Reachability Homology
and the
Magnitude-Path Spectral Sequence

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University of Edinburgh

joint work w/ Richard Hepworth

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Category Theory 2024
Santiago de Compostela

Three homological perspectives on directed graphs

Grigor'yan, Lin,
Muranov and Yau
2013

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$\boxed{\text{Path Homology}}$

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geometric analysis,
differential geometry

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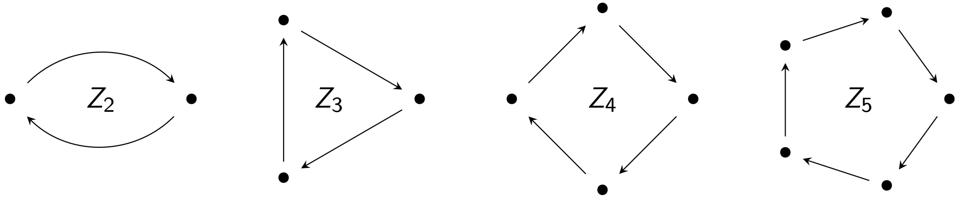
Path homology and reachability homology are also 'homotopy-invariant'.
All disagree on even very primitive classes of directed graphs.

Example

Directed cycles



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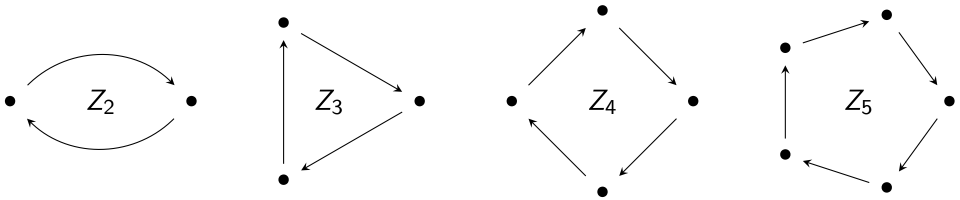

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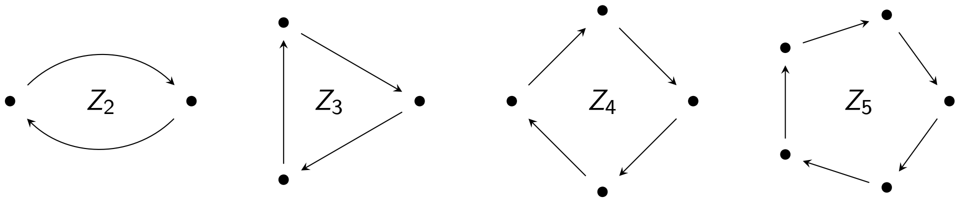

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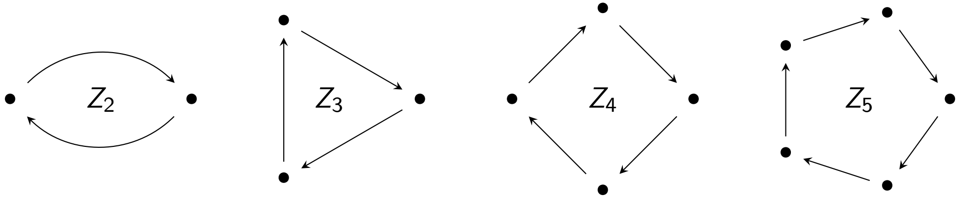


To **magnitude homology**, all the directed cycles are distinguishable.

To **path homology**, Z_2 looks ‘contractible’ and all the rest look ‘circle-like’.

To **reachability homology**, every directed cycle looks ‘contractible’.

1. **Reachability homology**

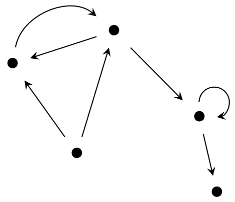
2. ‘Degrees’ of homotopy equivalence

3. The magnitude-path spectral sequence

Directed graphs

Definition A **directed graph** X consists of

- a set of vertices $V(X)$
- a set of edges $E(X) \subseteq V(X) \times V(X)$.



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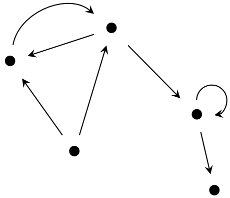

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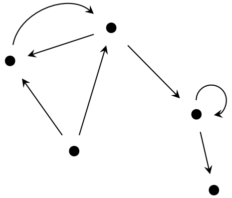

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The **shortest path metric** on $V(X)$ is the Lawvere metric

$$d(x, x') = \min\{n \mid \text{there is a path } x = x_0 \to x_1 \to \cdots \to x_n = x' \text{ in } X\}.$$

Reachability homology

$$\mathbf{DiGraph} \xrightarrow{\mathsf{Pre}} \mathbf{PreOrd} \xrightarrow{\mathsf{Nerve}} \mathbf{sSet} \xrightarrow{\mathbb{Z}\cdot -} \mathbf{sAb} \xrightarrow{N} \mathsf{Ch}(\mathbf{Ab}) \xrightarrow{H_*} \mathbf{Ab}^{\mathbb{N}}$$

Reachability homology

$$\begin{tikzcd} \mathbf{DiGraph} \arrow[r, "\mathrm{Pre}"] \arrow[rrrr, bend left, "\mathrm{RC}_*(-)"] & \mathbf{PreOrd} \arrow[r, "\mathrm{Nerve}"] & \mathbf{sSet} \arrow[r, "\mathbb{Z}\cdot -"] & \mathbf{sAb} \arrow[r, "N"] & \mathrm{Ch}(\mathbf{Ab}) \arrow[r, "H_*"] & \mathbf{Ab}^{\mathbb{N}} \end{tikzcd}$$

Definition (Hepworth & R., 2023) The **reachability complex** of a digraph X is

$$\mathrm{RC}_k(X) = \mathbb{Z} \cdot \{(x_0, x_1, \dots, x_k) \mid x_{i-1} \lneq x_i \text{ for every } i\}$$

with $\partial(x_0, \dots, x_k) = \sum (-1)^i (x_0, \dots, \widehat{x_i}, \dots, x_k)$.

Reachability homology

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$$\mathrm{RH}_*(X) = H_*(\mathrm{RC}(X)).$$

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Reachability homology has very strong homological properties.

The excision theorem

Theorem (Carranza *et al*, 2022)

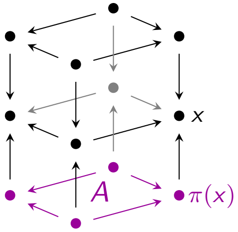
Path homology satisfies an excision theorem with respect to cofibrations ↴

Definition The **reach** of $A \subseteq X$ is the induced subgraph rA with

$$V(rA) = \{x \in V(X) \mid \text{there exists a path from } A \text{ to } x\}.$$

A **cofibration** is an inclusion $A \rightarrowtail X$ such that:

- There are no edges from $X \backslash A$ to A.
- For each $x \in V(rA)$ there is $\pi(x) \in V(A)$ such that, for all $v \in A$, we have $d(v, x) = d(v, \pi(x)) + d(\pi(x), x)$.



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Definition The **reach** of $A \subseteq X$ is the induced subgraph rA with

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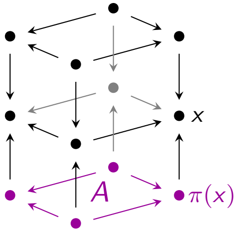
A **long cofibration** is an inclusion $A \rightarrowtail X$ such that:

- There are no edges from $X \backslash A$ to A.
- For each $x \in V(rA)$ there is $\pi(x) \in V(A)$ such that, for all $v \in A$, we have $d(v, x) < \infty \iff d(v, \pi(x)) < \infty$.



Theorem (Hepworth & R., 2023) Reachability homology can do better! It satisfies an excision theorem with respect to long cofibrations

The excision theorem

Theorem (Hepworth & R., 2024)

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & Y \arrow[d] \\ X \arrow[r] & X \cup_A Y \end{tikzcd}$$

Suppose that in this pushout the map $A \rightarrowtail X$ is a long cofibration. Then map induced on relative homology

$$\mathrm{RH}_*(X, A) \to \mathrm{RH}_*(X \cup_A Y, Y)$$

is an isomorphism.

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$$H_*(\mathrm{RC}(X)/\mathrm{RC}(A)) \to H_*(\mathrm{RC}(X \cup_A Y)/\mathrm{RC}(Y))$$

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Proof idea.

A long cofibration $A \rightarrowtail X$ is precisely a map of graphs that induces a Dwyer map $\mathrm{Pre}(A) \to \mathrm{Pre}(X)$ in **PreOrd**. The theorem follows easily from facts about Dwyer maps proved by Thomason in **Cat** *as a closed model category* (1980). □

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Homotopy Invariance Theorem If $f \rightsquigarrow g$, then $\mathrm{RH}_*(f) = \mathrm{RH}_*(g)$.

Proof. The long homotopy condition says that for all $x \in X$ we have $f(x) \leqslant g(x)$ in $\mathrm{Pre}(Y)$. So there's a natural transformation $\mathrm{Pre}(f) \Rightarrow \mathrm{Pre}(g)$, and thus a simplicial homotopy between the maps induced on the nerve. $\square$

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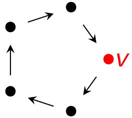
Example Every directed cycle is long-homotopy equivalent to a point: the inclusion of any vertex v admits a long homotopy to the identity.



This is much stronger than the homotopy-invariance enjoyed by path homology.

Summary so far

- It's easy to prove strong homological properties for $\mathrm{RH}_*(-)$. But it's a very insensitive theory: arguably its homotopy invariance is *too* strong.

Idea

Homotopy equivalence for graphs is not a matter of fact, but a matter of degree.

1. Reachability homology

2. **‘Degrees’ of homotopy equivalence**

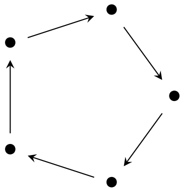
3. The magnitude-path spectral sequence

‘Degrees’ of homotopy equivalence

Definition Let $f, g\colon X \rightrightarrows Y$ be maps of graphs.

Fix $r \in \mathbb{N}$. We say **there is an r-homotopy** from f to g, and write $f \rightsquigarrow_r g$, if

$$d(f(x), g(x)) \leqslant r \text{ for all } x \in X.$$



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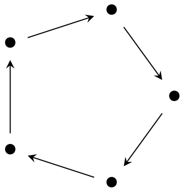
We say directed graphs X and Y are ***r*-homotopy equivalent**, and write $X \simeq_r Y$, if there exist maps $f\colon X \rightleftarrows Y : g$ such that

- $g \circ f$ is related to Id_X by a zig-zag of r-homotopies, and
- $f \circ g$ is related to Id_Y by a zig-zag of r-homotopies.



We say X is ***r*-contractible** if $X \simeq_r \bullet$.

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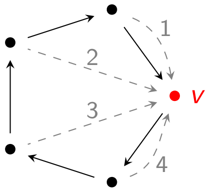
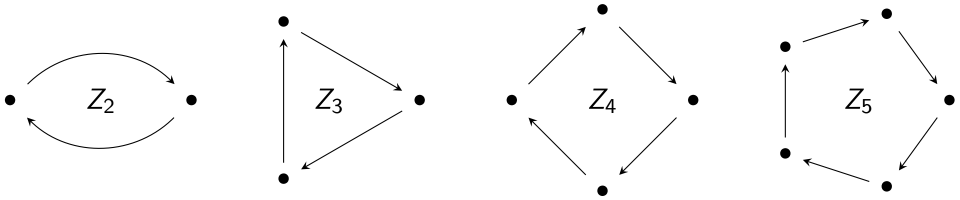


Example The directed n-cycle is r-contractible for every $r \geqslant n - 1$.

Example

Directed cycles



Proposition Fix $n \in \mathbb{N}$. Then $\begin{cases} \text{for } r < n-1 \text{ we have } Z_n \not\simeq_r Z_m \text{ whenever } m \neq n; \\ \text{for } r \geqslant n-1 \text{ we have } Z_n \simeq_r \bullet. \end{cases}$

Summary so far

- It's easy to prove strong homological properties for $\mathrm{RH}_*(-)$. But it's a very insensitive theory.

- For directed graphs, homotopy equivalence is a matter of degree: for each $r \in \mathbb{N}$ we can think about r-homotopy equivalence, getting weaker as r grows.

Question

Pick your preferred degree of homotopy equivalence—say degree r.

Is there a homology theory that can help us to distinguish directed graphs up to r-homotopy equivalence?

1. Reachability homology

2. ‘Degrees’ of homotopy equivalence

3. **The magnitude-path spectral sequence**

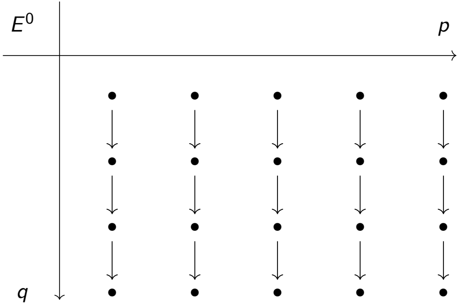
Spectral sequences

A **spectral sequence** is a sequence $E^0, E^1, E^2, \ldots$ of bigraded modules. We call these the **pages** of the sequence.

Each page E^r carries a differential

$$\partial^r_{pq} : E^r_{pq} \to E^r_{p-r,q+r-1}$$

and the modules on page E^{r+1} are obtained by taking homology on E^r.



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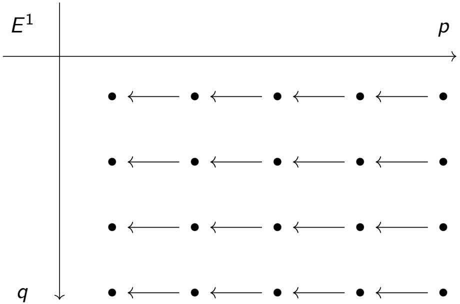

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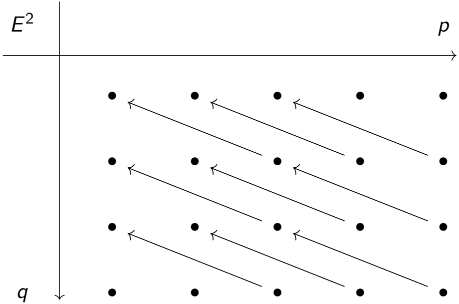

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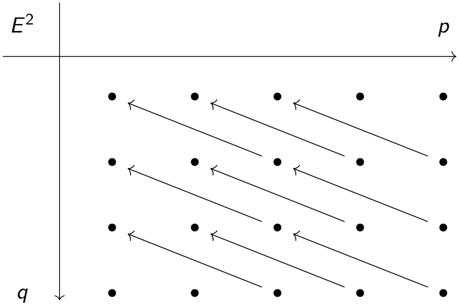

Example Any chain complex C_* with a filtration F—a sequence of subcomplexes

$$\cdots \subseteq F_{p-1}C_* \subseteq F_pC_* \subseteq \cdots \subseteq C_*$$

—determines a spectral sequence $E^*(C)$ for which $E^0_{pq}(C) = F_pC_{p+q}/F_{p-1}C_{p+q}$.

Functoriality and homotopy invariance of spectral sequences

Functoriality

A chain map $\phi\colon (C_*, F) \to (D_*, F')$ is **filtered** if $\phi(F_\ell C_*) \subseteq F'_\ell D_*$ for every ℓ.

A filtered chain map $\phi\colon (C_*, F) \to (D_*, F')$ induces a **map of spectral sequences**

$$E^\bullet(\phi)\colon E^\bullet(C) \to E^\bullet(D)$$

i.e. for every r a map of bigraded chain complexes $E^r(\phi)\colon E^r(C) \to E^r(D)$.

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Homotopy invariance

Let $\phi, \psi\colon (C_*, F) \to (D_*, F')$ be filtered chain maps. A chain homotopy h from ϕ to ψ is called a **chain r-homotopy** if $h(F_\ell D_*) \subseteq F_{\ell+r} D_*$ for every ℓ.

If there exists a chain r-homotopy from ϕ to ψ, then $E^r(\phi)$ and $E^r(\psi)$ are chain homotopic and $E^s(\phi) = E^s(\psi)$ for all $s > r$.

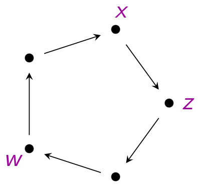
Filtering reachability chains by length

The **reachability complex** can be equivalently described as:

$$\mathrm{RC}_k(X) = \mathbb{Z} \cdot \{(x_0, x_1, \dots, x_k) \mid x_{i-1} \neq x_i \text{ and } {\color{red} x_{i-1} \leqslant x_i} \text{ for every } i\}$$

with differential $\partial(x_0, \dots, x_k) = \sum(-1)^i(x_0, \dots, \widehat{x_i}, \dots, x_k)$.

Example (w, x, z) is a generator of $\mathrm{RC}_2(Z_5)$.



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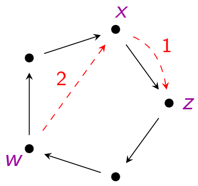

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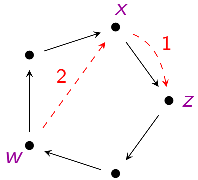
$$\mathrm{RC}_k(X) = \mathbb{Z} \cdot \{(x_0, x_1, \ldots, x_k) \mid x_{i-1} \neq x_i \text{ for every } i, \text{ and } \sum_{i=1}^{k} d(x_{i-1}, x_i) < \infty\}$$

with differential $\partial(x_0, \ldots, x_k) = \sum(-1)^i(x_0, \ldots, \widehat{x_i}, \ldots, x_k)$.

Example (w, x, z) is a generator of $\mathrm{RC}_2(Z_5)$.



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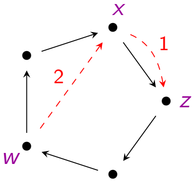

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$\mathrm{RC}_*(X)$ can be filtered by the **length** of its generators:

$$F_\ell(\mathrm{RC}_k(X)) = \mathbb{Z} \cdot \left\{ (x_0, x_1, \ldots, x_k) \mid x_{i-1} \neq x_i \text{ for every } i, \text{ and } \sum_{i=1}^{k} d(x_{i-1}, x_i) \leqslant \ell \right\}.$$

Thanks to the triangle inequality, ∂ respects the filtration.

Example (w, x, z) is a generator of $F_3(\mathrm{RC}_2(Z_5))$, but not of $F_2(\mathrm{RC}_2(Z_5))$.

The magnitude-path spectral sequence

Definition (Di *et al*, 2023) The **magnitude-path spectral sequence** (MPSS) of a digraph X is the spectral sequence $E^\bullet(X)$ associated to $(\mathrm{RC}_*(X), F_*)$.

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Observation (Hepworth & Willerton, 2015)
$E^0(X)$ is the magnitude chain complex $\mathrm{MC}_{**}(X)$.

$$\begin{tikzcd}
\mathrm{MC}_{00} & \mathrm{MC}_{11} \arrow[d] & \mathrm{MC}_{22} \arrow[d] & \mathrm{MC}_{33} \arrow[d] \\
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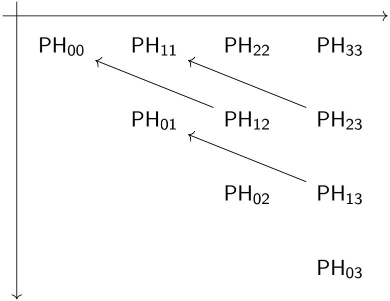

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By construction $E^\bullet(X) \Rightarrow \mathrm{RH}_*(X)$ under mild conditions on X.

Infinitely many homology theories!

Functoriality

For each $r \geqslant 0$, page $E^r(-)$ of the MPSS is a functor $\mathbf{DiGraph} \to \mathbf{Ab}^{\mathbb{N}\times\mathbb{N}}$.

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For each $r \geqslant 0$, page $E^{r+1}(-)$ of the MPSS is invariant under r-homotopy equivalence.

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In particular, these results hold for magnitude homology & bigraded path homology.

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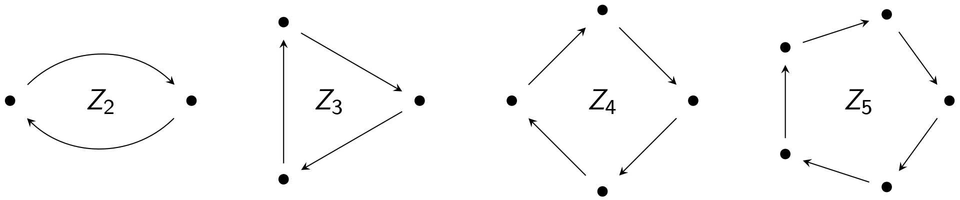
Example

Directed cycles



Theorem (Hepworth & R., 2024)

$E^r(Z_m)$ is trivial for every $m \leqslant r$, and $E^r(Z_m) \ncong E^r(Z_n)$ for $r \leqslant m < n$.

In particular, bigraded path homology distinguishes the directed m-cycles for all $m \geqslant 2$.

Summary

- It's easy to prove strong homological properties for reachability homology. But it's a very insensitive theory.

- For directed graphs, homotopy equivalence is a matter of degree: for each $r \in \mathbb{N}$ we can think about r-homotopy equivalence, getting weaker as r grows.

- The magnitude-path spectral sequence provides a spectrum of homology theories for directed graphs, interpolating between magnitude homology and reachability homology. For each $r \in \mathbb{N}$, page $r + 1$ of the MPSS is an r-homotopy invariant.

- Page E^2 is bigraded path homology. It shares the homotopy invariance of path homology, but is strictly finer: it distinguishes directed cycles of different lengths.

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- Page E^2 is bigraded path homology. It shares the homotopy invariance of path homology, but is strictly finer: it distinguishes directed cycles of different lengths.

- Not in this talk: There is a cofibration category structure on **DiGraph** whose weak equivalences are maps inducing isomorphisms on bigraded path homology. We expect to be able to describe such a structure for every page of the MPSS.

Thank you.

References

- Asao. Magnitude homology and path homology. *Bulletin of the London Mathematical Society*, 55(1):375–398, 2023; preprint arXiv:2201.08047, 2022.
- Asao. Magnitude and magnitude homology of filtered set enriched categories. arXiv:2303.05677, 2023.
- Carranza, Doherty, Kapulkin, Opie, Sarazola & Wong. Cofibration category of digraphs for path homology. *Algebraic Combinatorics*, to appear; preprint arXiv:2212.12568, 2022.
- Di, Ivanov, Mukoseev & Zhang. On the path homology of Cayley digraphs and covering digraphs. *Journal of Algebra* 653:156–199, 2024.
- Hepworth & Roff. The reachability homology of a directed graph. arXiv:2312.01378, 2023
- Hepworth & Roff. Bigraded path homology and the magnitude-path spectral sequence. arXiv:2404.06689, 2024
- Ivanov. Nested homotopy models of finite metric spaces and their spectral homology. arXiv:2312.11878, 2023.

Towards a ‘nested’ formal homotopy theory for directed graphs

A **cofibration category** is ‘one half of a model category’.

It is a category equipped with weak equivalences and cofibrations, satisfying axioms.

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Theorem (Hepworth & R., 2024)

DiGraph carries a cofibration category structure in which

- weak equivalences are maps inducing isomorphisms on bigraded path homology;
- cofibrations are those defined in Carranza *et al* (2022).

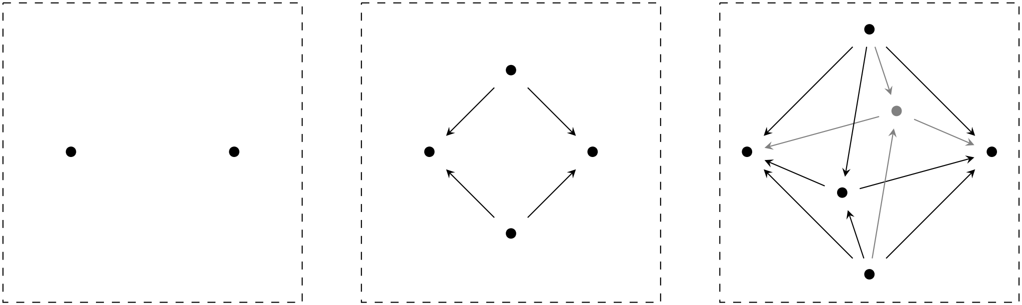
This structure is strictly finer than that for path homology given by Carranza *et al*: for instance, it distinguishes all the directed cycles Z_n for $n \geqslant 2$.

Proof. Combines all the homological properties of bigraded path homology. □

We expect a similar structure for every page of the MPSS.

Example

Spheres



Definition For each $n \geqslant 0$, let $\mathbb{S}^n$ be the face poset of the cell-decomposition of the n-sphere into hemispheres. Let $\mathbb{S}^\infty$ be the colimit of $\mathbb{S}^0 \hookrightarrow \mathbb{S}^1 \hookrightarrow \cdots \hookrightarrow \mathbb{S}^n \hookrightarrow \cdots$.

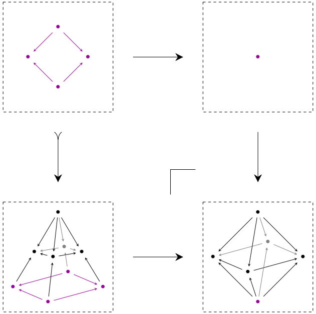
Example

Spheres

Theorem (Hepworth & R., 2024)

Let $n \geqslant 1$. Then $\mathrm{PH}_{k,\ell}(\mathbb{S}^n) = 0$ for $k \neq \ell$, while

$$\mathrm{PH}_{k,k}(\mathbb{S}^n) \cong \begin{cases} R & \text{if } k = 0, n \\ 0 & \text{otherwise.} \end{cases}$$



Proof sketch. $\mathbb{S}^n$ is the pushout of the maps $\mathrm{Cone}(\mathbb{S}^{n-1}) \leftarrowtail \mathbb{S}^{n-1} \rightarrow \bullet$. Write down the Mayer–Vietoris sequence and use the fact that $\mathrm{Cone}(\mathbb{S}^{n-1}) \simeq_1 \bullet$ to see that $\mathrm{PH}_{k,\ell}(\mathbb{S}^n) \cong \mathrm{PH}_{k-1,\ell}(\mathbb{S}^{n-1})$. Now induct on n. $\square$

Filtered colimits and the infinite sphere

Question The infinite topological sphere is contractible. What about $\mathbb{S}^\infty$?

Theorem (Hepworth & R., 2024; Di *et al*, 2023)
Every page of the MPSS is a finitary functor: it preserves filtered colimits.

Corollary Bigraded path homology sees $\mathbb{S}^\infty$ as contractible:

$$\mathrm{PH}_{k,\ell}(\mathbb{S}^\infty) = \begin{cases} R & k = \ell = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since $\mathrm{PH}_{*,*}(-)$ is finitary, we have

$$\mathrm{PH}_{k,\ell}(\mathbb{S}^\infty) = \mathrm{PH}_{k,\ell}(\mathrm{colim}_{\mathbb{N}}(\mathbb{S}^n)) \cong \mathrm{colim}_{\mathbb{N}}(\mathrm{PH}_{k,\ell}(\mathbb{S}^n)).$$

For each n, the map $i_* \colon \mathrm{PH}_{k,\ell}(\mathbb{S}^n) \to \mathrm{PH}_{k,\ell}(\mathbb{S}^{n+1})$ is zero except when $k = \ell = 0$. $\square$