \mathrm{hom}_{\mathsf{N}}(b_1+b_2,y)\times S(y,c_1+c_2)\\
&\xrightarrow{\int \mathrm{id}\times\circ\times\mathrm{id}} \int^{x,y} T(a_1+a_2,x)\times \mathrm{hom}_{\mathsf{N}}(x,y)\times S(y,c_1+c_2)\\
&\cong \int^{b} T(a_1+a_2,b)\times S(b,c_1+c_2)
\end{aligned}
$$

Bringing it all together

Lemma

The associator and unitors for profunctor composition are monoidal natural transformations.

Lemma

$(\mathbf{MonCAT}_{\mathbf{lax}}(\mathsf{N}^{\mathsf{op}} \times \mathsf{N}, \mathsf{Set}), \diamond, \hom_{\mathsf{N}})$ *is a monoidal category.*

Theorem

Monoids in $(\mathbf{MonCAT}_{\mathbf{lax}}(\mathsf{N}^{\mathsf{op}} \times \mathsf{N}, \mathsf{Set}), \diamond, \hom_{\mathsf{N}})$ *are equivalent to PROs.*

The upside: My generalization replaces Set, and thus clearly explains in what sense it is actually a generalization!