Iterated magnitude homology

(arXiv:2309.00577)

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Magnitude 2023
Osaka, December 2023

Plan

1. Magnitude homology
2. Enriched groups
3. Iterated magnitude homology
4. Iterated magnitude homology of enriched groups

Part I

Magnitude homology

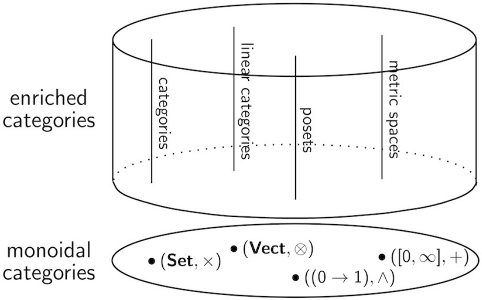
Enriched categories

Yesterday, in Tom's talk:

Enriched categories

A monoidal category is a category $\mathbf{V}$ equipped with some kind of product.

A category enriched in $\mathbf{V}$ is like an ordinary category, with a set/class of objects, but the 'hom-sets' $\mathrm{Hom}(A, B)$ are now objects of $\mathbf{V}$.



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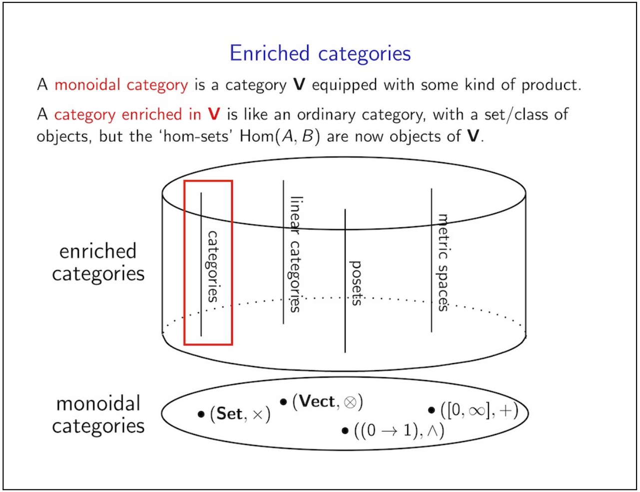

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The category of categories and functors is **Cat**.

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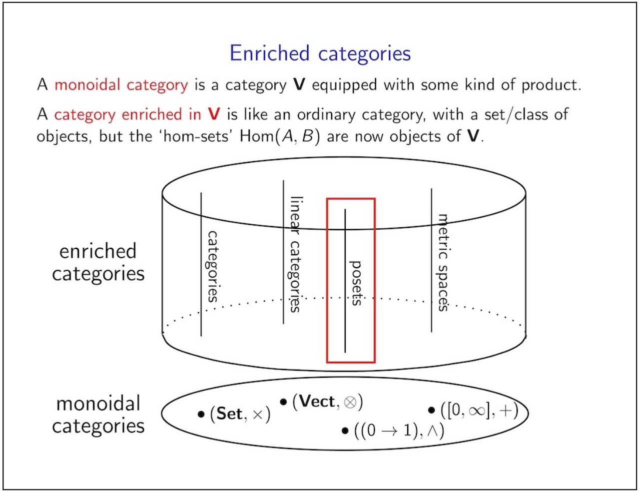

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The category of posets and monotone maps is **Poset**.

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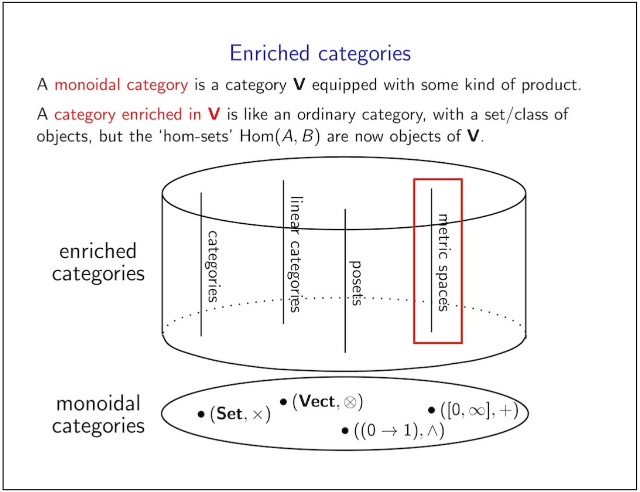

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The category of categories and functors is **Cat**.

The category of posets and monotone maps is **Poset**.

The category of metric spaces and 1-Lipschitz maps is **Met**.

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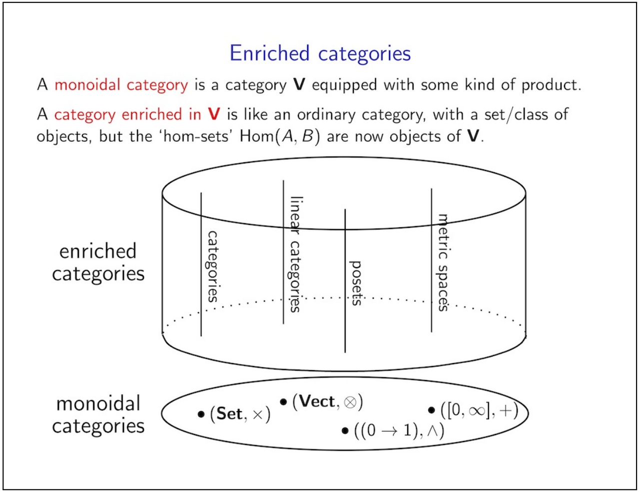

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The category of categories and functors is **Cat**.

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The category of metric spaces and 1-Lipschitz maps is **Met**.

Each of these is itself a monoidal category.

Categorifying magnitude

Magnitude

Given. . .

- a monoidal category $(\mathcal{V}, \otimes)$
- a ring $(R, +, \cdot)$
- a 'size homomorphism' $|-| : (\mathcal{V}, \otimes) \to (R, \cdot)$

$$\Big\downarrow \text{linear algebra}$$

Magnitude for finite $\mathcal{V}$-categories

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Magnitude homology

$$\mathcal{V}\mathbf{Cat} \xrightarrow{MB^{\Sigma}} [\Delta^{\mathsf{op}}, \mathbb{A}] \xrightarrow{C} \mathrm{Ch}(\mathbb{A}) \xrightarrow{H_\bullet} \mathbb{A}^{\mathbb{N}}$$

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Definition (Leinster & Shulman, 2017, after Hepworth & Willerton, 2015)

Let $\Sigma : \mathcal{V} \to \mathbb{A}$ be a strong symmetric monoidal functor. The **magnitude nerve** of a $\mathcal{V}$-category $\mathbf{X}$ is given for $n \in \mathbb{N}$ by

$$MB_n^{\Sigma}(\mathbf{X}) = \bigoplus_{x_0,\ldots,x_n \in \mathbf{X}} \Sigma\mathbf{X}(x_0, x_1) \otimes \cdots \otimes \Sigma\mathbf{X}(x_{n-1}, x_n)$$

with face maps δ^i induced by composition in $\mathbf{X}$ and terminal maps in $\mathcal{V}$.

Magnitude homology

$$\begin{tikzcd} \mathcal{V}\mathbf{Cat} \arrow[r, "MB^{\Sigma}"] \arrow[rr, bend right, "MC^{\Sigma}"'] & {[\Delta^{\mathrm{op}}, \mathbb{A}]} \arrow[r, "C"] & \mathrm{Ch}(\mathbb{A}) \arrow[r, "H_\bullet"] & \mathbb{A}^{\mathbb{N}} \end{tikzcd}$$

Definition (Leinster & Shulman, 2017, after Hepworth & Willerton, 2015)

The **magnitude complex** of $\mathbf{X}$ has $MC_n^{\Sigma}(\mathbf{X}) = MB_n^{\Sigma}(\mathbf{X})$, with boundary maps

$$\partial_n : MC_n^{\Sigma}(\mathbf{X}) \to MC_{n-1}^{\Sigma}(\mathbf{X})$$

given by $\partial_n = \sum_{i=0}^{n} (-1)^i \delta^i$.

Magnitude homology

$$\begin{tikzcd}
\mathcal{V}\mathbf{Cat} \arrow[r, "MB^{\Sigma}"] \arrow[rr, bend right, "MC^{\Sigma}"'] \arrow[rrr, bend left, "MH^{\Sigma}"] & {[\Delta^{\mathrm{op}}, \mathbb{A}]} \arrow[r, "C"] & \mathrm{Ch}(\mathbb{A}) \arrow[r, "H_\bullet"] & \mathbb{A}^{\mathbb{N}}
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The **magnitude homology** of $\mathbf{X}$ is $MH_\bullet^{\Sigma}(\mathbf{X}) = H_\bullet(MC^{\Sigma}(\mathbf{X}))$.

Magnitude homology for categories, posets and groups

Small categories are categories enriched in **Set**. We take the size of a set to be its cardinality and the size functor $\Sigma : \mathbf{Set} \to \mathbf{Ab}$ to be the free abelian group functor.

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The magnitude complex of a category $\mathbf{X}$ is then given in degree $n \geqslant 0$ by

$$MC_n^{\Sigma}(\mathbf{X}) = \mathbb{Z} \cdot \{(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n) \mid x_i, f_i \text{ in } \mathbf{X}\}.$$

The differential is $\partial_n = \sum_{i=1}^{n-1} (-1)^i \delta_i$ where δ_i is induced by composing f_i with f_{i+1}.

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So, by standard facts:

- If $\mathbf{C}$ is a category then $MH_{\bullet}^{\Sigma}(\mathbf{C})$ is the homology of its classifying space.
- If $\mathbf{P}$ is a poset then $MH_{\bullet}^{\Sigma}(\mathbf{P})$ is the homology of its order complex.
- If $\mathbf{G}$ is a group then $MH_{\bullet}^{\Sigma}(\mathbf{G})$ is is ordinary group homology.

Magnitude homology for metric spaces

For a metric space X the magnitude complex is an $[0,\infty]$-graded chain complex:

$$MC_n^{\ell}(X) = \mathbb{Z} \cdot \left\{ (x_0, \ldots, x_n) \mid x_i \in X \text{ and } x_i \neq x_{i+1}, \text{ and } \sum_{i=0}^{n-1} d(x_i, x_{i+1}) = \ell \right\}$$

for $n \in \mathbb{N}$ and $\ell \in [0,\infty]$, with $\partial_n = \sum_{i=1}^{n-1} (-1)^i \delta_i$ where

$$\delta_i(x_0, \ldots, x_n) = \begin{cases} (x_0, \ldots, \widehat{x_i}, \ldots x_n) & \text{if } d(x_{i-1}, x_i) + d(x_i, x_{i+1}) = d(x_{i-1}, x_{i+1}) \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{tikzcd} \mathbf{Met} \arrow[r, "MB^*"] \arrow[rrr, bend left, "MH_\bullet^*"] \arrow[rr, bend right, red, "MC^*"'] & {[\Delta^{\mathrm{op}}, \mathbf{Ab}^{[0,\infty]}]} \arrow[r, "C"] & \mathrm{Ch}(\mathbf{Ab}^{[0,\infty]}) \arrow[r, "H_\bullet"] & \mathbf{Ab}^{[0,\infty]\times\mathbb{N}} \end{tikzcd}$$

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Basic theorem (Leinster & Shulman) Call $(x,y) \in X\times X$ an **adjacent pair** if $x \neq y$ and there is <u>no</u> point $z \neq x,y$ such that $d(x,z)+d(z,y) = d(x,y)$. Then

$$MH_1^\ell(X) = \mathbb{Z}\cdot\{\text{adjacent pairs } (x,y) \mid d(x,y) = \ell.\}$$

Part II

Enriched groups

Groups with structure

Often a group comes equipped with interesting additional structure. For instance...

- A **partially ordered group** is a group G equipped with a partial order $\leqslant$ such that if $g \leqslant h$ then $gk \leqslant hk$ and $kg \leqslant kh$ for all $k \in G$.

  Example Every Coxeter group is partially ordered by the Bruhat order.

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- A **norm** on a group G is a function $|-| : G \to \mathbb{R}$ satisfying
  - $|g| \geqslant 0$ for all $g \in G$ and $|e| = 0$
  - $|gh| \leqslant |g| + |h|$ for all $g, h \in G$.

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  Examples Any generating set $S \subseteq G$ determines a word-length norm on G.

  Asao (2023) uses a normed fundamental group to classify metric fibrations.

Enriched groups

Definition

Let $(\mathcal{V}, \otimes, I)$ be a symmetric monoidal category. A **group enriched in $\mathcal{V}$**, or **$\mathcal{V}$-group**, is a one-object $\mathcal{V}$-category whose underlying ordinary category is a group.

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If $\mathcal{V} = \mathbf{Cat}$, **Poset** or **Met**, a $\mathcal{V}$-group is an object G of $\mathcal{V}$ equipped with $\mathcal{V}$-morphisms

- $m : G \otimes G \to G$ (multiplication)
- $e : I \to G$ (selecting the identity element $e \in G$)

$$\begin{tikzcd} G \otimes G \otimes G \arrow[r, "\mathrm{Id}\otimes m"] \arrow[d, "m\otimes\mathrm{Id}"'] & G \otimes G \arrow[d, "m"] \\ G \otimes G \arrow[r, "m"] & G \end{tikzcd}$$

making these diagrams commute in $\mathcal{V}$ $\lrcorner\!\uparrow$

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\begin{tikzcd}
G \arrow[r, "((-)^{-1}, \mathrm{Id})"] \arrow[d, "(\mathrm{Id}, (-)^{-1})"'] \arrow[dr, "e"] & G \times G \arrow[d, "m"] \\
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and a function $(-)^{-1} : \mathrm{ob}(G) \to \mathrm{ob}(G)$ making this diagram commute in **Set** ↗

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Example Every group object in a Cartesian category $\mathcal{V}$ is a group enriched in $(\mathcal{V}, \times)$. But enriched groups are more general.

Poset-groups and **Met**-groups

Example Every partially ordered group $(G, \leqslant)$ is a group enriched in $(\mathbf{Poset}, \times)$.

Exercise The map $(-)^{-1} : G \to G$ is monotone if and only if $g \leqslant h$ implies $g = h$. So only the trivial partial order makes G a group object in $(\mathbf{Poset}, \times)$.

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Example Every normed group $(G, |-|)$ carries a metric specified by

$$d(g, h) = |h^{-1}g|.$$

This gives an enrichment in $(\mathbf{Met}, \times_{\ell 1})$ if and only if $|-|$ is conjugation-invariant.

Exercise The map $(\mathrm{Id}, (-)^{-1}) : G \to G \times_{\ell_1} G$ is 1-Lipschitz if and only if $d(g, h) = 0$ for all g, h. So only the 'indiscrete' metric makes G a group object in $(\mathbf{Met}, \times_{\ell_1})$.

Strict 2-groups

Definition A strict 2-group is a group object in $(\mathbf{Cat}, \times)$.

Theorem (Mac Lane & Whitehead, 1950) Strict 2-groups classify homotopy 2-types.

$$\begin{tikzcd} \boxed{\text{Strict 2-groups}} \arrow[r, bend left, "\mathbb{B}(-)"] \arrow[r, phantom, "\simeq"] & \boxed{\begin{array}{c}\text{Spaces } X \text{ such that} \\ \pi_n(X) = 0 \text{ for } n > 2\end{array}} \arrow[l, bend left] \end{tikzcd}$$

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- objects the elements of the group G
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- the functor $m : G \times G \to G$ defined on objects by multiplication in G and on arrows by multiplication in $N \ltimes G$.

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Theorem (Mac Lane & Whitehead) For any $N \lhd G$ we have $\pi_1(\mathbb{B}(\mathbf{G}_N)) \cong G/N$.

Part III

Iterated magnitude homology

Taking enrichment into account

Observation In each of these examples, G has a 'second-order' enrichment.

$$\begin{tikzcd}
\boxed{\text{Strict 2-group}} \arrow[d, hook] & \boxed{\text{Partial ordering}} \arrow[d, hook] & \boxed{\text{Conjugation-invariant norm}} \arrow[d, equal] \\
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For these, 2-category theory provides a notion of classifying space.

The classifying space of a 2-category $\mathbf{X}$

The Duskin or Street approach

Define a simplicial set $\triangle\mathbf{X}$ by

$[n] = (0 \to 1 \to \cdots \to n)$

$$\triangle\mathbf{X}_n = \mathbf{BiCat}_{\mathrm{NLax}}([n], \mathbf{X}).$$

bicategories and
normal lax 2-functors

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The Segal approach

take the classifying space of each hom-category in $\mathbf{X}$

$$\begin{tikzcd}
\mathbf{2Cat} \arrow[d, "B_*"] \arrow[ddd, bend left, "B"] \\
\mathbf{TopCat} \arrow[d, "N"] \\
{[\Delta^{\mathrm{op}}, \mathbf{Top}]} \arrow[d, "|-|"] \\
\mathbf{Top}
\end{tikzcd}$$

The classifying space of a 2-category $\mathbf{X}$

The Duskin or Street approach

Define a simplicial set $\triangle\mathbf{X}$ by

$[n] = (0 \to 1 \to \cdots \to n)$

$$\triangle\mathbf{X}_n = \mathbf{BiCat}_{\mathrm{NLax}}([n], \mathbf{X}).$$

bicategories and
normal lax 2-functors

Call the topological space $|\triangle\mathbf{X}|$ the **classifying space** of $\mathbf{X}$.

The Segal approach

take the classifying space of each hom-category in $\mathbf{X}$

take the *internal nerve* of the **Top**-category $B_*\mathbf{X}$

$$\begin{tikzcd}
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geometrically realize

Call $B\mathbf{X}$ the **classifying space** of $\mathbf{X}$.

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\end{tikzcd}$$

Call $B\mathbf{X}$ the **classifying space** of $\mathbf{X}$.

Theorem (Bullejos & Cegarra, 2003) There's a natural equivalence $B\mathbf{X} \simeq |\triangle\mathbf{X}|$.

The classifying space of a 2-category $\mathbf{X}$

The Duskin or Street approach

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$$\begin{tikzcd}
\mathbf{2Cat} \arrow[d, "B_*"] \arrow[ddd, bend left=60, "B"] \\
\mathbf{TopCat} \arrow[d, "N"] \\
[\Delta^{\mathrm{op}}, \mathbf{Top}] \arrow[d, "|-|"] \\
\mathbf{Top}
\end{tikzcd}$$

Call $B\mathbf{X}$ the **classifying space** of $\mathbf{X}$.

Proof Compares both constructions to the diagonal of the bisimplicial ‘double nerve’.

The double **magnitude** nerve

Let $(\mathcal{V}, \otimes, I)$ be semicartesian and $\Sigma : \mathcal{V} \to \mathbb{A}$ a strong symmetric monoidal functor.

Proposition The magnitude nerve defines a strong symmetric monoidal functor

$$MB^{\Sigma} : (\mathcal{V}\mathbf{Cat}, \otimes_{\mathcal{V}}) \to ([\Delta^{\mathrm{op}}, \mathbb{A}], \otimes_{pw})$$

so we can employ it as a size functor.

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Definition The **double magnitude nerve** of a $\mathcal{V}\mathbf{Cat}$-category $\mathbf{X}$ is

$$MB^{MB^{\Sigma}}(\mathbf{X}) \in [\Delta^{\mathsf{op}}, [\Delta^{\mathsf{op}}, \mathbb{A}]] = [\Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}}, \mathbb{A}].$$

Iterated magnitude homology

$$
\begin{tikzcd}
2\mathcal{V}\mathbf{Cat} \arrow[r, "MB^{MB^{\Sigma}}"] \arrow[rr, bend right, "IMB"'] & {[\Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}}, \mathbb{A}]} \arrow[r, "\mathrm{diag}"] & {[\Delta^{\mathsf{op}}, \mathbb{A}]} \arrow[r, "C"] & \mathrm{Ch}(\mathbb{A}) \arrow[r, "H_\bullet"] & \mathbb{A}^{\mathbb{N}}
\end{tikzcd}
$$

Definition The **iterated magnitude nerve** of a $\mathcal{V}\mathbf{Cat}$-category $\mathbf{X}$ is

$$
IMB(\mathbf{X}) = \mathrm{diag}\left(MB^{MB^{\Sigma}}(\mathbf{X})\right).
$$

Iterated magnitude homology

$$\begin{tikzcd}
2\mathcal{V}\mathbf{Cat} \arrow[r, "MB^{MB^{\Sigma}}"] \arrow[rr, bend right, "IMB"'] \arrow[rrrr, bend left, "IMH", red] & {[\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathbb{A}]} \arrow[r, "\mathrm{diag}"] & {[\Delta^{\mathrm{op}}, \mathbb{A}]} \arrow[r, "C"] & \mathrm{Ch}(\mathbb{A}) \arrow[r, "H_\bullet"] & \mathbb{A}^{\mathbb{N}}
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$$IMH_\bullet(\mathbf{X}) = H_\bullet C(IMB(\mathbf{X})).$$

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Theorem For any 2-category $\mathbf{X}$, $IMH_\bullet(\mathbf{X})$ is the homology of its classifying space.

Iterated magnitude homology

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$$IMH_\bullet(\mathbf{X}) = H_\bullet C(IMB(\mathbf{X})).$$

Corollary For any strict 2-group $\mathbf{G}$ we have $IMH_\bullet(\mathbf{G}) \cong H_\bullet(\mathbb{B}(\mathbf{G}))$.

Part IV

Iterated magnitude homology of enriched groups

The iterated magnitude homology of a **Cat**-group

A **Cat**-group $\mathbf{G}$ has a category of elements with objects $g, h, \ldots$ and morphisms $\begin{array}{c} g \\ \Downarrow\alpha \\ h \end{array}$.

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A **Cat**-group $\mathbf{G}$ has a category of elements with objects $g, h, \ldots$ and morphisms $\begin{array}{c} g \\ \Downarrow\alpha \\ h \end{array}$.

Definition The **connected components** of $\mathbf{G}$ are the elements of $\pi_0(\mathbf{G}) = \mathrm{ob}(\mathbf{G})/\sim$ where $\sim$ is the equivalence relation generated by "$g \sim h$ if there's a morphism $g \Rightarrow h$".

Lemma The set $\{g \mid g \sim e\}$ is a normal subgroup, so $\pi_0(\mathbf{G})$ is a group.

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Theorem We have $IMH_1(\mathbf{G}) = (\pi_0(\mathbf{G}))_{\mathrm{ab}}$, the abelianization of $\pi_0(\mathbf{G})$.

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Sketch proof $IMH_\bullet(G)$ is isomorphic to the total homology of this double complex ↴

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\mathbb{Z} \arrow[d, "\partial^v"] & \mathbb{Z}\cdot\left\{\begin{array}{c} g_0 \\ \Downarrow\alpha_1 \\ h_0 \end{array}\right\} \arrow[l, "\partial^h"'] \arrow[d, "\partial^v"] & \mathbb{Z}\cdot\left\{\begin{array}{c|c} g_0 & g_1 \\ \Downarrow\alpha_1 & \Downarrow\alpha_2 \\ h_0 & h_1 \end{array}\right\} \arrow[l, "\partial^h"'] \arrow[d, "\partial^v"] & \cdots \arrow[l] \\
\mathbb{Z} & \mathbb{Z}\cdot\left\{ g_0 \right\} \arrow[l, "\partial^h"'] & \mathbb{Z}\cdot\left\{ g_0 \mid g_1 \right\} \arrow[l, "\partial^h"'] & \cdots \arrow[l]
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Facts about spectral sequences imply that

$$IMH_1(\mathbf{G}) \cong H^h H^v(C_{01}).$$

$$\begin{tikzcd}
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\mathbb{Z} & \boxed{\mathbb{Z}\cdot\{\, g_0 \,\}} \arrow[l, "\partial^h"'] & \mathbb{Z}\cdot\{\, g_0 \mid g_1 \,\} \arrow[l, "\partial^h"'] & \cdots \arrow[l]
\end{tikzcd}$$

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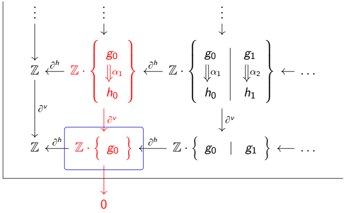
Vertical homology imposes the quotient by $\sim$.



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Facts about spectral sequences imply that

$$IMH_1(\mathbf{G}) \cong H^h H^v(C_{01}).$$

Vertical homology imposes the quotient by $\sim$.
Horizontal homology abelianizes. □

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\mathbb{Z} \arrow[d, "\partial^v"] & \mathbb{Z}\cdot\left\{\begin{smallmatrix} g_0 \\ \Downarrow\alpha_1 \\ h_0 \end{smallmatrix}\right\} \arrow[l, "\partial^h"'] \arrow[d, "\partial^v"] & \mathbb{Z}\cdot\left\{\begin{smallmatrix} g_0 \\ \Downarrow\alpha_1 \\ h_0 \end{smallmatrix} \middle| \begin{smallmatrix} g_1 \\ \Downarrow\alpha_2 \\ h_1 \end{smallmatrix}\right\} \arrow[l, "\partial^h"'] \arrow[d, "\partial^v"] & \cdots \arrow[l] \\
\mathbb{Z} & \boxed{\mathbb{Z}\cdot\{\, g_0 \,\}} \arrow[l, "\partial^h"'] \arrow[d] & \mathbb{Z}\cdot\{\, g_0 \mid g_1 \,\} \arrow[l, "\partial^h"'] & \cdots \arrow[l] \\
 & 0 & &
\end{tikzcd}$$

Normal subgroups and partial orders

Corollary I Let G be a group and N a normal subgroup of G. Then

$$IMH_1(\mathbf{G}_N) \cong (G/N)_{\mathrm{ab}}\,.$$

We can also deduce this from the fact that $\pi_1(\mathbf{G}_N) \cong G/N$ via the Hurewicz theorem.

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Definition The positive cone of a preordered group $(G, \leqslant)$ is $P_{\leqslant} = \{g \in G \mid e \leqslant g\}$. This is a normal subgroup if and only if $\leqslant$ is symmetric.

Corollary II Let $\mathbf{G} = (G, \leqslant)$ be a partially ordered group. Let $\sim$ be the equivalence relation generated by $\leqslant$. Then $P_{\sim} = \{g \in G \mid e \sim g\}$ is a normal subgroup of G, and

$$IMH_1(\mathbf{G}) \cong (G/P_{\sim})_{\mathrm{ab}}.$$

The iterated magnitude complex of a **Met**-group

Let $\mathbf{G} = (G, d)$ be a **Met**-group. Its iterated magnitude complex is $[0, \infty]$-graded, with

$$IMC_n^{\ell}(\mathbf{G}) = \mathbb{Z} \cdot \left\{ \begin{bmatrix} g_{10} & \cdots & g_{n0} \\ \vdots & & \vdots \\ g_{1n} & \cdots & g_{nn} \end{bmatrix} \mid g_{ij} \in G \text{ and } \sum_{i=1}^{n} \sum_{j=0}^{n-1} d(g_{ij}, g_{i,j+1}) = \ell \right\}.$$

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The boundary map is $\partial_n = \sum_{k=1}^{n-1} (-1)^k \delta_k$, where

$$\delta_k \begin{bmatrix} g_{10} & \cdots & g_{n0} \\ \vdots & & \vdots \\ g_{1n} & \cdots & g_{nn} \end{bmatrix} = \begin{bmatrix} g_{10} & \cdots & g_{k0}g_{k+1,0} & \cdots & g_{n0} \\ \vdots & & & & \vdots \\ \widehat{g_{1k}} & & \widehat{\cdots} & & \widehat{g_{nk}} \\ \vdots & & & & \vdots \\ g_{1n} & \cdots & g_{kn}g_{k+1,n} & \cdots & g_{nn} \end{bmatrix}$$

if this preserves the sum of the column-lengths, and 0 otherwise.

The iterated magnitude homology of a **Met**-group $\mathbf{G} = (G, d)$

Definition An element $g \in G$ is **primitive** if for all $h \in G$ we have

$$d(g, e) < d(g, h) + d(h, e).$$

Example

For a word metric with respect to $S \subseteq G$, the primitive elements are the elements of S.

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Theorem

In real grading 0, the magnitude homology of $\mathbf{G}$ is the ordinary group homology of G:

$$IMH^0_\bullet(\mathbf{G}) \cong H_\bullet(G).$$

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Theorem
In real grading 0, the magnitude homology of $\mathbf{G}$ is the ordinary group homology of G:

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In real gradings $\ell > 0$ we have $IMH^\ell_0(\mathbf{G}) = IMH^\ell_1(\mathbf{G}) = 0$ and

$$IMH^\ell_2(\mathbf{G}) = \mathbb{Z} \cdot \{\text{conjugacy classes of primitive elements of norm } \ell\}.$$

Sketch of the proof for $\ell > 0$

In each grading $\ell > 0$, we have $IMH^{\ell}_{\bullet}(\mathbf{G}) \cong H_{\bullet}(\mathrm{Tot}(C^{\ell}_{\bullet\bullet}))$ where $C^{\ell}_{\bullet\bullet}$ looks like this ↴

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 & \mathbb{Z} \cdot \left\{ \begin{bmatrix} g_0 \\ h_0 \\ k_0 \end{bmatrix} \right\} \arrow[l, "\partial^h_{02}"'] \arrow[d, "\partial^v_{11}"] & \mathbb{Z} \cdot \left\{ \begin{bmatrix} g_0 & g_1 \\ h_0 & h_1 \\ k_0 & k_1 \end{bmatrix} \right\} \arrow[l, "\partial^h_{12}"'] \arrow[d, "\partial^v_{21}"] & \cdots \arrow[l] \\
0 & \mathbb{Z} \cdot \left\{ \begin{bmatrix} g_0 \\ h_0 \end{bmatrix} \right\} \arrow[l, "\partial^h_{01}"'] \arrow[d, "\partial^v_{10}"] & \mathbb{Z} \cdot \left\{ \begin{bmatrix} g_0 & g_1 \\ h_0 & h_1 \end{bmatrix} \right\} \arrow[l, "\partial^h_{11}"'] \arrow[d, "\partial^v_{20}"] & \cdots \arrow[l] \\
0 & 0 & 0 &
\end{tikzcd}$$

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Since $C^{\ell}_{02} = C^{\ell}_{20} = 0$, we have

$$IMH^{\ell}_{2}(\mathbf{G}) \cong H^{h}H^{v}(C^{\ell}_{11}).$$

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 & \mathbb{Z}\cdot\left\{\begin{bmatrix} g_0 \\ h_0 \\ k_0 \end{bmatrix}\right\} \arrow[l, "\partial^h_{02}"'] \arrow[d, "\partial^v_{11}"] & \mathbb{Z}\cdot\left\{\begin{bmatrix} g_0 & g_1 \\ h_0 & h_1 \\ k_0 & k_1 \end{bmatrix}\right\} \arrow[l, "\partial^h_{12}"'] \arrow[d, "\partial^v_{21}"] & \cdots \arrow[l] \\
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0 & 0 & 0 &
\end{tikzcd}$$

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$$IMH^{\ell}_{2}(\mathbf{G}) \cong H^{h}H^{v}(C^{\ell}_{11}).$$

Column 1 is $MC^{\ell}_{\bullet}(G)$ for the metric space G, so

$$H^{v}(C^{\ell}_{11}) \cong \left\{ \begin{array}{l} \text{adjacent pairs } (g,h) \\ \text{such that } d(g,h) = \ell \end{array} \right\}.$$

Exercise Elements g and h are adjacent if and only if gh^{-1} is primitive.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 & \mathbb{Z}\cdot\left\{\begin{bmatrix} g_0 \\ h_0 \\ k_0 \end{bmatrix}\right\} \arrow[l, "\partial^h_{02}"'] \arrow[d, "\partial^v_{11}"] & \mathbb{Z}\cdot\left\{\begin{bmatrix} g_0 & g_1 \\ h_0 & h_1 \\ k_0 & k_1 \end{bmatrix}\right\} \arrow[l, "\partial^h_{12}"'] \arrow[d, "\partial^v_{21}"] & \cdots \arrow[l] \\
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0 & 0 & 0 &
\end{tikzcd}$$

Sketch of the proof for $\ell > 0$

In each grading $\ell > 0$, we have $IMH_\bullet^\ell(\mathbf{G}) \cong H_\bullet(\mathrm{Tot}(C_{\bullet\bullet}^\ell))$ where $C_{\bullet\bullet}^\ell$ looks like this ↴

Since $C_{02}^\ell = C_{20}^\ell = 0$, we have

$$IMH_2^\ell(\mathbf{G}) \cong H^h H^v(C_{11}^\ell).$$

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$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 & \mathbb{Z}\cdot\left\{\begin{bmatrix} g_0 \\ h_0 \\ k_0 \end{bmatrix}\right\} \arrow[l, "\partial^h_{02}"'] \arrow[d, "\partial^v_{11}"] & \mathbb{Z}\cdot\left\{\begin{bmatrix} g_0 & g_1 \\ h_0 & h_1 \\ k_0 & k_1 \end{bmatrix}\right\} \arrow[l, "\partial^h_{12}"'] \arrow[d, "\partial^v_{21}"] & \cdots \arrow[l] \\
0 & \boxed{\mathbb{Z}\cdot\left\{\begin{bmatrix} g_0 \\ h_0 \end{bmatrix}\right\}} \arrow[l, "\partial^h_{01}"'] \arrow[d, "\partial^v_{10}"] & \mathbb{Z}\cdot\left\{\begin{bmatrix} g_0 & g_1 \\ h_0 & h_1 \end{bmatrix}\right\} \arrow[l, "\partial^h_{11}"'] \arrow[d, "\partial^v_{20}"] & \cdots \arrow[l] \\
0 & 0 & 0 &
\end{tikzcd}$$

Exercise Elements g and h are adjacent if and only if gh^{-1} is primitive.

Finally, taking horizontal homology H^h identifies conjugate elements. □

Summary

Various valuable structures on groups are instances of second-order enrichment.

$$
\begin{tikzcd}
\boxed{\text{Strict 2-group}} \arrow[d, hook] & \boxed{\text{Partial ordering}} \arrow[d, hook] & \boxed{\text{Conjugation-invariant norm}} \arrow[d, equal] \\
\boxed{\begin{array}{c}\text{Enrichment in}\\ \text{the category of}\\ \textbf{Set}\text{-categories}\end{array}} & \boxed{\begin{array}{c}\text{Enrichment in}\\ \text{the category of}\\ (0 \to 1)\text{-categories}\end{array}} \arrow[l, hook'] & \boxed{\begin{array}{c}\text{Enrichment in}\\ \text{the category of}\\ [0, \infty]\text{-categories}\end{array}}
\end{tikzcd}
$$

Summary

Various valuable structures on groups are instances of second-order enrichment.

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\end{tikzcd}$$

For these, Mac Lane and Whitehead provide a classifying space.

Summary

Various valuable structures on groups are instances of second-order enrichment.

$$
\begin{tikzcd}
\boxed{\text{Strict 2-group}} \arrow[d, hook] & & \boxed{\text{Partial ordering}} \arrow[d, hook] & & \boxed{\text{Conjugation-invariant norm}} \arrow[d, equal] \\
\boxed{\begin{array}{c}\text{Enrichment in}\\ \text{the category of}\\ \mathbf{Set}\text{-categories}\end{array}} & & \boxed{\begin{array}{c}\text{Enrichment in}\\ \text{the category of}\\ (0 \to 1)\text{-categories}\end{array}} \arrow[ll, hook'] & & \boxed{\begin{array}{c}\text{Enrichment in}\\ \text{the category of}\\ [0, \infty]\text{-categories}\end{array}}
\end{tikzcd}
$$

For these, 2-category theory provides a classifying space.

Summary

Various valuable structures on groups are instances of second-order enrichment.

$$
\begin{tikzcd}
\boxed{\text{Strict 2-group}} \arrow[d, hook] & \boxed{\text{Partial ordering}} \arrow[d, hook] & \boxed{\text{Conjugation-invariant norm}} \arrow[d, equal] \\
\boxed{\begin{array}{c}\text{Enrichment in} \\ \text{the category of} \\ \mathbf{Set}\text{-categories}\end{array}} & \boxed{\begin{array}{c}\text{Enrichment in} \\ \text{the category of} \\ (0 \to 1)\text{-categories}\end{array}} \arrow[l, hook'] & \boxed{\begin{array}{c}\text{Enrichment in} \\ \text{the category of} \\ [0, \infty]\text{-categories}\end{array}}
\end{tikzcd}
$$

$\uparrow$

For these, iterated magnitude homology captures ‘the homology of a classifying space’.

Summary

Various valuable structures on groups are instances of second-order enrichment.

$$\begin{tikzcd}
\boxed{\text{Strict 2-group}} \arrow[d, hook] & \boxed{\text{Partial ordering}} \arrow[d, hook] & \boxed{\text{Conjugation-invariant norm}} \arrow[d, equal] \\
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\end{tikzcd}$$

For these, iterated magnitude homology captures 'the homology of a classifying space'.

For a group G with a conjugation-invariant norm, $IMH_\bullet(G)$ is sensitive to the topology of the ordinary classifying space and the geometry of the group under the norm.

Thank you.

References

- Asao. Classification of metric fibrations. arXiv:2307.04387 (2023).
- Bullejos and Cegarra. On the geometry of 2-categories and their classifying spaces. *K-Theory* 29 (2003).
- Duskin. Simplicial matrices and the nerves of weak n-categories I: Nerves of bicategories. *Theory and Applications of Categories* 9(10):198–308 (2002).
- Hepworth and Willerton. Categorifying the magnitude of a graph. *Homology, Homotopy and Applications* 19 (2017).
- Leinster and Shulman. Magnitude homology of enriched categories and metric spaces. *Algebraic and Geometric Topology* 21 (2021).
- Mac Lane and Whitehead. On the 3-type of a complex. *Proceedings of the National Academy of Sciences* 36(1):41–48 (1950).
- Segal. Classifying spaces and spectral sequences. *Publications Mathématiques de l'IHÉS*, 34:105–112 (1968).