Practical use of stochastic models for spatial climate and weather reconstruction

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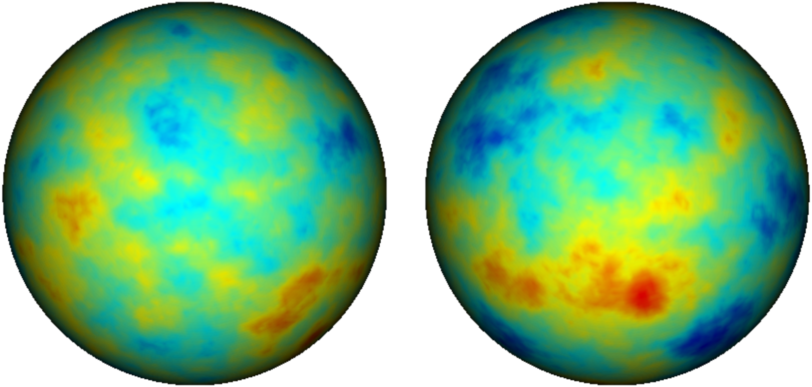
Spatial statistics on the globe

Input: Spatially and temporally irregular temperature measurements

Output: Weather and climate reconstruction, with proper uncertainty estimates

Stochastic non-stationary spatio-temporal models



Stationary models are likely inadequate, and non-stationary covariances typically lead to intractable computations
Solution: Use stochastic PDE models inspired by physics

Hierarchical spatial models

Hierarchical models

$\boldsymbol{\theta}$	Model parameters
$\boldsymbol{x}\|\boldsymbol{\theta}$	Latent processes, spatial or spatio-temporal fields
$\boldsymbol{y}\|\boldsymbol{\theta},\boldsymbol{x}$	Measured data

Classical spatial models

Spatial field: $x(\boldsymbol{u}),\ \boldsymbol{u} \in \mathbb{R}^d, \quad \{x(\boldsymbol{u}_i)\} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$

Spatial covariance: $\Sigma_{i,j} = \mathrm{Cov}(x(\boldsymbol{u}_i), x(\boldsymbol{u}_j))$

Measurements: $y_i = \boldsymbol{B}_i\boldsymbol{\beta} + x(\boldsymbol{u}_i) + \epsilon_i, \quad \boldsymbol{\epsilon}|\boldsymbol{x} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_\epsilon)$

Covariance $\boldsymbol{\Sigma}$: Explicit global dependence

Precision $\boldsymbol{Q} = \boldsymbol{\Sigma}^{-1}$: Explicit local, implicit global dependence

Describing spatial dependence

The Matérn covariance family on $\mathbb{R}^d$

$$\text{Cov}(x(\mathbf{0}), x(\boldsymbol{u})) = \sigma^2 \frac{2^{1-\nu}}{\Gamma(\nu)} (\kappa\|\boldsymbol{u}\|)^\nu K_\nu(\kappa\|\boldsymbol{u}\|)$$

Scale $\kappa > 0$, smoothness $\nu > 0$, variance $\sigma^2 > 0$

Whittle (1954, 1963): Matérn as SPDE solution

Matérn fields are stationary solutions to the SPDE

$$\left(\kappa^2 - \Delta\right)^{\alpha/2} x(\boldsymbol{u}) = \mathcal{W}(\boldsymbol{u}), \quad \alpha = \nu + d/2$$

$\sigma^2 = \frac{\Gamma(\nu)}{\Gamma(\alpha)\kappa^{2\nu}(4\pi)^{d/2}}$, Laplacian $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial u_i^2}$

Piecewise linear Markov models

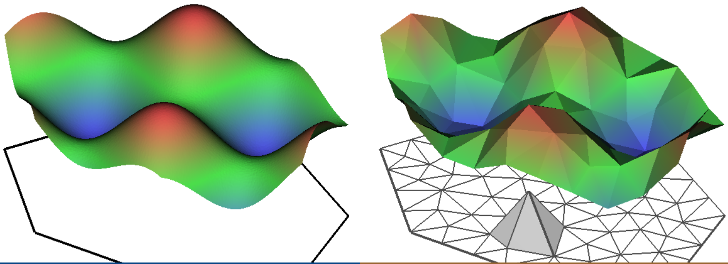
Continuous Markovian spatial models (Lindgren et al, 2011)

Local basis: $x(\boldsymbol{u}) = \sum_k \psi_k(\boldsymbol{u}) x_k$

Basis weights: $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q}_x^{-1})$, sparse $\boldsymbol{Q}$

Measurements: $\boldsymbol{y} = \boldsymbol{B}\boldsymbol{\beta} + \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\epsilon}$, $\boldsymbol{\epsilon}|\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q}_{y|x}^{-1})$

Posterior: Local observations $\Longrightarrow$ Markovian posterior for $\boldsymbol{x}$



The best piecewise linear approximation $\sum_k \psi_k(\boldsymbol{u})\, x_k$

Projection of the SPDE: Linear systems of equations ($\alpha = 2$)

$$\sum_j (\kappa^2 \underbrace{\langle \psi_i, \psi_j \rangle}_{\boldsymbol{C}_{ij}} + \underbrace{\langle \psi_i, -\Delta \psi_j \rangle}_{\boldsymbol{G}_{ij}}) x_j \overset{D}{=} \langle \psi_i, \mathcal{W} \rangle \quad \text{jointly for all } i.$$

$\boldsymbol{C}$ and $\boldsymbol{G}$ are as sparse as the triangulation neighbourhood

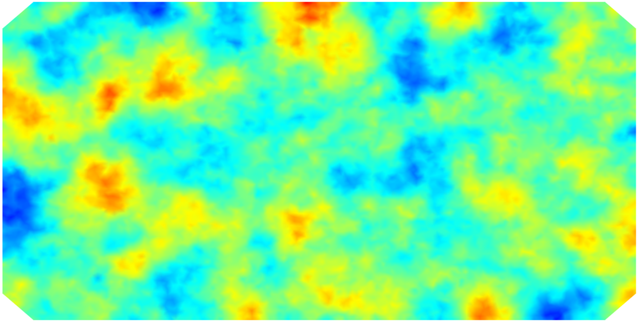
Constructing the precision matrices

$\boldsymbol{K} = \kappa^2 \boldsymbol{C} + \boldsymbol{G}$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3, 4, \ldots$
$\boldsymbol{K}\boldsymbol{x}$	$\mathcal{N}(\boldsymbol{0}, \boldsymbol{K})$	$\mathcal{N}(\boldsymbol{0}, \boldsymbol{C})$	$\mathcal{N}\left(\boldsymbol{0}, \boldsymbol{C}\boldsymbol{Q}_{\boldsymbol{x},\alpha-2}^{-1}\boldsymbol{C}\right)$
$\boldsymbol{Q}_{\boldsymbol{x},\alpha}$	$\boldsymbol{K}$	$\boldsymbol{K}^\top \boldsymbol{C}^{-1} \boldsymbol{K}$	$\boldsymbol{K}^\top \boldsymbol{C}^{-1} \boldsymbol{Q}_{\boldsymbol{x},\alpha-2} \boldsymbol{C}^{-1} \boldsymbol{K}$

Simulations with precisions via finite element calculations

The approach can in a straightforward way be extended to oscillating, anisotropic, non-stationary, non-separable spatio-temporal, and multivariate fields on manifolds.

$$(\kappa^2 - \Delta)(\tau\, x(\boldsymbol{u})) = \mathcal{W}(\boldsymbol{u}), \quad \boldsymbol{u} \in \mathbb{R}^d$$



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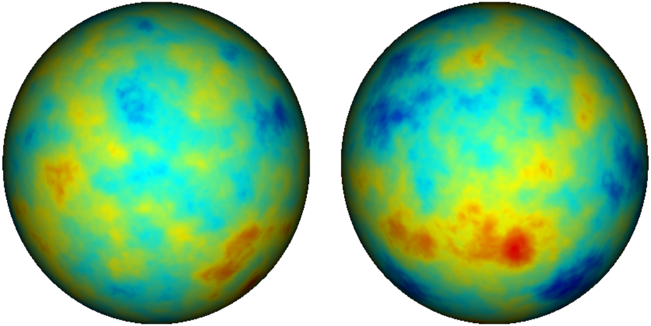

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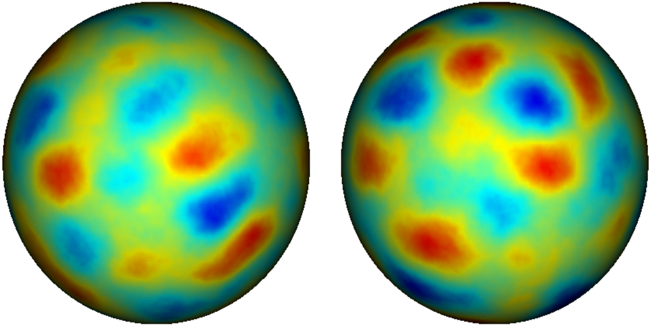
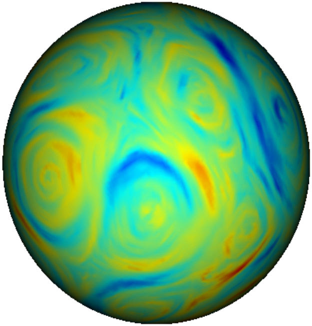
$$(\kappa^2 \, e^{i\pi\theta} - \Delta)(\tau \, x(\boldsymbol{u})) = \mathcal{W}(\boldsymbol{u}), \quad \boldsymbol{u} \in \Omega$$

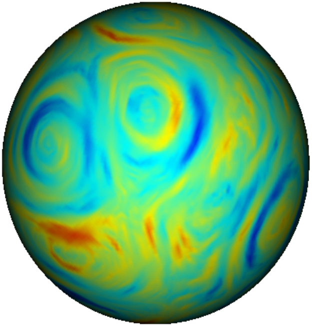


Simulations with precisions via finite element calculations

The approach can in a straightforward way be extended to oscillating, anisotropic, non-stationary, non-separable spatio-temporal, and multivariate fields on manifolds.

$$(\kappa_{\boldsymbol{u}}^2 + \nabla \cdot \boldsymbol{m}_{\boldsymbol{u}} - \nabla \cdot \boldsymbol{M}_{\boldsymbol{u}} \nabla)(\tau_{\boldsymbol{u}} x(\boldsymbol{u})) = \mathcal{W}(\boldsymbol{u}), \quad \boldsymbol{u} \in \Omega$$

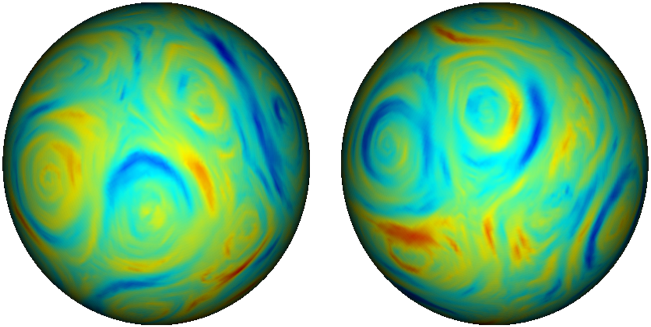




Simulations with precisions via finite element calculations

The approach can in a straightforward way be extended to oscillating, anisotropic, non-stationary, non-separable spatio-temporal, and multivariate fields on manifolds.

$$\left(\frac{\partial}{\partial t} + \kappa^2_{\boldsymbol{u},t} + \nabla \cdot \boldsymbol{m}_{\boldsymbol{u},t} - \nabla \cdot \boldsymbol{M}_{\boldsymbol{u},t} \nabla\right) (\tau_{\boldsymbol{u},t} x(\boldsymbol{u}, t)) = \mathcal{E}(\boldsymbol{u}, t), \quad (\boldsymbol{u}, t) \in \Omega \times \mathbb{R}$$



Bayesian inference with sparse precisions

Conditional distribution in a Gaussian model

$$\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{\mu}_x, \boldsymbol{Q}_x^{-1}), \quad \boldsymbol{y}|\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{Q}_{y|x}^{-1})$$

$$\boldsymbol{x}|\boldsymbol{y} \sim \mathcal{N}(\boldsymbol{\mu}_{x|y}, \boldsymbol{Q}_{x|y}^{-1})$$

$$\boldsymbol{Q}_{x|y} = \boldsymbol{Q}_x + \boldsymbol{A}^T \boldsymbol{Q}_{y|x} \boldsymbol{A}$$

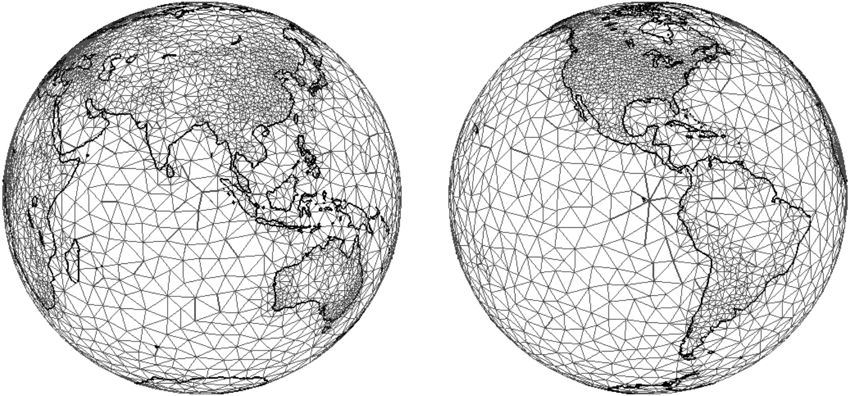
$$\boldsymbol{\mu}_{x|y} = \boldsymbol{\mu}_x + \boldsymbol{Q}_{x|y}^{-1} \boldsymbol{A}^T \boldsymbol{Q}_{y|x}(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{\mu}_x)$$

Direct Bayesian inference with INLA (`r-inla.org`)

$$p(\boldsymbol{\theta}|\boldsymbol{y}) \propto \left.\frac{p(\boldsymbol{\theta})p(\boldsymbol{x}|\boldsymbol{\theta})p(\boldsymbol{y}|\boldsymbol{x},\boldsymbol{\theta})}{p_G(\boldsymbol{x}|\boldsymbol{y},\boldsymbol{\theta})}\right|_{\boldsymbol{x}=\boldsymbol{x}^*}$$

$$p(\boldsymbol{x}_i|\boldsymbol{y}) \propto \int p_G(\boldsymbol{x}_i|\boldsymbol{y},\boldsymbol{\theta})p(\boldsymbol{\theta}|\boldsymbol{y})\,\mathrm{d}\boldsymbol{\theta}$$

Triangulation partly adapted to the data density



Aim: A framework for using a spatio-temporal stochastic model in combination with different data sources.
Current input modules are GHCNv2, ICOADS (gridded) and Antarctica.

Linear model for weather observations

Weather = Climate + Anomaly

$$\mathbf{z} \sim \mathsf{N}(0, \mathbf{Q}_z^{-1}) \quad \text{(climate: space-time model)}$$

$$z(t, \mathbf{s}) = \sum_k B_k(t) \mathbf{z}_k(\mathbf{s}) \quad \text{(basis function representation)}$$

$$\mathbf{a} \sim \mathsf{N}(0, \mathbf{I} \otimes \mathbf{Q}_a^{-1}) \quad \text{(anomaly: spatial model, indep. in time)}$$

$$w(t, \mathbf{s}) = a(t, \mathbf{s}) + z(t, \mathbf{s}) \quad \text{(weather)}$$

$$y_i = \text{altitude effect } + w(t_i, \mathbf{s}_i) + \epsilon_i \quad \text{(observations)}$$

$$\epsilon \sim \mathsf{N}(0, \mathbf{Q}_\epsilon^{-1})$$

$$\mathbf{y} = \boldsymbol{h}\beta_h + \mathbf{A}(\mathbf{a} + (\mathbf{B} \otimes \mathbf{I})\mathbf{z}) + \epsilon$$

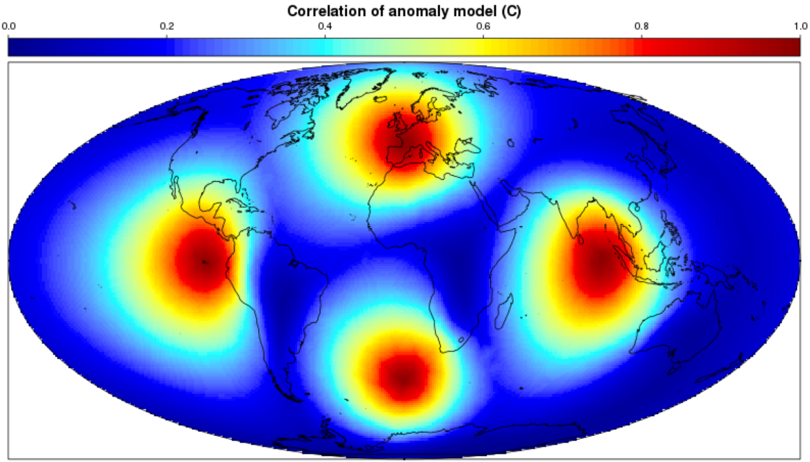
Stochastic weather anomaly and climate model

Non-stationary spatial SPDE

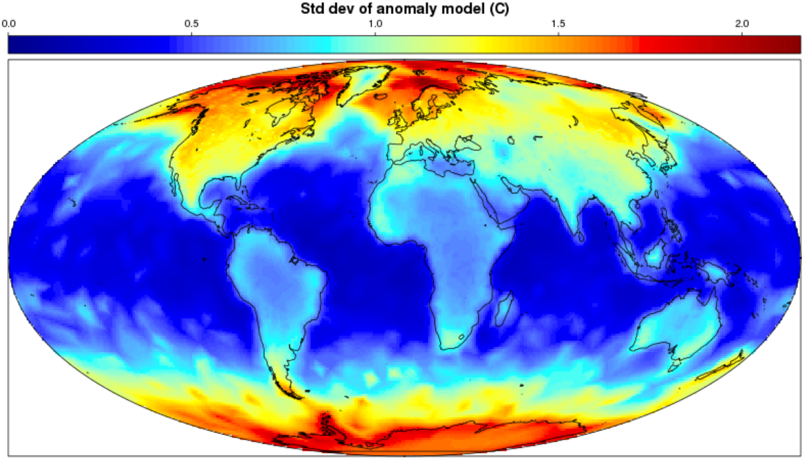
$$(\kappa(\mathbf{s})^2 - \Delta)(\tau(\mathbf{s})a(\mathbf{s})) = \mathcal{W}(\mathbf{s})$$
$$\log \kappa(\mathbf{s}) = \sum B_k^{\kappa}(\mathbf{s})\theta_k$$
$$\log \tau(\mathbf{s}) = \sum B_k^{\tau}(\mathbf{s})\theta_k$$

Simplified heat equation with spatially correlated noise

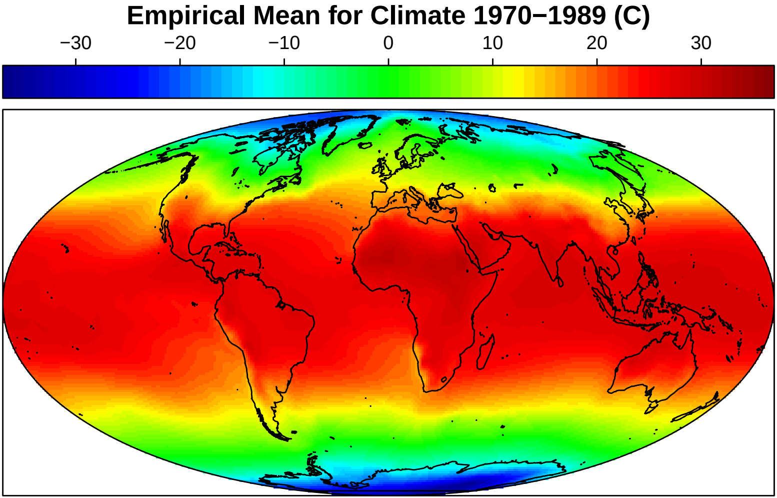
$$\gamma_t \dot{z}(\mathbf{s}, t) - \Delta z(\mathbf{s}, t) = \gamma_s^{-1/2} \mathcal{E}(\mathbf{s}, t)$$
$$\mathcal{E}(\mathbf{s}, \delta t) - \gamma_{\mathcal{E}} \Delta \mathcal{E}(\mathbf{s}, \delta t) = \mathcal{W}_{\mathcal{E}}(\mathbf{s}, \delta t)$$



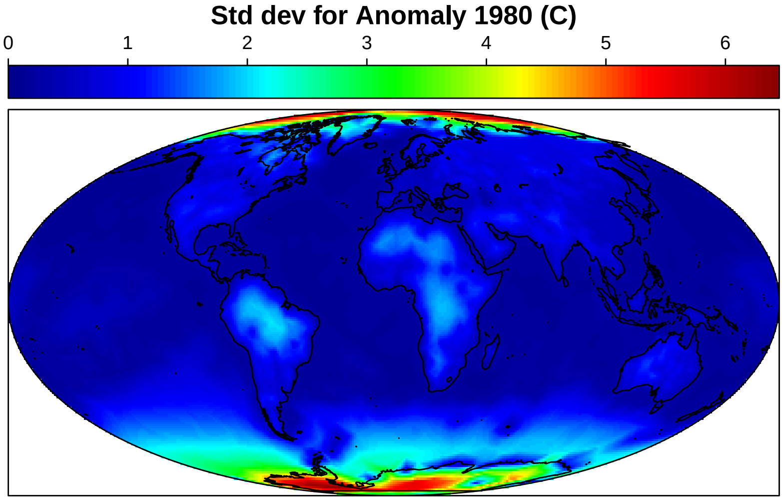
Preliminary estimate of correlations for the anomaly model



Preliminary estimate of standard deviations for the anomaly model

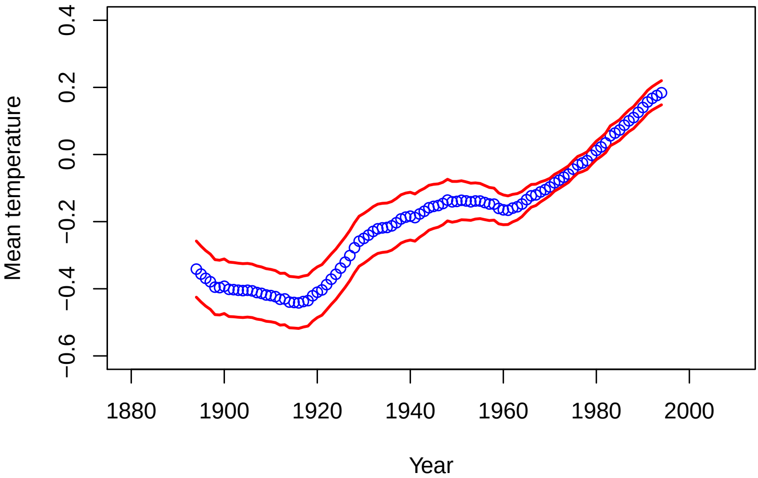


Older results without temporal climate model



Older results without temporal climate model

30 year running average anomalies (global)



Older results without temporal climate model

The lack of temporal climate leads to sensitity to coverage dropout

Challenges

Data and uncertainties

- Pre-homogenization and between-data-source calibration is assumed.
- Homogenization uncertainties could be used if available.
- ICOADS: Better (=any) model for the grid-box uncertainties is needed.

Computations

- The combined climate and anomaly model needs an iterative method based on the model structure, for the spatio-temporal reconstruction.
- Parameter estimation is done stepwise for climate and anomalies.
- Analysis done in blocks of 31 years, each shifted by 5 years so that inconsistencies can be detected.

Conclusion

Spatio-temporal stochastic modelling and estimation for historical climate and weather reconstruction is challenging but possible.

References

- F. Lindgren, H. Rue and J. Lindström (2011), *An explicit link between Gaussian fields and Gaussian Markov random fields: the stochastic partial differential equation approach (with discussion)*, Journal of the Royal Statistical Society, Series B, 73(4), 423–498.
- D. Simpson, F. Lindgren and H. Rue (2012), *In order to make spatial statistics computationally feasible, we need to forget about the covariance function*, Environmetrics, 23: 65–74.
- `http://www.r-inla.org/`

Point process on a complex domain