TraDE-OPT
TRAINING DATA-DRIVEN EXPERTS IN OPTIMIZATION
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Finding global solutions for a class of possibly nonconvex QCQP problems through the S-lemma

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Giovanni Bruccola
Systems Research Institute
Polish Academy of Sciences
Department of Modelling and Optimization of Dynamical Systems
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QCQP and S-QCQP: under which assumptions are the KKT conditions necessary and sufficient for optimality?

Possibly nonconvex QCQP problem for $x \in \mathbb{R}^n$, $A_J, A_k \in S^n, b_J, b_k \in \mathbb{R}^n$ and $c_J, c_k \in \mathbb{R}\ \forall\, k = 1, ..., m$ is:

$$Min_{\, x\in\mathbb{R}^n} \quad J(x) := x^T A_J x + 2b_J^T x + c_J$$
$$s.t. \quad f_k(x) := x^T A_k x + 2b_k^T x + c_k \le 0,$$
$$k = 1, ..., m$$

(QCQP)

When the matrices $A_J, A_k\ \forall\, k$ take the form $A_J = a_J I, A_k = a_k I$, and $a_J, a_k \in \mathbb{R}$, we have S-QCQP

$$Min_{\, x\in\mathbb{R}^n} \quad J(x) := a_j \|x\|^2 + 2b_J^T x + c_J$$
$$s.t. \quad f_k(x) := a_k \|x\|^2 + 2b_k^T x + c_k \le 0,$$
$$k = 1, ..., m$$

(S-QCQP)

A point x^* feasible for (QCQP) is a KKT point if there exist γ_k, $k \in 1, .., m$ not all null s.t.

$$(i)\; A_J + \sum_{k=1}^{m} \gamma_k A_k \succeq 0, \quad (ii)\; \nabla(J + \sum_{k=1}^{m} \gamma_k f_k)(x^*) = 0 \quad (iii)\; \gamma_k f_k(x^*) = 0 \;\; \forall\, k$$

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Literature review

- In the literature it is proved that KKT are necessary and sufficient for:
  1. QCQP with $m = 1$,
  2. QCQP with $m = 2$, $n > 2$ $\exists\, \gamma_1, \gamma_2$ s.t. $\gamma_1 H_1 + \gamma_2 H_2 \succ 0$,
  3. Z-matrices QCQP with any number of constraints.
- Consider the matricial form of $f_k(x) = \begin{pmatrix} x \\ 1 \end{pmatrix}^T H_k \begin{pmatrix} x \\ 1 \end{pmatrix}, \quad H_k := \begin{pmatrix} A_k & b^k \\ b_k^T & c_k \end{pmatrix}$
- H_k is a Z-matrix if it has non positive off diagonal elements.

I.Pólik and T.Terlaky, A survey of the S-Lemma, SIAM Rev., 49 (2007), 371-418.

V.Jeyakumar,G.M.Lee and G. Y. Li, Alternative Theorems for Quadratic Inequality Systems and Global Quadratic Optimization. SIAM Journal on Optimization. (2009);20(2):983-1001.

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Introduction: The assumption and the set Ω_0

- $\exists \gamma \in \mathbb{R}^m_+ \setminus \mathbf{0}_m$ such that $A_J + \sum_{k=1}^{m} \gamma_k A_k \succeq 0$.
- There exists a global minimum x^* of problem (QCQP).
- We define Ω_0 as

$$\Omega_0 := \{(f_0(x), ..., f_m(x)) | x \in \mathbb{R}^n\} + int \mathbb{R}^{m+1}_+ \tag{1}$$

(c.f. [1]), where

$$f_0(x) := J(x) - J(x^*) = x^T A_J x + 2b_J^T x - (x^*)^T A_J x^* + 2b_J^T x^* \tag{2}$$

$$f_k(x) := x^T A_k x + 2b_k^T x + c_k \;\forall\, k \in \{1, ..., m\} \tag{3}$$

V.Jeyakumar,G.M.Lee and G. Y. Li, Alternative Theorems for Quadratic Inequality Systems and Global Quadratic Optimization. SIAM Journal on Optimization. (2009);20(2):983-1001.

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Part I: The S-lemma

- The general S-Lemma establishes under which assumptions, exactly one between the following two statements holds:
  1. $\exists x \in \mathbb{R}^n$ such that $f_k(x) < 0 \;\; \forall k \in \{0, ..., m\}$
  2. $(\exists \gamma \in \mathbb{R}_+^{m+1} \backslash \mathbf{0}_{m+1}) \sum_{k=0}^{m} \gamma_k f_k(x) \geq 0 \, \forall x \in \mathbb{R}^n$
- Under the current assumptions the general S-Lemma may fail for $m > 2$, [1].
- We need also to assume that Ω_0 **is convex**.

I.Pólik and T.Terlaky, A survey of the S-Lemma, SIAM Rev., 49 (2007), 371-418.

Part I: General S-Lemma, Sketch of the proof

- $\Omega_0 := \{(f_0(x), ..., f_m(x))|x \in \mathbb{R}^n\} + int\mathbb{R}_+^{m+1}$

[Not(1.)⇒(2.)] ⟷ $f_k(x) < 0 \ \ \forall\, k \in \{0, ..., m\}$ has no solution ⟷ $\Omega_0 \cap \ (-\text{int}\mathbb{R}_+^{m+1}) = \emptyset$

- We can apply the convex separation theorem, [1].
- Hence there exists an Hyperplane which properly separates $int\,\Omega_0$ and $(-\text{int}\mathbb{R}_+^{m+1})$.

R. T. Rockafellar. Convex Analysis Princeton University Press. Princeton, NJ. (1970).

Part I: KKT conditions for QCQP

Theorem

Let x^* be a global minimizer of (QCQP). *Let the set Ω_0 be convex.*

1. *The following Fritz-John conditions are necessary for optimality, i.e. there exists a vector $(\gamma_0, ..., \gamma_m) \in \mathbb{R}_+^{m+1} \backslash \mathbf{0}_{m+1}$ such that*

$$(i)\ \ \nabla(\gamma_0 J + \sum_{k=1}^{m} \gamma_k f_k)(x^*) = 0\ \ (ii)\ \gamma_k f_k(x^*) = 0\ \ k \in \{1, ..., m\}\ \ (iii)\ \gamma_0 A_J + \sum_{k=1}^{m} \gamma_k A_k \succeq 0$$

2. *If there exists a point $x_0 \in \mathbb{R}^n$ such that $f_k(x_0) < 0\ \forall\ \ k \in \{1, .., m\}$, the following KKT conditions are necessary and sufficient for optimality, i.e. $\exists(\gamma_1, ..., \gamma_m) \in \mathbb{R}_+^m \backslash \mathbf{0}_m$ such that*

$$(i)\ \ \nabla(J + \sum_{k=1}^{m} \gamma_k f_k)(x^*) = 0\ \ (ii)\ \gamma_k f_k(x^*) = 0\ \ k \in \{1, ..., m\}\ \ (iii)\ A_J + \sum_{k=1}^{m} \gamma_k A_k \succeq 0$$

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Part I: KKT for QCQP, Sketch of the proof of point 1

1. Let $f_0(x) := J(x) - J(x^*)$. Since x^* is a global minimizer of (QCQP), $f_0(x) \geq 0$ for every x feasible for (QCQP).
2. The system $f_k(x) < 0 \; k = 0, ..., m$ has no solution.
3. We apply the generalized S-Lemma. There exists $(\gamma_0, ..., \gamma_m) \in \mathbb{R}_+^{m+1} \backslash \mathbf{0}_{m+1}$ such that

$$\gamma_0 f_0(x) + \sum_{k=1}^{m} \gamma_k f_k(x) \geq 0 \Rightarrow \; \gamma_0 J(x) + \sum_{k=1}^{m} \gamma_k f_k(x) \geq \gamma_0 J(x^*) \; \forall x \in \mathbb{R}^n$$

4. At this point, we can get the Fritz-John necessary optimality conditions with some calculations.

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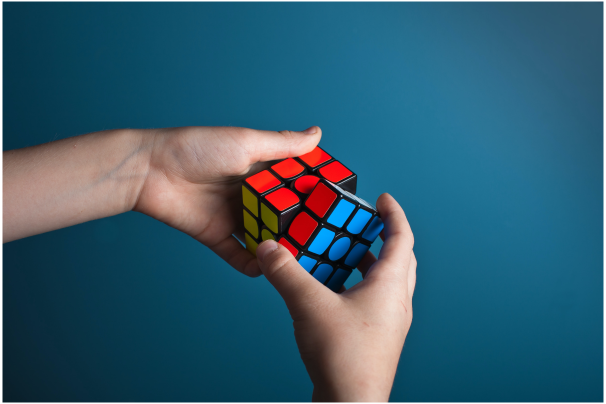
Part II: KKT for S-QCQP



- In Part I, we proved for any QCQP that if Ω_0 is convex then the KKT conditions are necessary and sufficient for global optimality.
- To ”complete the puzzle”, we need to show that Ω_0 is convex for S-QCQP.

Part II: KKT for S-QCQP

Theorem

Consider problem (S-QCQP) *with* $m+1<n$. *The set* $\Omega_0 := \{(f_0(x), ..., f_m(x)) | x \in \mathbb{R}^n\} + int\mathbb{R}_+^{m+1}$ *is convex.*

Sketch of the proof

1. Take any $v, w \in \Omega_0$. This means that $\exists\, x_v, x_w$ s.t. for any component v_k, w_k of v, w

$$f_k(x_v) < v_k; \quad f_k(x_w) < w_k \quad \forall\, k \in \{0, ..., m\} \tag{4}$$

2. For any $\lambda \in (0,1]$, we have to show

$$\begin{aligned} &\lambda v + (1-\lambda) w \in \Omega_0 \\ &\exists\, \tilde{x}\; s.t.\; f_k(\tilde{x}) \leq \lambda f_k(x_v) + (1-\lambda) f_k(x_w) < \lambda v + (1-\lambda) w \;\; \forall\, k \in \{0, ..., m\} \end{aligned} \tag{5}$$

3. It is possible to find $\tilde{x} \in \mathcal{S}^n := \{x \in \mathbb{R}^n \mid \|x\|^2 = \lambda \|x_v\|^2 + (1-\lambda)\|x_w\|^2\}$ which proves (5).

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Part III: Convex relaxations

(S-QCQP) problem:

$$Min_{x\in\mathbb{R}^n}\ \ J(x) := a_J x^T x + 2b_J^T x + c_J$$
$$s.t.\ \ f_k(x) := a_k x^T x + 2b_k^T x + c_k \leq 0,$$
$$k = 1, ..., m$$

The convex relaxation (CR):

$$Min\ \ J(x,y) := a_J^T y + 2b_J^T x$$
$$s.t.\ f(x,y) := a_k^T y + 2b_k^T x + c_k \leq 0\ k = 1, ..., m$$
$$x_i^2 - y_i \leq 0,\ i = 1, ..., n$$

- Consider $(\bar{x}, \bar{y})$ is the global minimum of (CR) and x^* is the global minimum of (S-QCQP).

(CR) is exact $\longleftrightarrow$ $J(\bar{x}, \bar{y}) = J(x^*)$

- (CR) is equivalent to the SDP relaxation and the SOCP relaxation;
- the SDP relaxation and the SOCP relaxation are exact.

Thank You for Your Attention

- The paper is available at http://arxiv.org/abs/2206.00618

Contact:

- giovanni.bruccola@ibspan.waw.pl
- +48 667 458 415
- +39 393 866 42 53