Magnitude, diversity, homology

A survey, with questions

Emily Roff
University of Edinburgh

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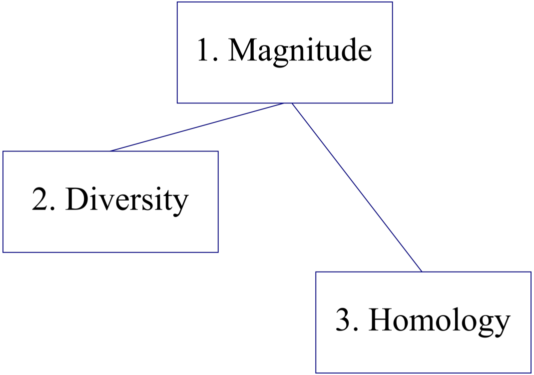
Plan



Part I

Magnitude

The magnitude of a finite metric space

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Lemma If Z_X is invertible then $|X| = \sum_{x,y \in X} Z_X^{-1}(x, y)$.

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The magnitude function of a finite metric space

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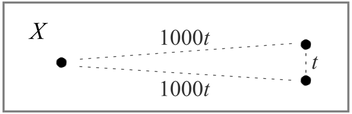

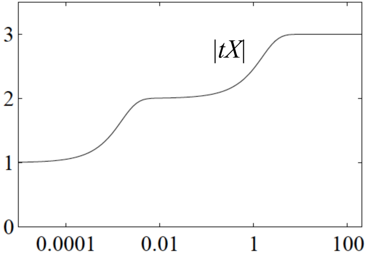

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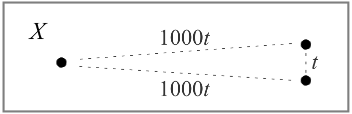

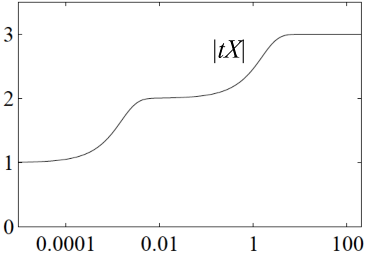

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Question Can a magnitude function be defined for larger metric spaces?

Positive definite spaces

Definition (Meckes, 2010) A metric space X is **positive definite** if, for all finite subspaces $Y \subseteq X$, the matrix Z_Y is positive definite.

Examples Spheres with geodesic metric, hyperbolic space, subsets of Euclidean space.

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Proposition (Leinster, 2010) Let X be a finite positive definite space. Then

$$|X| = \sup \frac{\left(\sum_{x \in X} \mathbf{v}(x)\right)^2}{\mathbf{v}^T Z_X \mathbf{v}}$$

where the supremum is over $\mathbf{v} \neq \mathbf{0}$ in $\mathbb{R}^X$. (Proof Cauchy–Schwarz inequality. $\square$)

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where the supremum is over $\mathbf{v} \neq \mathbf{0}$ in $\mathbb{R}^X$. (Proof Cauchy–Schwarz inequality. $\square$)

Corollary For any finite positive definite space X we have $|X| = \sup\{|Y| \mid Y \subseteq X\}$.

Idea Let this formulation *define* the magnitude of larger metric spaces.

The magnitude of a compact metric space

Proposition (Leinster & Meckes, 2016)

The quantity $S(X) = \sup\{|Y| \mid \text{finite } Y \subseteq X\}$ is lower semicontinuous as a function on the class of positive definite metric spaces with the Gromov–Hausdorff topology.

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So there is a canonical way to extend magnitude to compact positive definite spaces:

Definition (Leinster & Meckes, 2016)

Let X be a compact positive definite metric space. The **magnitude** of X is

$$|X| = \sup\{|Y| \mid \text{finite } Y \subseteq X\}.$$

The **magnitude function** of X is the the function $t \mapsto |tX|$.

Magnitude via weight measures

Given a compact metric space X, let $M(X) = \{\text{finite Borel measures on } X\}$.

For $t \geqslant 0$, define $\mathcal{Z}_X(t) : M(X) \to C(X)$ by

$$\mathcal{Z}_X(t)(\mu)(x) = \frac{1}{t} \int_X e^{-td(x,y)} \mathrm{d}\mu(y).$$

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Definition (Willerton, 2010)

A **weight measure** for tX is a solution μ_t to the equation $t\mathcal{Z}_X(t)(\mu_t) = 1$.

Proposition (Meckes, 2010)

If tX is positive definite and admits a weight measure, then $|tX| = \mu_t(X)$.

The magnitude function carries geometric information

Theorem (Barceló & Carbery, 2015)

Let $X \subset \mathbb{R}^n$ be a nonempty compact set. Then $|tX| \to 1$ as $t \to 0$ and

$$\frac{|tX|}{t^n} \to \frac{\mathrm{Vol}(X)}{n!\omega_n} \text{ as } t \to +\infty.$$

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Theorem (Gimperlein & Goffeng, 2017)

Let $X \subset \mathbb{R}^n$ be a compact, smooth domain, with n odd. Then

$$|tX| \sim \frac{1}{n!\omega_n} \sum_{j=0}^{\infty} c_j t^{n-j} \text{ as } t \to +\infty$$

where c_0, c_1 and c_2 record the volume, surface area, and total mean curvature of X.

Magnitude via weight distributions

Gimperlein, Goffeng & Louca show that when $X \subset \mathbb{R}^n$ is a compact domain, $\mathcal{Z}_X(t)$ is a pseudodifferential operator which extends to a certain space H of distributions on X.*

*Specifically, the Sobolev space $\dot{H}^{-\frac{n+1}{2}}(X)$.

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Proposition (Gimperlein, Goffeng & Louca, 2022, via Meckes, 2015)

Let $X \subseteq \mathbb{R}^n$ be a compact domain. Then

$$|tX| = \langle u_t, 1 \rangle_X$$

where $u_t \in H$ is the unique distributional solution to $t\mathcal{Z}_X(t)(u_t) = 1$ on X.

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They construct an approximate inverse to $\mathcal{Z}_X(t)$ and thus can compute magnitude as

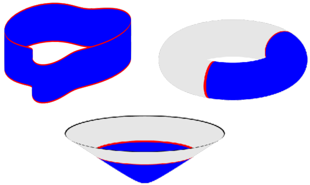
$$|tX| = \frac{1}{t} \langle \mathcal{Z}_X(t)^{-1}(1), 1 \rangle_X.$$

*Specifically, the Sobolev space $\dot{H}^{-\frac{n+1}{2}}(X)$.

Recent geometric results

Theorem (Gimperlein & Goffeng, 2021)

For a smooth, compact domain X in odd dimensions, the asymptotics of $|tX|$ determine the Willmore energy of the boundary ∂X.



Theorems (Gimperlein, Goffeng & Louca, 2022)

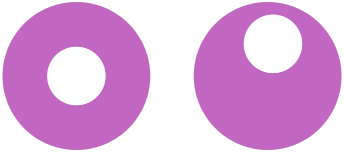
- For nice enough domains, magnitude satisfies an asymptotic inclusion-exclusion principle.
- You can 'magnitude the shape of a ball'!
- When $n = 2$ or n is odd, magnitude characterizes domains with constant mean curvature.

. . . and lots more exciting stuff!

Open questions

What is the geometric content of the magnitude function?

In general, $|tX|$ does not determine X up to isometry. Meckes has showed that these two spaces have the same magnitude function $\rightarrow$



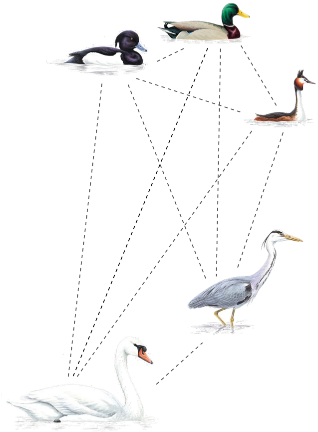
- If X and Y are such that $|tX|$ and $|tY|$ coincide, what can we say about them?
- Can you magnitude the shape of convex drums? Star-shaped drums?
- Can one compute the magnitude function exactly for interesting domains?
- The magnitude function extends to a meromorphic function on the complex plane. Do the poles of this function carry geometric information?
- What can be said about the small-t asymptotics of $|tX|$?

Part II

Diversity

Ecological connections

Let X be a (finite) set of biological species and d a metric on X recording differences among species.



Bird illustrations from the RSPB.

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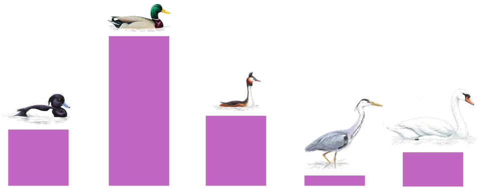
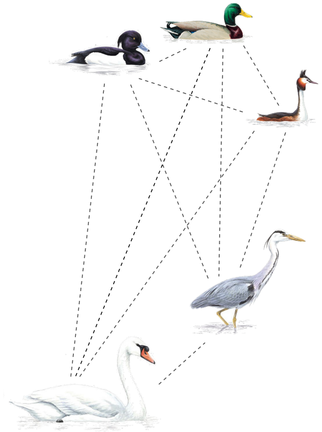
An ecological community comprising members of species in X can be modelled by a probability distribution $\mathbf{p}$ on X, where $\mathbf{p}(x)$ is the relative abundance of species x in the community.





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Typicality and diversity

The matrix Z_X records the similarity between each pair of species in X. So

$$(Z_X \mathbf{p})(x) = \sum_{y \in X} e^{-d(x,y)} \mathbf{p}(y)$$

tells us the expected similarity between a member of species x and an individual chosen at random from $\mathbf{p}$. Call this the **typicality** of members of species x in $\mathbf{p}$.

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Idea A community in which the average member is highly typical is homogenous.

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Idea The less homogenous a community is, the more diverse we should consider it.

Definition (temporary!) The **diversity** of a probability distribution $\mathbf{p}$ on X is

$$\frac{1}{\sum_{x \in X} (Z_X \mathbf{p})(x) \mathbf{p}(x)}.$$

The denominator is the mean typicality of members of the community modelled by $\mathbf{p}$.

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Theorem Let X be a finite positive definite space admitting a non-negative weight vector $\mathbf{v}$. Then $\mathbf{p} = \mathbf{v}/|X|$ maximizes diversity on X, and $\mathrm{MaxDiv}(X) = |X|$.

An uncountable family of diversity indices

Leinster & Cobbold observe that the weighted arithmetic mean of atypicality is not the only way to quantify diversity. We should consider weighted power means of all orders! Their approach unifies many diversity indices used in practice by ecologists.

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Definition (Leinster & Roff, 2019, following Leinster & Cobbold, 2012)

Let X be a compact metric space and μ a probability measure on X. For $q \in [0, \infty)$ not equal to 1, the **diversity of order q** of μ is

$$D_q(\mu) = \left(\int \left(\int e^{-d(x,y)} \mathrm{d}\mu(x) \right)^{q-1} \mathrm{d}\mu(y) \right)^{1/(1-q)}.$$

At $q = 1, \infty$ this expression takes its limiting values.

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Examples $\quad D_2(\mu)$ is 1/(the expected proximity of a μ-random pair of points).

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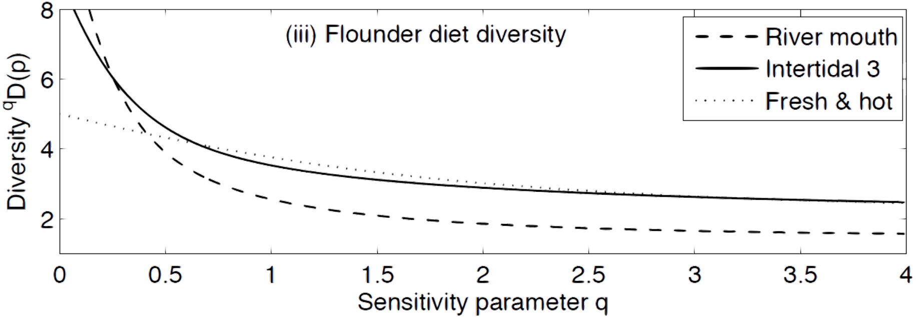
At $q = 1, \infty$ this expression takes its limiting values.

Examples $D_\infty(\mu)$ is $1/$(the ess.sup. of the 'typicality function' of μ on X).

The parameter q matters!



Leinster & Cobbold, *Measuring Diversity...*, Ecology 93 (2012)

A maximization theorem

Theorem (Leinster & Roff, 2019)

Let X be a non-empty compact metric space.

1. There exists a probability measure μ on X that maximizes D_q for all q at once.
2. The **maximum diversity** $D_{\max}(X) = \sup_\mu D_q(\mu)$ is independent of $q \in [0, \infty]$.

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Theorem (Leinster & Roff, 2019)

Let X be a non-empty positive definite compact metric space admitting a positive weight measure μ. Then $D_{\max}(X) = |X|$.

These results extend those proved for finite spaces by Leinster & Meckes in 2015.

Open questions

What do diversity-maximizing measures look like?

What do weight distributions look like?

Even in the finite setting, diversity-maximizing measures are typically not uniform.

Weight measures and diversity-maximizing measures on finite spaces have been used in boundary-detection algorithms (e.g. Bunch *et al*, 2021).

But in the compact setting, even on very familiar spaces, we know little about them!

Part III

Homology

Combinatorial connections

Let $(A, \leqslant)$ be a finite poset.

The **incidence algebra** $\mathbb{I}(A)$ is the algebra of functions $A \times A \xrightarrow{f} \mathbb{Q}$ satisfying $f(a, b) = 0$ unless $a \leqslant b$. Multiplication in $\mathbb{I}(A)$ is by convolution.

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The **zeta function** of A is $\zeta \in \mathbb{I}(A)$ defined by

$$\zeta(a, b) = \begin{cases} 1 & a \leqslant b \\ 0 & \text{otherwise.} \end{cases}$$

If ζ is invertible, ζ^{-1} is the **Möbius function** of A.

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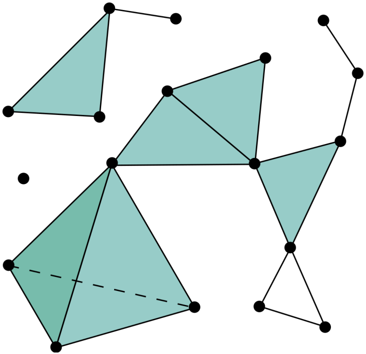

Graphic from Wikipedia.

Theorem Let A be the poset of simplices in a finite simplicial complex S. Then

$$\sum_{a,b \in A} \zeta^{-1}(a, b) = \chi(S).$$

The Euler characteristic of a category

More generally, let $\mathbf{C}$ be a finite category. Write $\mathrm{ob}(\mathbf{C})$ for its set of objects.

The **zeta matrix** of $\mathbf{C}$ is the $\mathrm{ob}(\mathbf{C}) \times \mathrm{ob}(\mathbf{C})$ matrix $Z_{\mathbf{C}}$ defined by

$$Z_{\mathbf{C}}(a, b) = \#\{\text{arrows } a \to b \text{ in } \mathbf{C}\}.$$

If $Z_{\mathbf{C}}$ is invertible over $\mathbb{Q}$, call $Z_{\mathbf{C}}^{-1}$ the **Möbius matrix** of $\mathbf{C}$.

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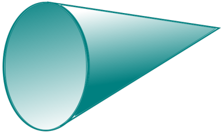
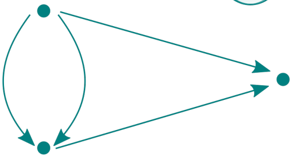


Theorem (Leinster, 2006)

If $Z_{\mathbf{C}}$ is invertible, then

$$\sum_{a,b \in \mathrm{ob}(\mathbf{C})} Z^{-1}(a, b) = \chi(\mathbb{B}\mathbf{C})$$

where $\mathbb{B}\mathbf{C}$ is the **classifying space** of $\mathbf{C}$.



A common generalization

Posets

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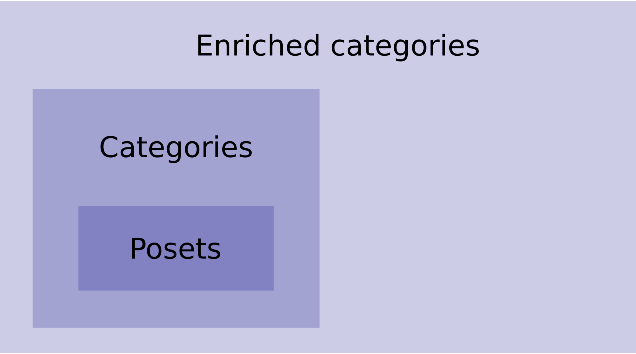

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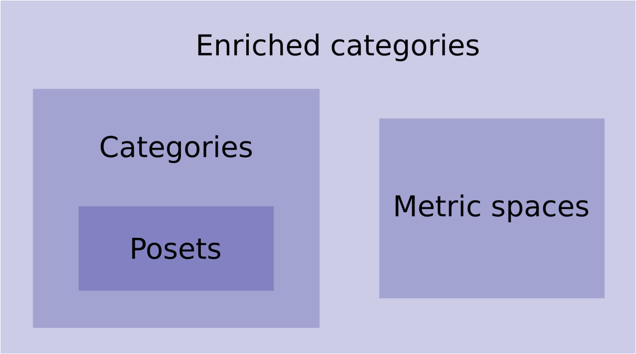

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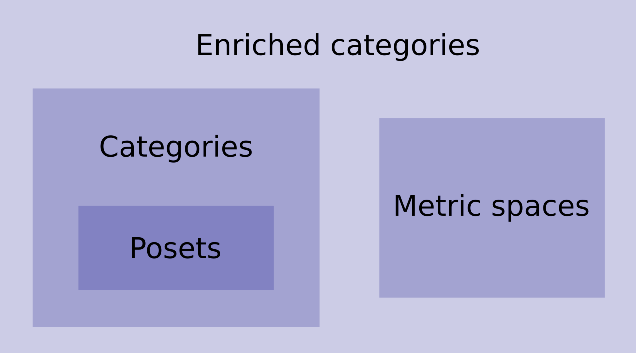
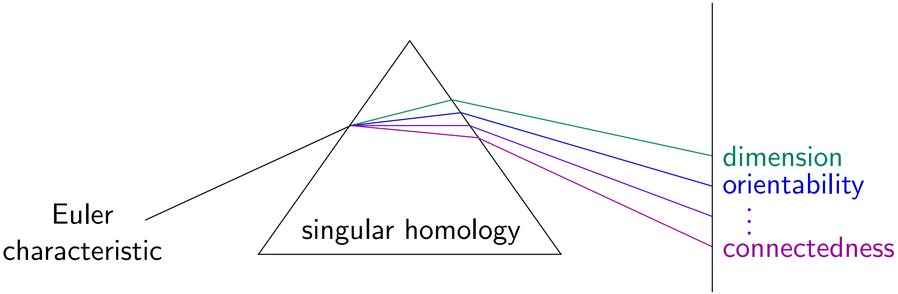


You can define **magnitude** for enriched categories. It specializes to Euler characteristic for ordinary categories and to the magnitude function for finite metric spaces.

Categorifying magnitude



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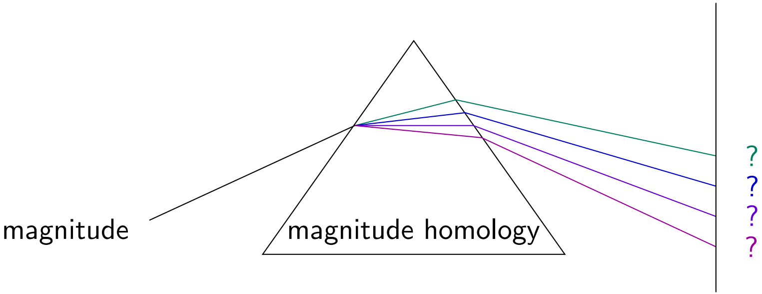

Novikov series

Let $\mathbb{Z}[q^{\mathbb{R}_+}] = \{a_0 q^{\ell_0} + \cdots + a_n q^{\ell_n} \mid a_i \in \mathbb{Z}, \ell_i \in [0, \infty)\}$. This ring carries a valuation—

$$v(p) = \text{the minimal exponent with non-zero coefficient in } p$$

—and thus a metric: $d(p, r) = e^{-v(p-r)}$. It's also an integral domain.

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Cauchy completing and taking the field of fractions yields a square

$$\begin{array}{ccc} \mathbb{Z}[q^{\mathbb{R}_+}] & \xhookrightarrow{c_1} & \mathbb{Z}[[q^{\mathbb{R}_+}]] \\ {\scriptstyle f_1}\big\downarrow & & \big\downarrow{\scriptstyle f_2} \\ \mathbb{Q}(q^{\mathbb{R}}) & \xhookrightarrow{c_2} & \mathbb{Q}((q^{\mathbb{R}})) \end{array}$$

The field $\mathbb{Q}((q^{\mathbb{R}}))$ is the field of **Novikov series**: generalized formal Laurent series.

Formal magnitude for finite spaces

Let X be a finite metric space. Its **formal similarity matrix** is defined by

$$Z(x, y) = q^{d(x,y)}.$$

Lemma Every formal similarity matrix is invertible over $\mathbb{Q}((q^{\mathbb{R}}))$.

Proof.
All diagonal entries in Z are 1 and off-diagonal entries are q^{ℓ} for some $\ell > 0$. So $\det(Z)$ is a generalized polynomial with constant term 1 and thus a unit in $\mathbb{Z}[[q^{\mathbb{R}}]]$. □

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The **formal magnitude** of X is the Novikov series $\mathrm{Mag}(X) = \sum_{x,y \in X} Z^{-1}(x, y)$.

Theorem
For every finite metric space X and all $t \in [0, \infty)$ we have $|tX| = \mathrm{Mag}(X)|_{q=e^{-t}}$. □

A combinatorial formula for the coefficients

Theorem (Leinster, 2014*)

Let X be a finite metric space. Then $\mathrm{Mag}(X) = \sum_{\ell \in [0,\infty)} a_\ell q^\ell$ where

$$a_\ell = \sum_{k=0}^{\infty} (-1)^k \#\{(x_0, \ldots, x_k) \mid x_i \in X, x_i \neq x_{i+1} \text{ and } d(x_0, x_1) + \cdots + d(x_{k-1}, x_k) = \ell\}.$$

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Idea (Hepworth & Willerton, 2015)

Each coefficient in $\mathrm{Mag}(X)$ is the Euler characteristic of a chain complex.

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Magnitude homology

Definition (Leinster & Shulman, 2017, following Hepworth & Willerton, 2015)

The **magnitude chain complex** of a metric space X is a real-graded chain complex of vector spaces. In grading $\ell \in [0, \infty)$ and degree $k \in \mathbb{N}$ it's given by

$$MC_k^\ell(X) = \mathbb{Z} \cdot \{(x_0, \ldots, x_k) \mid x_i \neq x_{i+1} \text{ and } d(x_0, x_1) + \cdots + d(x_{k-1}, x_k) = \ell\}.$$

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The **magnitude homology** of X is defined by $MH_k^\ell(X) = H_k(MC_\bullet^\ell(X))$.

Theorem (Leinster & Shulman, 2017)

For finite metric spaces, magnitude homology recovers formal magnitude:

$$\chi(MH_\bullet^*(X)) := \sum_{i \geqslant 0} (-1)^i \mathrm{rk}(MH_i^*(X)) = \mathrm{Mag}(X).$$

Geometric content of magnitude homology

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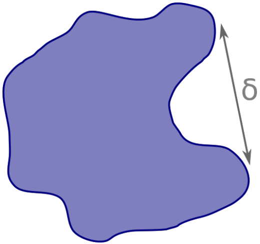

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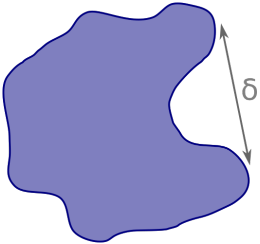

Theorem (Kaneta & Yoshinaga, 2018)
Let X be a closed subset of Euclidean space which is *not* convex. Then $MH_1^*(X)$ records the 'diameter' of the largest 'dent' in X.

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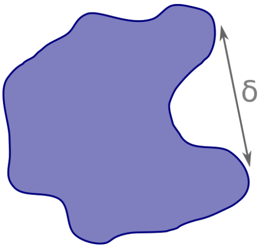

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Theorem (Asao, 2022)
The magnitude homology of a graph is closely related to its path homology.

Magnitude via magnitude homology

We are beginning to prove new results about magnitude using magnitude homology.

For example...

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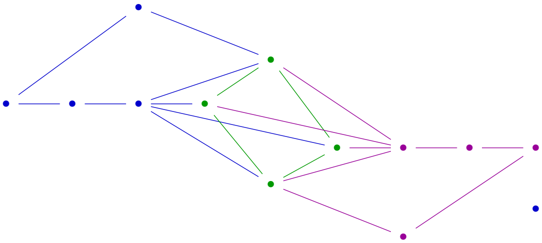

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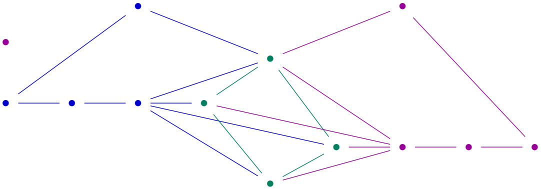

Theorem (Roff, 2022)

Let X and Y be graphs which differ by a sycamore twist. Then $\mathrm{Mag}(X) = \mathrm{Mag}(Y)$.

The Proof is homological—related to, but independent from, an excision theorem.

Open question

Can we categorify the magnitude function for compact metric spaces?

The magnitude homology of a compact space does not recover its magnitude function.

Problem is: we don't know how to extend formal magnitude to compact spaces.

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Problem is: we don't know how to extend formal magnitude to compact spaces.

Things that don't work:

- Taking suprema: $\mathbb{Q}((q^{\mathbb{R}}))$ is an ordered field, but not Dedekind complete!
- Taking limits: $\mathrm{Mag}(-)$ is not continuous with respect to the valuation metric!

Open question

Can we categorify the magnitude function for compact metric spaces?

The magnitude homology of a compact space does not recover its magnitude function.

Problem is: we don't know how to extend formal magnitude to compact spaces.

Things that might work:

- Treat $\mathbb{Q}((q^{\mathbb{R}}))$ as a space of distributions (Hepworth).
- Treat $\mathbb{Q}((q^{\mathbb{R}}))$ as the stalk at zero of a certain sheaf of functions (me).
- Mimic Meckes's 'weighting space' approach over $\mathbb{Q}((q^{\mathbb{R}}))$ (me).

Summary

- What is the geometric content of the magnitude function?
  - If X and Y are such that $|tX| \sim |tY|$, what can we say about them?
  - Can you magnitude the shape of a convex drum?
  - Do the poles of the magnitude function carry interesting information?
  - What can be said about the small-t asymptotics of $|tX|$?
- What do diversity-maximizing measures and weight distributions look like?
- How should the formal magnitude of a compact space be defined?

Thank you.

References

- Asao. Magnitude homology and path homology. arXiv:2201.08047, 2022; *Bulletin of the LMS* (to appear).
- Barceló and Carbery. On the magnitudes of compact sets in Euclidean spaces. arXiv:1507.02502, 2015; *American Journal of Mathematics* 140, 2018.
- Bunch *et al*. Weighting vectors for machine learning: numerical harmonic analysis applied to boundary detection. arXiv:2106.00827, 2021.
- Gimperlein and Goffeng. On the magnitude function of domains in Euclidean space. arXiv:1706.06839, 2017; *American Journal of Mathematics* 143, 2021.
- Gimperlein and Goffeng. The Willmore energy and the magnitude of Euclidean domains. arXiv:2109.10097, 2021; to appear in *Proceedings of the AMS*.
- Gimperlein, Goffeng and Louca. The magnitude and spectral geometry. arXiv:2201.11363, 2022.

References

- Hepworth and Willerton. Categorifying the magnitude of a graph. arXiv:1505.04125, 2015; *Homology, Homotopy and Applications* 19, 2017.
- Kaneta and Yoshinaga. Magnitude homology of metric spaces and order complexes. arXiv:1803.04247, 2018; *Bulletin of the LMS* 53(3), 2021.
- Leinster. The Euler characteristic of a category. arXiv:math.CT/0610260, 2006; *Documenta Mathematica* 13, 2008.
- Leinster. The magnitude of metric spaces. arXiv:1012.5857, 2010; *Documenta Mathematica* 18, 2013.
- Leinster. The magnitude of a graph. arXiv:1401.4623, 2014; *Mathematical Proceedings of the Cambridge Philosophical Society* 166, 2019.
- Leinster and Cobbold Measuring diversity: the importance of species similarity. *Ecology* 93, 2012.

References

- Leinster and Meckes Maximizing diversity in biology and beyond. arXiv:1512.06314, 2015; *Entropy* 18, 2016.
- Leinster and Meckes The magnitude of a metric space: from category theory to geometric measure theory. arXiv:1606.00095, 2016; in Nicola Gigli (ed.), *Measure Theory in Non-Smooth Spaces*, de Gruyter Open, 2017.
- Leinster and Roff. The maximum entropy of a metric space. arXiv:1908.11184, 2019; *The Quarterly Journal of Mathematics* 72(4), 2021.
- Leinster and Shulman. Magnitude homology of enriched categories and metric spaces. arXiv:1711.00802, 2017; *Algebraic and Geometric Topology* 21 (2021).
- Meckes. Positive definite metric spaces. arXiv:1012.5863, 2010; *Positivity* 17, 2013.

References

- Meckes. Magnitude, diversity, capacities, and dimensions of metric spaces. arXiv:1308.5407, 2013; *Potential Analysis* 42, 2015.
- Roff. Magnitude, homology, and the Whitney twist. arXiv:2211.02520, 2022.
- Willerton. On the magnitude of spheres, surfaces and other homogeneous spaces. arXiv:1005.4041, 2010; *Geometriae Dedicata* 168, 2014.
- Willerton. Spread: a measure of the size of metric spaces. arXiv:1209.2300, 2012; *International Journal of Computational Geometry and Applications* 25, 2015.