Morita Categories

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Topological Field Theories

Definition

A topological field theory (TFT) valued in a (higher) category $\mathcal{S}$ is a symmetric monoidal functor

$$Z : \mathrm{Bord}_n \to \mathcal{S}$$

(∞, n)-cat

⚠ Possibly truncated.

Definition

A monoidal category is a category $\mathcal{C}$ equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and

- a distinguished object $\mathbb{1}_{\mathcal{C}}$
- natural isomorphisms $\lambda : \mathbb{1}_{\mathcal{C}} \otimes - \to \mathrm{Id}_{\mathcal{C}}, \rho : - \otimes \mathbb{1}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$
- a natural transformation $\alpha : (- \otimes -) \otimes - \to - \otimes (- \otimes -)$

satisfying the pentagon axiom and triangle axiom for all objects.

$$\begin{tikzcd}
(A \otimes \mathbb{1}) \otimes B \arrow[rr] \arrow[dr] & \circlearrowright & A \otimes (\mathbb{1} \otimes B) \arrow[dl] \\
& A \otimes B &
\end{tikzcd}$$

$$\begin{tikzcd}
& (A \otimes B) \otimes (C \otimes D) \arrow[dr] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur] \arrow[d] & \circlearrowright & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[r] & A \otimes ((B \otimes C) \otimes D) \arrow[ur] &
\end{tikzcd}$$

$F : \mathcal{C} \to \mathcal{D}$ monoidal if

$F(V \otimes W) \simeq F(V) \otimes F(W).$

Definition

A bicategory is

- a collection of objects
- between any two objects X, Y a category $B(X, Y)$ of 1-morphisms, such that $B(X, X)$ has a distinguished object 1_X
- a functor $\circ : B(Y, Z) \times B(X, Y) \to B(X, Z)$, called horizontal composition
- for $f \in B(Y, Z), g \in B(X, Y), h \in B(W, X)$ a natural transformation $\alpha_{f,g,h} : (f \circ g) \circ h \to f \circ (g \circ h)$

Horizontal composition is required to satisfy the pentagon axiom.

Proposition

A monoidal category is the same as a bicategory with a single object.

Definition

An (m, n)-category has

0. objects
1. 1-morphisms

   $\ldots$

m. m-morphisms

and morphisms of level $n < k \leq m$ are invertible.

Definition

For $\mathcal{C}$ a monoidal (m, n)-category, $B\mathcal{C}$ is the $(m+1, n+1)$-category with

- a single object $*$
- $End_{B\mathcal{C}}(*) = \mathcal{C}$

Also: $M\mathcal{C} = \begin{cases} \text{objects: 1-morphisms of } \mathcal{C} \\ \text{1-morphisms: 2-morphisms} \\ \vdots \end{cases}$

$M \dashv B$.

$$\mathrm{Hom}_{N\text{-cats}}(h_N\mathcal{C}, \mathcal{D}) \cong \mathrm{Hom}_{m\text{-cats}}(\mathcal{C}, \iota_m\mathcal{D}).$$

$N \le m$.

Definition

For $\mathcal{C}$ an ~~d~~ (m, n)-category, $h_N(\mathcal{C})$ is the N-category with

0. objects: objects of $\mathcal{C}$
1. 1-morphisms: 1-morphisms of $\mathcal{C}$

   ...

N. N-morphisms: isomorphism classes of N-morphisms of $\mathcal{C}$.

$\mathcal{D}$ an N-category, $\iota_m\mathcal{D} = \left\{\begin{array}{l} \text{objects of } \mathcal{D} \\ \vdots \\ N\text{-morphisms } f, g, \ldots \\ \mathrm{Hom}(f,g) = \{\bullet\} \\ \vdots \end{array}\right.$ m

$h_N \dashv \iota_m$.

⚠ Possibly factor through $h_N \mathrm{Bord}_n$ if $\mathcal{S}$ is an N-cat.

Definition

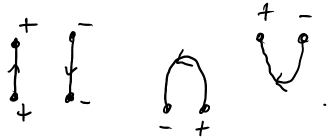
A topological field theory (TFT) valued in a (higher) category $\mathcal{S}$ is a symmetric monoidal functor

$$Z : \mathrm{Bord}_n \to \mathcal{S}$$

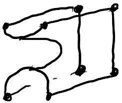
$\otimes = \amalg$.

Objects: oriented points $\overset{+}{\bullet}$ $\overset{-}{\bullet}$

1-morphisms: diff classes of oriented bordisms.



2-morphisms: bordisms of bordisms.



⋮

n-morphisms: n-manifolds with corners.

$k > n$: diffeos. + isotopies.

Example $n=1$, $\mathcal{S} = \underline{Vect}$

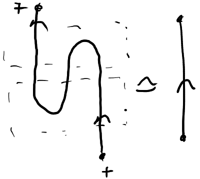
$Z(\cdot^{+}) = V$, $Z(\cdot^{-}) = W$.

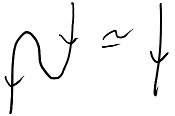
$Z\Big($ 

$\Big) = \mathrm{ev} : W \otimes V \longrightarrow k$

$Z\Big($ 

$\Big) = \mathrm{coev} : k \longrightarrow V \otimes W$





$$\begin{tikzcd} V \arrow[d, "\mathrm{coev}\otimes 1"'] \arrow[dr, "1"] & \\ V\otimes W\otimes V \arrow[r, "1\otimes \mathrm{ev}"'] & V \end{tikzcd}$$

$$\begin{tikzcd} W \arrow[d, "1\otimes\mathrm{coev}"'] \arrow[dr, "1"] & \\ W\otimes V\otimes W \arrow[r, "\mathrm{ev}\otimes 1"'] & W \end{tikzcd}$$

$\Bigg\}$ V has a left dual $V^* \cong W$ in $\underline{Vect}$.

Example

<u>Question</u>: Is it enough to choose a dualizable object?

Dualizability

Definition

A functor $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint if there exists a functor $F : \mathcal{C} \to \mathcal{D}$ and natural transformations $\epsilon : FG \to 1, \eta : 1 \to GF$ such that

$$\begin{tikzcd} G \arrow[d, "\eta \circ 1"'] \arrow[dr, "1"] & \\ GFG \arrow[r, "1 \circ \epsilon"'] & G \end{tikzcd} \qquad \begin{tikzcd} F \arrow[d, "1 \circ \eta"'] \arrow[dr, "1"] & \\ FGF \arrow[r, "\epsilon \circ 1"'] & F \end{tikzcd}$$

Definition

A 1-morphism $g : D \to C$ in a bicategory has a left adjoint if there exists a 1-morphism $f : C \to D$ and 2-morphisms $\epsilon : fg \to 1, \eta : 1 \to gf$ such that

$$\begin{tikzcd} g \arrow[d, "\eta \circ 1"'] \arrow[dr, "1"] & \\ gfg \arrow[r, "1 \circ \epsilon"'] & g \end{tikzcd} \qquad \begin{tikzcd} f \arrow[d, "1 \circ \eta"'] \arrow[dr, "1"] & \\ fgf \arrow[r, "\epsilon \circ 1"'] & f \end{tikzcd}$$

Definition

Let $\mathcal{C}$ be a monoidal (∞, n)-category. Then

0 - dualizable

- an object has has duals if it does in $h_1\mathcal{C}$
- a 1-morphism has adjoints if it does in $h_2\mathcal{C}$
- a k-morphism has adjoints if it does as a 1-morphism in the appropriate $(\infty, n-k-1)$-category $\Omega^{k-1}\mathcal{C}$
- an object is k-dualizable if it is $(k-1)$-dualizable and the counit and unit have adjoints
- an object is fully dualizable if it is n-dualizable.

Definition

Let $\mathcal{C}$ be a monoidal (∞, n)-category. Then

- an object has has duals if it does in $h_1\mathcal{C}$
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Definition

An n-framing of an n-manifold M is a trivialization of TM.

Cobordism Hypothesis

$$Z : \mathrm{Bord}_n^{fr} \longrightarrow \underline{\mathcal{S}}$$

Theorem

There is a correspondence

$$\{\text{ Framed TFTs with target } \mathcal{S} \ \} \xrightarrow{\sim} \{\text{ Fully dualizable objects of } \mathcal{S} \ \}$$

given by evaluation on a point $Z \mapsto Z(*)$.

Morita categories: a motivating example

$1 + 1 = 2$

The prototype Morita category is $\mathrm{Alg}_1(\mathrm{Vect})$, which has

0. Objects: associative algebras $A, B, \ldots$
1. 1-morphisms: (A, B)-bimodules
   - Composition of ${}_A M_B$ and ${}_B N_C$ is given by $M \otimes_B N$
2. 2-morphisms: bimodule homomorphisms.

E_1 algebra: $\mathrm{Disk}_1 \longrightarrow S$

Morita Categories

$Alg_3(\underline{Cat})$

Definition

Given a symmetric monoidal m-category $\mathcal{S}$, the category $\mathrm{Alg}_n(\mathcal{S})$ is the $(n+m)$-category with

0. Objects: E_n-algebras in $\mathcal{S}$, $A, B, \ldots$
1. 1-morphisms: E_{n-1}-algebras in (A, B)-bimodules, $R, S, \ldots$
2. 2-morphisms: E_{n-2}-algebras in (R, S)-bimodules

   $\ldots$

n. n-morphisms: bimodules $M, N, \ldots$

$n+1$. $(n+1)$-morphisms: 1-morphisms of bimodules in $\mathcal{S}$

   $\ldots$

$n+m$. $(n+m)$-morphisms: m-morphisms in $\mathcal{S}$.

Such categories are called **Morita categories**.

Factorization homology

$\exists$ Morita categories $\mathrm{Alg}_n(\mathcal{S})$

objects: functors $\mathrm{Disk}_n \longrightarrow \mathcal{S}$

(∞, n)

Definition

Factorization homology is defined as a left Kan extension.

$$\begin{tikzcd} \mathrm{Disk}_n \arrow[r, "A"] \arrow[d, hook] & \mathcal{S} \\ \mathrm{Mfld}_n \arrow[ur, dashed] & \end{tikzcd} \qquad M \longmapsto \int_M A$$

Scheimbauer: constructs TFTs valued in $\mathrm{Alg}_n(\mathcal{S})$

using Fact. Hom. (independently of C.H.).

$n=2$: $A \in Alg_2(Cat)$

$$Z_A(\Sigma) = \int_\Sigma A .$$

Brochier-Jordan-Snyder: fully dualizable objects in $Alg_3(Cat)$ (e.g. $Rep_q G$)

C.H. $\Rightarrow$ TFT $\bar{Z}_A$.

Cooke: $\int_\Sigma A \simeq SkCat_q(\Sigma)$ $A = Rep_q G$.

Q: $\bar{Z}_{Rep_q G}(M^3) \simeq Sk_{Rep_q G}(M)$?