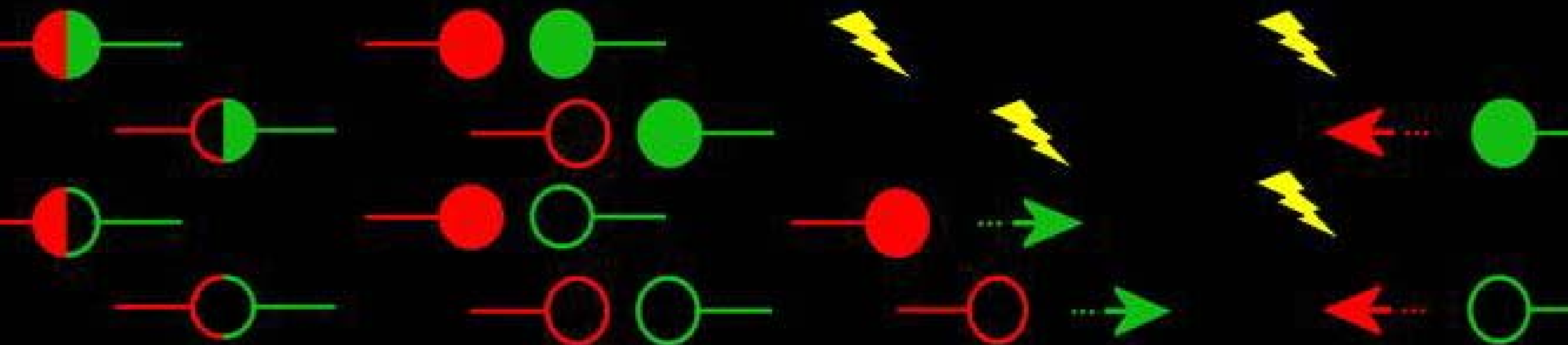

Different duals in conic optimization: closure can tighten the duality gap

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joint work with:

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J. Liu, W. Schachinger, G. Uchida

COA Workshop Edinburgh, 8 June 2018

Overview

1. Duality principles

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2. **Conic complications**

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4. A hierarchy of duals; tightening the gap
5. Empirical evidence on multi-dim.knapsack problems

Linear optimization over cones and duality

Consider linear problem over convex cone $\mathcal{K}$

$$\inf_{\mathbf{x}\in\mathcal{K}} \{\langle \mathbf{c}, \mathbf{x}\rangle : \langle \mathbf{a}_0, \mathbf{x}\rangle = 1, \langle \mathbf{a}_i, \mathbf{x}\rangle = 0, i \in [1:m]\}$$

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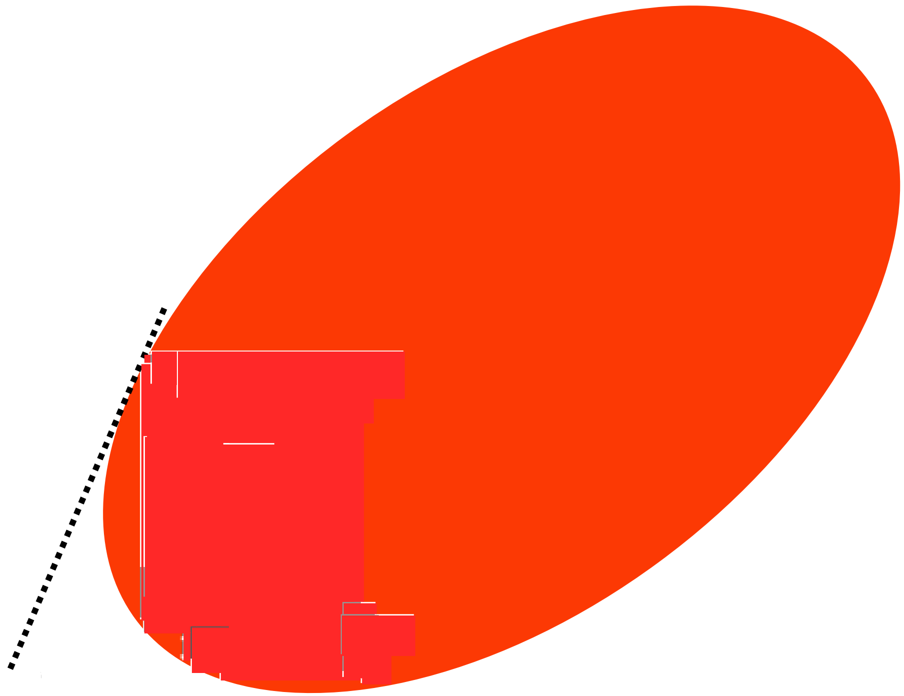
where $\mathcal{B}^* = \left\{\mathbf{s} \in \mathbb{R}^d : \langle \mathbf{s}, \mathbf{x}\rangle \geq 0\,, \text{ all } \mathbf{x} \in \mathcal{B}\right\}$.



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Let $\mathcal{K}$ be a convex cone of symmetric $n \times n$ matrices $\mathbf{X} = \mathbf{X}^\top$.

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However, when departing from LP, strong duality may fail.

Familiar cases: LP, SDP, and beyond

Departing? Indeed, for

$\mathcal{K} = \mathcal{N}_n = \left\{ \mathbf{X} = \mathbf{X}^\top n \times n : \mathbf{X} \geq \mathbf{O} \right\} \ldots$ LP, barrier: $-\sum_{i,j} \log X_{ij}$,

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Why bother?

Copositive optimization (COP)

COP encodes many NP-hard problems [Dür'10, Burer'12, B.'12].

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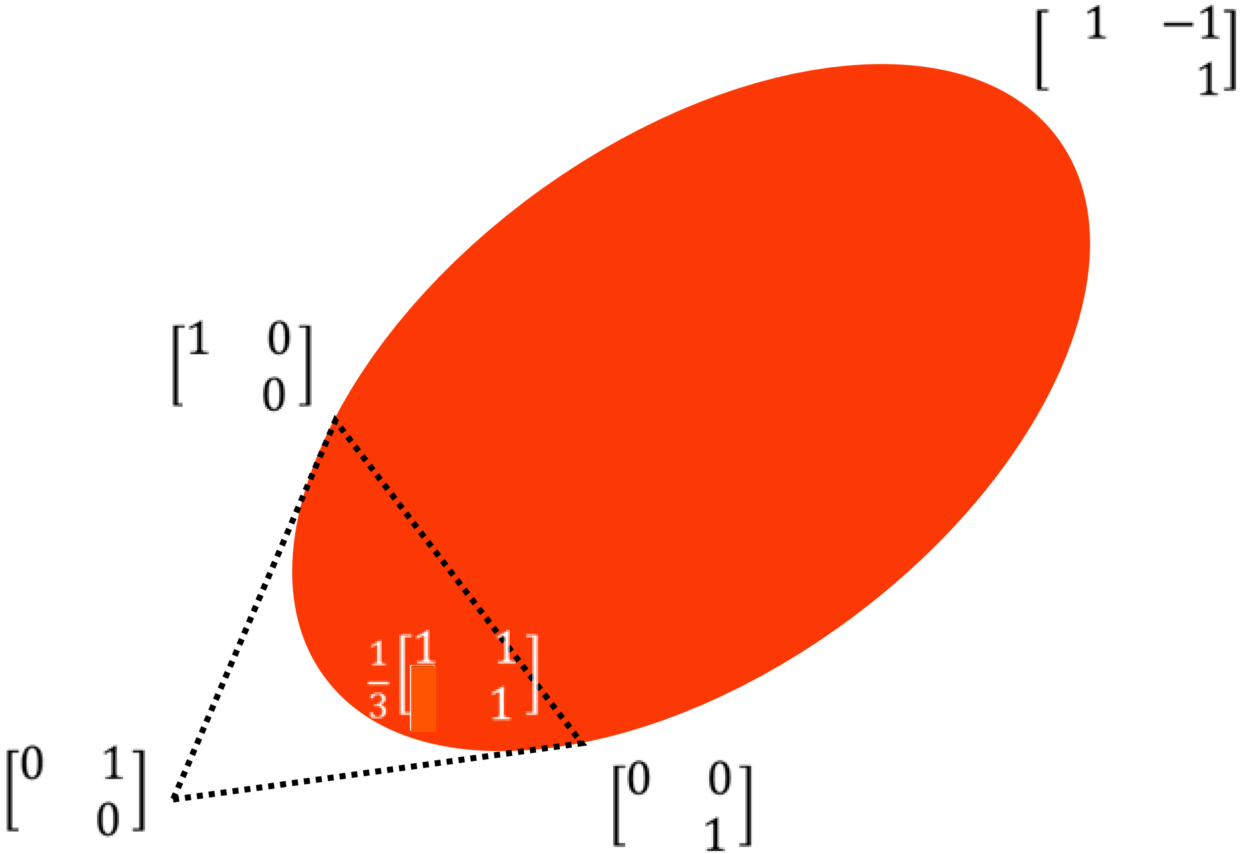
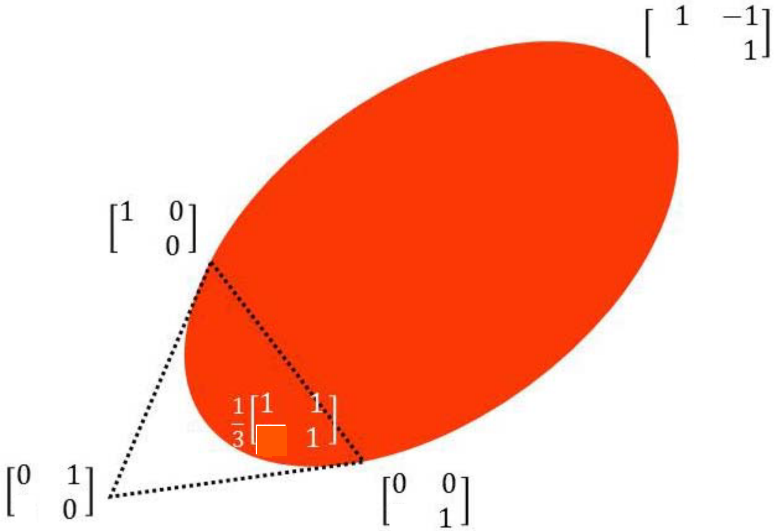


Semidefinite cone $\mathcal{P}$



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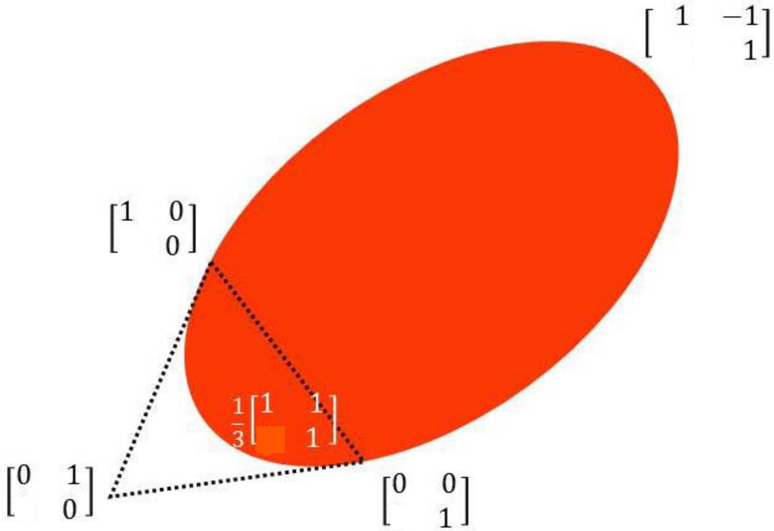

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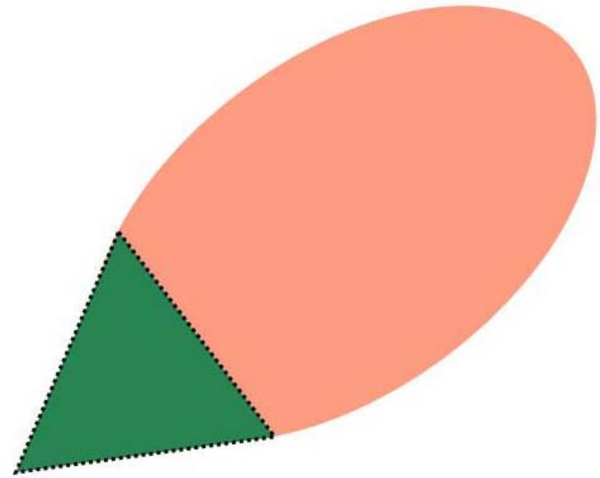

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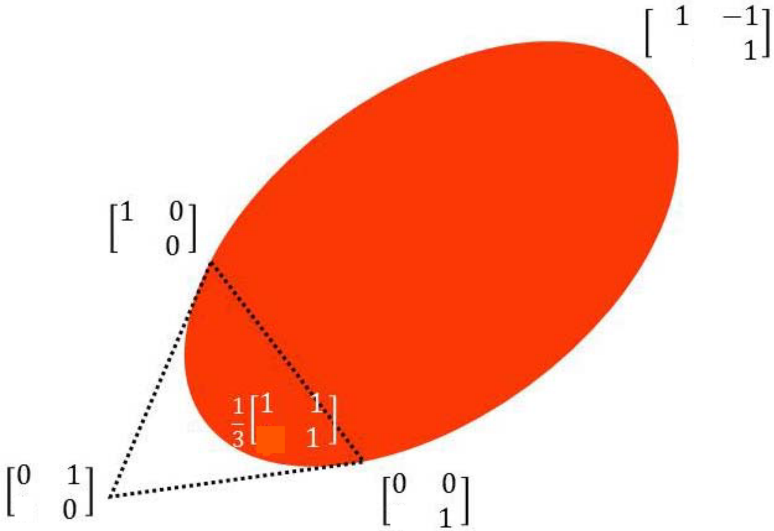

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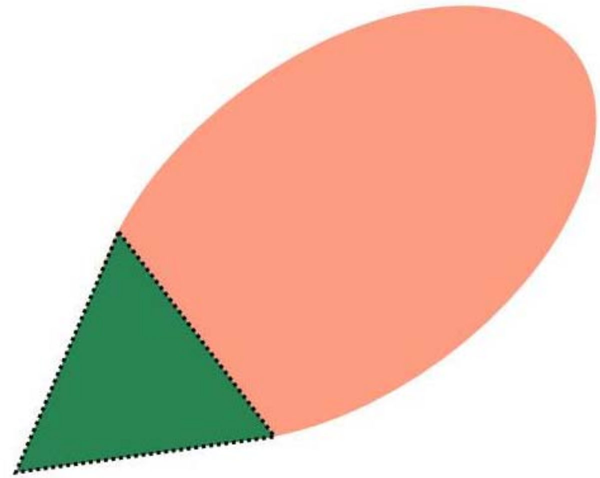

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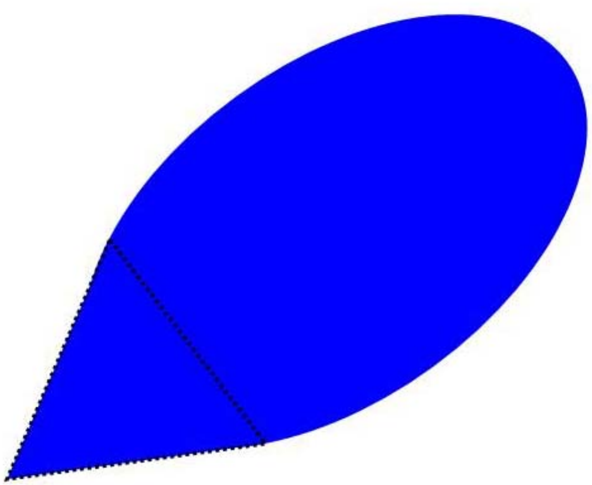

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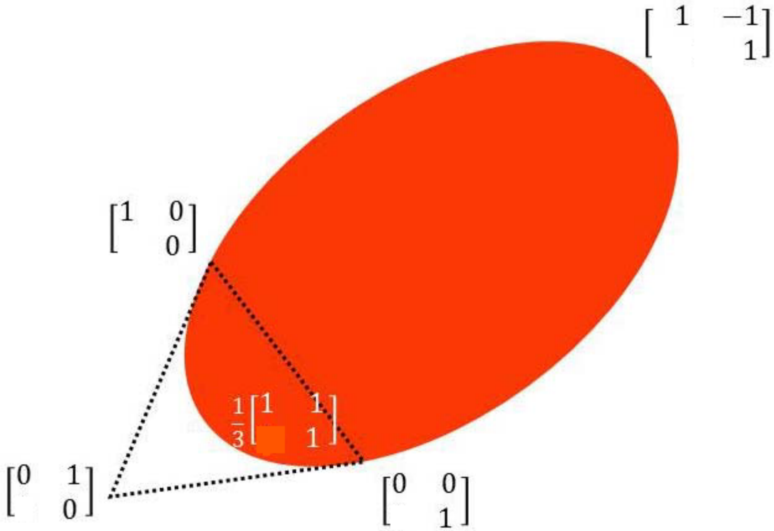

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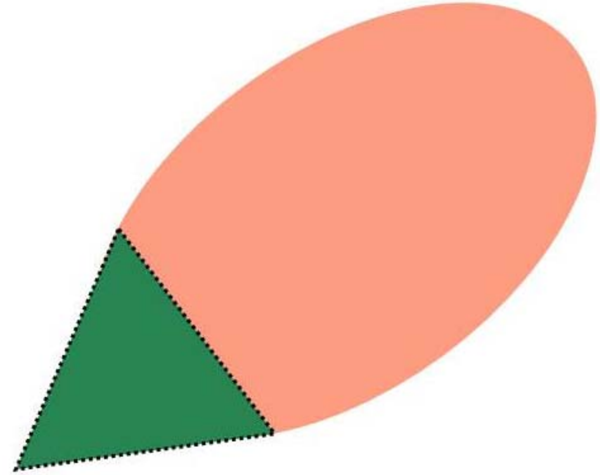

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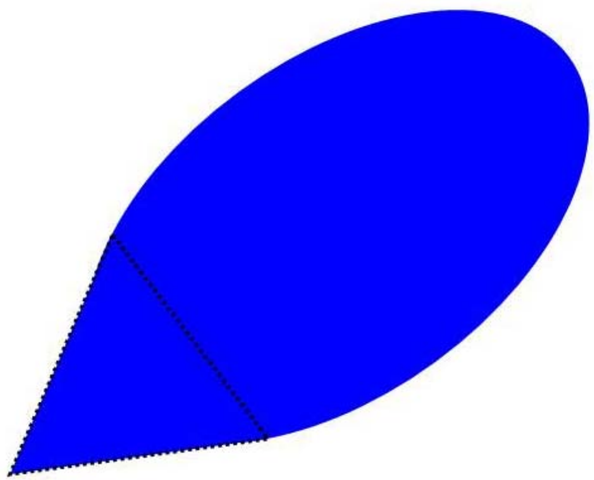

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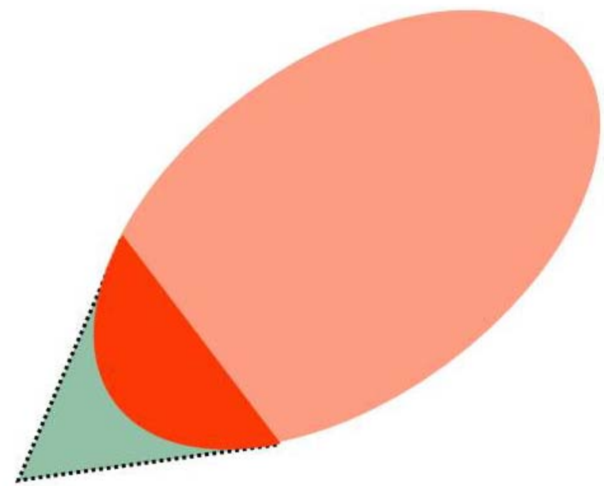

Completely positive cone $\mathcal{K}$

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Primal/dual pair in (COP) with conic duality:

$$\begin{aligned} p^* &= \inf \left\{\langle \mathbf{C}, \mathbf{X}\rangle : \langle \mathbf{A}_0, \mathbf{X}\rangle = 1, \mathbf{X} \in \mathcal{L} \cap \mathcal{C}_n\right\}, \\ d^* &= \sup \left\{y_0 : \mathbf{C} - \textstyle\sum_{i=0}^m y_i \mathbf{A}_i \in \mathcal{C}_n{}^*\right\} \end{aligned}$$

with $\mathcal{L} = \{\mathbf{A}_1, \ldots, \mathbf{A}_m\}^\perp$.

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Strict primal feasibility: $\{\mathbf{X} \in \mathcal{L} : \langle \mathbf{A}_0, \mathbf{X} \rangle = 1\} \cap \text{int } \mathcal{C}_n \neq \emptyset$

implies zero duality gap and dual attainability, and

strict dual feasibility $\left\{y_0 : \mathbf{S} = \mathbf{C} - \sum_{i=0}^{m} y_i \mathbf{A}_i \in \text{int } \mathcal{C}_n^*\right\} \neq \emptyset$

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Same is true for COP, too. Why ? Problems with addition ...

Minkowski summation

... looks innocent: take two closed convex sets $\mathcal{B}$ and $\mathcal{C}$, consider

$$\mathcal{B} + \mathcal{C} = \{\mathbf{B} + \mathbf{C} : \mathbf{B} \in \mathcal{B}, \mathbf{C} \in \mathcal{C}\}$$

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Example. $\mathcal{B} = \left\{ \begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix} : t \in \mathbb{R} \right\}$ and $\mathcal{C} = \mathcal{P}_2$.

$$\begin{bmatrix} -k & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} k & -1 \\ -1 & 1/k \end{bmatrix}$$

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Example will return in various attires [B./Schachinger/Uchida'12]:

In COP, almost anything can happen, too !

Exclude doubly infeasible cases where both

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Possible attainability/duality gap constellations for COP:

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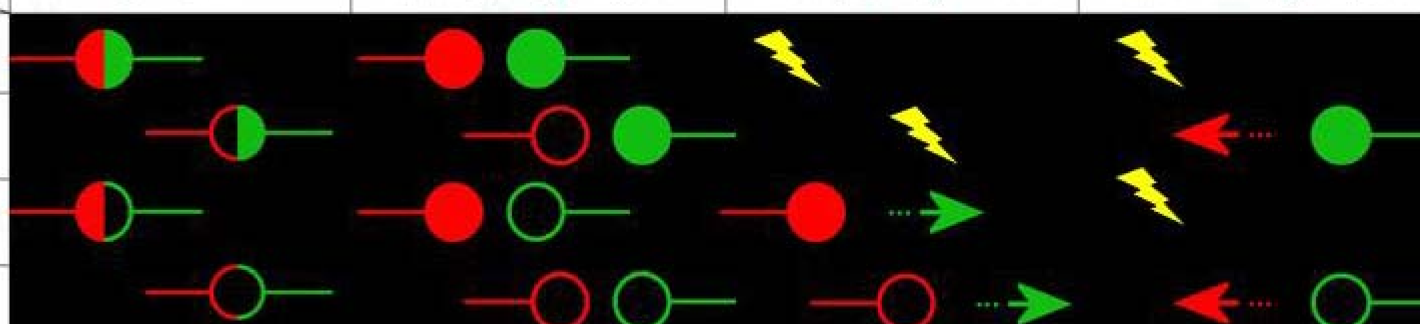
Possible attainability/duality gap constellations for COP:

duality gap / attained	zero $d^* = p^* \in \mathbb{R}$	finite positive $-\infty < d^* < p^* < \infty$	infinite $-\infty < d^* < p^* = \infty$	infinite $-\infty = d^* < p^* < \infty$
both attained				
p^* attained, d^* not attained				
p^* not attained, d^* attained				
neither attained				



duality gap / attained	zero $d^* = p^* \in \mathbb{R}$	finite positive $-\infty < d^* < p^* < \infty$	infinite $-\infty < d^* < p^* = \infty$	infinite $-\infty = d^* < p^* < \infty$
both attained				
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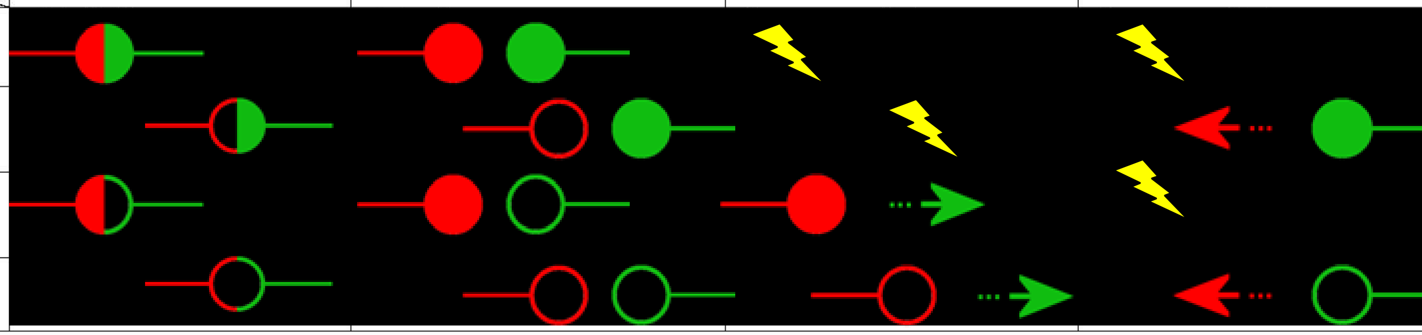

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both attained	StQP, both strictly f.			
p^* attained, d^* not attained	MStQP, dual strictly f.			
p^* not attained, d^* attained				
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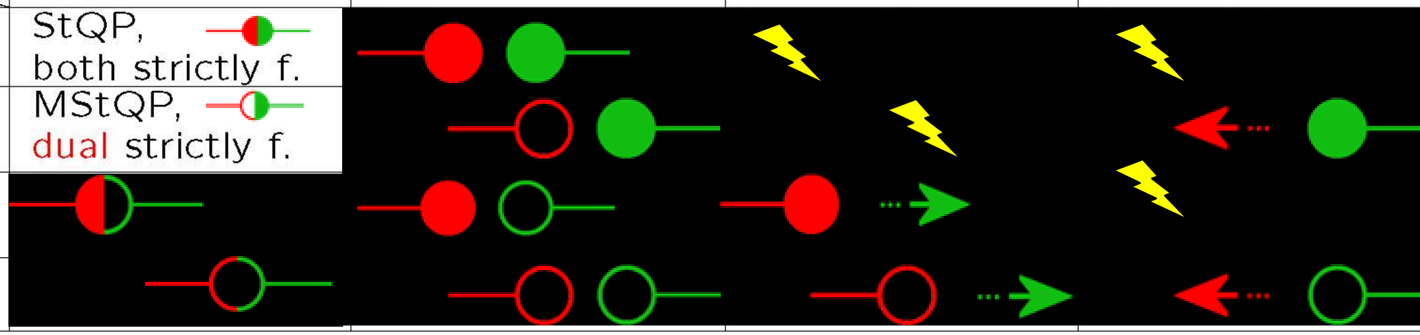


An example adapted from SDP

Here it works:

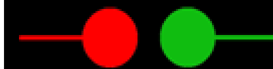
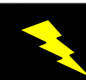
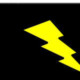
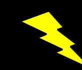
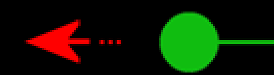
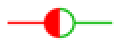
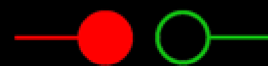
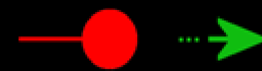
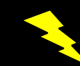
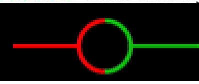
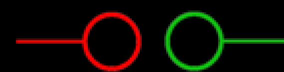
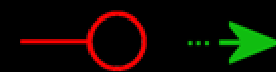
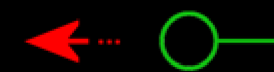
Example 1: $n = 2$, $m = 1$, $\langle C, X\rangle = x_{11}$, $\langle A_1, X\rangle = x_{12} + x_{21}$ and $b_1 = 2$. Then

$$d^* = \sup\left\{2y_1 : \begin{bmatrix} 1 & -y_1 \\ -y_1 & 0 \end{bmatrix} \in \mathcal{C}_2^*\right\} = 0$$

is attained for $y_1^* = 0$.

$$\begin{bmatrix} \frac{1}{k} & 1 \\ 1 & k \end{bmatrix} \in \mathcal{C}_2$$

is primally feasible X_k with $\langle C, X_k\rangle = \frac{1}{k} \searrow 0$ as $k \nearrow \infty$, so that $p^* = d^*$. But p^* cannot be attained since $x_{11} = 0$ conflicts with $x_{12} = 1$ and $X \in \mathcal{C}_2 \subset \mathcal{P}_2$.

duality gap / attained	zero $d^* = p^* \in \mathbb{R}$	finite positive $-\infty < d^* < p^* < \infty$	infinite $-\infty < d^* < p^* = \infty$	infinite $-\infty = d^* < p^* < \infty$
both attained	StQP,  both strictly f.			
p^* attained, d^* not attained	MStQP,  dual strictly f.			
p^* not attained, d^* attained	Ex.1,  primal strictly f.			
neither attained				

Constructing failure in dual attainability

Theorem 1. Given a COP instance $(\mathbf{A}, \mathbf{b}, C)$ in $\mathcal{C}_n/\mathcal{C}_n^*$, denote by $\mathcal{T}_\mathsf{d}(\mathbf{A}, \mathbf{b}, C)$ the following new COP instance in $\mathcal{C}_{n+2}/\mathcal{C}_{n+2}^*$:

$$\bar{C} = \begin{bmatrix} C & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & 0 & -1 \\ \mathbf{o}^\top & -1 & 0 \end{bmatrix} \quad \text{and} \quad \bar{A}_i = \begin{bmatrix} A_i & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & 0 & 0 \\ \mathbf{o}^\top & 0 & 0 \end{bmatrix}, \; 1 \le i \le m\,,$$

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Then

- $\mathcal{T}_\mathsf{d}(\mathbf{A}, \mathbf{b}, C)$ is feasible if and only if $(\mathbf{A}, \mathbf{b}, C)$ is feasible;

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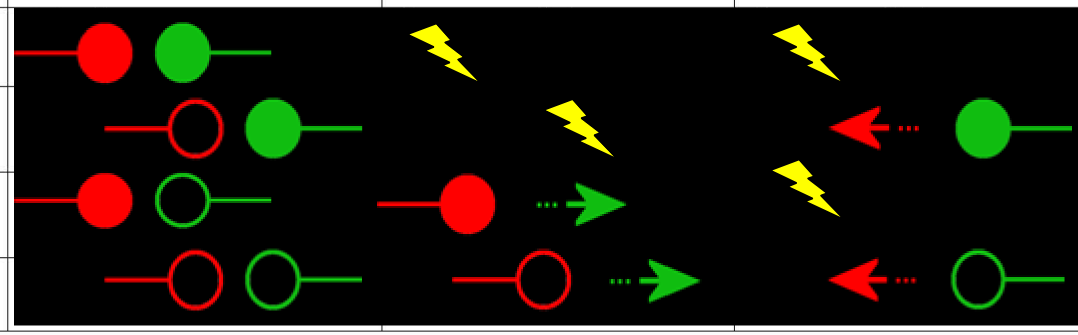
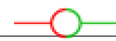
- d^* is never attained in $\mathcal{T}_\mathsf{d}(\mathbf{A}, \mathbf{b}, C)$.

duality gap / attained	zero $d^* = p^* \in \mathbb{R}$	finite positive $-\infty < d^* < p^* < \infty$	infinite $-\infty < d^* < p^* = \infty$	infinite $-\infty = d^* < p^* < \infty$
both attained	StQP,  both strictly f.			
p^* attained, d^* not attained	MStQP,  dual strictly f.			
p^* not attained, d^* attained	Ex.1,  primal strictly f.			
neither attained	Ex.1 & Thm.1 			

Another example adapted from SDP

Example 2: Here $n = 3$, $m = 2$, $\langle C, X \rangle = x_{33}$ whereas

$$\mathbf{A}X = \begin{bmatrix} x_{33} + 2x_{12} \\ x_{22} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then $p^* = \inf \{x_{33} : x_{33} + 2x_{12} = 1\,, x_{22} = 0\,, X \in \mathcal{C}_3\} = 1$, attained for an $X^* \in \mathcal{C}$ with all $x^*_{ij} = 0$ except $x^*_{33} = 1$.

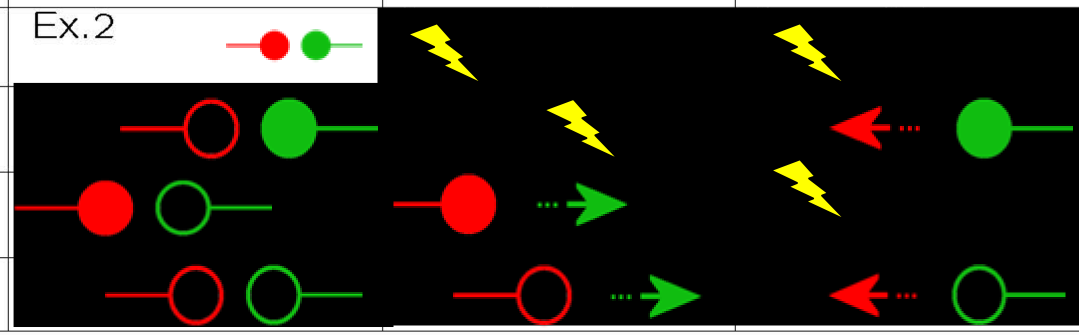
The dual reads

$$d^* = \sup \left\{ y_1 : \begin{bmatrix} 0 & -y_1 & 0 \\ -y_1 & -y_2 & 0 \\ 0 & 0 & 1 - y_1 \end{bmatrix} \in \mathcal{C}^*_3 \right\} = 0\,,$$

attained for $\mathbf{y}^* = \mathbf{o}$.

Theorem 1 above gives an instance $\mathcal{T}_{\mathsf{d}}(\mathbf{A}, \mathbf{b}, C)$ with the same $d^* < p^*$, but where d^* is not attained.

duality gap / attained	zero $d^* = p^* \in \mathbb{R}$	finite positive $-\infty < d^* < p^* < \infty$	infinite $-\infty < d^* < p^* = \infty$	infinite $-\infty = d^* < p^* < \infty$
both attained	StQP, both strictly f.	Ex.2		
p^* attained, d^* not attained	MStQP, dual strictly f.			
p^* not attained, d^* attained	Ex.1, primal strictly f.			
neither attained	Ex.1 & Thm.1			



attained \ duality gap	zero $d^* = p^* \in \mathbb{R}$	finite positive $-\infty < d^* < p^* < \infty$	infinite $-\infty < d^* < p^* = \infty$	infinite $-\infty = d^* < p^* < \infty$
both attained	StQP, both strictly f.	Ex.2		
p^* attained, d^* not attained	MStQP, dual strictly f.	Ex.2 & Thm.1		
p^* not attained, d^* attained	Ex.1, primal strictly f.			
neither attained	Ex.1 & Thm.1			

Constructing failure in primal attainability

Theorem 2. Given a COP instance $(\mathbf{A}, \mathbf{b}, C)$ in $\mathcal{C}_n/\mathcal{C}_n^*$, denote by $\mathcal{T}_\mathsf{p}(\mathbf{A}, \mathbf{b}, C)$ the following new COP instance in $\mathcal{C}_{n+2}/\mathcal{C}_{n+2}^*$:

$$\bar{C} = \begin{bmatrix} C & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & 1 & 0 \\ \mathbf{o}^\top & 0 & 0 \end{bmatrix} \quad \text{and} \quad \bar{A}_{m+1} = \begin{bmatrix} O & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & 0 & 1 \\ \mathbf{o}^\top & 1 & 0 \end{bmatrix},$$

all $\bar{A}_i$, $1 \le i \le m$, are A_i augmented by zeroes as in Thm. 1, and $\bar{\mathbf{b}} = \begin{bmatrix} \mathbf{b} \\ 2 \end{bmatrix} \in \mathbb{R}^{m+1}$. Then

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- d^* is attained in one problem iff it is in the other;

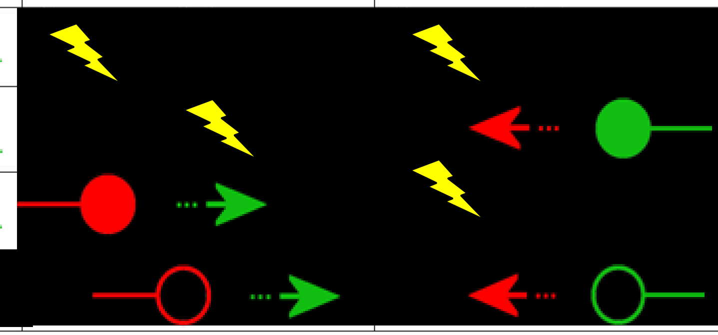
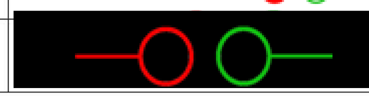
Constructing failure in primal attainability

Theorem 2. Given a COP instance $(\mathbf{A}, \mathbf{b}, C)$ in $\mathcal{C}_n/\mathcal{C}_n^*$, denote by $\mathcal{T}_{\mathsf{p}}(\mathbf{A}, \mathbf{b}, C)$ the following new COP instance in $\mathcal{C}_{n+2}/\mathcal{C}_{n+2}^*$:

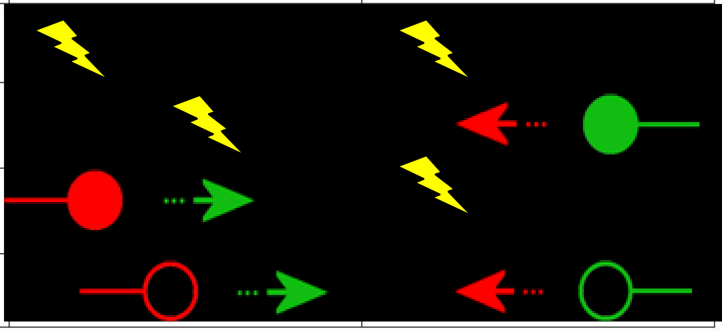
$$\bar{C} = \begin{bmatrix} C & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & 1 & 0 \\ \mathbf{o}^\top & 0 & 0 \end{bmatrix} \quad \text{and} \quad \bar{A}_{m+1} = \begin{bmatrix} O & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & 0 & 1 \\ \mathbf{o}^\top & 1 & 0 \end{bmatrix},$$

all $\bar{A}_i$, $1 \le i \le m$, are A_i augmented by zeroes as in Thm. 1, and $\bar{\mathbf{b}} = \begin{bmatrix} \mathbf{b} \\ 2 \end{bmatrix} \in \mathbb{R}^{m+1}$. Then

- $\mathcal{T}_{\mathsf{p}}(\mathbf{A}, \mathbf{b}, C)$ is feasible if and only if $(\mathbf{A}, \mathbf{b}, C)$ is feasible;
- the primal and dual optimal values remain the same;
- d^* is attained in one problem iff it is in the other;
- p^* is never attained in $\mathcal{T}_{\mathsf{p}}(\mathbf{A}, \mathbf{b}, C)$.

attained \ duality gap	zero $d^* = p^* \in \mathbb{R}$	finite positive $-\infty < d^* < p^* < \infty$	infinite $-\infty < d^* < p^* = \infty$	infinite $-\infty = d^* < p^* < \infty$
both attained	StQP, both strictly f.	Ex.2		
p^* attained, d^* not attained	MStQP, dual strictly f.	Ex.2 & Thm.1		
p^* not attained, d^* attained	Ex.1, primal strictly f.	Ex.2 & Thm.2		
neither attained	Ex.1 & Thm.1			

duality gap / attained	zero $d^* = p^* \in \mathbb{R}$	finite positive $-\infty < d^* < p^* < \infty$	infinite $-\infty < d^* < p^* = \infty$	infinite $-\infty = d^* < p^* < \infty$
both attained	StQP, both strictly f.	Ex.2		
p^* attained, d^* not attained	MStQP, dual strictly f.	Ex.2 & Thm.1		
p^* not attained, d^* attained	Ex.1, primal strictly f.	Ex.2 & Thm.2		
neither attained	Ex.1 & Thm.1	Ex.2 & Thms.1,2		



Constructing more failures

Summarizing: if $(\mathbf{A}, \mathbf{b}, C)$ is the instance of Example 2, then

- $(\mathbf{A}, \mathbf{b}, C)$ has $-\infty < d^* < p^* < \infty$ with both d^* and p^* attained,
- $\mathcal{T}_{\mathsf{d}}(\mathbf{A}, \mathbf{b}, C)$ has $-\infty < d^* < p^* < \infty$ with d^* not attained,
- $\mathcal{T}_{\mathsf{p}}(\mathbf{A}, \mathbf{b}, C)$ has $-\infty < d^* < p^* < \infty$ with p^* not attained,
- $\mathcal{T}_{\mathsf{p}}[\mathcal{T}_{\mathsf{d}}(\mathbf{A}, \mathbf{b}, C)]$ and $\mathcal{T}_{\mathsf{d}}[\mathcal{T}_{\mathsf{p}}(\mathbf{A}, \mathbf{b}, C)]$ have $-\infty < d^* < p^* < \infty$ with neither p^* nor d^* attained.

So the center column of the table is filled !

It remains to deal with infeasibility of one of the problems ...

duality gap / attained	zero $d^* = p^* \in \mathbb{R}$	finite positive $-\infty < d^* < p^* < \infty$	infinite $-\infty < d^* < p^* = \infty$	infinite $-\infty = d^* < p^* < \infty$
both attained	StQP, both strictly f.	Ex.2	impossible	
p^* attained, d^* not attained	MStQP, dual strictly f.	Ex.2 & Thm.1	impossible	
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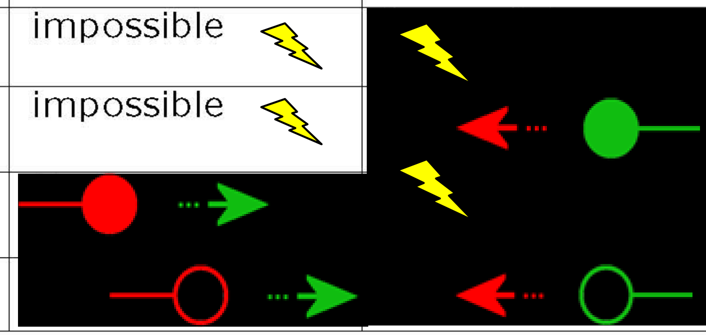
neither attained	Ex.1 & Thm.1	Ex.2 & Thms.1,2		



Infinite duality gaps – infeasible primal

Example 3: Here $n = 3$, $m = 2$, and $C = O$ whereas

$$\mathbf{A}X = \begin{bmatrix} 2x_{22} + 2x_{23} \\ 2x_{12} - 2x_{33} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

If $X \in \mathcal{C}_3$, then $x_{23} \geq 0$ and $x_{22} \geq 0$ imply $x_{22} = 0$, hence $x_{12} = 0$, hence $x_{33} = -1 < 0$, which is absurd. Hence primal is infeasible, $p^* = \infty$. Now look at dual with $\mathbf{b}^\top \mathbf{y} = 2y_2$. Since

$$C - \mathbf{A}^\top \mathbf{y} = \begin{bmatrix} 0 & -y_2 & 0 \\ -y_2 & -2y_1 & -y_1 \\ 0 & -y_1 & 2y_2 \end{bmatrix}$$

(look top-left!), $y_2 \leq 0$ for any $\mathbf{y} \in \mathbb{R}^2$ with $C - \mathbf{A}^\top \mathbf{y} \in \mathcal{C}_3^*$. Thus $\mathbf{y}^* = \mathbf{o}$ is dually feasible, thus optimal, and $d^* = 0$ is attained.

Theorem 1 gives an instance $\mathcal{T}_\mathrm{d}(\mathbf{A}, \mathbf{b}, C)$ with $0 = d^* < p^* = \infty$, but where d^* is not attained.

duality gap / attained	zero $d^* = p^* \in \mathbb{R}$	finite positive $-\infty < d^* < p^* < \infty$	infinite $-\infty < d^* < p^* = \infty$	infinite $-\infty = d^* < p^* < \infty$
both attained	StQP, both strictly f.	Ex.2	impossible	
p^* attained, d^* not attained	MStQP, dual strictly f.	Ex.2 & Thm.1	impossible	
p^* not attained, d^* attained	Ex.1, primal strictly f.	Ex.2 & Thm.2	Ex.3	
neither attained	Ex.1 & Thm.1	Ex.2 & Thms.1,2		

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both attained	StQP, both strictly f.	Ex.2	impossible	
p^* attained, d^* not attained	MStQP, dual strictly f.	Ex.2 & Thm.1	impossible	
p^* not attained, d^* attained	Ex.1, primal strictly f.	Ex.2 & Thm.2	Ex.3	
neither attained	Ex.1 & Thm.1	Ex.2 & Thms.1,2	Ex.3 & Thm.1	

Infinite duality gaps – infeasible dual

Example 4: Keep $\mathbf{A}$ from Example 3, but change $\mathbf{b} = \mathbf{o}$ now. Then any feasible X satisfies $x_{33} = 0$. Also change $c_{33} = -1$ now (rest zero). Then $X^* = O \in \mathcal{C}_3$ is optimal, so $p^* = 0$ is attained. However,

$$C - \mathbf{A}^\top \mathbf{y} = \begin{bmatrix} 0 & -y_2 & 0 \\ -y_2 & -2y_1 & -y_1 \\ 0 & -y_1 & -1 + 2y_2 \end{bmatrix} \in \mathcal{C}_3^*$$

is impossible, as still $y_2 \leq 0$, implying $-1 + 2y_2 \leq -1 < 0$, absurd. Hence $d^* = -\infty$.

Theorem 2 gives an instance $\mathcal{T}_{\mathsf{p}}(\mathbf{A}, \mathbf{b}, C)$ with $-\infty = d^* < p^* = 0$, but where p^* is not attained.

Now all table entries filled !

attained \ duality gap	zero $d^* = p^* \in \mathbb{R}$	finite positive $-\infty < d^* < p^* < \infty$	infinite $-\infty < d^* < p^* = \infty$	infinite $-\infty = d^* < p^* < \infty$
both attained	StQP, both strictly f.	Ex.2	impossible	impossible
p^* attained, d^* not attained	MStQP, dual strictly f.	Ex.2 & Thm.1	impossible	Ex.4
p^* not attained, d^* attained	Ex.1, primal strictly f.	Ex.2 & Thm.2	Ex.3	[redacted]
neither attained	Ex.1 & Thm.1	Ex.2 & Thms.1,2	Ex.3 & Thm.1	[redacted]

duality gap / attained	zero $d^* = p^* \in \mathbb{R}$	finite positive $-\infty < d^* < p^* < \infty$	infinite $-\infty < d^* < p^* = \infty$	infinite $-\infty = d^* < p^* < \infty$
both attained	StQP, both strictly f.	Ex.2	impossible	impossible
p^* attained, d^* not attained	MStQP, dual strictly f.	Ex.2 & Thm.1	impossible	Ex.4
p^* not attained, d^* attained	Ex.1, primal strictly f.	Ex.2 & Thm.2	Ex.3	impossible
neither attained	Ex.1 & Thm.1	Ex.2 & Thms.1,2	Ex.3 & Thm.1	Ex.4 & Thm.2

Restart: linear optimization over cones and duality

Consider linear problem over convex cone $\mathcal{K}$

$$\inf_{\mathbf{x}\in\mathcal{K}} \{\langle \mathbf{c},\mathbf{x}\rangle : \langle \mathbf{a}_0,\mathbf{x}\rangle = 1, \langle \mathbf{a}_i,\mathbf{x}\rangle = 0, i \in [1\!:\!m]\}$$

where $\{\mathbf{c},\mathbf{a}_i\} \cup \mathcal{K} \subset \mathbb{R}^d$.

Note: $\mathbf{A}\mathbf{x} = \mathbf{b} \iff (x_0 = 1)\ \&\ [-\mathbf{b}\,|\,\mathbf{A}] \begin{bmatrix} x_0 \\ \mathbf{x} \end{bmatrix} = \mathbf{o}$,

so all linear constraints can be homogenized except one.

Dual problem: let $\mathcal{L} = \{\mathbf{a}_1,\ldots,\mathbf{a}_m\}^{\perp}$, so primal/dual pair is

$$p^* = \inf_{\mathbf{x}\in\mathcal{L}\cap\mathcal{K}} \{\langle \mathbf{c},\mathbf{x}\rangle : \langle \mathbf{a}_0,\mathbf{x}\rangle = 1\} \quad \text{and} \quad \sup\{y_0 : \mathbf{c} - y_0\mathbf{a}_0 \in (\mathcal{L}\cap\mathcal{K})^*\}.$$

Let's be precise ...

Need to describe $(\mathcal{L} \cap \mathcal{K})^*$ but we only know

$$(\mathcal{L} \cap \mathcal{K})^* = \text{closure}(\mathcal{L}^* + \mathcal{K}^*)\,.$$

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Duality gap generated by ignoring closure (forgetting limits) !!

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Indeed have [B./Cheng/Dickinson/Lisser'17]:

Theorem (Slater is not needed):
Unless both problems are infeasible, there is zero duality gap,

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Caution: closure does not guarantee attainability.

Closure closes duality gap; does this help ?

Yes, e.g. if we have choices to describe primal feasibility $\mathbf{X} \in \mathcal{F}$:

let $\mathcal{L}_i$, $\mathcal{K}_i$ such that all $\mathcal{L}_i \cap \mathcal{K}_i = \mathcal{F}$ are all the same but cones for dual $\mathcal{L}_i^* + \mathcal{K}_i^*$ are different as i varies!

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Now, forgetting the closure, dual bounds can vary in quality, e.g., if $\mathcal{L}_1^* + \mathcal{K}_1^*$ is closed (zero gap) but $\mathcal{L}_2^* + \mathcal{K}_2^*$ is not.

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Quite abstract hope ? No, for MBQP-COP it works!

Mixed-binary QPs and COP reformulation

Consider mixed-binary quadratic problem under linear constraints:

$$q^* = \min \left\{ \mathbf{x}^\top \mathbf{Q} \mathbf{x} + 2\mathbf{c}^\top \mathbf{x} : \mathbf{x} \in \mathcal{Z}, x_j \in \{0, 1\}, j \in B \right\}$$

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and $B \subseteq [1\!:\!n]$ such that $z_j = 0$ for all $j \in B$, $\mathbf{z} \in \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}^\perp$.

Here $\mathbf{A}^\top = [\mathbf{a}_1, \ldots, \mathbf{a}_m]$ with $\mathbf{a}_i \in \mathbb{R}^n$ linearly independent.

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$$\widehat{\mathbf{Q}} = \begin{bmatrix} 0 & \mathbf{c}^\top \\ \mathbf{c} & \mathbf{Q} \end{bmatrix} \text{ and } \mathbf{R} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \mathbf{x}_0 & \mathbf{a}_{m+1} & \cdots & \mathbf{a}_n \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1-m)}.$$

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Ouff, enough preparations; but they pay in various ways !

Burer's result on COP for Mixed-Binary QPs

[Burer'09] established a first COP reformulation:

$$q^* = p_2^* = \min \left\{ \langle \mathbf{Q}, \mathbf{Y} \rangle : Y_{00} = 1 \, , \, \mathbf{Y} \in \mathcal{L}_2 \cap \mathcal{B}_1 \cap \mathcal{C}_{n+1} \right\}$$

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$$\begin{aligned} \mathcal{L}_2 &= \left\{ \mathbf{Y} = \begin{bmatrix} y_0 & \mathbf{x}^\top \\ \mathbf{x} & \mathbf{X} \end{bmatrix} : \mathbf{a}_i^\top \mathbf{x} = b_i y_0 \,,\, \mathbf{a}_i^\top \mathbf{X} \mathbf{a}_i = b_i^2 y_0 \,,\, i \in [1:m] \right\} \\ &= \{\mathbf{A}_1, \ldots, \mathbf{A}_{2m}\}^\perp \end{aligned}$$

representing the linear constraints

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$$\mathcal{B}_1 = \left\{ \mathbf{Y} = \begin{bmatrix} y_0 & \mathbf{x}^\top \\ \mathbf{x} & \mathbf{X} \end{bmatrix} : x_j = X_{jj} \, , \, j \in B \right\} = \left\{ \mathbf{B}_j : j \in B \right\}^\perp$$

the binarity constraints.

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Not relaxation $p_2^* \leq q^*$,

Burer's result on COP for Mixed-Binary QPs

[Burer'09] established a first COP reformulation:

$$q^* = p_2^* = \min \left\{ \langle \widehat{\mathbf{Q}}, \mathbf{Y} \rangle : Y_{00} = 1 \, , \, \mathbf{Y} \in \mathcal{L}_2 \cap \mathcal{B}_1 \cap \mathcal{C}_{n+1} \right\}$$

with

$$\begin{aligned} \mathcal{L}_2 &= \left\{ \mathbf{Y} = \begin{bmatrix} y_0 & \mathbf{x}^\top \\ \mathbf{x} & \mathbf{X} \end{bmatrix} : \mathbf{a}_i^\top \mathbf{x} = b_i y_0 \, , \, \mathbf{a}_i^\top \mathbf{X} \mathbf{a}_i = b_i^2 y_0 \, , i \in [1 : m] \right\} \\ &= \{ \mathbf{A}_1, \ldots, \mathbf{A}_{2m} \}^\perp \end{aligned}$$

representing the linear constraints and

$$\mathcal{B}_1 = \left\{ \mathbf{Y} = \begin{bmatrix} y_0 & \mathbf{x}^\top \\ \mathbf{x} & \mathbf{X} \end{bmatrix} : x_j = X_{jj} \, , \, j \in B \right\} = \left\{ \mathbf{B}_j : j \in B \right\}^\perp$$

the binarity constraints.

Not relaxation $p_2^* \leq q^*$, rather **convex** reformulation $p_2^* = q^*$ of mixed-binary **nonconvex** QP!

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$$\mathcal{L}_1 = \mathbf{R}\mathcal{S}^{n+1-m}\mathbf{R}^\top = \left\{\mathbf{Y} = \mathbf{R}\mathbf{U}\mathbf{R}^\top : \mathbf{U} = \mathbf{U}^\top \text{ has order } n+1-m\right\}$$

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$$p_1^* = \min\left\{\langle \mathbf{Q}, \mathbf{Y}\rangle : Y_{00} = 1\,,\, \mathbf{Y} \in \mathcal{L}_1 \cap \mathcal{B}_1 \cap \mathcal{C}_{n+1}\right\}.$$

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Note: $\mathcal{L}_1^\perp = \left\{\mathbf{S} = \mathbf{S}^\top : \mathbf{R}^\top\mathbf{S}\mathbf{R} = \mathbf{O}\right\}$.

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Repeat last method for B: replace $\mathcal{B}_1$ with

$$\mathcal{B}_2 = \left\{ \mathbf{Y} = \begin{bmatrix} y_0 & \mathbf{x}^\top \\ \mathbf{x} & \mathbf{X} \end{bmatrix} : \sum_{j \in B} (X_{jj} - x_j) = 0 \right\}$$

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Theorem [B./Cheng/Dickinson/Lisser'17]:

For all $\{(i,j),(r,s)\} \subset \{1,2,3\} \times \{1,2\}$, *we have*

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The same is true for the DNN approximation ($\mathcal{P}_d \cap \mathcal{N}_d$ *pro* $\mathcal{C}_d$).

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Tightest DNN dual:

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If $\mathcal{Z}$ is compact, then Slater holds, under suitable conditions also for unbounded $\mathcal{Z}$.

Application: purely binary QPs

More precisely, multi-dimensional knapsack problems:

$$\max_{\mathbf{x}\in\{0,1\}^n} \left\{ \mathbf{x}^\top \mathbf{Q}\mathbf{x} : {\mathbf{a}_i}^\top \mathbf{x} \le b_i, i \in [1:m] \right\}$$

with $\mathbf{a}_i \in \mathbb{R}^n_+$. Notoriously hard problems !

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Instances from Beasley OR-library with random $\mathbf{Q}$:

Table 1: Numerical result for the multidimensional knapsack problem using CP reformulations

Prime problem													
Orig prob		No merging		Merging linear		Merging binary		Merging Both		Reduced no merging		Reduced merging	
(n,m)	Opt val	Opt val	CPU	Opt val	CPU	Opt val	CPU	Opt val	CPU	Opt val	CPU	Opt val	CPU
(10,10)	13840	14876	0.9	–(-Inf)	1.3	14877	0.8	16156*	1.3	14852	0.5	14852	0.2
(20,10)	46922	48451	10.1	48792*	20.8	48453	10.9	50572*	28.0	48435	1.4	48435	1.4
(30,5)	48110	50890	54	51186*	120	50890	59	56723*	135	50854	10	50854	10
(40,5)	105154	110296	333	110809*(150)	721	110298	351	132268* (150)	767	110222	70	110222	68
(50,5)	206590	213470	2741	215141*	3413	213475	2477	228663*	2682	213330	558	213330	502
(60,5)	176100	181041	5425	–(150)	8779	181043	5386	– (150)	8894	180953	769	180953	748
(70,5)	318644	–	–	–	–	–	–	–	–	322884	2484	322884	2431
(80,5)	–	–	–	–	–	–	–	–	–	341745	5248	341745	5395

"–" means the problem can not be solved within three hours while "*" means the problem is not solved accurately. "150" means the algorithm reaches the maximum number of iterations set by Sedumi.

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