On dynamical system related to a primal-dual scheme for finding zeros of the sum of maximally monotone operators

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[1]based on the works done together with Raj Narayan Dhara, Leonid Minchenko, Krzysztof Rutkowski and Krzysztof Leśniewski

What is it about...

Primal dual method for convex optimization

Best approximation method for Kuhn-Tucker set

1.

Dynamical system related to the above

2.

- 'Bird eye' view at the solution method for solving the optimisation problem (Flam, '92)

$$\text{Minimize}_{x \in \mathcal{H}} f(x)$$

  via trajectories of the related dynamical system (solution method obtained under suitable discretization)

$$\frac{dx}{dt} = F(x), \quad \textit{with} \;\; F(x) \; \textit{depending on the solution method}$$

- defining the dynamical system related to the best approximation method, for convex optimization, for operator inclusion problems

Lipschtzness of the right-hand side

There is no global Lipschitzess of the right-hand side of the proposed dynamical system, **the right-hand side of the proposed dynamical system is only locally Lipschitz**

3.

Papers

1. EB, Anna Jezierska-Wesierska, Krzysztof Rutkowski, Proximal primal–dual best approximation algorithm with memory, Computational Optimization and Applications (2018) 71:767–794
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3. EB, Leonid Minchenko, Krzysztof Rutkowski, On Lipschitz-like continuity of a class of set-valued mappings, Volume 69, 2020 - Issue 12: Dedicated to the 65th birthday of Alexander Kruger, 2535-2549 ,
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5. EB, Krzysztof W. Leśniewski, Krzysztof Rutkowski, On tangent cone to systems of inequalities and equations in Hilbert spaces under relaxed constant rank condition, submitted, ESAIM: COCV Volume 27, 2021,
`https://doi.org/10.1051/cocv/2021004`

Contents

Problem statement

Operator inclusions problems

Let $\mathcal{H}, \mathcal{G}$ be Hilbert spaces, $A: \ \mathcal{H} \to \mathcal{H}$, $B: \ \mathcal{G} \to \mathcal{G}$ be maximally monotone operators and $L: \ \mathcal{H} \to \mathcal{G}$ be a linear, bounded continuous operator.
The primal inclusion problem is to find $u \in \mathcal{H}$ such that

$$0 \in Au + L^* BLu. \tag{P}$$

The dual inclusion problem to (P) is to find $v^* \in \mathcal{G}$ such that

$$0 \in -LA^{-1}(-Lv^*) + B^{-1}v^*. \tag{D}$$

A point $u \in \mathcal{H}$ solves (P) if and only if $v^* \in \mathcal{G}$ solves (D) and $(u, v^*) \in Z$, where point from the Kuhn Tucker set given by

$$Z := \{(u, v^*) \in \mathcal{H} \times \mathcal{G} \mid -L^* v^* \in Au \quad \text{and} \quad Lu \in B^{-1}v^*\}. \tag{set Z}$$

It is known that Z is a closed convex set. We assume $Z \neq \emptyset$.

The aim

Find a point $(\bar{x}, \bar{v}^*)$ from Z
under some conditions $\bar{x}$ solves (P) and $\bar{v}^*$ solves (D)

Relation to convex optimization problems - primal-dual framework

$A = \partial f$, $B = \partial g$ - subdifferentials of convex functions f and g,
$f : \mathcal{H} \to (-\infty, +\infty]$ $g : \mathcal{G} \to (-\infty, +\infty]$, proper, convex, l.s.c. with conjugates f^*, g^*

$L : \ \mathcal{H} \to \mathcal{G}$, a linear continuous operator with conjugate L^*

under some constraint qualification problem P corresponds to the minimization

$$\text{minimize}_{x \in \mathcal{H}} \ f(x) + g(Lx) \qquad (P)$$

the problem (D) corresponds to Fenchel-Rockafellar dual problem

$$\text{minimize}_{v^* \in \mathcal{G}} \ f^*(-L^* v^*) + g^*(v^*) \qquad (D)$$

and the associated Kuhn–Tucker set is the set Z coincides with (set Z)

$$Z = \{(x, v^*) \in \mathcal{H} \times \mathcal{G} \mid -L^* v^* \in \partial f(x) \ \text{ and } \ Lx \in \partial g^*(v^*)\}$$

The aim

Find a point $(\bar{x}, \bar{v}^*)$ from Z
under some conditions $\bar{x}$ solves (P) and $\bar{v}^*$ solves (D)

The approach

The idea of Eckstein and Svaiter is to construct halfspaces satisfying

$$Z \subset H_\varphi := \{(u, v^*) \in \mathcal{H} \times \mathcal{H} \mid \varphi(u, v^*) \leq 0\}$$

(in their original formulation $L = Id$), with

$$\varphi(u, v^*) := \langle u - b \mid b^* - v^*\rangle + \langle u - a \mid a^* + v^*\rangle, \quad (a, a^*) \in \text{gph}A,\ (b, b^*) \in \text{gph}B.$$

This idea has been continued by Zhang and Cheng, Alatoibi and Combettes and Shahzad.

J. Eckstein and B. F. Svaiter. “A family of projective splitting methods for the sum of two maximal monotone operators”. In: *Mathematical Programming* 111.1 (2008), pp. 173–199. ISSN: 1436-4646. DOI: 10.1007/s10107-006-0070-8. URL: http://dx.doi.org/10.1007/s10107-006-0070-8

Hui Zhang and Lizhi Cheng. “Projective splitting methods for sums of maximal monotone operators with applications”. In: *Journal of Mathematical Analysis and Applications* 406.1 (2013), pp. 323 –334. ISSN: 0022-247X. DOI: https://doi.org/10.1016/j.jmaa.2013.04.072

Inspiration - Eckstein and Svaiter, 2007 - decomposable separator

1. original problem: $0 \in Ax + Bx$ (no L and $\mathcal{H} = \mathcal{G}$)
2. extended solution set

$$S_e(A, B) = \{(x, v^*) \in \mathcal{H} \times \mathcal{H} \mid -v^* \in Ax \text{ and } v^* \in Bx\}$$

3. this is the Kuhn-Tucker set Z
4. Fact 1: $x \in (A + B)^{-1}(0) \Leftrightarrow \exists\, v^* \in \mathcal{H}$ s.t. $(x, v^*) \in S_e(A, B)$

$$\text{proof: } 0 \in Ax + Bx \equiv \exists \; v^* \in \mathcal{H} \quad -v^* \in Ax \text{ and } v^* \in Bx$$

5. Fact 2: $A, B : \mathcal{H} \rightrightarrows \mathcal{H}$, then $S_e(A, B)$ is closed and convex
6. Let $(b, b^*) \in \mathrm{Gph} B$ and $(a, a^*) \in \mathrm{Gph} A$ and let $\varphi : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$

$$\varphi(x, v^*) := \langle x - b, b^* - v^* \rangle + \langle x - a, a^* + v^* \rangle$$

7. Fact 3. Given $(b, b^*) \in \mathrm{Gph} B$ and $(a, a^*) \in \mathrm{Gph} A$. We have
   1. $S_e(A, B) \subset \{(x, v^*) \in \mathcal{H} \times \mathcal{H} \mid \varphi(x, v^*) \leq 0\}$
   2. additionally: φ is both continuous and affine,

$$\nabla \varphi = 0 \;\Leftrightarrow\; (b, b^*) \in S_e(A, B),\; b = a, a^* = -b^*$$

Successive Fejér approximations iterative scheme

Let $\mathcal{H}$, $\mathcal{G}$ be real Hilbert spaces and let Z be defined by (set Z). Let $\{H_n\}_{n\in\mathbb{N}} \subset \mathcal{H} \times \mathcal{G}$, be a sequence of convex closed sets such that $Z \subset H_n$, $n \in \mathbb{N}$. The projections $P_{H_n}(x)$ of any $x \in \mathcal{H} \times \mathcal{G}$ onto H_n are uniquely defined.

Algorithm 1 Generic primal-dual Fejér Approximation Iterative Scheme

1: Choose an initial point $x_0 \in \mathcal{H} \times \mathcal{G}$
2: Choose a sequence of parameters $\{\lambda_n\}_{n\geq 0} \in (0, 2)$
3: **for** $n = 0, 1 \ldots$ **do**
4: Choose H_n convex closed such that $Z \subset H_n$
5: $x_{n+1} = x_n + \lambda_n(P_{H_n}(x_n) - x_n)$
6: **end forreturn**

J. Eckstein and B. F. Svaiter. "A family of projective splitting methods for the sum of two maximal monotone operators". In: *Mathematical Programming* 111.1 (2008), pp. 173–199. ISSN: 1436-4646. DOI: 10.1007/s10107-006-0070-8. URL: http://dx.doi.org/10.1007/s10107-006-0070-8

Hui Zhang and Lizhi Cheng. "Projective splitting methods for sums of maximal monotone operators with applications". In: *Journal of Mathematical Analysis and Applications* 406.1 (2013), pp. 323 –334. ISSN: 0022-247X. DOI: https://doi.org/10.1016/j.jmaa.2013.04.072

Convergence result

Theorem

For any sequence generated by Iterative Scheme 1 the following hold:

1. $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{H} \times \mathcal{G}$ *is Fejér monotone with respect to the set* Z, *i.e*

$$\forall_{n\in\mathbb{N}}\ \forall_{z\in Z}\ \|x_{n+1} - z\| \leq \|x_n - z\|,$$

2. $\sum_{n=0}^{+\infty} \lambda_n(2-\lambda_n)\|P_{H_n}(x_n) - x_n\|^2 < +\infty,$
3. *if*

$$\forall x \in \mathcal{H} \times \mathcal{G}\ \forall\{k_n\}_{n\in\mathbb{N}} \subset \mathbb{N} \quad x_{k_n} \rightharpoonup x \implies x \in Z,$$

then $\{x_n\}_{n\in\mathbb{N}}$ *converges weakly to a point in* Z.

A. Alotaibi, P. L. Combettes, and N. Shahzad. "Solving Coupled Composite Monotone Inclusions by Successive Fejér Approximations of their Kuhn-Tucker Set.". In: *SIAM Journal on Optimization* 24.4 (2014), pp. 2076–2095. DOI: 10.1137/130950616. eprint: http://dx.doi.org/10.1137/130950616. URL: http://dx.doi.org/10.1137/130950616

$$H_{a_n,b_n^*} := \left\{ x \in \mathcal{H} \times \mathcal{G} \mid \left\langle x \mid s^*_{a_n,b_n^*} \right\rangle \leq \eta_{a_n,b_n^*} \right\},$$
$$s^*_{a_n,b_n^*} := (a_n^* + L^* b_n^*, b_n - La_n), \eta_{a_n,b_n^*} := \langle a_n \mid a_n^* \rangle + \langle b_n \mid b_n^* \rangle, \qquad (*)$$

with

$$a_n := J_{\gamma_n A}(p_n - \gamma_n L^* v_n^*), \quad b_n := J_{\mu_n B}(Lp_n + \mu_n v_n^*),$$
$$a_n^* := \gamma_n^{-1}(p_n - a_n) - L^* v_n^*, \quad b_n^* := \mu_n^{-1}(Lp_n - b_n) + v_n^*,$$

where for any maximally monotone operator D and constant $\xi > 0$, $J_{\xi D}(x) = (Id + \xi D)^{-1}(x)$.
Parameters $\mu_n, \gamma_n > 0$ are suitable defined.
It easy to see $H_{\varphi_n} = H_{a_n,b_n^*}$, where $\varphi_n = \varphi(a_n, b_n^*)$.

Best approximation iterative schemes

Here we study iterative best approximation schemes in the form of Algorithm 2. For any $x, y \in \mathcal{H} \times \mathcal{G}$ we define

$$H(x, y) := \{h \in \mathcal{H} \times \mathcal{G} \mid \langle h - y \mid x - y \rangle \leq 0\}.$$

As previously, let $\{H_n\}_{n \in \mathbb{N}} \subset \mathcal{H} \times \mathcal{G}$ be a sequence of closed convex sets, $Z \subset H_n$ for $n \in \mathbb{N}$.

Algorithm 2 Generic primal-dual best approximation iterative scheme

Choose an initial point $x_0 = (p_0, v_0^*) \in \mathcal{H} \times \mathcal{G}$
Choose a sequence of parameters $\{\lambda_n\}_{n \geq 0} \in (0, 1]$
for $n = 0, 1 \ldots$ **do**
  Fejérian step
  Choose H_n such that $Z \subset H_n$
  $x_{n+1/2} = x_n + \lambda_n (P_{H_n}(x_n) - x_n)$
  Let C_n be a closed convex set such that $Z \subset C_n \subset H(x_n, x_{n+1/2})$.
  Haugazeau step
  Choose C_n such that $Z \subset C_n \subset H_n$
  $x_{n+1} = P_{H(x_0, x_n) \cap C_n}(x_0)$
end forreturn

The choice of $C_n = H(x_n, x_{n+1/2})$ has been already investigated in

Abdullah Alotaibi, Patrick L. Combettes, and Naseer Shahzad. "Best approximation from the Kuhn-Tucker set of composite monotone inclusions". In: *Numer. Funct. Anal. Optim.* 36.12 (2015), pp. 1513–1532. ISSN: 0163-0563. DOI: 10.1080/01630563.2015.1077864. URL: http://dx.doi.org/10.1080/01630563.2015.1077864

Convergence

Theorem

Let Z be a nonempty closed convex subset of $\mathcal{H} \times \mathcal{G}$ and let $x_0 = (p_0, v_0^*) \in \mathcal{H} \times \mathcal{G}$. Let $\{C_n\}_{n\in\mathbb{N}}$ be any sequence satisfying $Z \subset C_n \subset H(x_n, x_{n+1/2})$, $n \in \mathbb{N}$. For the sequence $\{x_n\}_{n\in\mathbb{N}}$ generated by Algorithm 2 the following hold:

1. $Z \subset H(x_0, x_n) \cap C_n$ *for* $n \in \mathbb{N}$,
2. $\|x_{n+1} - x_0\| \geq \|x_n - x_0\|$ *for* $n \in \mathbb{N}$,
3. $\sum_{n=0}^{+\infty} \|x_{n+1} - x_n\|^2 < +\infty$,
4. $\sum_{n=0}^{+\infty} \|x_{n+1/2} - x_n\|^2 < +\infty$.
5. *If*

$$\forall x \in \mathcal{H} \times \mathcal{G}\ \forall \{k_n\}_{n\in\mathbb{N}} \subset \mathbb{N} \quad x_{k_n} \rightharpoonup x \implies x \in Z,$$

then $x_n \to P_Z(x_0)$.

Generic best approximation algorithm

Let $x_0 = (u_0, v_0^*) \in \mathcal{H} \times \mathcal{G}$.

1: **for** $n = 0, 1, \dots$ **do**

2:

$$(u_{n+1}, v_{n+1}^*) = P_{H_1(x_0,(u_n,v_n^*))\cap H_2(u_n,v_n^*)}(x_0)$$

3: **end for**

where, for suitably chosen s and η,

$$\begin{aligned} H_1(x_0,(u,v^*)) &:= \{h \in \mathcal{H}\times\mathcal{G} \mid \langle h-(u,v^*) \mid x_0-(u,v^*)\rangle \le 0\},\\ H_2(u,v^*) &:= \{h \in \mathcal{H}\times\mathcal{G} \mid \langle h \mid s(u,v^*)\rangle \le \eta(u,v^*)\}, \quad Z \subset H_2(u,v^*),\\ s &: \mathcal{H}\times\mathcal{G} \to \mathcal{H}\times\mathcal{G}, \quad \eta : \mathcal{H}\times\mathcal{G} \to \mathbb{R} \end{aligned}$$

Yves. Haugazeau. "Sur les inequations variationnelles et la minimisation de fonctionnelles convexes". French. PhD thesis. [S.l.]: [s.n.], 1968

Abdullah Alotaibi, Patrick L. Combettes, and Naseer Shahzad. "Best approximation from the Kuhn-Tucker set of composite monotone inclusions". In: *Numer. Funct. Anal. Optim.* 36.12 (2015), pp. 1513–1532. ISSN: 0163-0563. DOI: 10.1080/01630563.2015.1077864. URL: http://dx.doi.org/10.1080/01630563.2015.1077864

Inertial variants

$$(u_{n+1}, v_{n+1}^*) = P_{H_1(x_0,(u_n,v_n^*))\cap H_2(u_n,v_n^*)\cap H_3((u_n,v_n^*),(u_{n-1},v_{n-1}^*))}(x_0)$$

Lipschitzness of data

1. Let $(u, v^*) \in D := \mathrm{cl}B(\frac{x_0+P_Z(x_0)}{2}, \frac{\|x_0-P_Z(x_0)\|}{2})$. Then $P_Z(x_0) \in H_1(x_0, (u, v^*))$ and for all $(u, v^*) \notin D$, $P_Z(x_0) \notin H_1(x_0, (u, v^*))$
2. Let $\gamma, \mu \in \mathbb{R}_{++}$. Operator $\eta : \; D \to \mathbb{R}$, defined as

$$\eta(u, v^*) := \langle J_{\gamma A}(u - \gamma L^* v) \mid \frac{1}{\gamma}(u - \gamma L^* v - J_{\gamma A}(u - \gamma L^* v))\rangle \\ + \langle J_{\mu B}(Lu + \mu v^*) \mid L^*(Lu + \mu v^*)\rangle$$

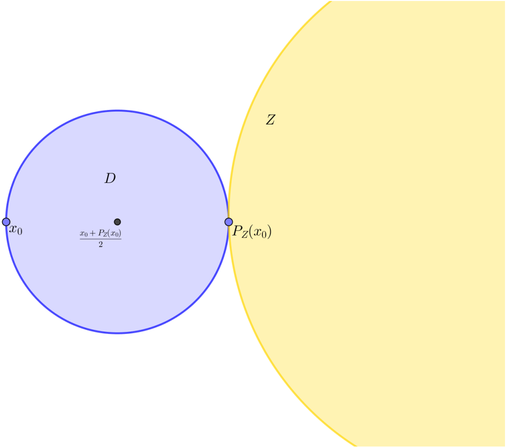
   is (locally) Lipschitz continuous on D.

3. Let $\gamma, \mu \in \mathbb{R}_{++}$. An operator $s : \; \mathcal{H} \times \mathcal{G} \to \mathcal{H} \times \mathcal{G}$ defined as

$$s(u, v^*) := \begin{bmatrix} \frac{1}{\gamma}(u - \gamma L^* v - J_{\gamma A}(u - \gamma L^* v)) + \frac{1}{\mu} L^*(Lu + \mu v^* - J_{\mu B}(Lu + \mu v^*)) \\ J_{\mu B}(Lu + \mu v^*) - L J_{\gamma A}(u - \gamma L^* v) \end{bmatrix}$$

   is Lipschitz continuous on $\mathcal{H} \times \mathcal{G}$.

4. $$H_1(x_0, (u, v^*)) \cap H_2(u, v^*) = \\ \left\{ x \in \mathcal{H} \times \mathcal{G} \;\middle|\; \begin{array}{rcl} \langle x \mid x_0 - (u, v^*)\rangle & \leq & \langle (u, v^*) \mid x_0 - (u, v^*)\rangle, \\ \langle x \mid s(u, v^*)\rangle & \leq & \eta(u, v^*) \end{array} \right\}$$


Z
D
x_0
$\frac{x_0 + P_Z(x_0)}{2}$
$P_Z(x_0)$

Optimization and Dynamical Systems

Dynamical systems and iterative solution schemes

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Dynamical system related to algorithm 2

We investigate, for any given $x_0, \bar{w} \in \mathcal{H} \times \mathcal{G}$, the following dynamical system, solution of which asymptotically approaches solution of (P)-(D),

$$\begin{aligned} &\dot{x}(t) = Q(\bar{w}, x(t), \mathbb{T}x(t)) - x(t), \quad t \geq 0, \\ &x(0) = x_0, \end{aligned} \tag{S}$$

where $\mathbb{T}: \mathcal{H} \times \mathcal{G} \to \mathcal{H} \times \mathcal{G}$, fixed point set of the operator $\mathbb{T}$ is Z, $\operatorname{Fix} \mathbb{T} = Z$, with Z defined by (set Z) and $Q : (\mathcal{H} \times \mathcal{G})^3 \to \mathcal{H} \times \mathcal{G}$,

$$Q(\bar{w}, b, c) := P_{H(\bar{w},b) \cap H(b,c)}(\bar{w}), \tag{1}$$

is the projection P of the element $\bar{w}$ onto the set $H(\bar{w}, b) \cap H(b, c)$ which is the intersection of two hyperplanes of the form

$$H(z_1, z_2) := \{h \in \mathcal{H} \times \mathcal{G} \mid \langle h - z_2 \mid z_1 - z_2 \rangle \leq 0\}, \quad z_1, z_2 \in \mathcal{H} \times \mathcal{G}. \tag{2}$$

In particular,

$$H(\bar{w}, b) = \{h \in \mathcal{H} \times \mathcal{G} \mid \langle h - b \mid \bar{w} - b \rangle \leq 0\}.$$

Under suitable discretization the system (S) leads to primal-dual best approximation scheme introduced for finding $(p, v^*) \in Z$ in

Abdullah Alotaibi, Patrick L. Combettes, and Naseer Shahzad. "Best approximation from the Kuhn-Tucker set of composite monotone inclusions". In: *Numer. Funct. Anal. Optim.* 36.12 (2015), pp. 1513–1532. ISSN: 0163-0563. DOI: 10.1080/01630563.2015.1077864. URL: http://dx.doi.org/10.1080/01630563.2015.1077864

Example 1

First order dynamical systems related to optimization problems have been discussed by many authors.
A natural assumption is that the vector field F is globally Lipschitz and consequently, the existence and uniqueness of solutions to the dynamical system is guaranteed by classical results (see e.g. the bok by Brezis) For instance, Abbas, Attouch and Svaiter considered the following system

$$\begin{aligned} &\dot{x}(t) + x(t) = \operatorname{prox}_{\mu\Phi}(x(t) - \mu B(x(t))), \\ &x(0) = x_0, \end{aligned} \tag{3}$$

where $\Phi : \mathcal{H} \to \mathbb{R} \cup +\infty$ is a proper, convex and lower semicontinuous function defined on a hilbert space $\mathcal{H}$, $B : \mathcal{H} \to \mathcal{H}$ is β-cocoercive operator and $\operatorname{prox}_{\mu\Phi} : \mathcal{H} \to \mathcal{H}$ is a proximal operator defined as

$$\operatorname{prox}_{\mu\Phi}(x) = \arg\min_{y \in \mathcal{H}} \{\Phi(y) + \frac{1}{2\mu}\|x - y\|^2\}.$$

Example 2

Radu Boț and Erno Csetnek, a dynamical system is studied in a form

$$\begin{aligned} &\dot{x}(t) = \lambda(t)(\mathbb{T}(x(t)) - x(t)), \quad t \geq 0 \\ &x(0) = x_0, \end{aligned} \tag{4}$$

where $\mathbb{T} : \mathcal{H} \to \mathcal{H}$ is a nonexpansive operator, $\lambda : [0, \infty) \to [0, 1]$ is a Lebesgue measurable function.
By taking $\mathbb{T} = J_{\gamma A}(Id - \gamma B)$, where $A : \mathcal{H} \to \mathcal{H}$ is a maximally monotone operator, Boț and Csetnek obtain the system

$$\begin{aligned} &\dot{x}(t) = \lambda(t)[J_{\gamma A}(x(t) - \gamma B(x(t))) - x(t)], \\ &x(0) = x_0. \end{aligned} \tag{5}$$

This system under special discretization leads to the forward-backward algorithm for solving operator inclusion problem in a form

$$\text{find } x \in \mathcal{H} \quad s.t. \quad 0 \in A(x) + B(x).$$

The problem

The most essential difference between (S)

$$\begin{aligned} &\dot{x}(t) = Q(\bar{w}, x(t), \mathbb{T}x(t)) - x(t), \quad t \geq 0, \\ &x(0) = x_0, \end{aligned} \tag{S}$$

and the systems (3), (4), (5) is that, in general, one cannot expect that the vector field Q given in (S) is globally Lipschitz with respect to variable x as it is the case of dynamical systems (3), (4) and (5).

The contribution related to (S) is as follows.

1. The existence and uniqueness of solutions to dynamical system (S) by studying a more general problem (DS-0).
2. Extendability of solutions to dynamical system (DS-0)
3. The behaviour at $+\infty$ of solutions to (DS-0)

Formulation of the system (S) in the general form (DS-0)

Let $\bar{w}, \bar{z} \in \mathcal{X}$ and the associated norm in Hilbert space $\mathcal{X}$ be defined as $\|\cdot\| = \sqrt{\langle \cdot \mid \cdot \rangle}$. Let $\mathcal{D} \subset \mathcal{X}$ be a closed convex subset of $\mathcal{X}$ such that $\bar{w}, \bar{z} \in \mathcal{D}$ and

$$\langle \bar{z} - x \mid \bar{w} - x \rangle \leq 0 \quad \text{for all } x \in \mathcal{D}. \tag{6}$$

Note that the condition (6) immediately implies that $\bar{w}$ and $\bar{z}$ are boundary points of the set $\mathcal{D}$.
Let r be such that $\|\bar{w} - \bar{z}\|^2 > r > 0$. We consider set $\hat{\mathcal{D}}$ related to $\mathcal{D}$ (see Figure 1):

$$\hat{\mathcal{D}} = \{x \in \mathcal{D} \mid \|x - \bar{w}\|^2 \geq r\}.$$

We consider the following Cauchy problem

$$\begin{aligned} &\dot{x}(t) = F(x(t)), \qquad t \geq t_0 \geq 0, \\ &x(t_0) = x_{00} \in \hat{\mathcal{D}} \setminus \{\bar{z}\}, \end{aligned} \tag{DS-0}$$

where $F : \hat{\mathcal{D}} \to \mathcal{X}$ is a continuous function on $\hat{\mathcal{D}}$ and locally Lipschitz on $\hat{\mathcal{D}} \setminus \{\bar{z}\}$ and bounded on $\hat{\mathcal{D}}$ ($\|F(x)\| \leq M$, $M > 0$, $x \in \hat{\mathcal{D}}$).

Pictures

Moreover, we assume:

1. $\bar{z}$ is the only zero point of F in $\hat{\mathcal{D}}$, i.e. $F(x) = 0$ iff $x = \bar{z}$.
2. for all $x \in \hat{\mathcal{D}}$, for all $h \in [0, 1]$ we have $x + hF(x) \in \hat{\mathcal{D}}$

Together with assumptions 1, 2 we also consider the following assumption, related to the projection[2]:

1. $\langle F(x) \mid \bar{w} - x \rangle \leq 0$ for all $x \in \hat{\mathcal{D}}$.

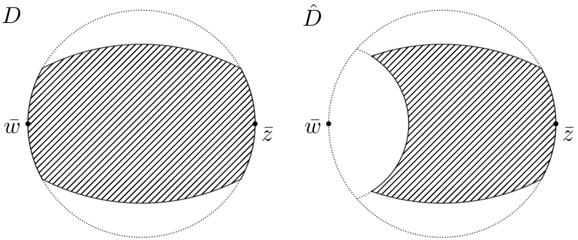


Figure: Illustration of the considered sets.

[2]Here, for $f(x) := F(x) + x$ (so that $F(x) = f(x) - x$) we have that $\bar{z} \in H(\bar{w}, f(x))$.

Remark

Remark

The motivation for considering a nonconvex set $\hat{\mathcal{D}}$ comes from the following observation. Consider $F: \mathcal{D} \to \mathcal{X}$ defined as

$$F(x) = P_{\mathbb{C}(x)}(\bar{w}), \tag{7}$$

where $P_{\mathbb{C}(x)}(\bar{w})$ is the projection of $\bar{w}$ onto $\mathbb{C}(x)$, $\mathbb{C}: \mathcal{D} \rightrightarrows \mathcal{X}$ is a multifunction given by $\mathbb{C}(x) = H(\bar{w}, x) \cap H(x, g(x))$ (see formula (2) for $H(\cdot, \cdot)$) and $g: \mathcal{X} \to \mathcal{X}$ satisfies $\bar{z} \in H(x, g(x))$ for all $x \in \mathcal{X}$. Under a suitable assumption on g, the function F given by (7) is locally Lipschitz on $\mathcal{D} \setminus \{\bar{w}, \bar{z}\}$ see last section below

Definitions

Definition

Let
$$\mathcal{T} = [t_0; T),\ t_0 < T \leq +\infty \text{ or } \mathcal{T} = [t_0; T],\ t_0 < T < +\infty.$$

Solution of

$$\begin{aligned} \dot{x}(t) &= F(x(t)), \qquad t \geq t_0 \geq 0, \\ x(t_0) &= x_{00} \in A \setminus \{\bar{z}\}, \end{aligned} \tag{DS-A}$$

where $F: \ A \to \mathcal{X}$, $A \subseteq \mathcal{X}$, on interval $\mathcal{T}$ is any function

$$\boxed{x(\cdot) \in C^1(\mathcal{T}, A)}$$

satisfying

1. initial condition $x(t_0) = x_0$;
2. equation $\dot{x}(t) = F(x(t))$ for all $t \in \mathcal{T}$, where the differentiation is understood in the sense of strong derivative on space $\mathcal{X}$, where at the boundary point of the interval $\mathcal{T}$, in the case when it belongs to $\mathcal{T}$, the differentiation is understood in the one-sided way.

Definitions

Definition

A solution $x(t)$ to problem (DS-0). on interval $\mathcal{T}_1 = [0, T]$ (or $\mathcal{T}_1 = [0, T)$) is called non-extendable if there is no solution $x_2(\cdot) \in C^1(\mathcal{T}_2, \hat{\mathcal{D}})$ on any interval $\mathcal{T}_2$ of this problem satisfying conditions:

1. $\mathcal{T}_2 \supsetneq \mathcal{T}_1$;
2. $\forall t \in \mathcal{T}_1, \quad x_2(t) = x(t)$.

Remark

If $x(t)$ is a solution of Cauchy problem (DS-0) on interval $\mathcal{T} = [0, T]$ (or $\mathcal{T} = [0, T)$), then restriction of $x(t)$ on any interval $\mathcal{T}_1 = [t_0, t_1] \subset \mathcal{T}$ (or $\mathcal{T}_1 = [t_0, t_1) \subset \mathcal{T}$) is a solution of Cauchy problem (DS-0) on $\mathcal{T}_1$ with initial condition $x_0 = x(t_0)$.

Existence uniqueness and extendibility of solutions to (DS-0)

Theorem (Existence and uniqueness)

Suppose that assumptions 1, 2 and 3 hold. There exists a unique solution of (DS-0) *on* $[t_0, +\infty)$.

Theorem (Behavior at $+\infty$)

Let $x(t)$ be a solution of (DS-0) *on* $[t_0, +\infty)$*. Assume that either*

1. $\mathcal{X}$ *is finite-dimensional and* $\lim_{t\to+\infty} x(t)$ *exists, or*
2. $\mathcal{X}$ *is infinite-dimensional and for every sequence* $\{t_n\}_{n\in\mathbb{N}}$, $t_n \to +\infty$

$$x(t_n) \rightharpoonup \tilde{x} \implies \tilde{x} = \bar{z}, \tag{8}$$

where $x(t)$ is a unique solution of (DS-0).

Then the trajectory $x(t)$ satisfies the condition $\lim_{t\to+\infty} x(t) = \bar{z}$*, where convergence is understood in the sense of the norm of $\mathcal{X}$.*

Projective dynamical system

Now we give an example of the system (DS-0): the projective dynamical system

$$\begin{aligned} \dot{x}(t) &= P_{\mathbb{C}(x)}(\bar{w}) - x, \\ x(t_0) &= x_0 \in \hat{\mathcal{D}},\ t_0 \geq 0, \end{aligned} \tag{PDS}$$

where $\mathbb{C}(x) :\ \mathcal{D} \rightrightarrows \mathcal{X}$ is a multifunction such that:

1. for all $x \in \mathcal{D}$, $\bar{z} \in \mathbb{C}(x)$ and $P_{\mathbb{C}(x)}(\bar{w}) = x$ iff $x = \bar{z}$,
3. $\langle P_{\mathbb{C}(x)}(\bar{w}) - x \mid \bar{w} - x\rangle \leq 0$ for all $x \in \mathcal{D}$,
4. for all $x \in \mathcal{D}$, $\mathbb{C}(x)$ is closed and convex.

Condition 4 ensures that the projection onto $\mathbb{C}(x)$, $x \in \mathcal{D}$ is uniquely defined and therefore 3 is equivalent to the condition:

$$\forall x \in \mathcal{D}\ \forall h \in \mathbb{C}(x) \quad \langle h - x \mid \bar{w} - x\rangle \leq 0.$$

Let us note that in this setting (1, 3, 4) assumption 2 is satisfied since for all $x \in \hat{\mathcal{D}}$ and for any $h \in [0, 1]$

$$\begin{aligned} & x + h(P_{\mathbb{C}(x)}(\bar{w}) - x) = (1 - h)x + hP_{\mathbb{C}(x)}(\bar{w}) \in \mathcal{D}, \\ & \|x + h(P_{\mathbb{C}(x)}(\bar{w}) - x) - \bar{w}\|^2 = \|x - \bar{w}\|^2 \\ & - 2h\langle P_{\mathbb{C}(x)}(\bar{w}) - x \mid \bar{w} - x\rangle + h^2\|P_{\mathbb{C}(x)}(\bar{w}) - x\|^2 \geq \|x - \bar{w}\|^2 \geq r, \end{aligned}$$

i.e. $x + h(P_{\mathbb{C}(x)}(\bar{w}) - x) \in \hat{\mathcal{D}}$.

Solvability of the projected dynamical system

As a consequence of Theorem 7 we can formulate the following theorem.

Theorem

Suppose that 1, 3 , 4 holds. Assume that $P_{\mathbb{C}(x)}(\bar{w})$ is locally Lipschitz continuous on $\hat{\mathcal{D}} \setminus \{\bar{z}\}$ and continuous on $\hat{\mathcal{D}}$. Then the system (PDS) *has a unique solution on $[t_0, +\infty)$.*

To investigate the local Lipschitzness of $P_{\mathbb{C}(x)}(\bar{w})$ on $\hat{\mathcal{D}} \setminus \{\bar{z}\}$ (and the continuity of $P_{\mathbb{C}(x)}(\bar{w})$ on $\hat{\mathcal{D}}$) one should take into account the form of multifunction $\mathbb{C}$. Behaviour of the projection of a given $\bar{w}$ onto polyhedral multifunction $\mathbb{C}$ given by a finite number of linear inequalities and equalities were investigated by EB& KR, AM0, 2020

For the multifunction $\mathbb{C}(x) = H(\bar{w}, x) \cap H(x, \mathbb{T}x)$, i.e. in the case when $P_{\mathbb{C}(x)}(\bar{w}) = Q(\bar{w}, x, \mathbb{T}x)$, we obtain the dynamical systems of the form (DS-0) related to different solution schemes depending on the choice of the operator $\mathbb{T}$.

Dynamical systems of the form (DS-0) related to different operators $\mathbb{T}$.

For $\mathbb{C}(x) = H(\bar{w}, x) \cap H(x, \mathbb{T}x)$, when $P_{\mathbb{C}(x)}(\bar{w}) = Q(\bar{w}, x, \mathbb{T}x)$, we obtain the dynamical systems of the form (DS-0) related to different solution schemes

1. When $\mathbb{T} = J_A$, $A : \mathcal{X} \to \mathcal{X}$ is maximally monotone we obtain dynamical system related to Haugazeau scheme for finding $x \in \mathcal{X}$ such that $0 \in Ax$
2. When $\mathbb{T} = (1/2)(Id + J_{\gamma A} \circ (Id - \gamma B))$, $A : \mathcal{X} \to \mathcal{X}$ is maximally monotone, $B : \ \mathcal{X} \to \mathcal{X}$ is β-cocoercive, $\gamma \in [0, 2\beta]$ we obtain dynamical system related to Haugazeau scheme for finding $x \in \mathcal{X}$ such that $0 \in Ax + Bx$
3. When $\mathbb{T}$ is defined as in For instance, in the primal-dual iterative scheme for finding zeros of sum of maximally monotone operators the operator $\mathbb{T} : \mathcal{H} \times \mathcal{G} \to \mathcal{H} \times \mathcal{G}$ is defined as

$$\begin{aligned} &\mathbb{T}(x) = P_{H(x)}(x), \\ &H(x) := \{h \in \mathcal{H} \times \mathcal{G} \mid \langle h \mid s^*(x) \rangle \le \eta(x)\}, \end{aligned} \tag{9}$$

and, for any $x = (p, v^*) \in \mathcal{H} \times \mathcal{G}$,

$$\left.\begin{aligned} &s^*(x) := (a^*(x) + L^* b^*(x), b(x) - La(x)); \\ &\eta(x) := \langle a(x) \mid a^*(x) \rangle + \langle b(x) \mid b^*(x) \rangle; \\ &a(x) := J_{\gamma A}(p - \gamma L^* v^*), \quad b(x) := J_{\mu B}(Lp + \mu v^*); \\ &a^*(x) := {}^{\gamma}A(p - \gamma L^* v^*), \quad b^*(x) := {}^{\mu}B(Lp + \mu v^*), \ \gamma, \mu \in (0,1), \end{aligned}\right\} \tag{10}$$

where $J_A : \ \mathcal{H} \to \mathcal{H}$ is the resolvent of operator A, $J_A = (Id - A)^{-1}$, and, for any $\gamma > 0$, ${}^{\gamma}A : \mathcal{H} \to \mathcal{H}$ is Yosida approximation of A, ${}^{\gamma}A = \frac{1}{\gamma}(Id - J_{\gamma} A)$.
(10) we obtain dynamical system related to Haugazeau scheme for finding $(p, v^*) \in \mathcal{H} \times \mathcal{G}$ such that $0 \in Ap + B(Lp)$ and $0 \in -LA^{-1}(-Lv^*) + B^{-1}v^*$

Local Lipschitzness of projection onto moving sets

Lipschitzness of projections onto moving polyhedral sets

General Parametric Problem

$$\begin{array}{l}\text{minimize } \varphi_0(x,p) \text{ subject to } x \in X \\ \varphi_i(x,p) \le 0, \quad i = 1,...,m\end{array}$$

all the functions $\varphi_i : X \times P \to \bar{\mathbb{R}}$ are C^2 around the reference point $(\bar{x}, \bar{p})$, Mordukhovich, Nghia (2014), Ralph, Dempe (1995), and the references therein.

Dontchev, Rockafellar, Implicit Functions and Solution Mappings, Springer 2009, 2014

Mangasarian-Fromowitz, constant rank constraint qualifications

Lipschitzness of projection onto moving polyhedral sets

Let $\mathcal{X}$ be a Hilbert space and $D \subset \mathcal{X}$. Let $\mathbb{C}: D \rightrightarrows \mathcal{X}$ set-valued mapping given as $\mathbb{C} := C(p)$,

$$C(p) := \left\{ x \in \mathcal{X} \mid \begin{array}{ccc} \langle x \mid g_i(p) \rangle & = & f_i(p), & i \in I_1 \\ \langle x \mid g_i(p) \rangle & \leq & f_i(p), & i \in I_2 \end{array} \right\}, \ C(p) \neq \emptyset, \ p \in D,$$

where $f_i(p): \ D \to \mathbb{R}$, $g_i(p): \ D \to \mathcal{X}$, $i \in I_1 \cup I_2$ are locally Lipschitz functions on D.

Let $x_0 \in D$. For $p \in D$ the function $G(p) = P_{C(p)}(x_0)$ is well defined.

Finding $P_{C(p)}(x_0)$ is equivalent to finding $y \in C(p)$ which solves variational inequality

$$\langle x_0 - y \mid x - y \rangle \leq 0 \quad \text{for all } x \in C(p). \tag{VI}$$

Let

$$P(x_0, p) := \{x \in \mathcal{X} \mid x_0 \in x + \partial_x h(x, p)\},$$

where stands $\partial_x h(x, p)$ for the partial limiting subdifferential of h with respect to x. If $h(x, p) = \iota_{C(p)}(x)$, where ι is the indicator function of $C(p)$, then

$$P(x_0, p) = P_{C(p)}(x_0) = (N_{C(p)} + I)^{-1}(x_0),$$

where $N_{C(p)}$ is the normal cone to $C(p)$. The case where

$$C(p) := \{x \in \mathbb{R}^n \mid \ \langle x \mid g_i \rangle \ \leq \ f_i(p), \ \ i \in I_2 \ \}, \ C(p) \neq \emptyset, \ p \in D,$$

was investigated e.g. in

N. D. Yen. "Lipschitz Continuity of Solutions of Variational Inequalities with a Parametric Polyhedral Constraint". In: *Mathematics of Operations Research* 20.3 (1995), pp. 695–708. ISSN: 0364765X, 15265471. URL: http://www.jstor.org/stable/3690178

General stability result

Theorem (Mordukhovich, Nghia, Pham)

Let $\bar{p} \in D$, $\bar{v} \in \mathcal{H}$ and $\bar{x} = P(\bar{v}, \bar{p})$. Suppose that

(A) *the mapping $p \to epi(ind(\cdot, p))$ is Lipschitz-like around $(\bar{p}, (\bar{x}, ind(\bar{x}, \bar{p}))) = (\bar{p}, (\bar{x}, 0))$.*

Then the following conditions are equivalent.

(I) *The graphical subdifferential mapping $Gr : \ D \rightrightarrows \mathcal{H} \times \mathcal{H}$ defined as*

$$Gr(p) = \{(x, x') \mid x \in C(p), \ x' \in N(x, C(p))\} \tag{11}$$

is Lipschitz-like around $(\bar{p}, \bar{x}, \bar{v} - \bar{x})$.

(II) *There exist a neighbourhood $W(\bar{v})$ of $\bar{v}$ and a neighbourhood $U(\bar{p})$ of $\bar{p}$ so that the estimate*

$$\|(v_1 - v_2) - 2\kappa_0[P(v_1, p_1) - P(v_2, p_2)]\| \leq \|v_1 - v_2\| + \ell^0\|p_1 - p_2\| \tag{12}$$

holds for all $(v_1, p_1), (v_2, p_2) \in W \times U$ with some positive constants κ_0 and ℓ^0.

[1] B. S. Mordukhovich, T. T. A. Nghia, and D. T. Pham. "Full Stability of General Parametric Variational Systems". In: *Set-Valued and Variational Analysis* (2018). ISSN: 1877-0541. DOI: 10.1007/s11228-018-0474-7

Relaxed constant rank constraint qualification (RCRCQ)

For any $(p, x) \in D \times \mathcal{H}$ let $I_p(x) := \{i \in I_1 \cup I_2 \mid \langle x \mid g_i(p)\rangle = f_i(p)\}$ be the active index set for $p \in D$ at $x \in \mathcal{H}$.

Definition

The *Relaxed constant rank constraint qualification* (RCRCQ) is satisfied at $(\bar{x}, \bar{p})$, $\bar{x} \in C(\bar{p})$, if there exists a neighbourhood $U(\bar{p})$ of $\bar{p}$ such that, for any index set J, $I_1 \subset J \subset I_{\bar{p}}(\bar{x})$, for every $p \in U(\bar{p})$ the system of vectors $\{g_i(p), i \in J\}$ has constant rank. Precisely, for any J, $I_1 \subset J \subset I_{\bar{p}}(\bar{x})$

$$\text{rank}(g_i(p,), i \in J) = \text{rank}(g_i(\bar{p}), i \in J) \quad \text{for all } p \in U(\bar{p}).$$

L. Minchenko and S. Stakhovski. "Parametric Nonlinear Programming Problems under the Relaxed Constant Rank Condition". In: *SIAM Journal on Optimization* 21.1 (2011), pp. 314–332. DOI: 10.1137/090761318. eprint: https://doi.org/10.1137/090761318. URL: https://doi.org/10.1137/090761318

This definition has been introduced in finite dimensional case by Minchenko and Stakhovski for more general set-valued mappings

$$H(p) := \left\{ x \in \mathbb{R}^n \mid \begin{array}{ccc} \xi_i(p,x) & = & 0, & i \in I_1 \\ \xi_i(p,x) & \leq & 0, & i \in I_2 \end{array} \right\},$$

where $\xi_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $i \in I_1 \cup I_2$ are continuously differentiable functions with respect to variable x. Non-parametric versions of constant rank qualifications has been studied by Kruger and Minchenko and Outrata, Andreani and Haeser and Schuverdt and Silva.

R-regularity of a set-valued mapping

Let $\mathbb{C}: D \rightrightarrows \mathcal{H}$ be a multifunction defined as $\mathbb{C}(p) := C(p)$, where

$$C(p) = \left\{ x \in \mathcal{H} \;\middle|\; \begin{array}{ll} \langle x \mid g_i(p) \rangle = f_i(p), & i \in I_1, \\ \langle x \mid g_i(p) \rangle \leq f_i(p), & i \in I_2 \end{array} \right\}, \tag{13}$$

and $f_i: D \to \mathbb{R}$, $g_i: D \to \mathcal{H}$, $i \in I_1 \cup I_2$, $I_1 = \{1, \ldots, m\}$, $I_2 = \{m+1, \ldots, n\}$ are locally Lipschitz on D.

Definition

Multifunction $\mathbb{C}: D \rightrightarrows \mathcal{H}$ given by (13) is said to be R-regular at a point $(\bar{p}, \bar{x})$, if for all (p, x) in a neighbourhood of $(\bar{p}, \bar{x})$,

$$dist\,(x, C(p)) \leq \alpha \max\{0, \ |\langle x \mid g_i(p) \rangle - f_i(p)|, \ i \in I_1, \ \langle x \mid g_i(p) \rangle - f_i(p), \ i \in I_2\}$$

for some $\alpha > 0$.

Theorem

Let $\mathcal{H}$ be a Hilbert space, $f_i: D \to \mathbb{R}$, $g_i: D \to \mathbb{R}$ be locally Lipschitz on $D \subset \mathcal{X}$. If the set-valued mapping $\mathbb{C}: D \rightrightarrows \mathcal{H}$ given by (13) is R-regular at $(\bar{p}, \bar{x})$, $\bar{p} \in D$, $\bar{x} \in C(\bar{p})$, $\bar{x} \in \liminf_{p \to \bar{p}} C(p)$, then $\mathbb{C}$ is Lipschitz-like at $(\bar{p}, \bar{x})$

Lagrange multipliers

Let $p \in D$, $w \in \mathcal{H}$, $w \notin \mathbb{C}(p)$. Projection of w onto set $\mathbb{C}(p)$ is defined as

$$P_{\mathbb{C}(p)}(w) = \arg\min_{x \in \mathbb{C}(p)} \|w - x\|. \tag{14}$$

Denote $G_i(x,p) = \langle x \mid g_i(p)\rangle - f_i(p)$, $i \in I_1 \cup I_2$ and

$$G(x,p) = [G_i(x,p)]_{i=1,\dots,n},$$

Let $\lambda \in \mathbb{R}^n$ and

$$L_w(p,x,\lambda) := \|x - w\| + \langle \lambda \mid G(x,p)\rangle,$$

The set of Lagrange multipliers corresponding to (14) are defined as

$$\Lambda_w(p,x) := \{\lambda \in \mathbb{R}^n \mid \nabla_x L_w(p,x,\lambda) = 0 \text{ where, for } i \in I_2,\ \lambda_i \geq 0,\ \lambda_i G_i(x,p) = 0\},$$

Then

$$\nabla_x L_w(p, P_{\mathbb{C}(p)}(w), \lambda) = \frac{P_{\mathbb{C}(p)}(w) - w}{\|P_{\mathbb{C}(p)}(w) - w\|} + \sum_{i=1}^{n} \lambda_i g_i(p), \tag{15}$$

If $w \notin \mathbb{C}(p)$, condition $\nabla_x L_w(p, P_{\mathbb{C}(p)}(w), \lambda) = 0$ is equivalent to the following

$$\frac{w - P_{\mathbb{C}(p)}(w)}{\|P_{\mathbb{C}(p)}(w) - w\|} = \sum_{i=1}^{n} \lambda_i g_i(p) \quad \Leftrightarrow \quad w - P_{\mathbb{C}(p)}(w) = \sum_{i=1}^{n} \hat{\lambda}_i g_i(p), \tag{16}$$

where $\hat{\lambda}_i = \lambda_i \|P_{\mathbb{C}(p)}(w) - w\|$, $i = 1,\dots,n$.

RCRCQ and Lagrange multipliers

We start with the proposition which relates RCRCQ condition to the boundedness (with respect to p, w) of Lagrange multiplier set

$$\Lambda_w^M(p, P_{C(p)}(w)) := \{\lambda \in \Lambda_w(p, P_{C(p)}(w)) \mid \sum_{i=1}^{n} |\lambda_i| \leq M\}.$$

Theorem

Let multifunction $\mathbb{C}$ *given by* (13) *satisfy RCRCQ at* $(\bar{x}, \bar{p})$, $\bar{x} \in C(\bar{p})$. *Assume that* $\bar{x} \in \liminf_{p \to \bar{p}} \mathbb{C}(p)$. *Then there exist numbers* $M > 0$, $\delta > 0$, $\delta_0 > 0$ *such that*

$$\Lambda_w^M(p, P_{\mathbb{C}(p)}(w)) \neq \emptyset \quad \text{for } p \in \bar{p} + \delta_0\mathbb{B},\ w \in \bar{x} + \delta\mathbb{B},\ w \notin \mathbb{C}(p).$$

Boundedness of multipliers and R-regularity

In the next proposition we relate the boundedness of the Lagrange multiplier set $\Lambda_w^M(p, P_{\mathbb{C}(p)}(w))$ to the R-regularity of $\mathbb{C}$ at $(\bar{p}, \bar{x})$. For sets $\mathbb{C}(p)$ given as solution sets to parametric systems of nonlinear equations and inequalities in finite-dimensional spaces this fact has been already proved in [10, Theorem 2].

Theorem

Let $\bar{p} \in D$, $\bar{x} \in \mathbb{C}(\bar{p})$ and $\bar{x} \in \liminf_{p \to \bar{p}} \mathbb{C}(p)$. Assume that there exist numbers $M > 0$, $\delta_1 > 0$, $\delta_2 > 0$ such that

$$\Lambda_w^M(p, P_{\mathbb{C}(p)}(w)) := \{\lambda \in \Lambda_w(p, P_{\mathbb{C}(p)}(v)) \mid \sum_{i=1}^{n} |\lambda_i| \leq M\} \neq \emptyset$$

for all $p \in (\bar{p} + \delta_1 \mathbb{B}) \cap S$ and for all $w \in (\bar{x} + \delta_2 \mathbb{B})$, $w \notin \mathbb{C}(p)$. Then the multifuction $\mathbb{C}$ is R-regular at $(\bar{x}, \bar{p})$.

Main results

Let $\bar{p} \in D$. The lower Kuratowski limit is defined as

$$\liminf_{p \to \bar{p}} C(p) := \{x \in \mathcal{X} \mid \forall p_k \to \bar{p} \; \exists \; x_k \in C(p_k) \quad s.t. \; x_k \to x\}$$

Now we show that, if the multifunction $\mathbb{C}$ is R-regular at $(\bar{p}, \bar{x})$, then $\mathbb{C}$ is Lipschitz-like at $(\bar{p}, \bar{x})$.

Theorem

Let $\mathcal{H}$, $\mathcal{G}$ be Hilbert spaces and $f_i : \ D \to \mathbb{R}$, $g_i : \ D \to \mathcal{H}$ are locally Lipschitz at $\bar{p} \in D$. If the set-valued mapping $\mathbb{C} : \ D \rightrightarrows \mathcal{H}$, given by (13), is R-regular at $(\bar{p}, \bar{x})$, $\bar{x} \in \mathbb{C}(\bar{p})$, then $\mathbb{C}$ is Lipschitz-like at $(\bar{p}, \bar{x})$

Theorem

Let $\mathcal{H}, \mathcal{G}$ be Hilbert spaces. Let multifunction $\mathbb{C} : \ D \rightrightarrows \mathcal{H}$, given by (13), satisfy RCRCQ at $(\bar{p}, \bar{x})$, $\bar{x} \in \mathbb{C}(\bar{p})$ and the functions $f_i : \ D \to \mathbb{R}$, $g_i : \ D \to \mathcal{H}$, $i \in I_1 \cup I_2$, be locally Lipschitz at $\bar{p} \in D$. Assume that $\bar{x} \in \liminf_{p \to \bar{p}} \mathbb{C}(p)$. Then $\mathbb{C}$ is Lipschitz-like at $(\bar{p}, \bar{x})$.

Lipschitzness of the projection - consequence of theorem of Mordukhovich, Nghia, Pham

For any $(p,x) \in D \times \mathcal{H}$, $I_p(x) := \{i \in I_1 \cup I_2 \mid \langle x - f_i(p) \mid g_i(p)\rangle = 0\}$ is the active index. For any index set L, $L \subset I_{\bar{p}}(\bar{x}) \setminus (I_1 \cup \bar{K})$ satisfying

$$g_i(\bar{p}),\ i \in I_1 \cup \bar{K} \cup L, \text{ linearly independent} \tag{17}$$

$$C_L(p) = \left\{ x \in \mathcal{H} \;\middle|\; \begin{array}{ll} \langle x \mid g_i(p)\rangle = f_i(p), & i \in I_1 \cup \bar{K} \cup L, \\ \langle x \mid g_i(p)\rangle \le f_i(p), & i \in I_2 \setminus (\bar{K} \cup L) \end{array} \right\}.$$

Theorem (Main result)

Let $\bar{p} \in D$, $\bar{x} \in C(\bar{p})$, $\bar{v} \notin C(\bar{p})$, $\bar{x} = P_{C(\bar{p})}(\bar{v})$ and:

1. $$\bar{v} - \bar{x} = \sum_{i \in I_1 \cup \bar{K}} \bar{\lambda}_i g_i(\bar{p}),\ \bar{\lambda}_i > 0,\ i \in \bar{K} \subset I_{\bar{p}}(\bar{x}) \cap I_2,$$

   where $g_i(\bar{p})$, $i \in I_1 \cup \bar{K}$ are linearly independent,
2. *(RCRCQ) is satisfied at $(\bar{p}, \bar{x})$,*
3. $\bar{x} \in \liminf_{p \to \bar{p}} C_L(p)$ *for any L satisfying (17).*

Then there exist a neighborhood W of $\bar{v}$ and U of $\bar{p}$ such that the Lipschitzian estimate

$$\|(v_1 - v_2) - 2\kappa_0[P_{C(p_1)}(v_1) - P_{C(p_2)}(v_2)]\| \le \|v_1 - v_2\| + \ell^0\|p_1 - p_2\| \tag{18}$$

holds for all $(v_1, p_1), (v_2, p_2) \in W \times U$ with some positive constants κ_0 and ℓ^0.

Consequence for the vector field related to proximal primal-dual dynamical system

Consequence for the vector field related to proximal primal-dual dynamical system

Let $I_1 = \emptyset$, $I_2 = \{1, 2\}$, $\bar{z} = P_Z(x_0)$. Let $x_0 \in \mathcal{H} \times \mathcal{G}$, $\bar{p} \in D \setminus \{x_0, P_Z(x_0)\}$ and

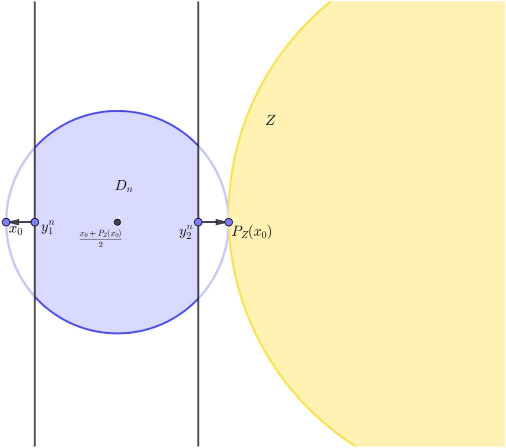
$$C(p) := C(u, v^*) := H_1(x_0, (u, v^*)) \cap H_2(u, v^*) =$$

$$\left\{ x \in \mathcal{H} \times \mathcal{G} \;\middle|\; \begin{array}{rcl} \langle x \mid x_0 - (u, v^*) \rangle & \leq & \langle (u, v^*) \mid x_0 - (u, v^*) \rangle, \\ \langle x \mid s(u, v^*) \rangle & \leq & \eta(u, v^*) \end{array} \right\}$$

Let $g_1(p) := g_1(u, v^*) := x_0 - (u, v^*)$, $g_2(p) := g_2(u, v^*) := s(u, v^*)$. Then $g_i(\bar{p})$, $i \in I_{\bar{p}}(P_{C(\bar{p})})$ are linearly independent and the set-valued mapping $\mathbb{C}$ satisfies (RCRCQ) at $(P_{C(\bar{p})}, \bar{p})$.

Remark

Let us note that $g_1(x_0) = g_1(u_0, v_0^*) = 0$ and $g_2(P_Z(x_0)) = g_2(\bar{u}, \bar{v}^*) = s(\bar{u}, \bar{v}^*) = 0$ and at points $x_0, P_Z(x_0)$ (RCRCQ) does not hold.



Z
D_n
x_0
y_1^n
$\frac{x_0 + P_Z(x_0)}{2}$
y_2^n
$P_Z(x_0)$

General case -Lipschitz-likeness of the constraint set under (RCRCQ)

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{G}$ be a normed space.
Let us consider a set-valued mapping $\mathcal{F} : \mathcal{G} \rightrightarrows \mathcal{H}$, defined as $\mathcal{F}(p) := F(p)$, where

$$F(p) := \{x \in \mathcal{H} \mid h_i(p, x) \leq 0,\ i \in I,\ h_i(p, x) = 0,\ i \in I_0\}, \tag{19}$$

where $p \in \mathcal{G}$ is a parameter, $x \in \mathcal{H}$ stands for the decision variable, $I = \{1, \ldots, m\}$, $I_0 = \{m + 1, \ldots, n\}$ (we admit the case $I_0 = \emptyset$).
Functions $h_i : \ \mathcal{G} \times \mathcal{H} \to \mathbb{R}$, $i = 1, \ldots, n$, are assumed to be (jointly) continuous together with their partial gradients with respect to x, $\nabla_x h_i$, $i = 1, \ldots, n$.

Relaxed constant rank condition in the general case

Let $P \subset \mathcal{G}$, $X \subset \mathcal{H}$ and $I(p,x) := \{i \in I \mid h_i(p,x) = 0\}$ be the set of indices of active inequality constraints at $(p,x) \in grF$.

Definition

1. The set-valued mapping $\mathcal{F}$ satisfies the Relaxed Constant Rank Constraint Qualification, or shortly, RCRCQ (relative to $P \times X$) at $(p^0, x^0) \in grF$, if for any index set $K \subset I(p^0, x^0)$

$$\operatorname{rank}\{\nabla_x h_i(p,x) : i \in I_0 \cup K\} = \operatorname{rank}\{\nabla_x h_i(p^0,x^0) : i \in I_0 \cup K\}$$

for $(p,x) \in P \times X$ from a neighbourhood of (p^0, x^0)

2. The set $F(p^0)$ satisfies RCRCQ at $x^0 \in F(p^0)$ if for any index set $K \subset I(p^0, x^0)$

$$\operatorname{rank}\{\nabla_x h_i(p^0,x) : i \in I_0 \cup K\} = \operatorname{rank}\{\nabla_x h_i(p^0,x^0) : i \in I_0 \cup K\}$$

for all x in a neighbourhood of x^0.

Thank you for your attention!