Modularity and Monads

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What kind of category theory do I do?

Monads

- Way of adding structure to objects and morphisms

$$\underbrace{(T}_{\text{Endofunctor}}, \underbrace{\mu, \eta}_{\text{Natural transformations}}) \ + \text{axioms}$$

$$T : \mathbf{C} \to \mathbf{C}$$

$$\mu : TT \Rightarrow T$$

$$\eta : \mathrm{id}_{\mathbf{C}} \Rightarrow T$$

- Handling of side-effects in programming languages

Modularity

- Structures that are essentially-monads-but-not-quite

- “Bunch” of monads with indexing

$$T_i : \mathbf{C} \to \mathbf{C}$$

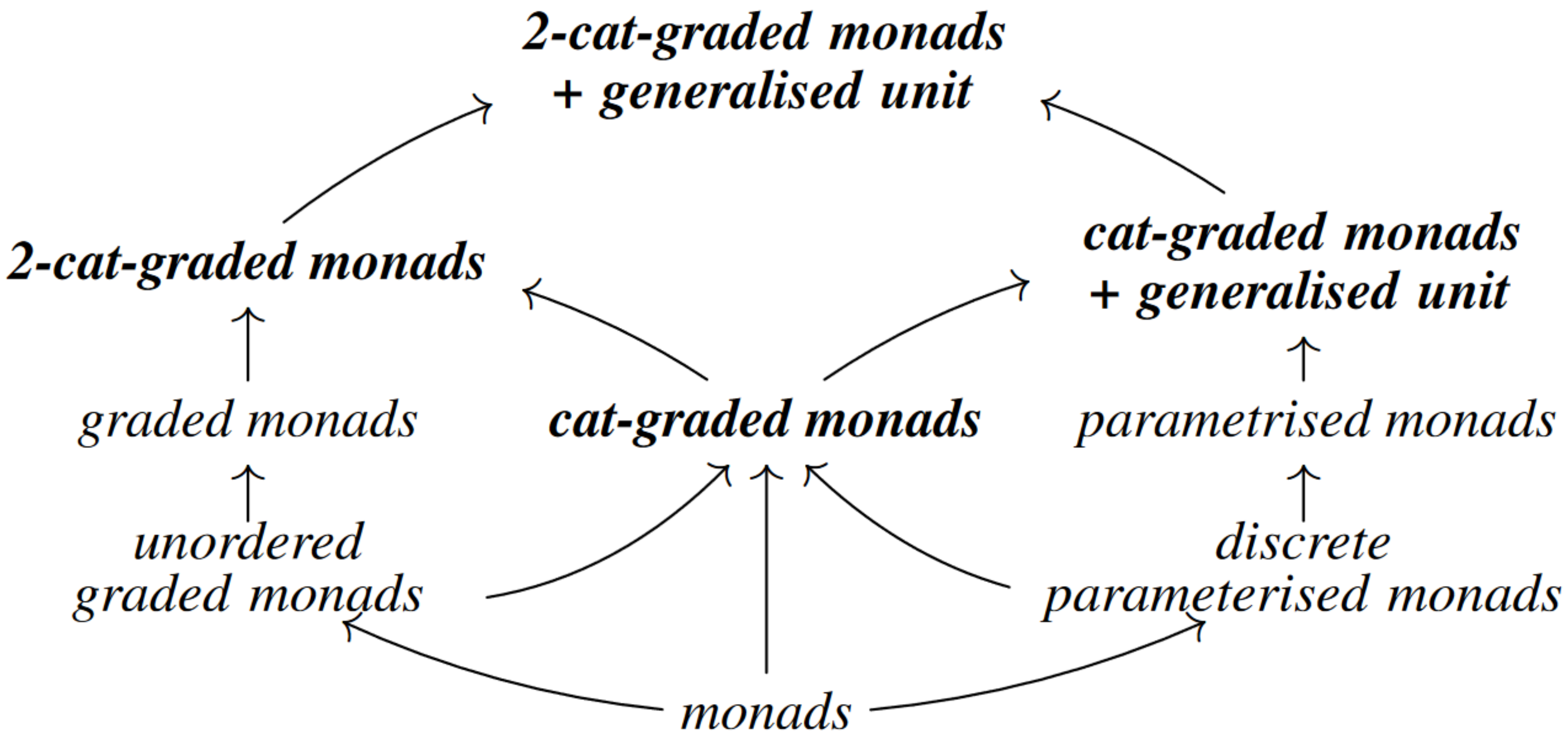
$$\mu_i : T_i T_i \Rightarrow T_i$$

$$\eta_i : \mathrm{id}_{\mathbf{C}} \Rightarrow T_i$$

- What is the nature of i ?
- Can we have i and j ?
- Same for all components?

	Indexed Monad	Graded Monad	Parameterised Monad	Category-Graded Monad
	$\mathbf{M}$ monoidal category $\mathbf{C}$ category	$\mathbf{M}$ monoidal category $\mathbf{C}$ category	$\mathbf{S}$ small category $\mathbf{C}$ category with finite products	$\mathbf{S}$, $\mathbf{C}$ categories
	Functor	Lax Monoidal Functor	Functor	Lax Functor
	$\mathbf{M} \longrightarrow \mathbf{Monads}(\mathbf{C})$ $m \longmapsto (T_m, \eta_m, \mu_m)$ $f : m \to n \longmapsto T_f : T_m \to T_n$	$(\mathbf{M}, \otimes, I) \longrightarrow ([\mathbf{C}, \mathbf{C}], \circ, \mathrm{id}_{\mathbf{C}})$ $m \longmapsto T_m : \mathbf{C} \to \mathbf{C}$ $f : m \to n \longmapsto T_f : T_m \to T_n$	$T : \mathbf{S}^{\mathrm{op}} \times \mathbf{S} \times \mathbf{C} \longrightarrow \mathbf{C}$	$\mathbf{I}^{\mathrm{op}} \longrightarrow \mathbf{Endo}(\mathbf{C})$ $I \longmapsto \mathbf{C}$ $f : I \to J \longmapsto T_f : \mathbf{C} \to \mathbf{C}$
		$\mu_{m,n} : T_m \circ T_n \longrightarrow T_{m \otimes n}$ $\eta : \mathrm{id}_{\mathbf{C}} \longrightarrow T_I$	$\mu_{S_1,S_2,S_3,A} : T(S_1, S_2, T(S_2, S_3, A)) \longrightarrow T(S_1, S_3, A)$ $\eta_{S,A} : A \longrightarrow T(S, S, A)$	$\mu_{f,g} : T_f T_g \longrightarrow T(g \circ f)$ $\eta_I : \mathrm{id}_{T_I} \longrightarrow T(\mathrm{id}_I)$
Indexed by	Objects of monoidal category	Objects of monoidal category	Pairs of indices	Morphisms of a category
Applications		Graded effect systems	Computation starts in a state in S_1 and ends in a state in S_2.	Index is a computation (i.e. program or proof)

Connections



- Indexed induces graded: tensor unit is unital
- Graded induces indexed: if the tensor product has codiagonals

Orchard, Dominic, Philip Wadler, and Harley Eades III. "Unifying graded and parameterised monads."

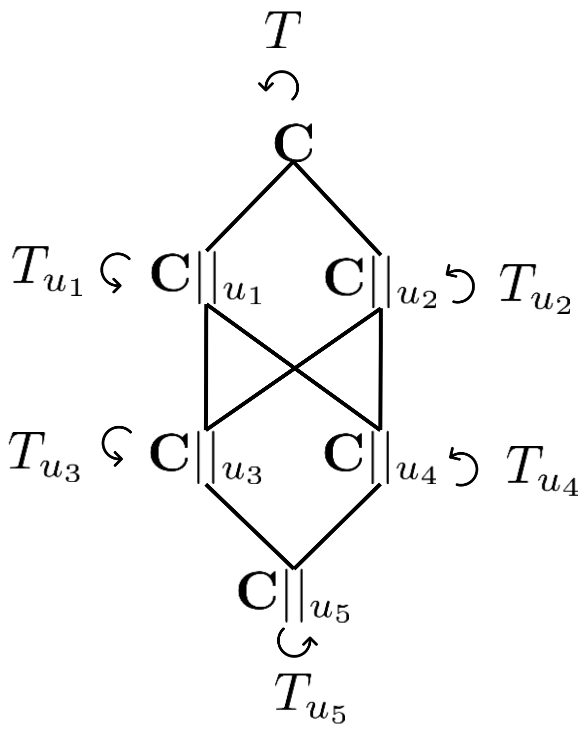
Constantin, Carmen, Nuiok Dicaire, and Chris Heunen. "Localisable Monads."

Another notion...

- In any monoidal category $\mathbf{C}$, we can define categories $\mathbf{C}||_u$
- Indexing over central idempotents:
  - morphism $u : U \to I$
- **Localisable monad**: A monad is localisable at U if we have maps

$$\mathrm{st}_{A,U} : T(A) \otimes U \to T(A \otimes U)$$

- Characterisation as particular (formal) monad in a 2-category



How can this be connected?

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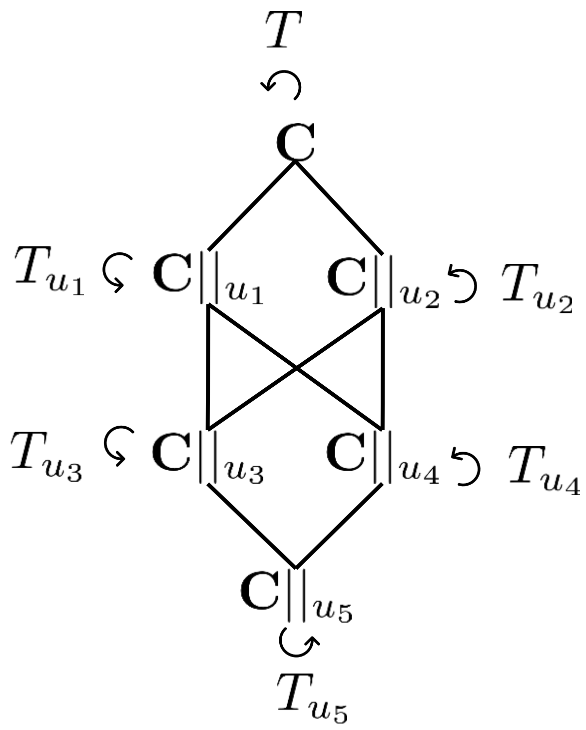
$\leftrightarrows$



Indexing over central idempotents

In conclusion...

- This is "The kind of category theory that I do"

AND

- I'll keep you updated!

Thank you