Generalized Gauss Inequalities via Semidefinite Programming

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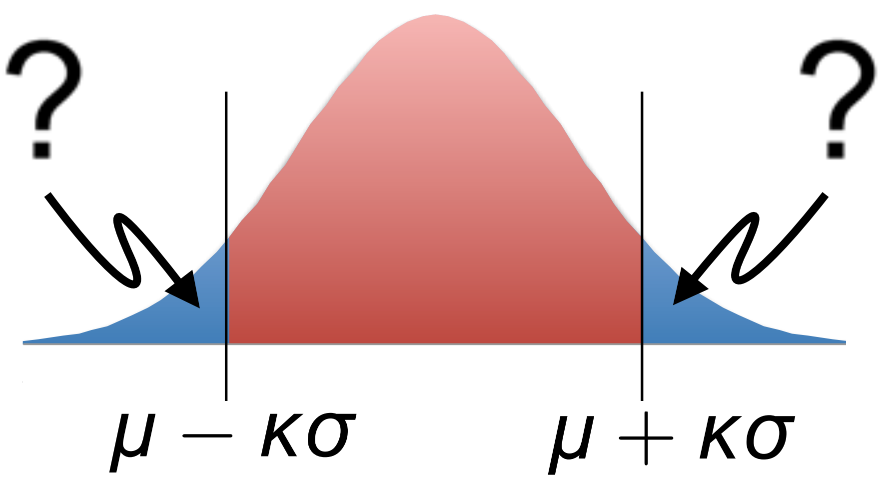
Bounding Univariate Tail Probabilities



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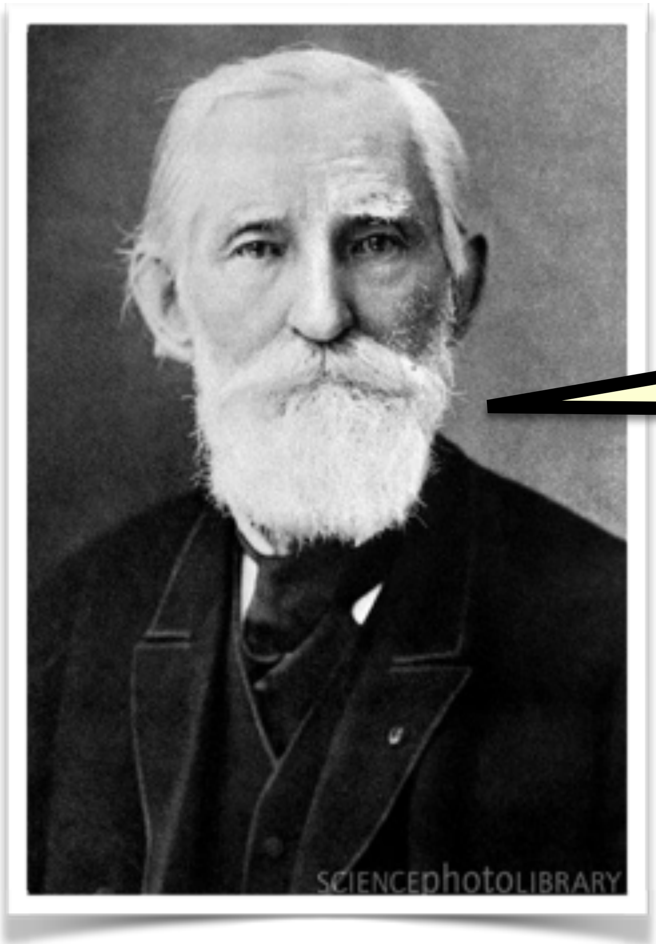
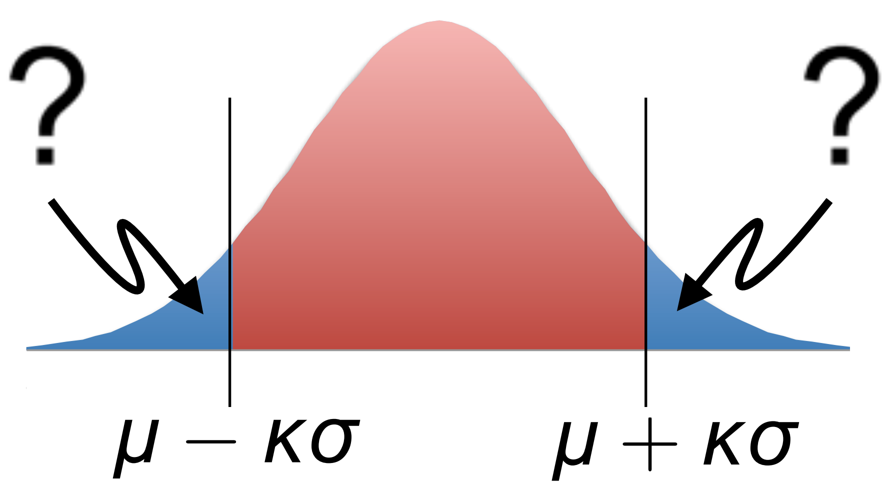


Pafnuty Chebyshev
(1821–1894)

Chebyshev (1867): For $\xi \sim (\mu, \sigma)$

$$\mathbb{P}(|\xi - \mu| \geq \kappa\sigma) \leq \frac{1}{\kappa^2} \quad (\kappa \geq 1)$$



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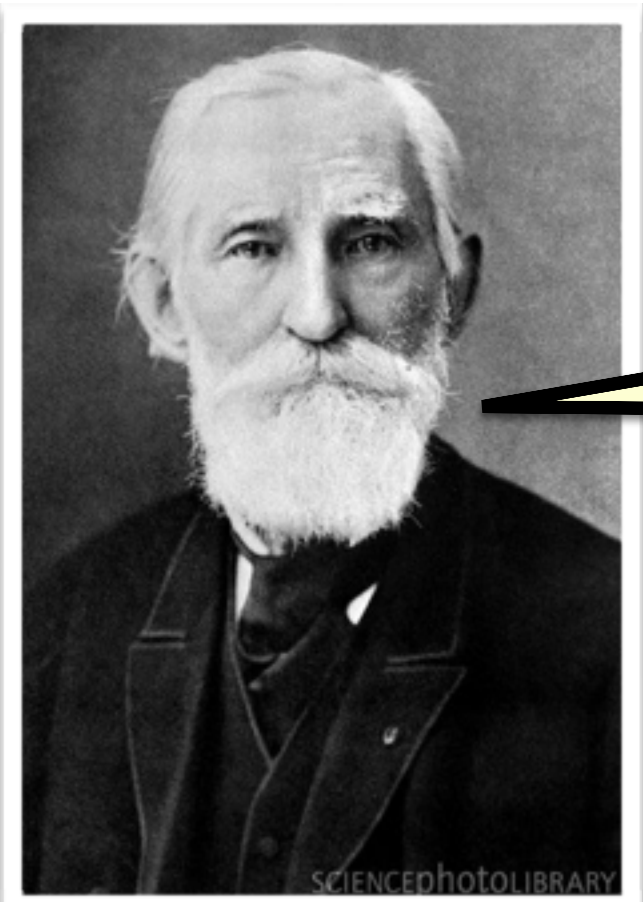

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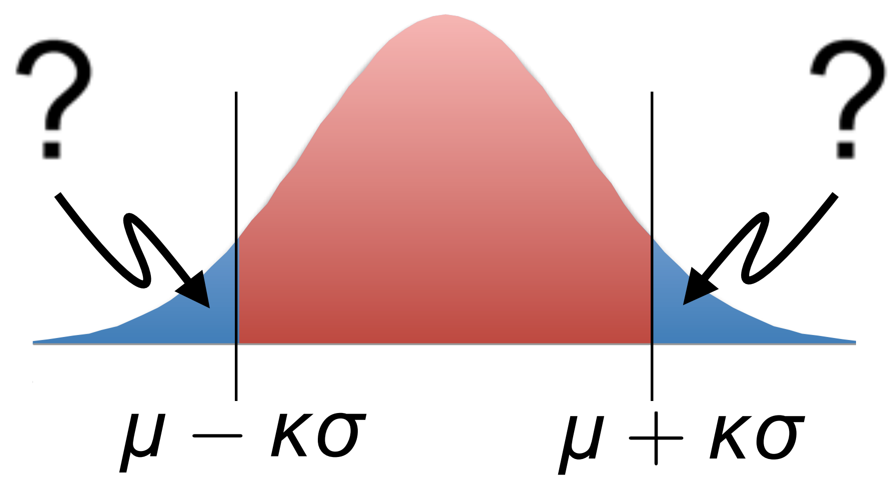


Carl Friedrich Gauss
(1777–1855)

Gauss (1823): For $\xi \sim (\mu, \sigma)$ unimodal

$$\mathbb{P}(|\xi - \mu| \geq \kappa\sigma) \leq \frac{4}{9\kappa^2} \quad (\kappa \geq \frac{2}{\sqrt{3}})$$

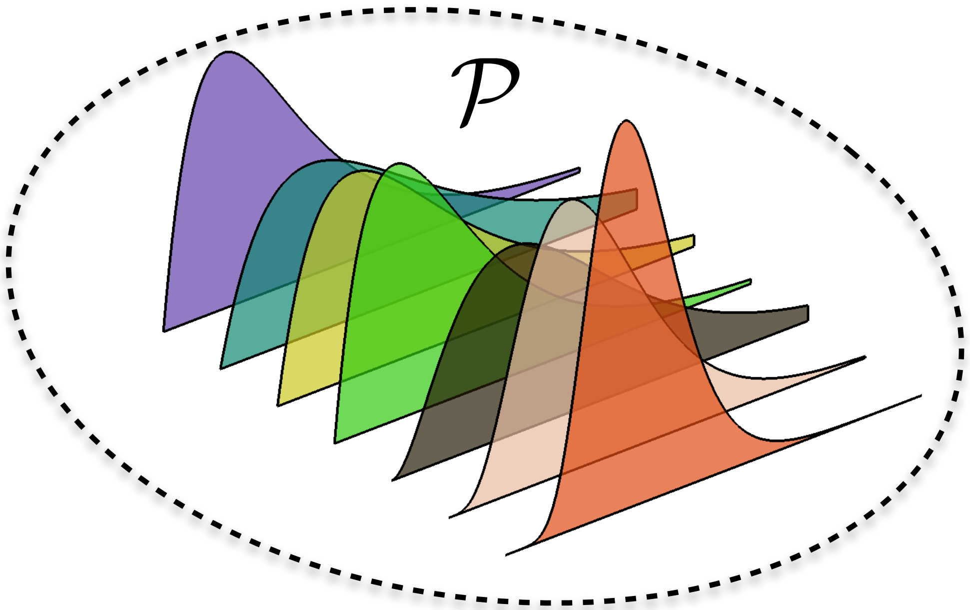
Optimization Perspective

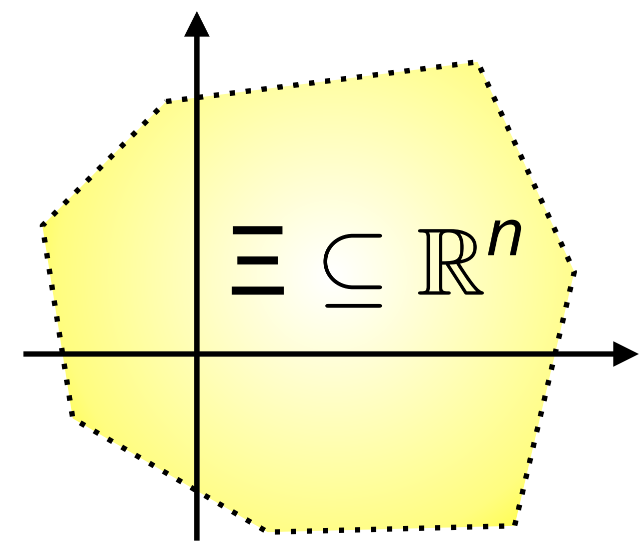
Worst-case probability problems:

$$\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}(\xi \notin \Xi) \;=\; ?$$

ambiguity set

open polyhedron





$\Xi = \{\xi : a_i^\top \xi < b_i \; \forall i = 1, \ldots, k\}$

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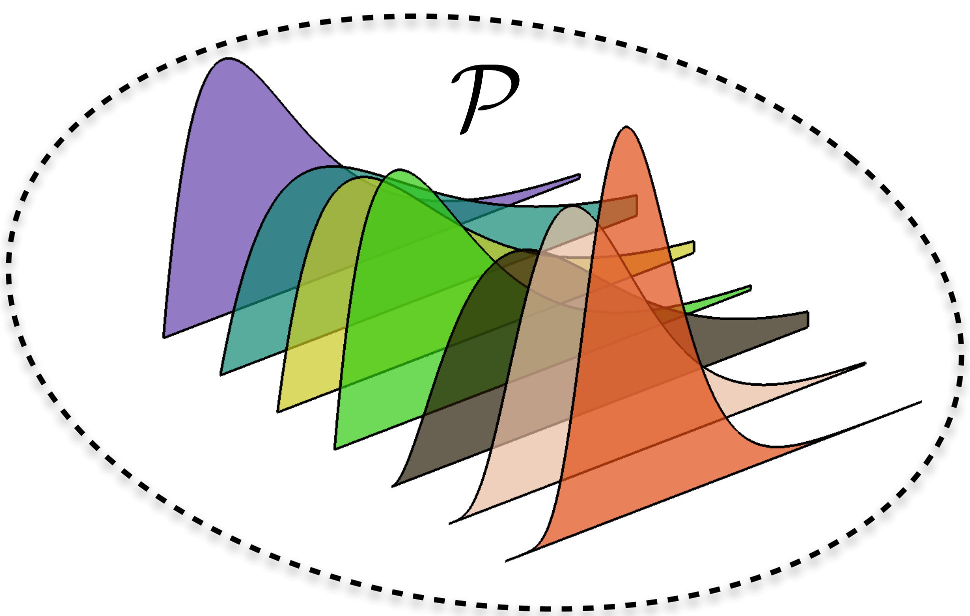

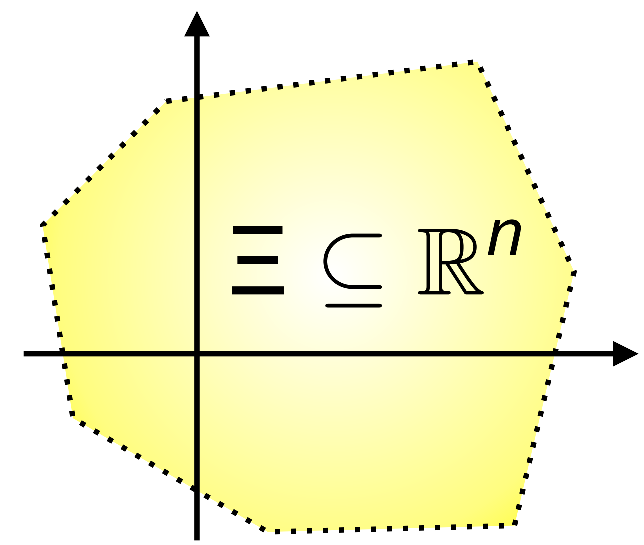

$$\Xi = \{\xi : a_i^\top \xi < b_i \; \forall i = 1, \ldots, k\}$$

Chebyshev: $\mathcal{P}$ = set of all distributions with mean μ and variance σ^2

$$\Xi = \{\xi : \mu - \kappa\sigma < \xi < \mu + \kappa\sigma\}$$

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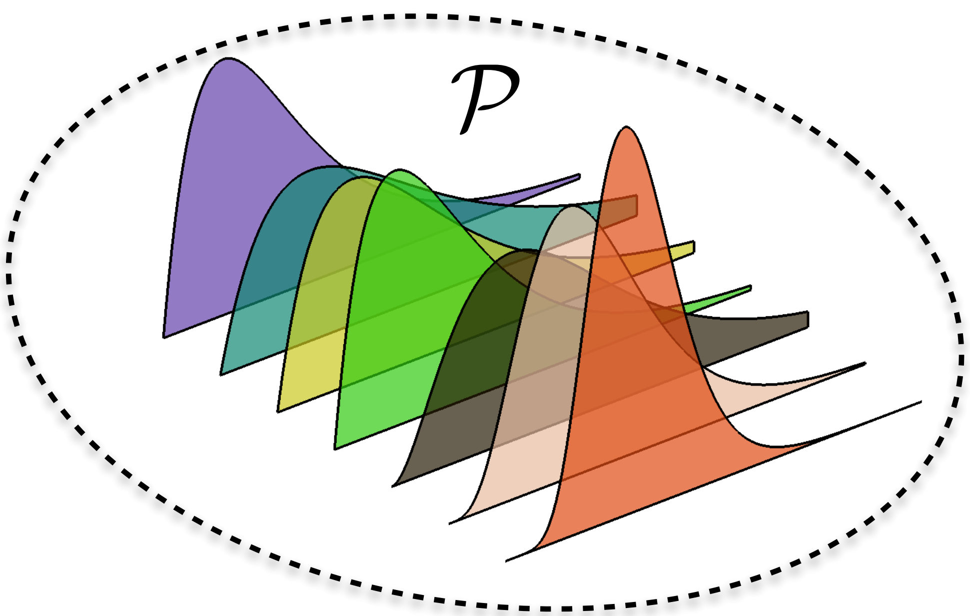

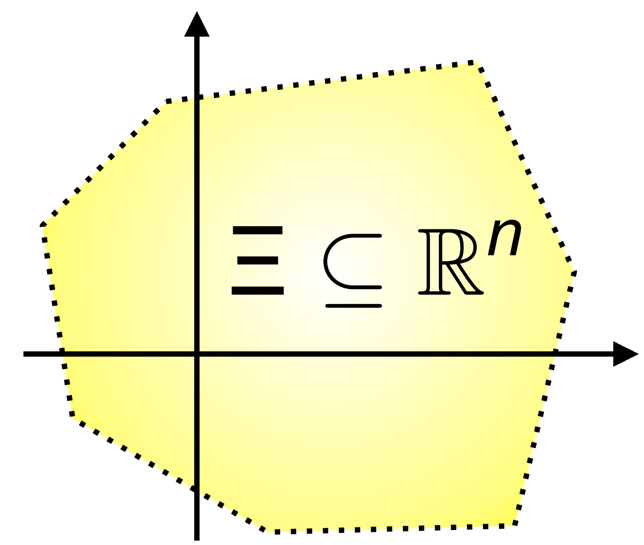

$$\Xi = \{\xi : a_i^\top \xi < b_i \ \forall i = 1, \ldots, k\}$$

Gauss: $\mathcal{P}$ = set of all unimodal distributions with mean μ and variance σ^2

$$\Xi = \{\xi : \mu - \kappa\sigma < \xi < \mu + \kappa\sigma\}$$

Multivariate Chebyshev Bound

$$\mathcal{P}(\mu, S) = \left\{\mathbb{P} : \mathbb{E}_{\mathbb{P}}(\xi) = \mu,\ \mathbb{E}_{\mathbb{P}}(\xi\xi^\top) = S\right\}$$

Multivariate Chebyshev Bound

$$\mathcal{P}(\mu, S) = \{\mathbb{P} : \mathbb{E}_{\mathbb{P}}(\xi) = \mu,\ \mathbb{E}_{\mathbb{P}}(\xi\xi^\top) = S\}$$

Vandengerghe, Boyd, Comanor (2007):

$$\begin{aligned} \sup_{\mathbb{P}\in\mathcal{P}(\mu,S)} \mathbb{P}(\xi \notin \Xi) \quad = \quad \max \quad & \textstyle\sum_{i=1}^k \lambda_i \\ \text{s.t.} \quad & a_i^\top z_i \geq b_i\lambda_i \quad \forall i = 1,\dots,k \\ & \textstyle\sum_{i=1}^k \begin{pmatrix} Z_i & z_i \\ z_i^\top & \lambda_i \end{pmatrix} \preceq \begin{pmatrix} S & \mu \\ \mu^\top & 1 \end{pmatrix} \\ & \begin{pmatrix} Z_i & z_i \\ z_i^\top & \lambda_i \end{pmatrix} \succeq 0 \quad \forall i = 1,\dots,k \end{aligned}$$

Multivariate Gauss Bound

$$\mathcal{P}_\star(\mu, S) = \left\{\mathbb{P} \text{ star-unimodal} : \mathbb{E}_\mathbb{P}(\xi) = \mu,\ \mathbb{E}_\mathbb{P}(\xi\xi^\top) = S\right\}$$

Multivariate Gauss Bound

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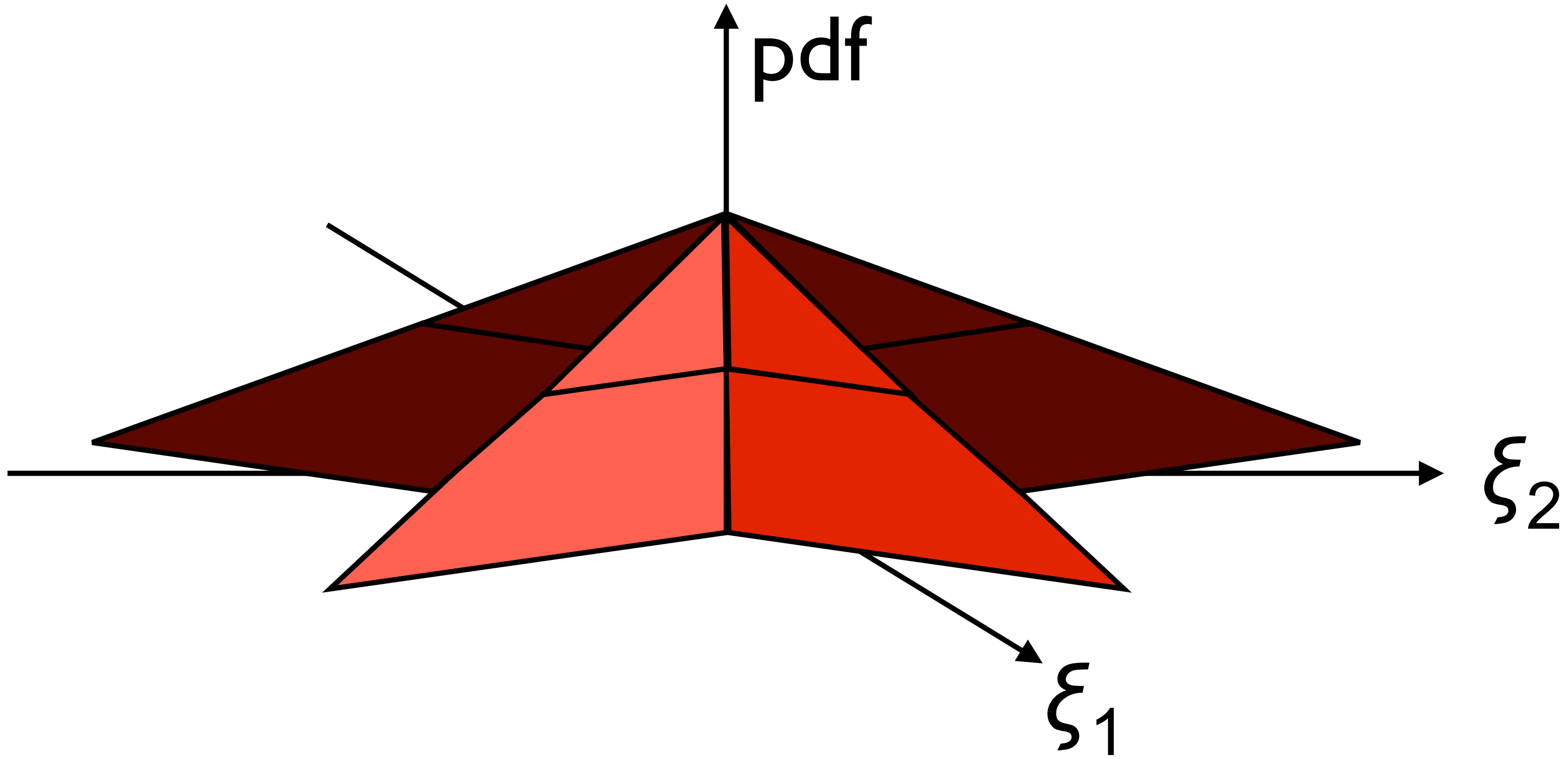
Definition: If $\mathbb{P}$ has a continuous pdf f, then $\mathbb{P}$ is star-unimodal if $f(t\xi)$ is non-increasing in $t > 0$ for all $\xi \neq 0$.



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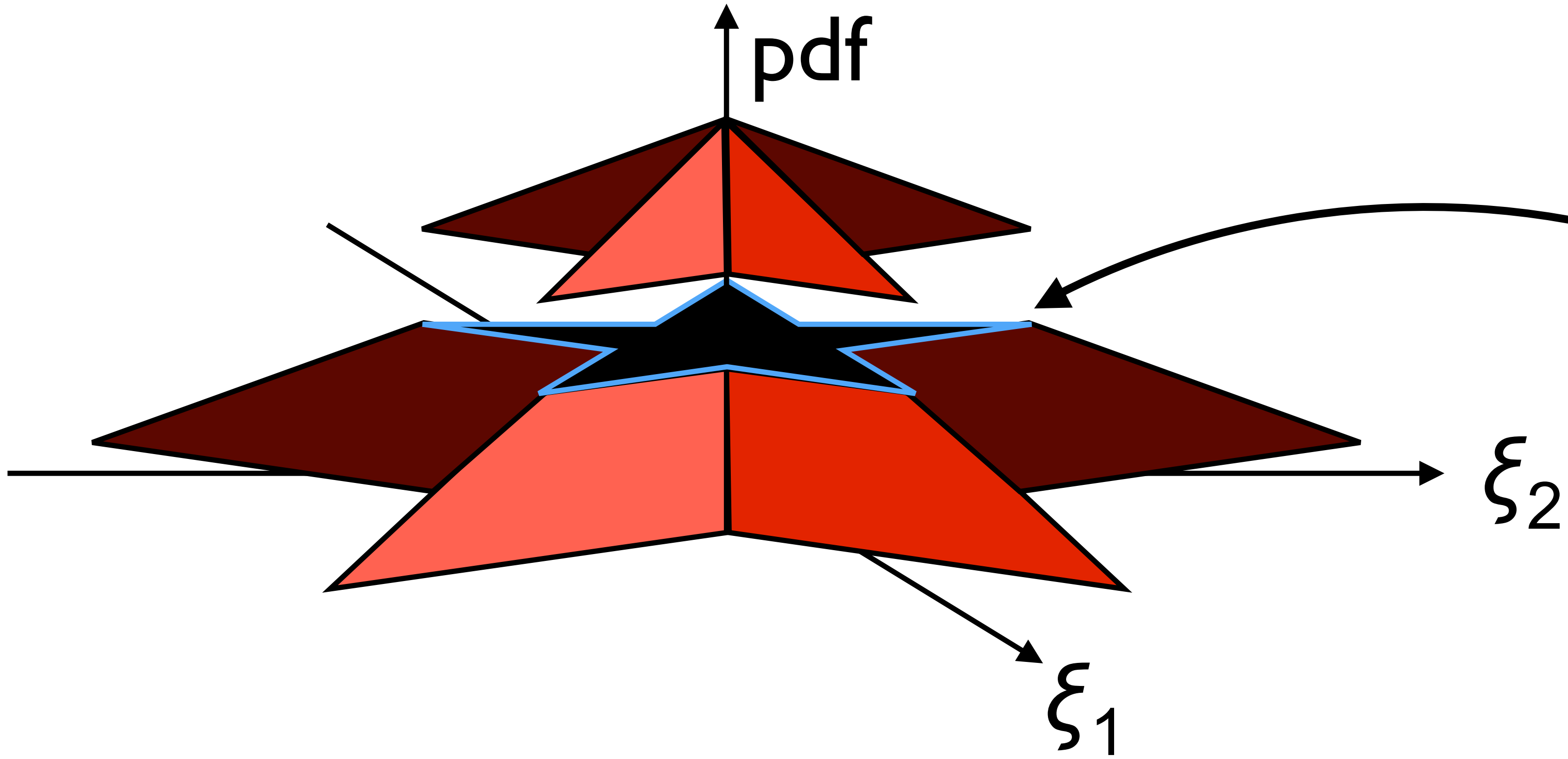

Superlevel sets of pdf are star-shaped with center 0

Multivariate Gauss Bound

$$\mathcal{P}_\star(\mu, S) = \left\{\mathbb{P} \text{ star-unimodal} : \mathbb{E}_\mathbb{P}(\xi) = \mu,\ \mathbb{E}_\mathbb{P}(\xi\xi^\top) = S\right\}$$

Van Parys, Goulart, K. (2014):

$$\begin{array}{rcll} \sup\limits_{\mathbb{P}\in\mathcal{P}_\star(\mu,S)} \mathbb{P}(\xi \notin \Xi) & = & \max & \sum_{i=1}^k \lambda_i - \tau_i \\ & & \text{s.t.} & a^\top z_i \geq 0,\ \tau_i \geq 0 \quad \forall i = 1,\ldots,k \\ & & & \tau_i (a_i^\top z_i)^n \geq \lambda_i^{n+1} b_i^n \quad \forall i = 1,\ldots,k \\ & & & \sum_{i=1}^k \begin{pmatrix} Z_i & z_i \\ z_i^\top & \lambda_i \end{pmatrix} \preceq \begin{pmatrix} \frac{n+2}{n} S & \frac{n+1}{n}\mu \\ \frac{n+1}{n}\mu^\top & 1 \end{pmatrix} \\ & & & \begin{pmatrix} Z_i & z_i \\ z_i^\top & \lambda_i \end{pmatrix} \succeq 0 \quad \forall i = 1,\ldots,k \end{array}$$

State-of-the-Art: The Duality Method

- Worst-case probability problem is an infinite-dimensional LP;
- Dual LP is a semi-infinite LP/robust optimization problem;
- Dual can often be reformulated as a tractable conic program:
  - $*$ Bertsimas & Popescu (2005): SOS techniques
  - $*$ Popescu (2005): SOS techniques and Choquet theory;
  - $*$ Boyd, Vandenberghe & Comanor (2007): $\mathcal{S}$-lemma;
  - $*$ Delage & Ye (2010): Ellipsoid method;
  - $*$ Zymler, K. & Rustem (2013): Farkas lemma and $\mathcal{S}$-lemma;
  - $*$ etc.

Outline

- Different Degrees of Unimodality
- Choquet Representations
    - The Set of All Distributions
    - The Set of Star-unimodal Distributions
    - The Set of α-Unimodal Distributions
- Generalized Probability Bounds
    - Feasibility Conditions
    - Structure of Extremal Distributions
    - SDP Reformulation
- Extensions

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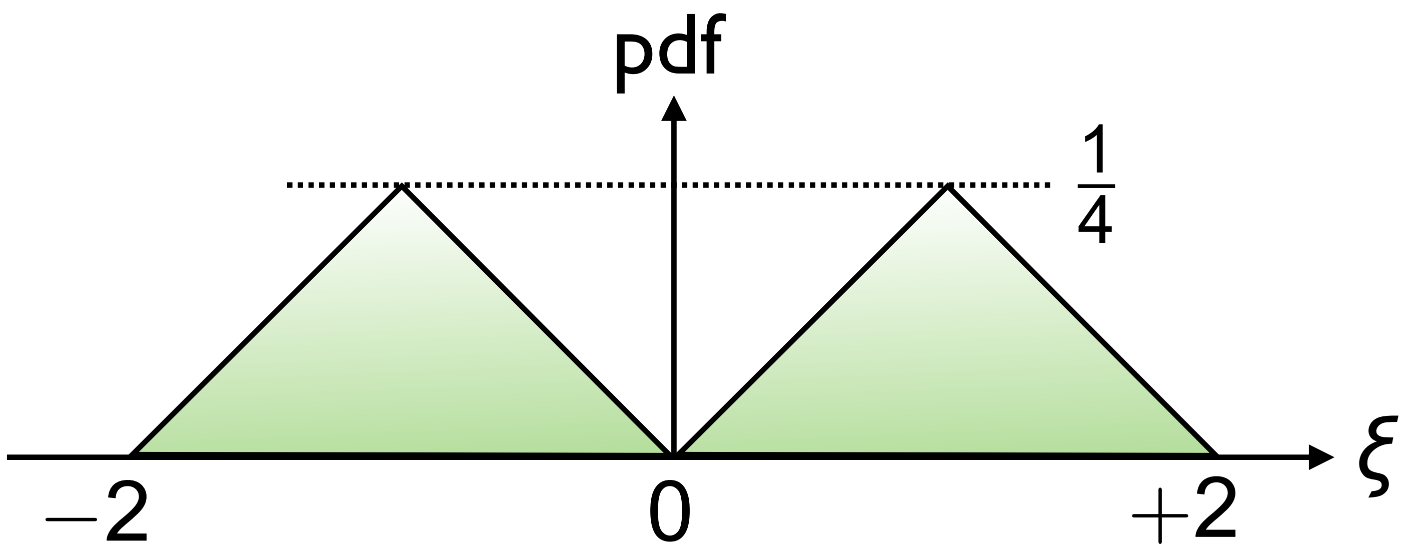
Different Degrees of Unimodality

Definition: If $\mathbb{P}$ has a continuous pdf f, then $\mathbb{P}$ is α-unimodal if $t^{n-\alpha} f(t\xi)$ is non-increasing in $t > 0$ for all $\xi \neq 0$.

Example: $n = 1$, $\alpha = 2$



Definition: $\mathcal{P}_\alpha$ = weak closure of the set of all α-unimodal distributions.

α-Unimodal Probability Bounds

Goal: Compute $\sup_{\mathbb{P} \in \mathcal{P}_\alpha(\mu, S)} \mathbb{P}(\xi \notin \Xi)$

Ambiguity set: $\mathcal{P}_\alpha(\mu, S) = \mathcal{P}_\alpha \cap \mathcal{P}(\mu, S)$

structural information

moment information

α-Unimodal Probability Bounds

Goal: Compute $\sup_{\mathbb{P} \in \mathcal{P}_\alpha(\mu, S)} \mathbb{P}(\xi \notin \Xi)$

Ambiguity set: $\mathcal{P}_\alpha(\mu, S) = \mathcal{P}_\alpha \cap \mathcal{P}(\mu, S)$

Special cases: $\mathrm{cl}(\cup_{\alpha=1}^{\infty} \mathcal{P}_\alpha) = \mathcal{P}_\infty =$ set of all distributions
$\Longrightarrow$ Generlized Chebyshev bounds

$\mathcal{P}_n =$ set of all star-unimodal distributions
$\Longrightarrow$ Generalized Gauss bounds

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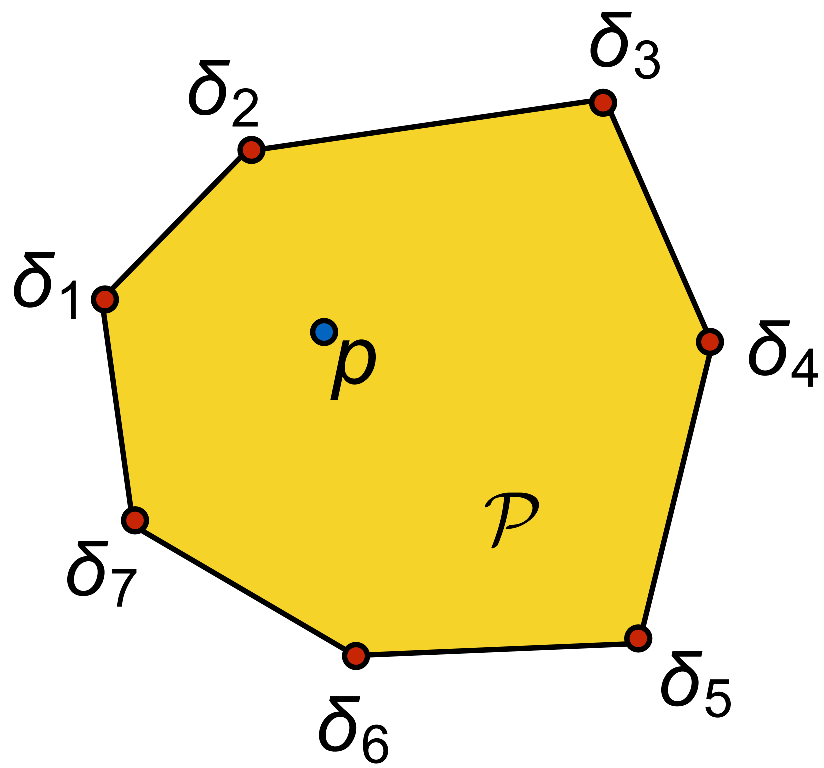
Choquet Representations

Minkowski: Every point in a compact convex set $\mathcal{P} \subset \mathbb{R}^n$ is the mean of a distribution on ex$\mathcal{P}$.



$$p = \sum_i m_i \delta_i$$

probability distribution on ex $\mathcal{P}$

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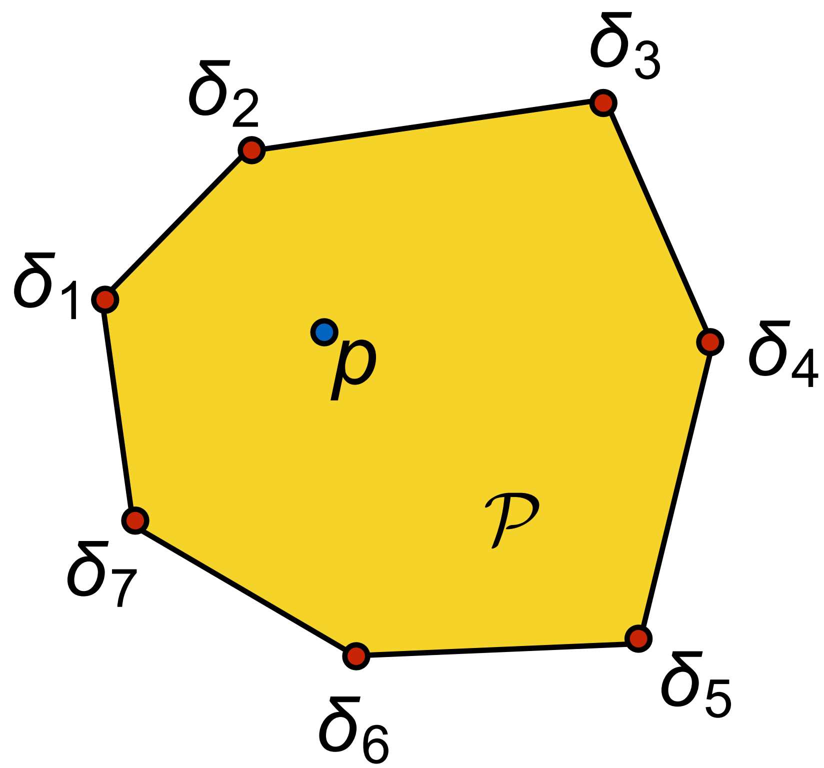

$$p = \sum_i m_i \delta_i$$

probability distribution on ex $\mathcal{P}$

A convex set $\mathcal{P}$ of distributions admits a Choquet representation if $\forall \mathbb{P} \in \mathcal{P}$ there is a distribution m on ex $\mathcal{P}$ with:

$$\mathbb{P}(\cdot) = \int_{\text{ex}\,\mathcal{P}} \delta(\cdot)\, m(\mathrm{d}\delta)$$

mixture distribution

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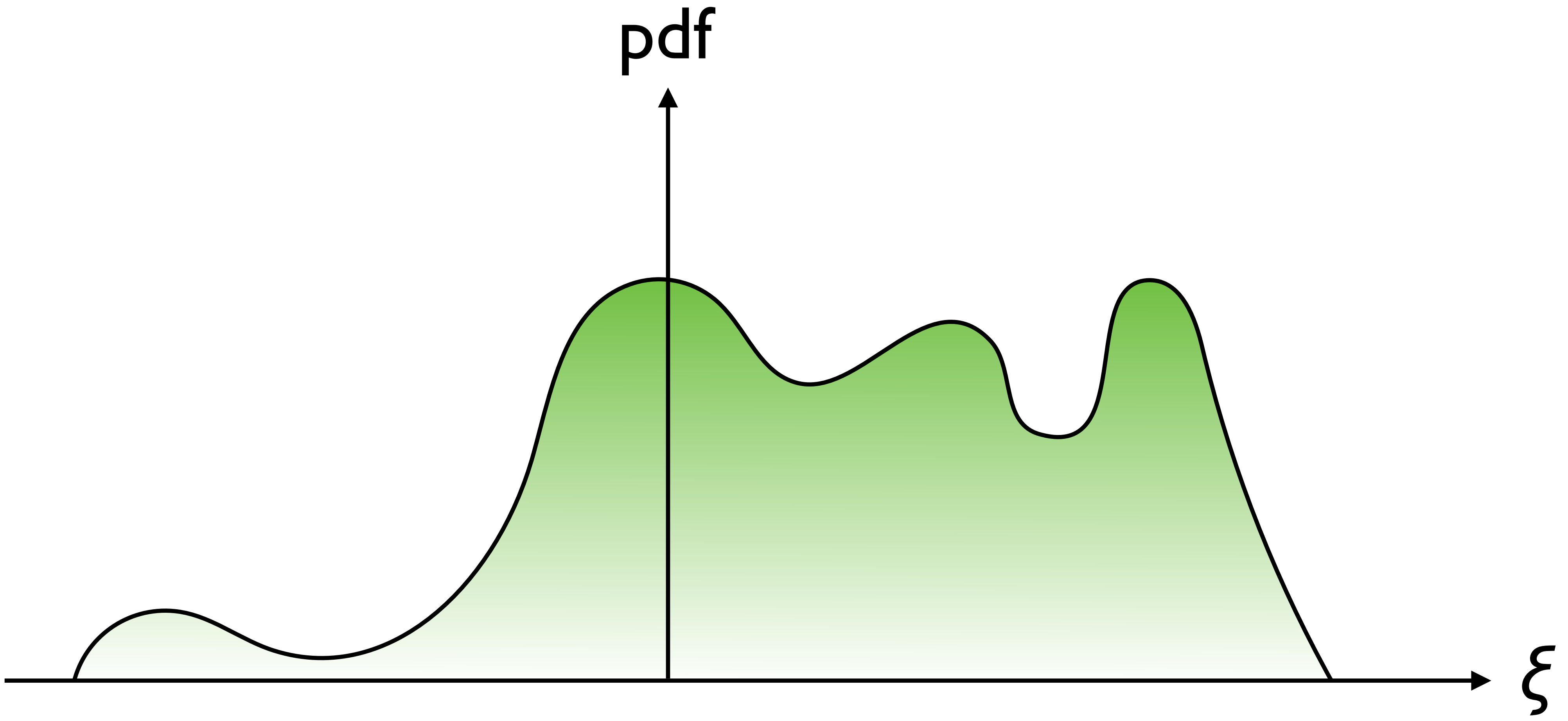
The Set of all Distributions



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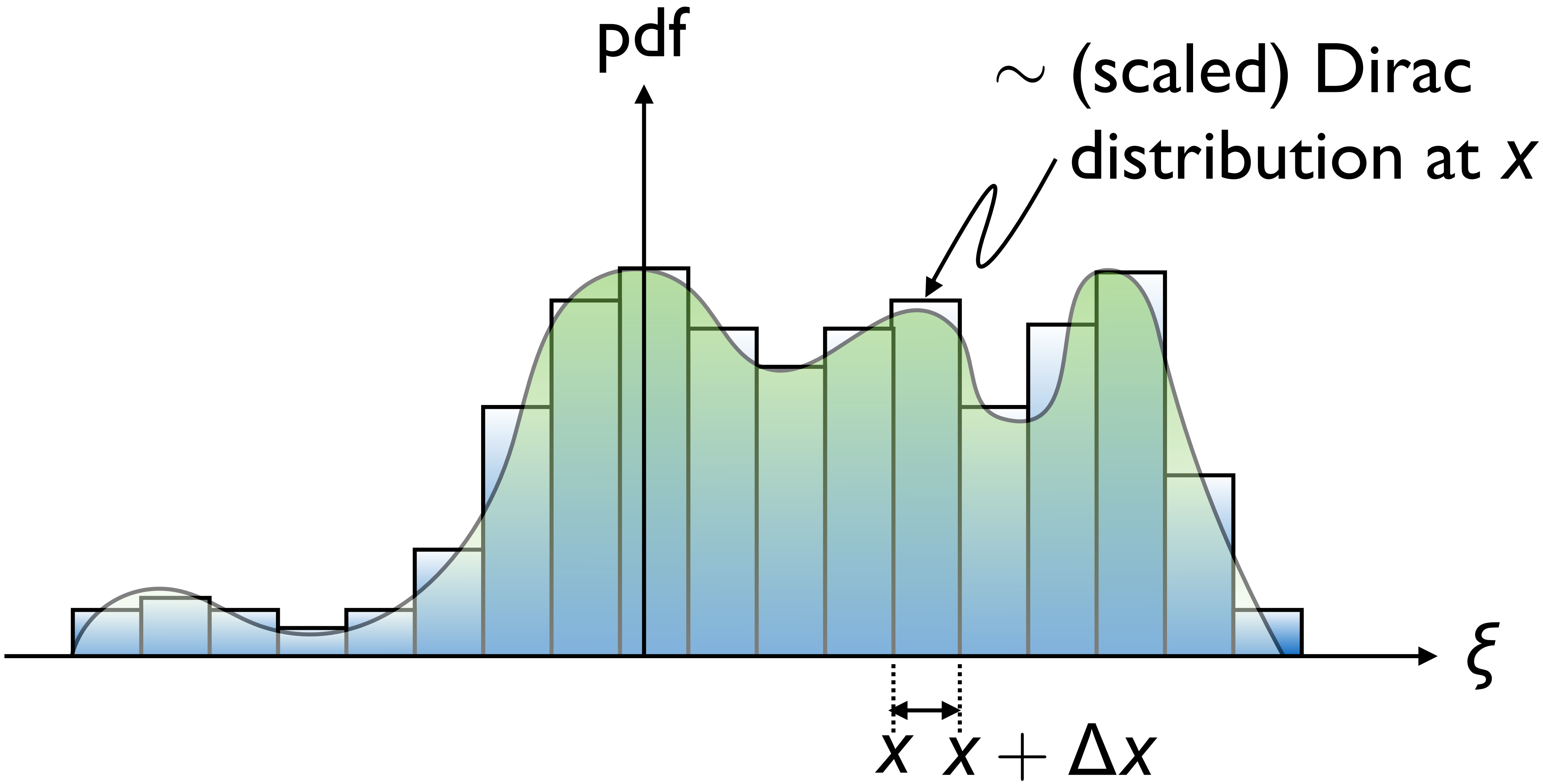

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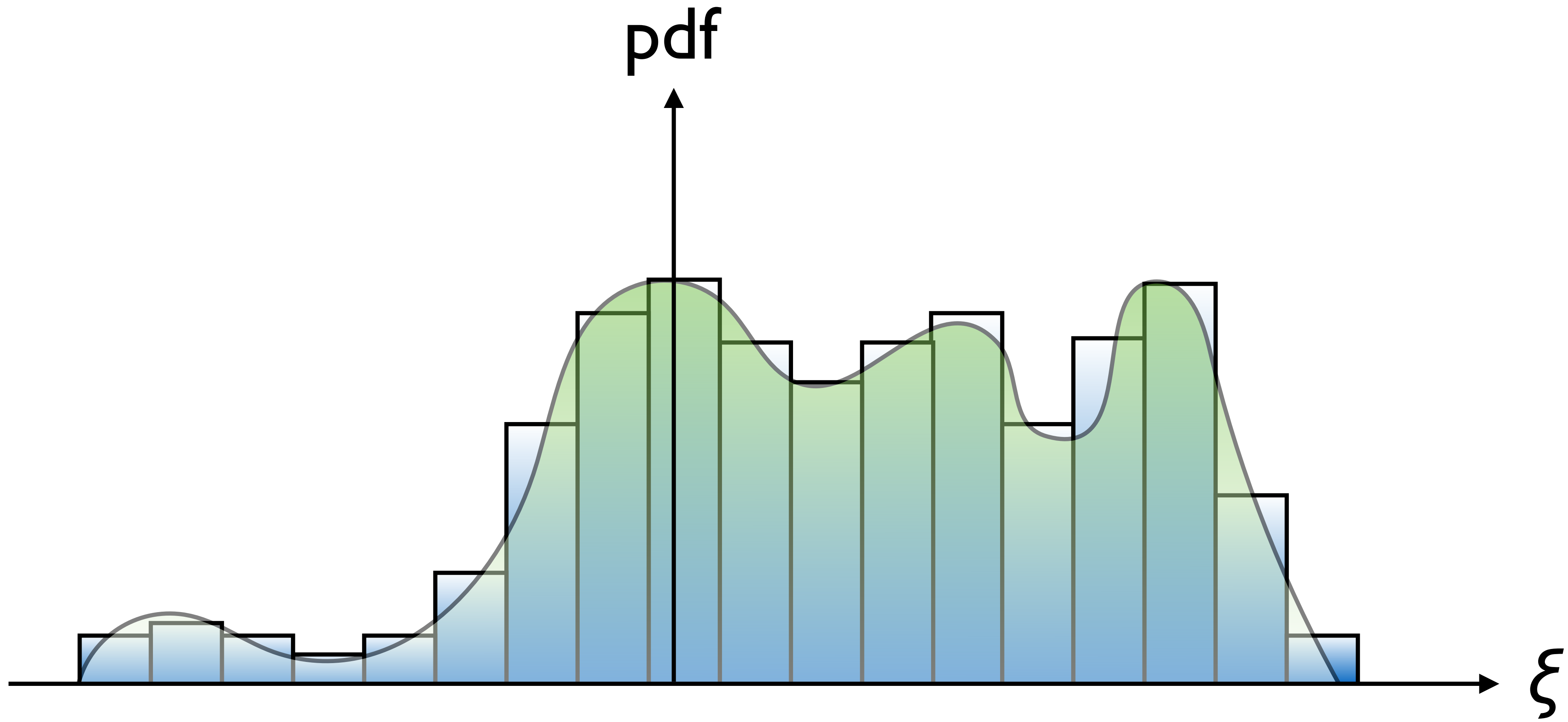

$$\mathbb{P}(\cdot) = \int_{\mathbb{R}^n} \delta_x(\cdot)\, \mathbb{P}(\mathrm{d}x)$$

Dirac distribution at x

mixture distribution $m \equiv \mathbb{P}$

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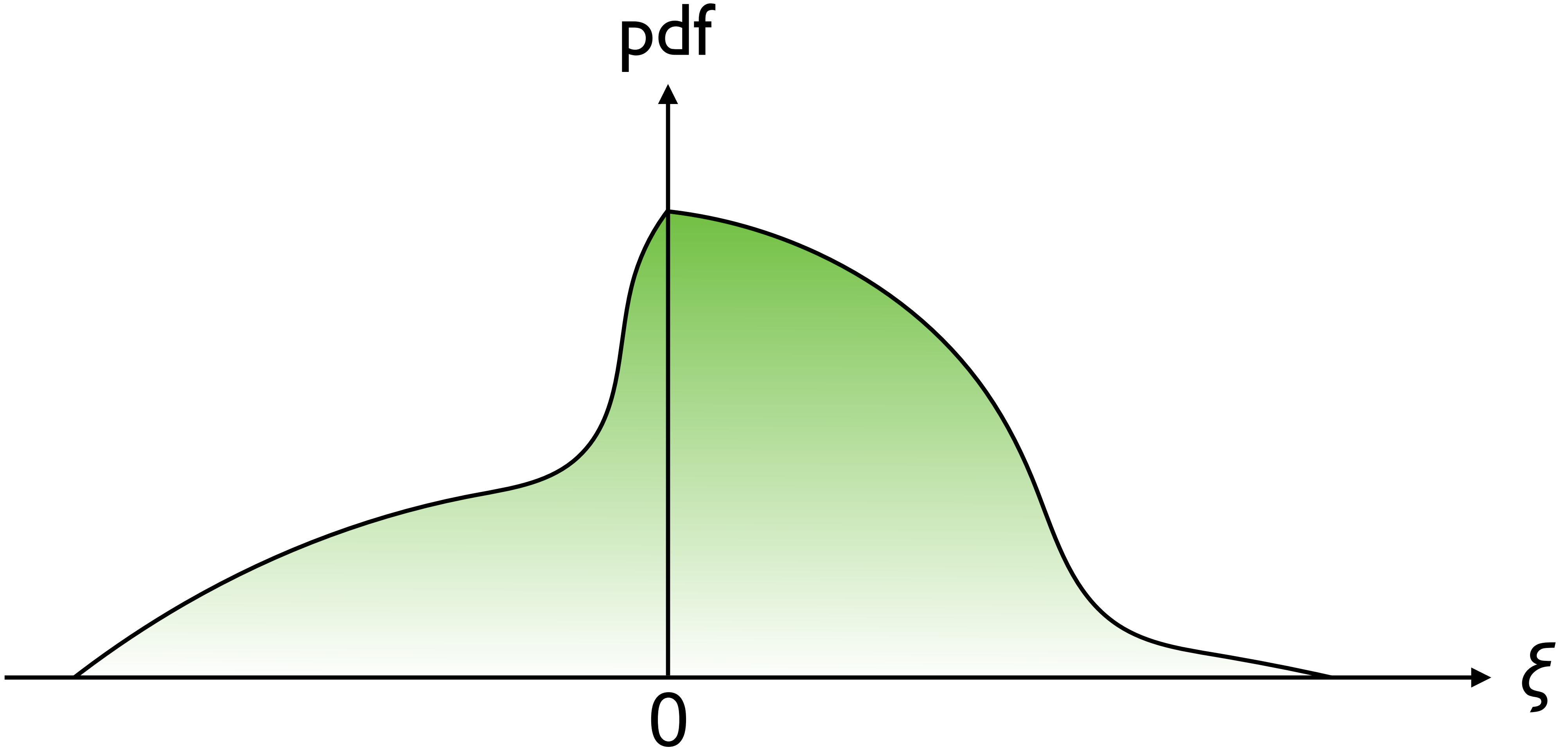
Unimodal Univariate Distributions



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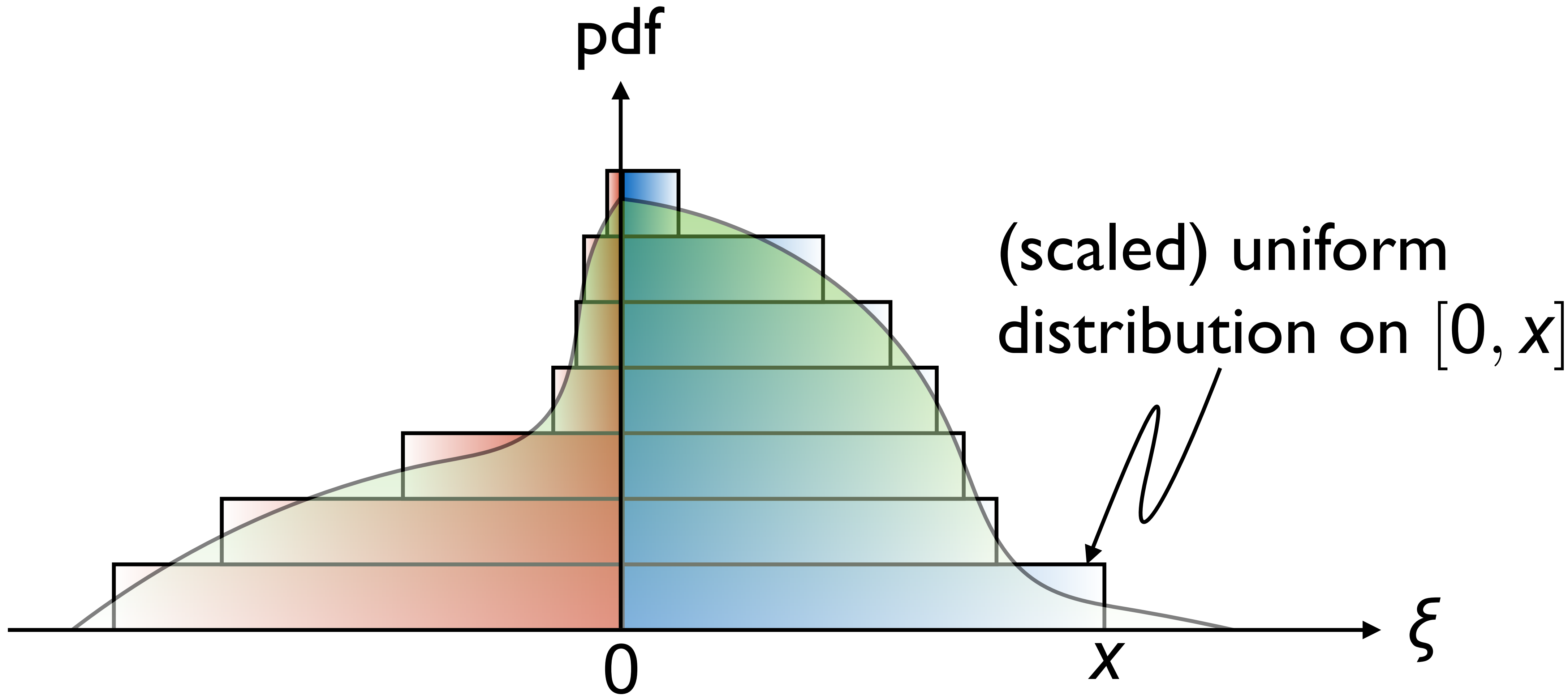

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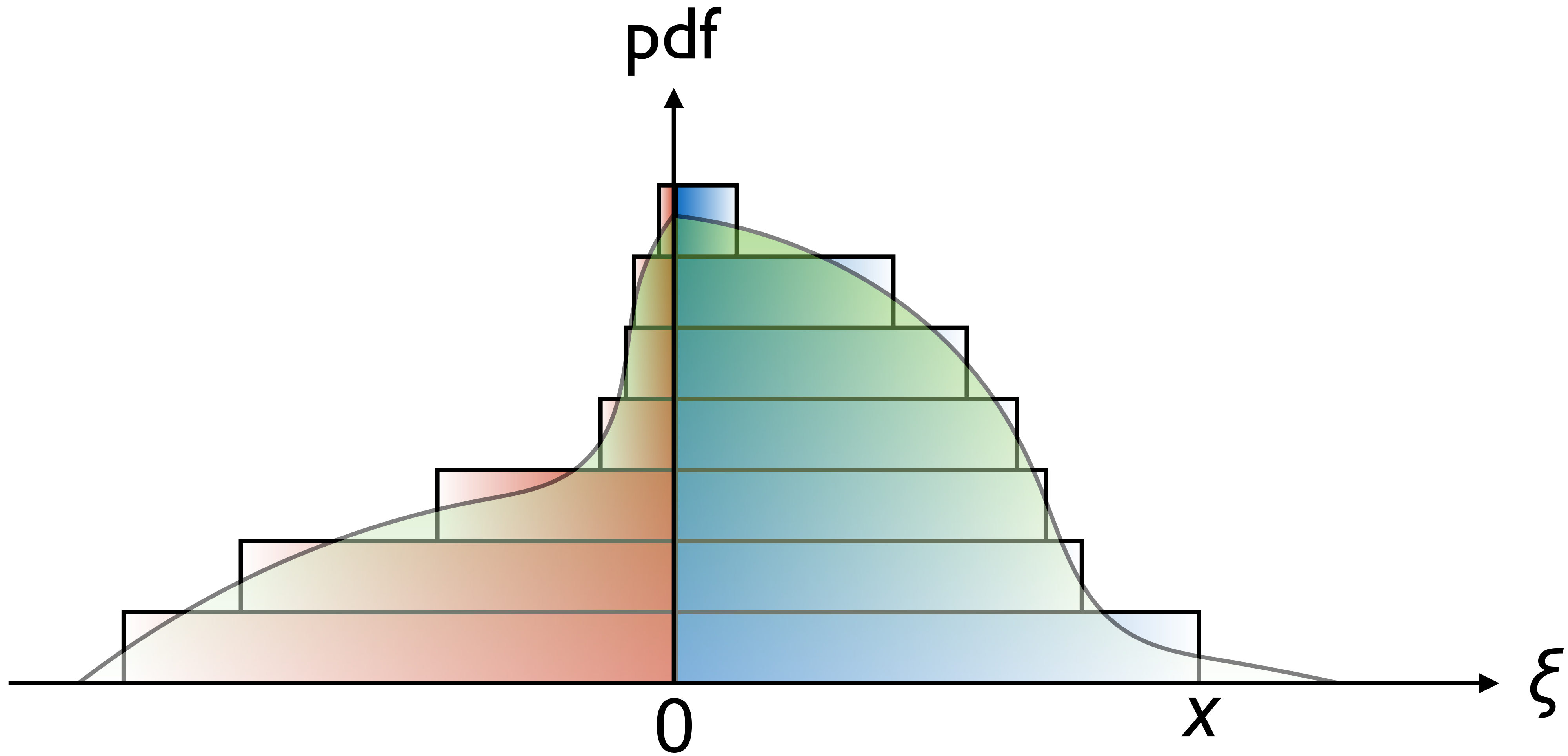
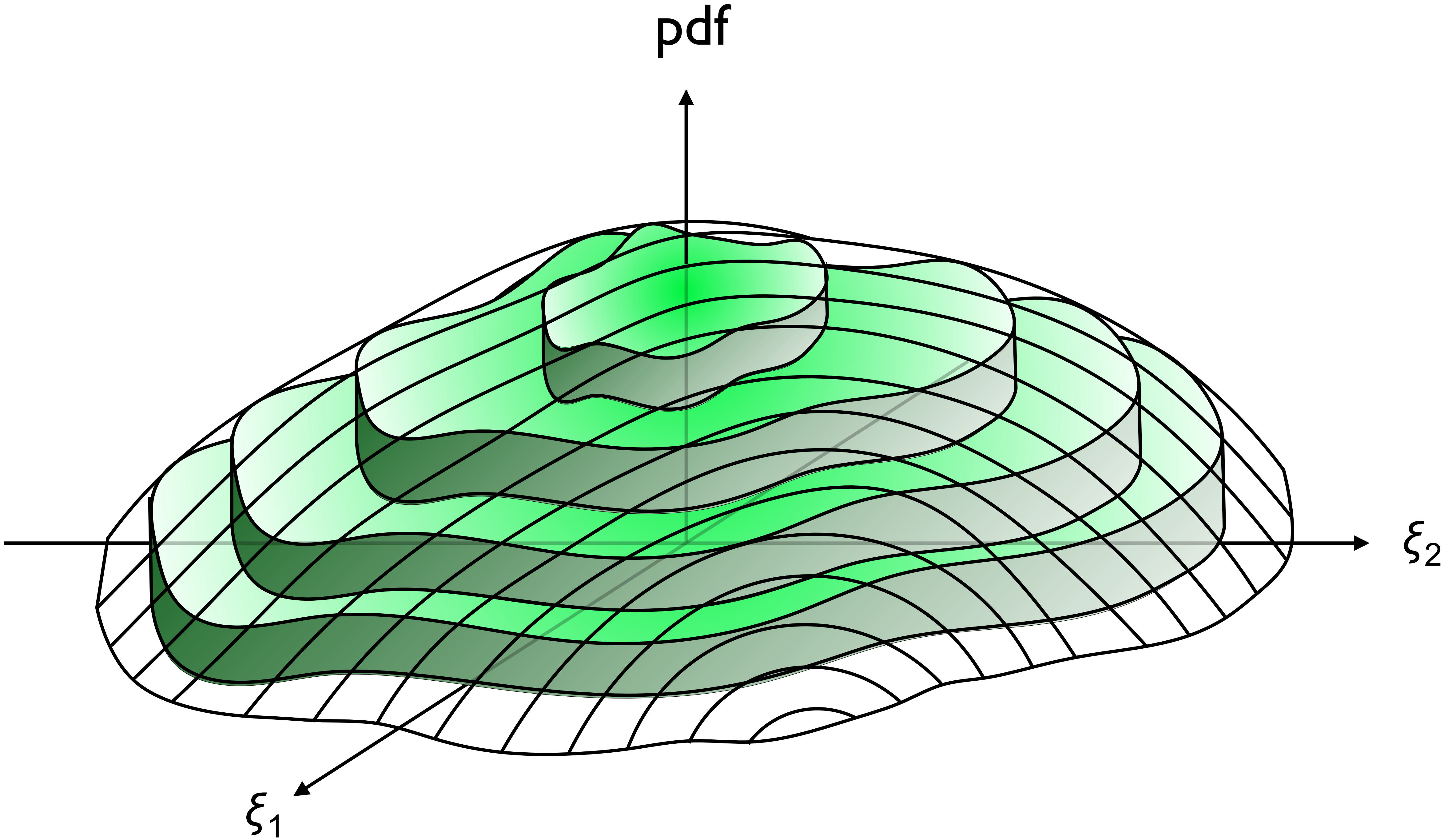


Khinchine (1938): $\mathbb{P}(\cdot) = \int_{-\infty}^{+\infty} \delta^1_{[0,x]}(\cdot)\, m(\mathrm{d}x)$

uniform distribution on $[0, x]$

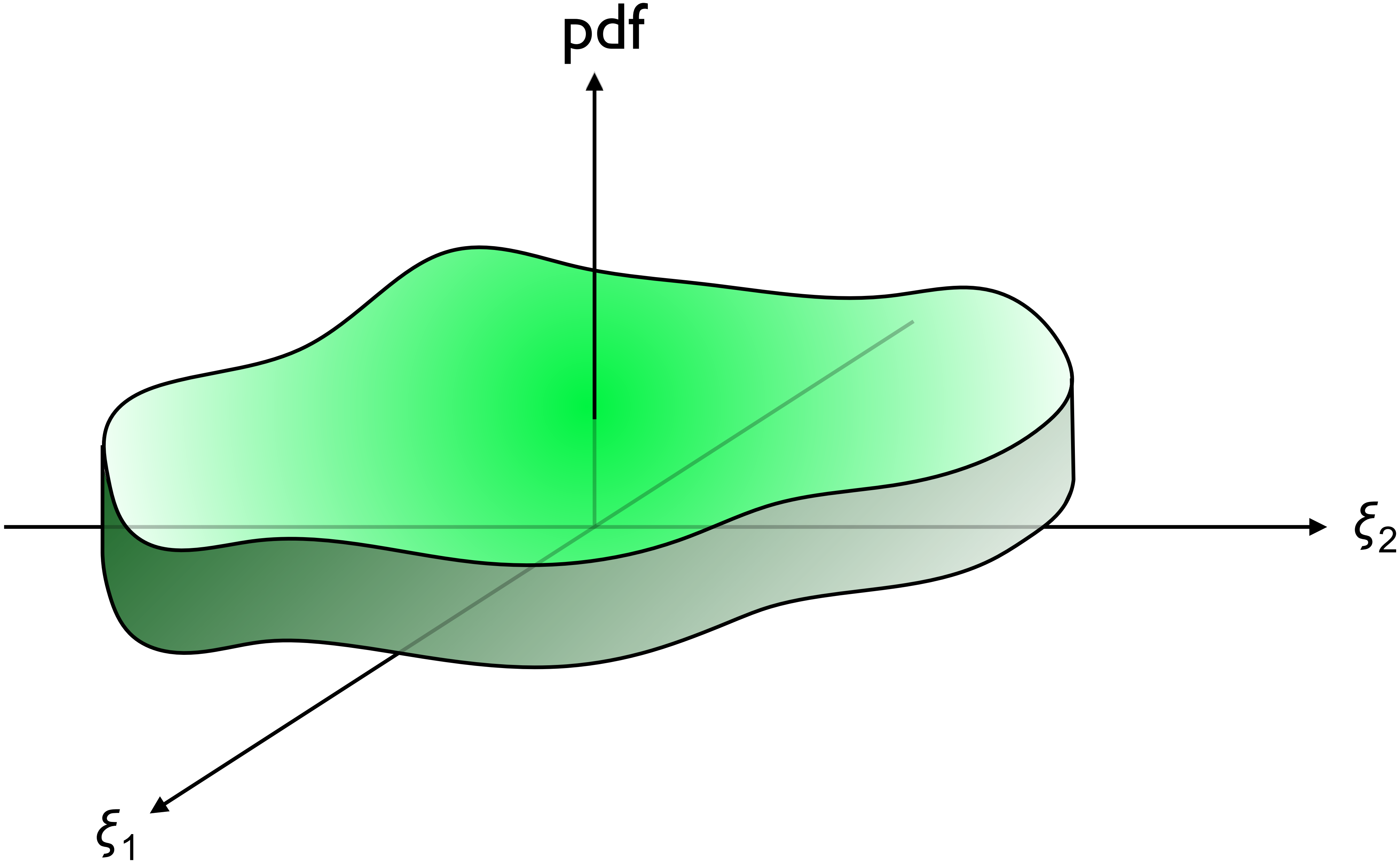
mixture distribution

Unimodal Bivariate Distributions

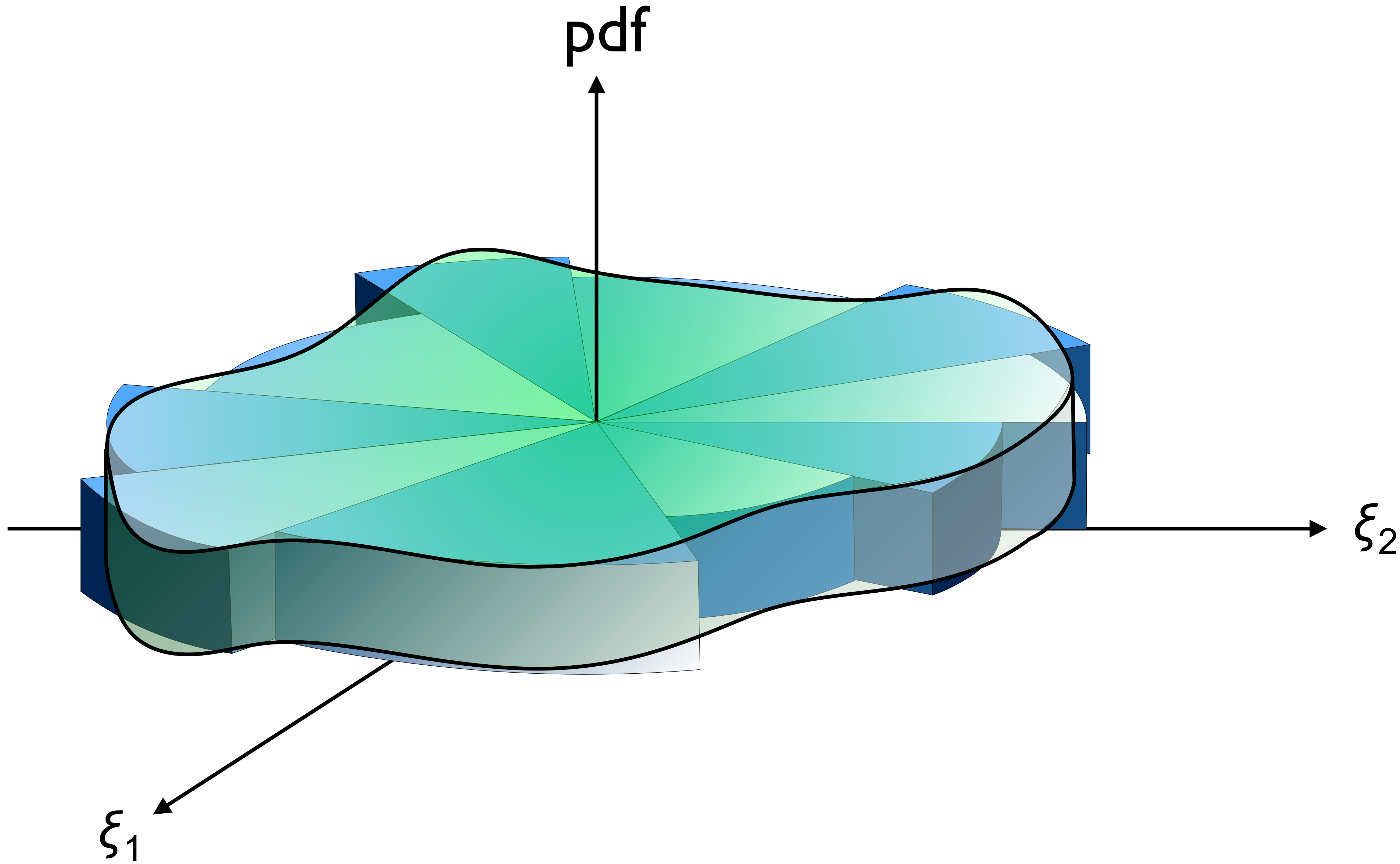


Multivariate unimodal distributions can be decomposed into uniform distributions on star-shaped sets.

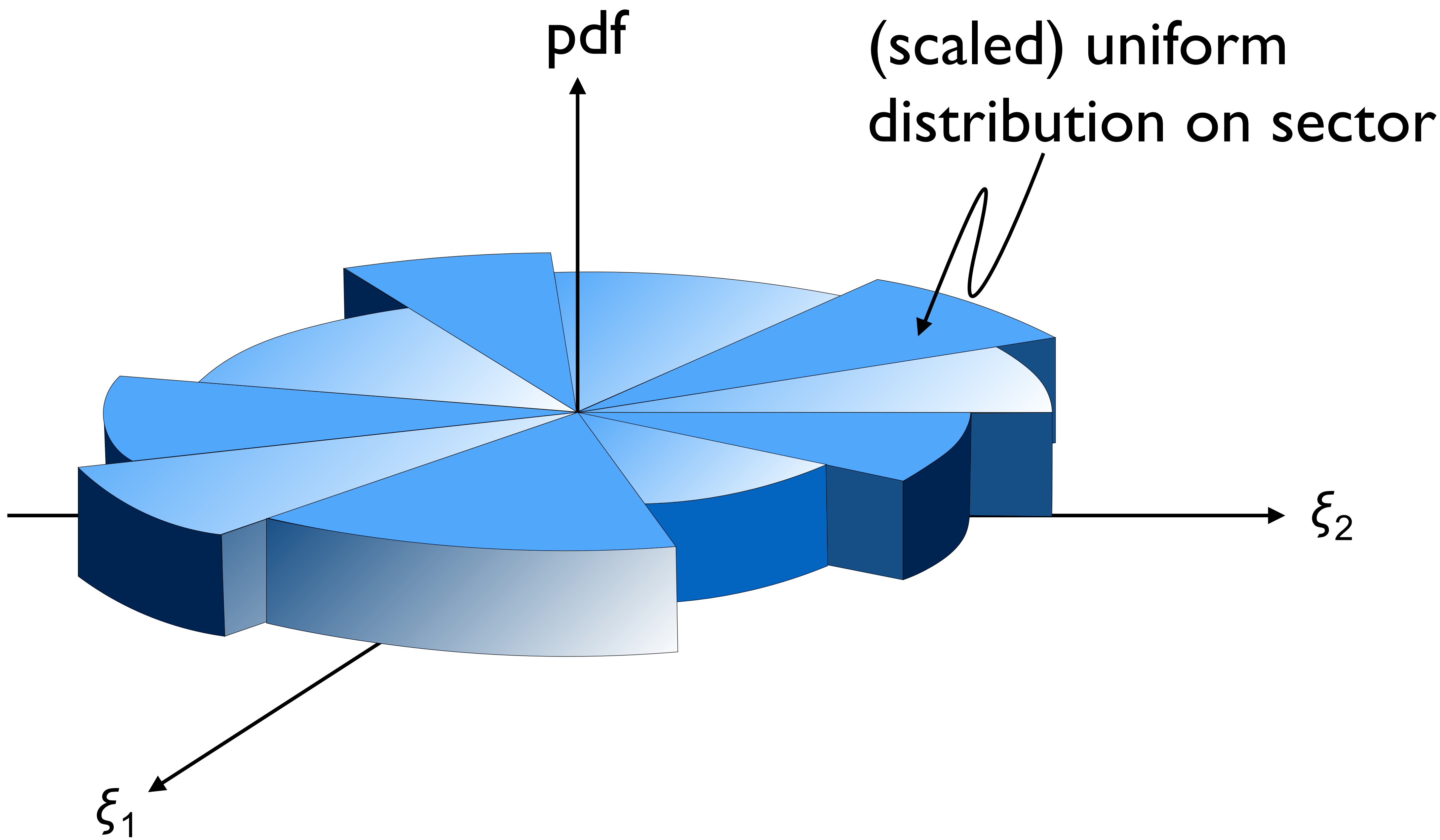
Uniform Distributions on Star-shaped Sets



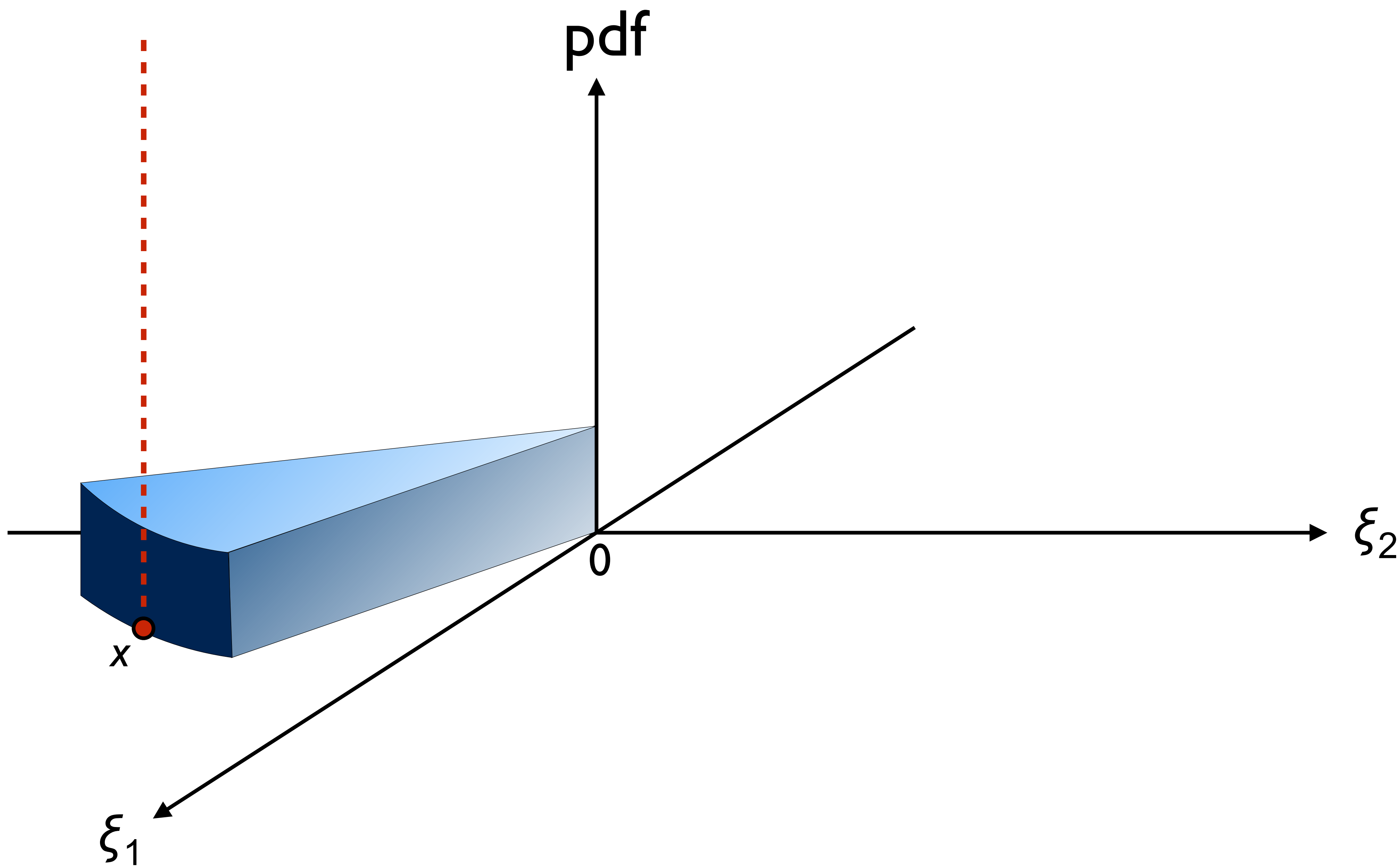
Uniform Distributions on Star-shaped Sets



Uniform Distributions on Star-shaped Sets



Radial Unimodal Distributions



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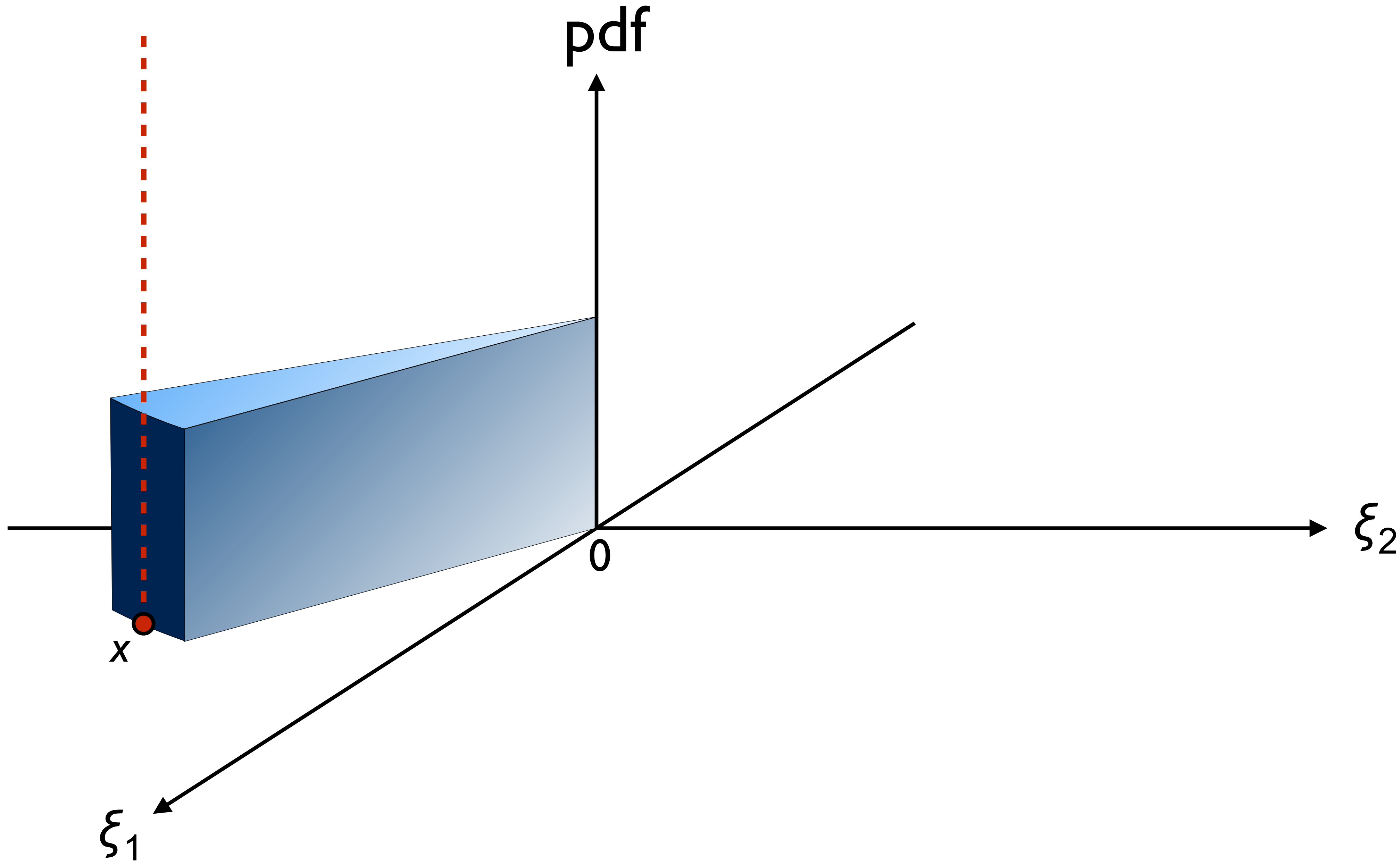

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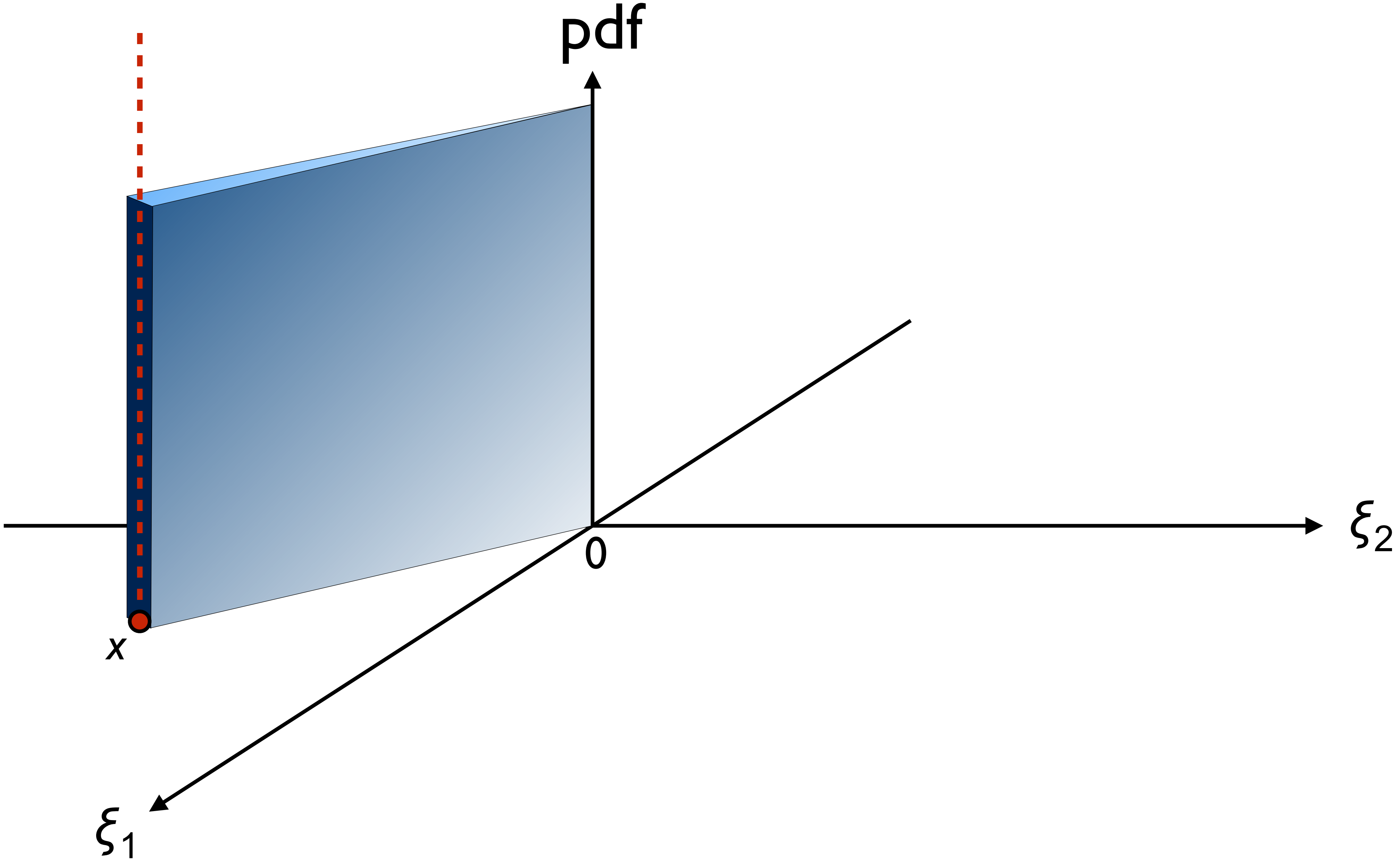

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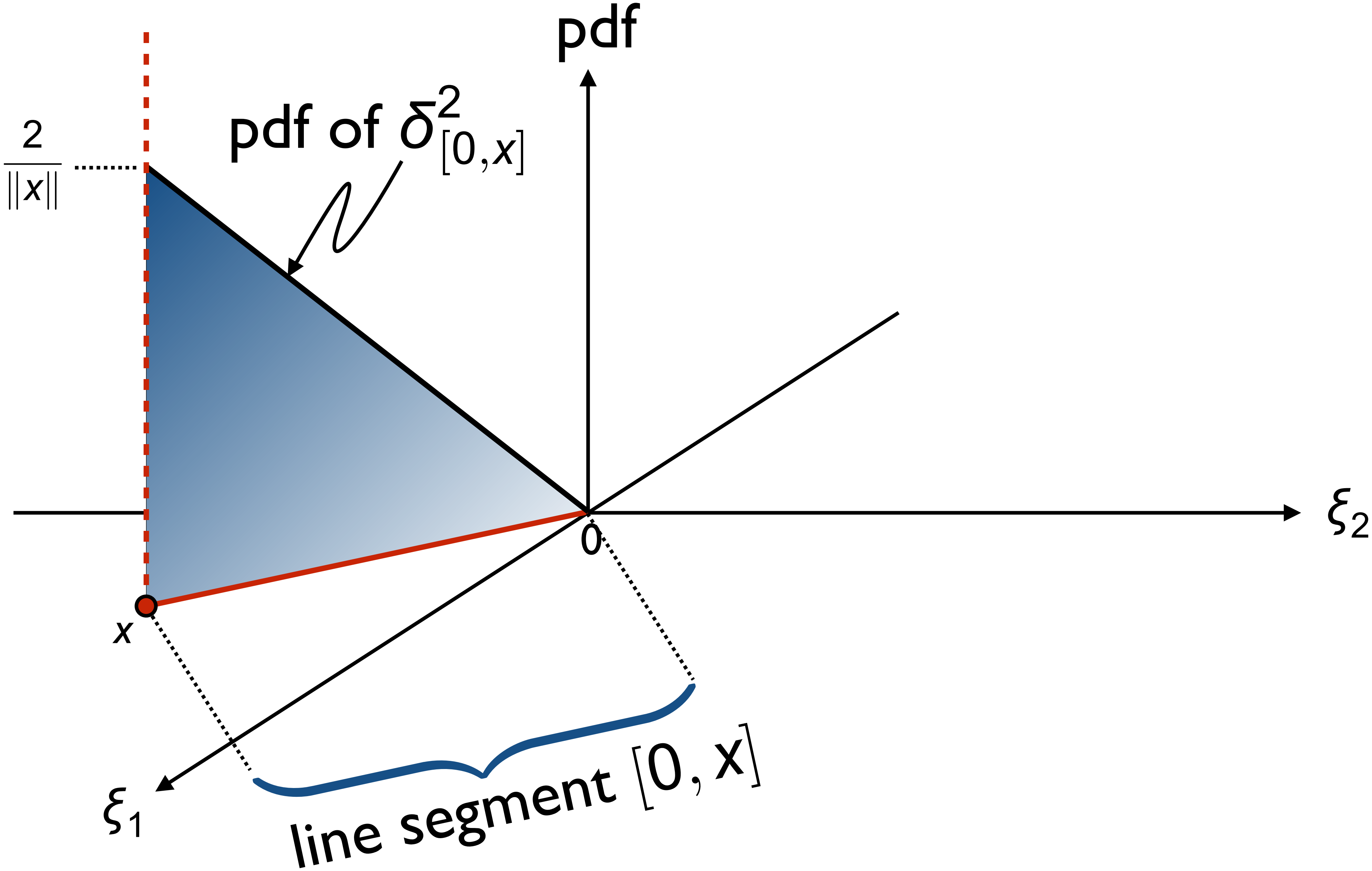

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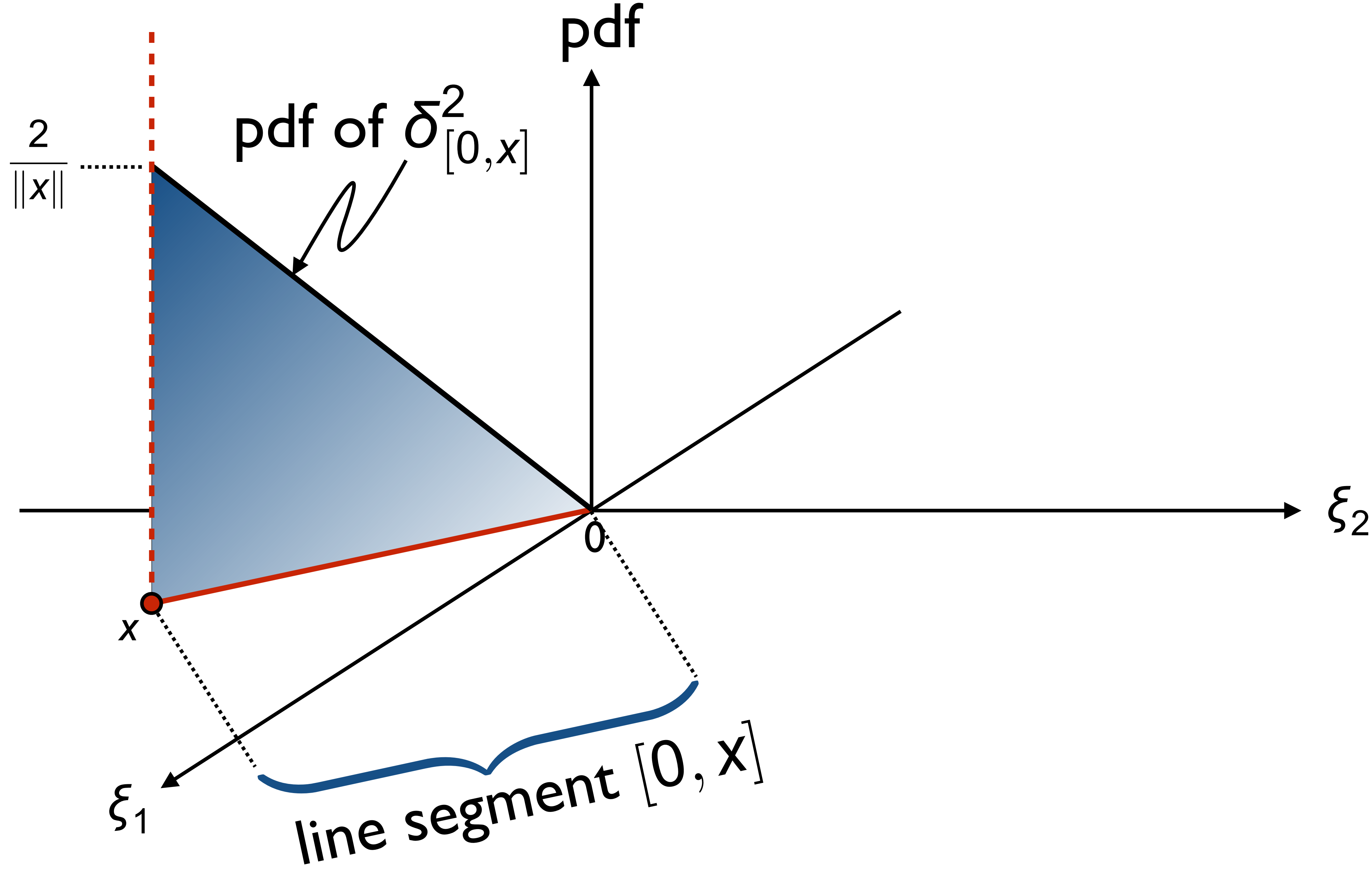
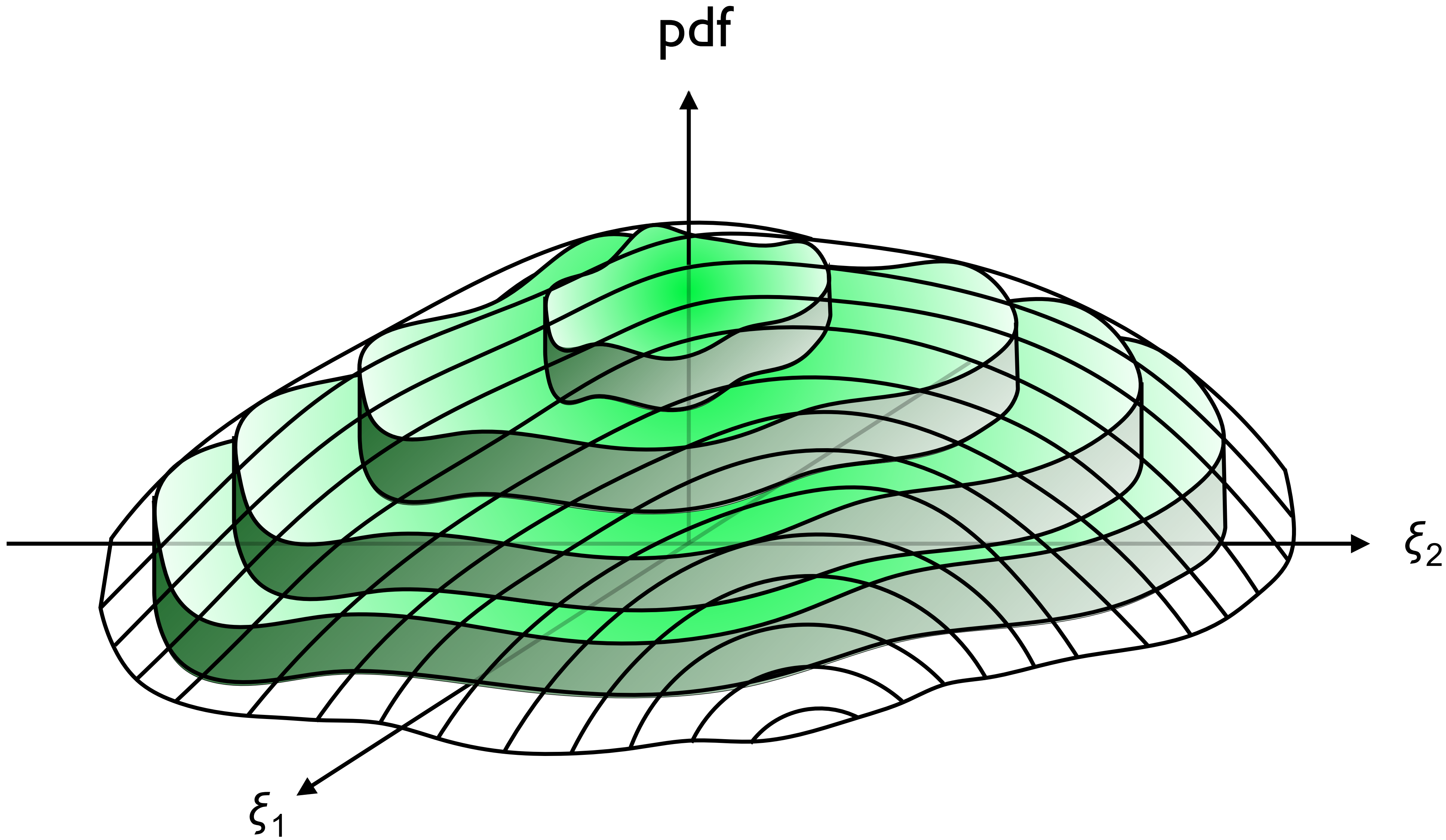


$\delta^2_{[0,x]}$ is the distribution on $[0, x]$ with $\delta^2_{[0,x]}([0, tx]) = t^2 \; \forall t \in [0, 1]$.

Unimodal Bivariate Distributions



$$\mathbb{P}(\cdot) = \int_{-\infty}^{+\infty} \delta^2_{[0,x]}(\cdot)\, m(\mathrm{d}x)$$

radial unimodal distribution on $[0, x]$

mixture distribution

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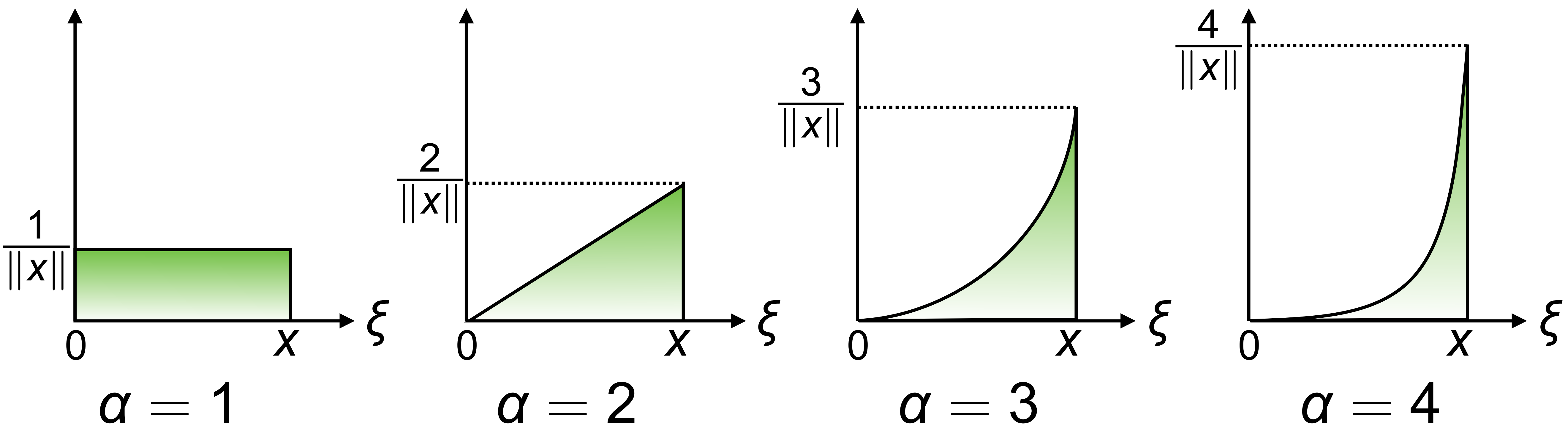
Radial α-Unimodal Distributions

$\delta^{\alpha}_{[0,x]}$ is the distribution on $[0,x]$ with $\delta^{\alpha}_{[0,x]}([0,tx]) = t^{\alpha}\ \forall t \in [0,1]$.



Dharmadhikari, Joag-Dev (1988):

$$\mathbb{P} \in \mathcal{P}_\alpha \iff \exists! m \in \mathcal{P}_\infty :\ \mathbb{P}(\cdot) = \int_{\mathbb{R}^n} \delta^{\alpha}_{[0,x]}(\cdot)\, m(\mathrm{d}x)$$

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Main Result

Theorem: If $0 \in \Xi$, then $\sup_{\mathbb{P} \in \mathcal{P}_\alpha(\mu, S)} \mathbb{P}(\xi \notin \Xi)$ is equivalent to:

$$
\begin{array}{rl}
\max & \sum_{i=1}^{k} \lambda_i - \tau_i \\
\text{s.t.} & a^\top z_i \geq 0, \;\; \tau_i \geq 0 \quad \forall i = 1, \ldots, k \\
& \tau_i (a_i^\top z_i)^\alpha \geq \lambda_i^{\alpha+1} b_i^\alpha \quad \forall i = 1, \ldots, k \\
& \sum_{i=1}^{k} \begin{pmatrix} Z_i & z_i \\ z_i^\top & \lambda_i \end{pmatrix} \preceq \begin{pmatrix} \frac{n+2}{n} S & \frac{n+1}{n} \mu \\ \frac{n+1}{n} \mu^\top & 1 \end{pmatrix} \\
& \begin{pmatrix} Z_i & z_i \\ z_i^\top & \lambda_i \end{pmatrix} \succeq 0 \quad \forall i = 1, \ldots, k
\end{array}
$$

$\alpha \to \infty$: Generalized Chebyshev Bound

$\alpha = n$: Generalized Gauss Bound

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Feasibility Conditions

$$\mathcal{P}_\alpha(\mu, S) \neq \emptyset \quad \Longleftrightarrow \quad \begin{pmatrix} \frac{\alpha+2}{\alpha} S & \frac{\alpha+1}{\alpha}\mu \\ \frac{\alpha+1}{\alpha}\mu^\top & 1 \end{pmatrix} \succeq 0$$

Proof: $\mathbb{P} \in \mathcal{P}_\alpha$ iff $\mathbb{P}(\cdot) = \int \delta^\alpha_{[0,x]}(\cdot)\, m(\mathrm{d}x)$ for $m \in \mathcal{P}_\infty$.

Feasibility Conditions

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Proof: $\mathbb{P} \in \mathcal{P}_\alpha$ iff $\mathbb{P}(\cdot) = \int \delta^\alpha_{[0,x]}(\cdot)\, m(\mathrm{d}x)$ for $m \in \mathcal{P}_\infty$.

$$\begin{pmatrix} S & \mu \\ \mu^\top & 1 \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{\alpha+2} S_m & \frac{\alpha}{\alpha+1}\mu_m \\ \frac{\alpha}{\alpha+1}\mu_m^\top & 1 \end{pmatrix}$$

Feasibility Conditions

$$\mathcal{P}_\alpha(\mu, S) \neq \emptyset \quad \Longleftrightarrow \quad \begin{pmatrix} \frac{\alpha+2}{\alpha} S & \frac{\alpha+1}{\alpha}\mu \\ \frac{\alpha+1}{\alpha}\mu^\top & 1 \end{pmatrix} \succeq 0$$

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$$\begin{pmatrix} \frac{\alpha+2}{\alpha} S & \frac{\alpha+1}{\alpha}\mu \\ \frac{\alpha+1}{\alpha}\mu^\top & 1 \end{pmatrix} = \begin{pmatrix} S_m & \mu_m \\ \mu_m^\top & 1 \end{pmatrix}$$

Covariance matrix $S_m - \mu_m \mu_m^\top$ must be psd. ■

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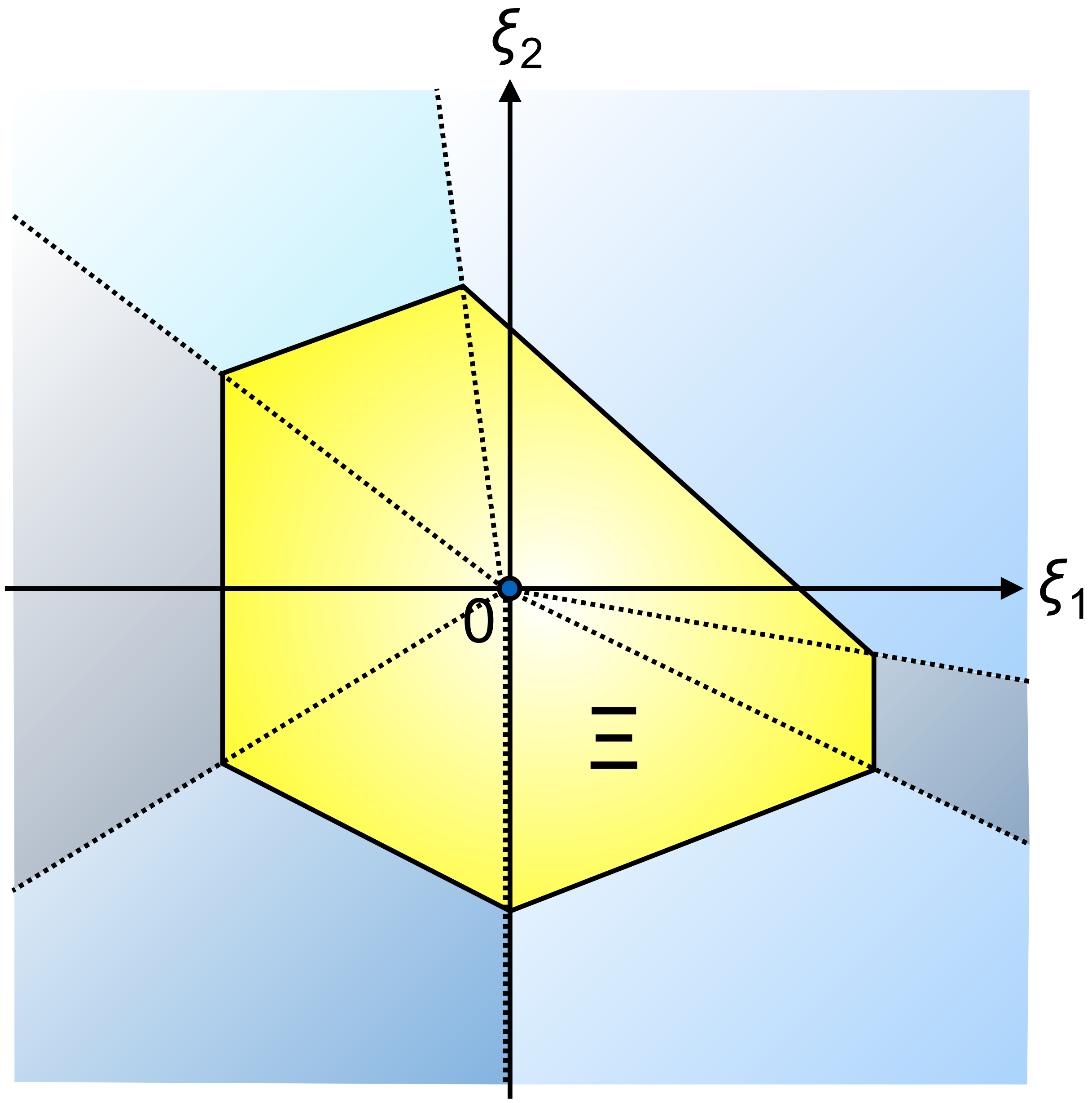
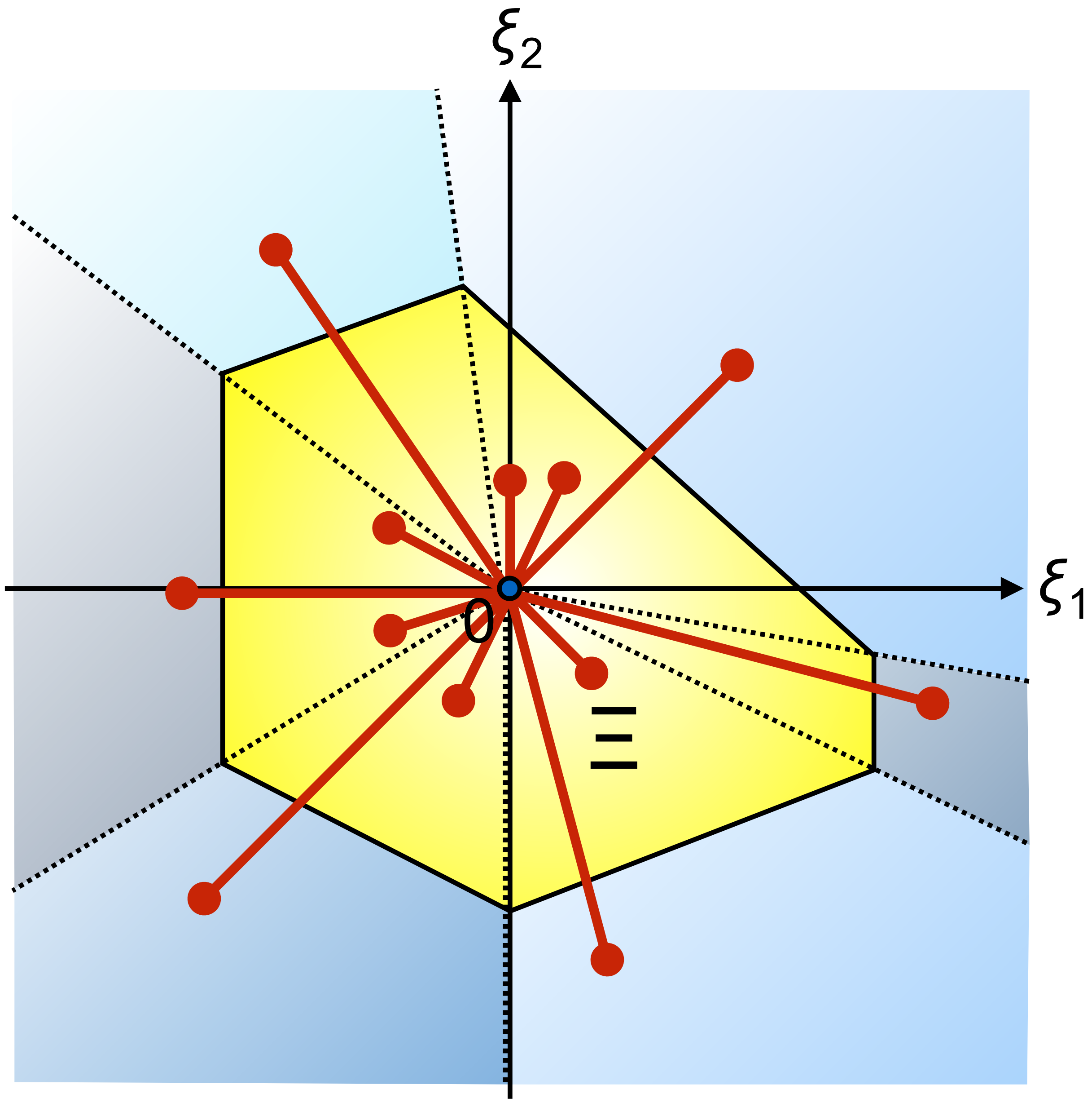
Structure of the Extremal Distributions

Partition $\mathbb{R}^n$ into k cones



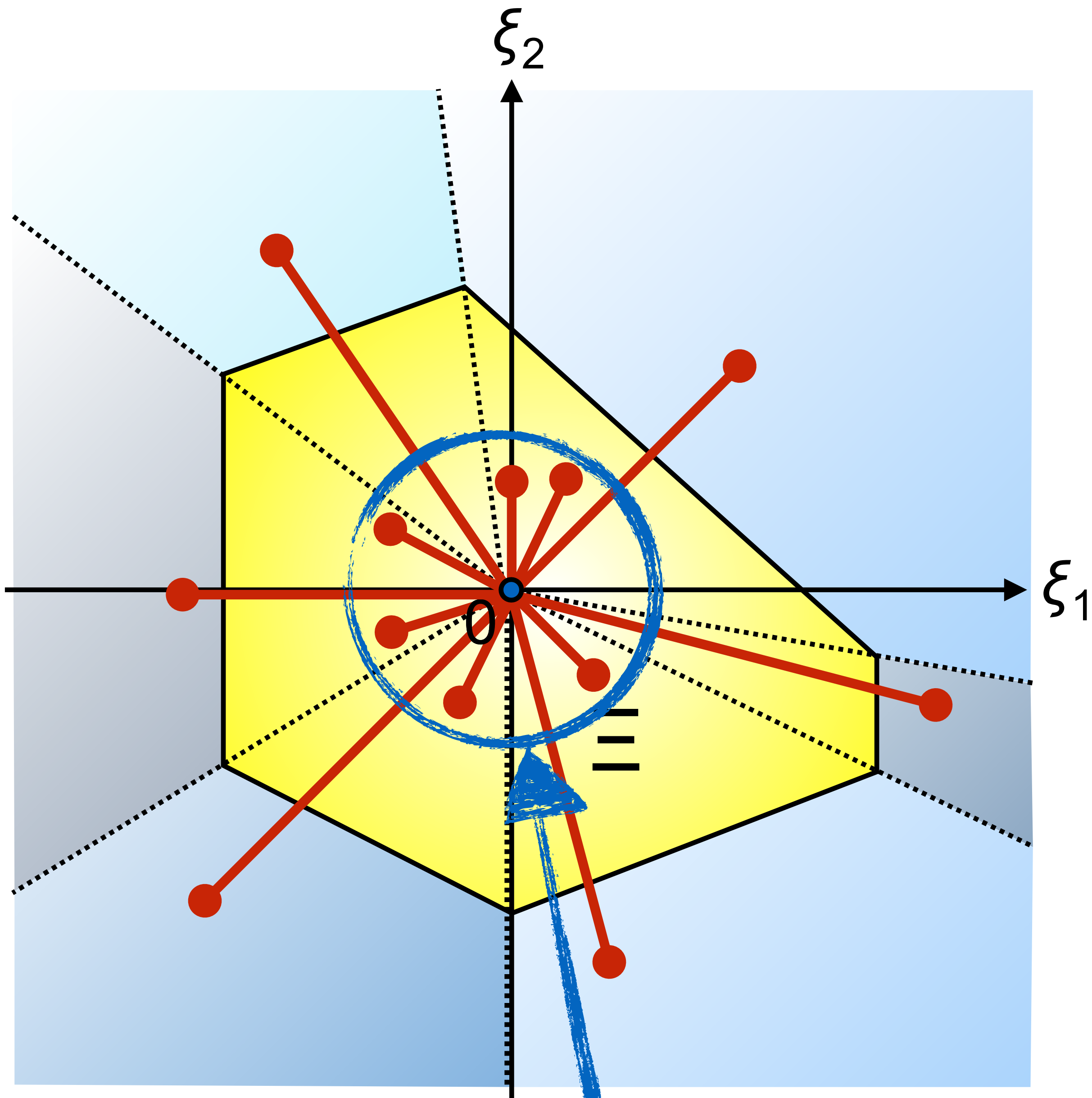
Structure of the Extremal Distributions



Extremal distribution: $$\mathbb{P}^* = \lambda_0 \cdot \mathbb{P}_0 + \sum_{i=1}^{k} \lambda_i \cdot \delta^{\alpha}_{[0,x_i]}$$

Structure of the Extremal Distributions

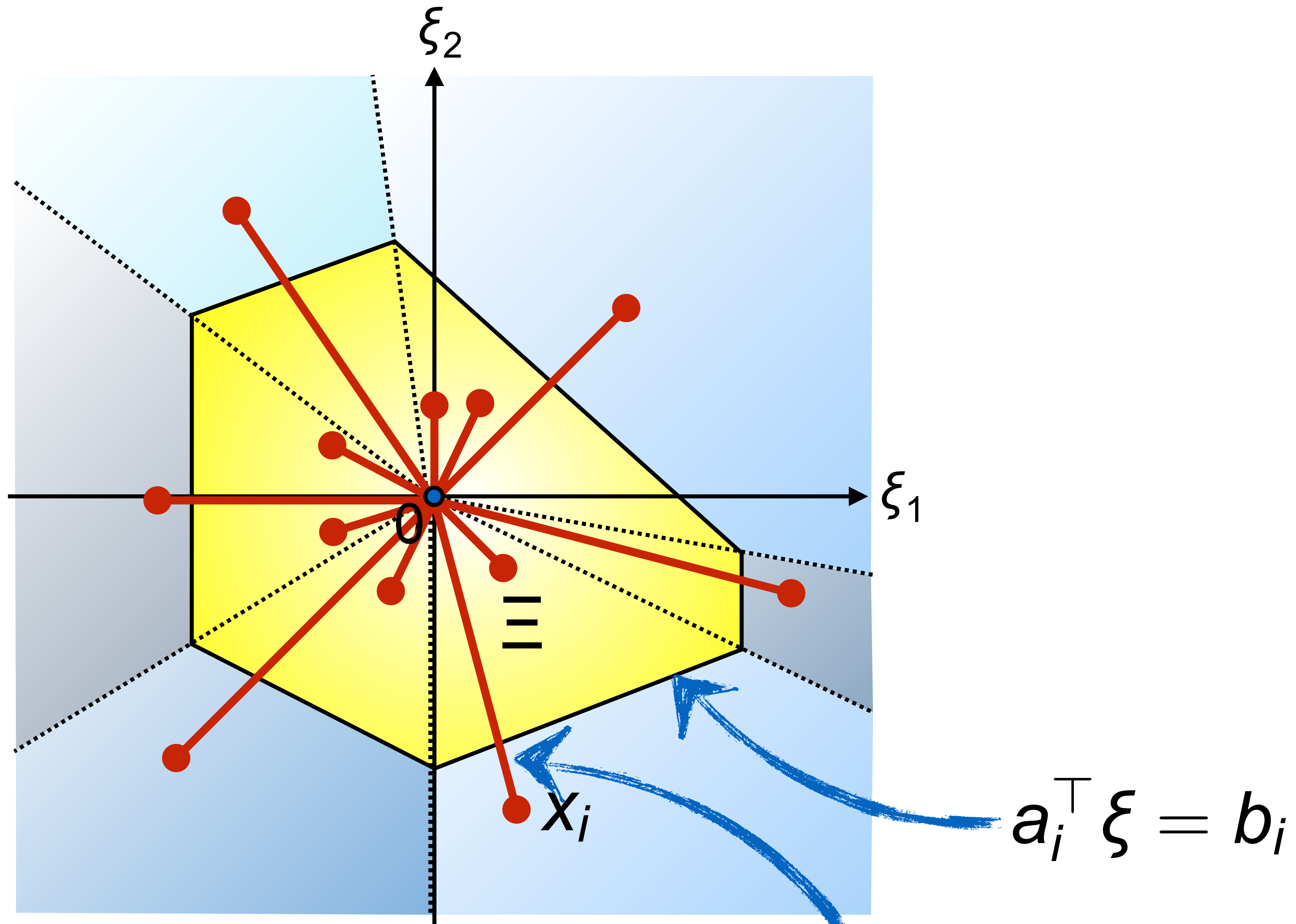
"Slack" distribution on Ξ



Extremal distribution: $\mathbb{P}^* = \lambda_0 \cdot \mathbb{P}_0 + \sum_{i=1}^{k} \lambda_i \cdot \delta^{\alpha}_{[0,x_i]}$

Structure of the Extremal Distributions

Radial α-unimodal distribution for sector of *i*th facet



Extremal distribution: $\mathbb{P}^* = \lambda_0 \cdot \mathbb{P}_0 + \sum_{i=1}^{k} \lambda_i \cdot \delta^{\alpha}_{[0,x_i]}$

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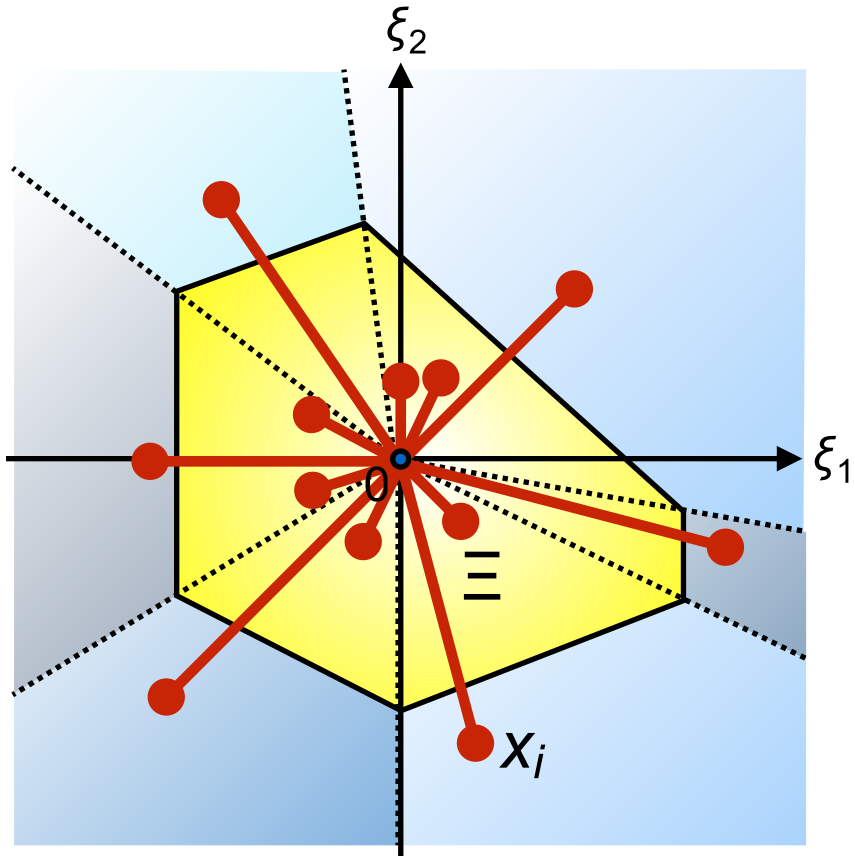
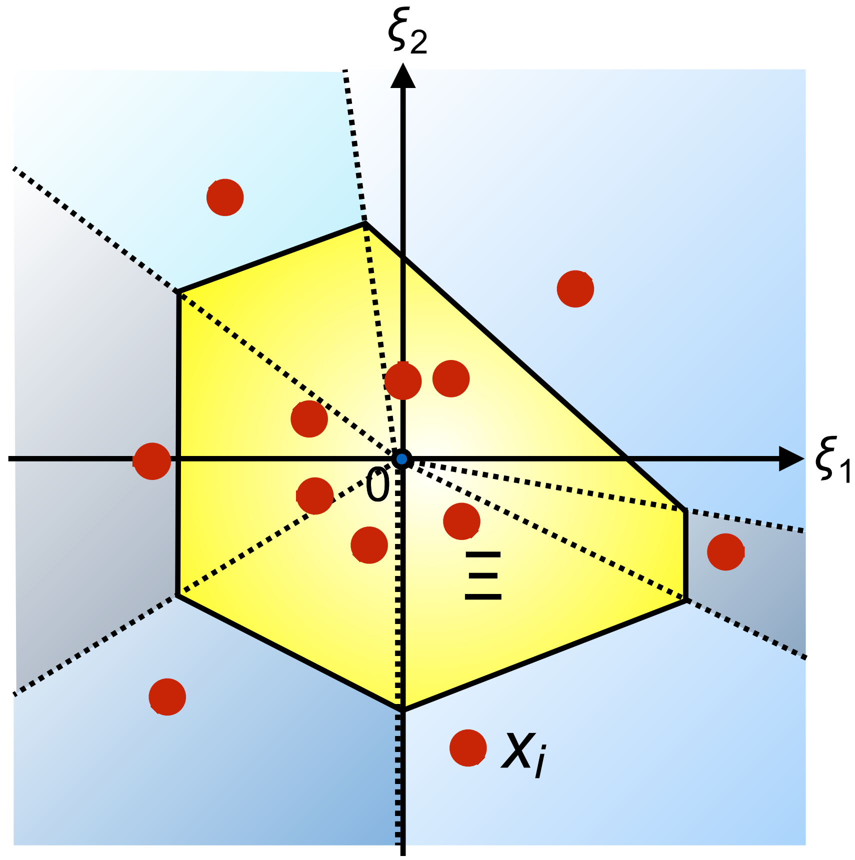
Moment Conditions

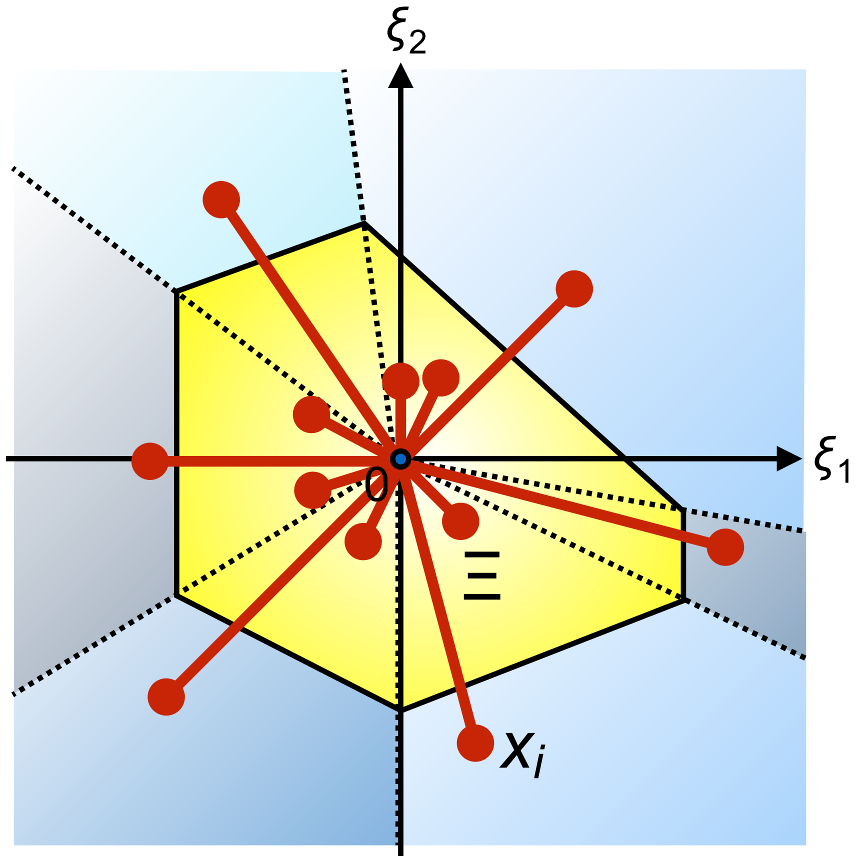


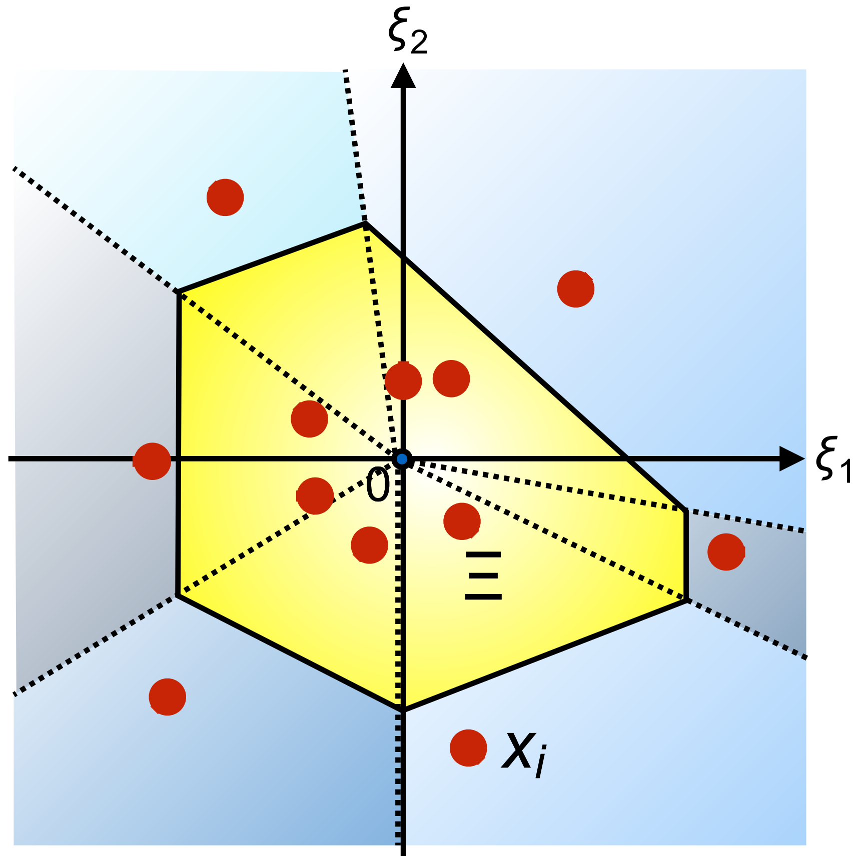


$$\mathbb{P}^* = \lambda_0 \cdot \mathbb{P}_0 + \sum_{i=1}^{k} \lambda_i \cdot \delta^{\alpha}_{[0,x_i]}$$

$$m^* = \lambda_0 \cdot m_0 + \sum_{i=1}^{k} \lambda_i \cdot \delta_{x_i}$$

Moment Conditions

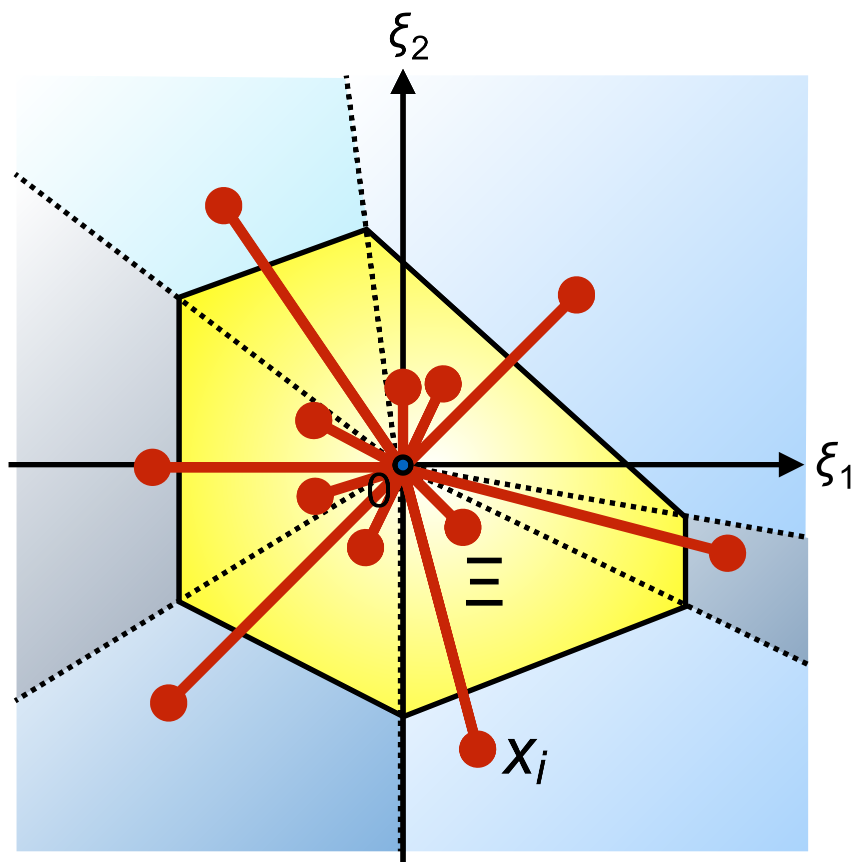


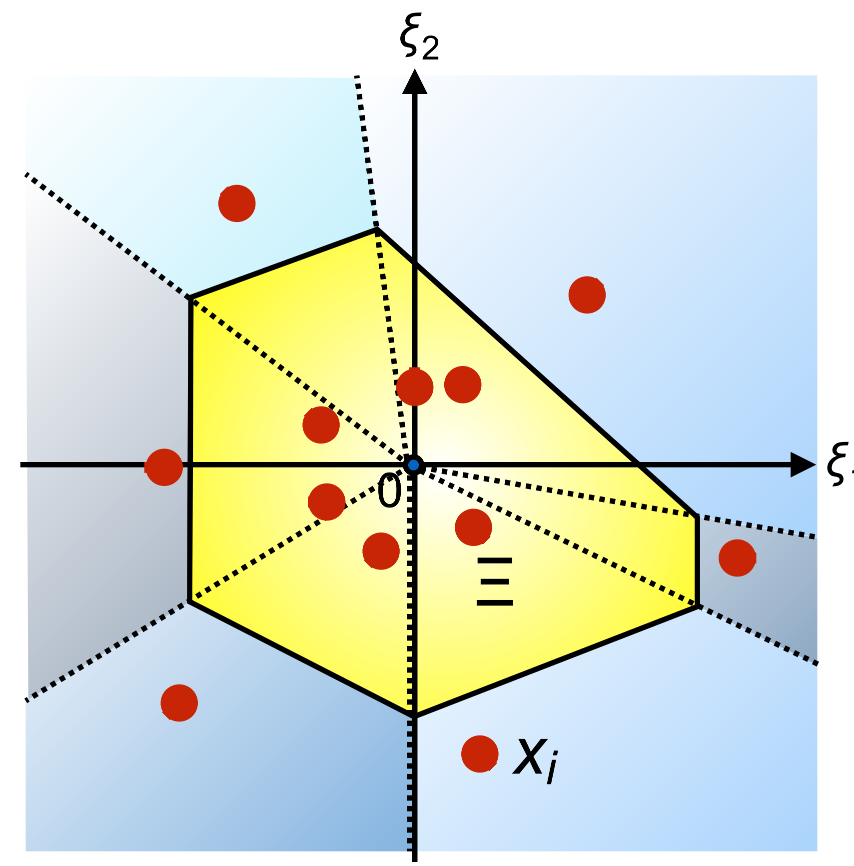


$$\mathbb{P}^* = \lambda_0 \cdot \mathbb{P}_0 + \sum_{i=1}^{k} \lambda_i \cdot \delta^{\alpha}_{[0,x_i]} \qquad m^* = \lambda_0 \cdot m_0 + \sum_{i=1}^{k} \lambda_i \cdot \delta_{x_i}$$

$$\mathbb{E}_{m^*}\left[\begin{pmatrix} xx^\top & x \\ x^\top & 1 \end{pmatrix}\right] = \begin{pmatrix} \frac{\alpha+2}{\alpha} S & \frac{\alpha+1}{\alpha}\mu \\ \frac{\alpha+1}{\alpha}\mu^\top & 1 \end{pmatrix}$$

Moment Conditions

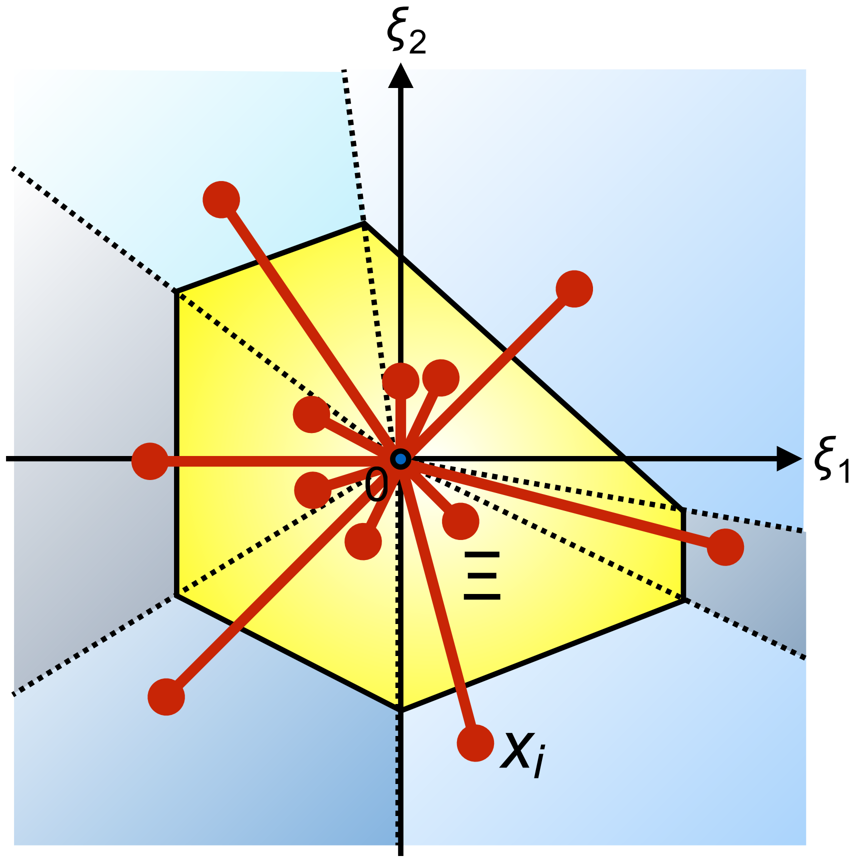


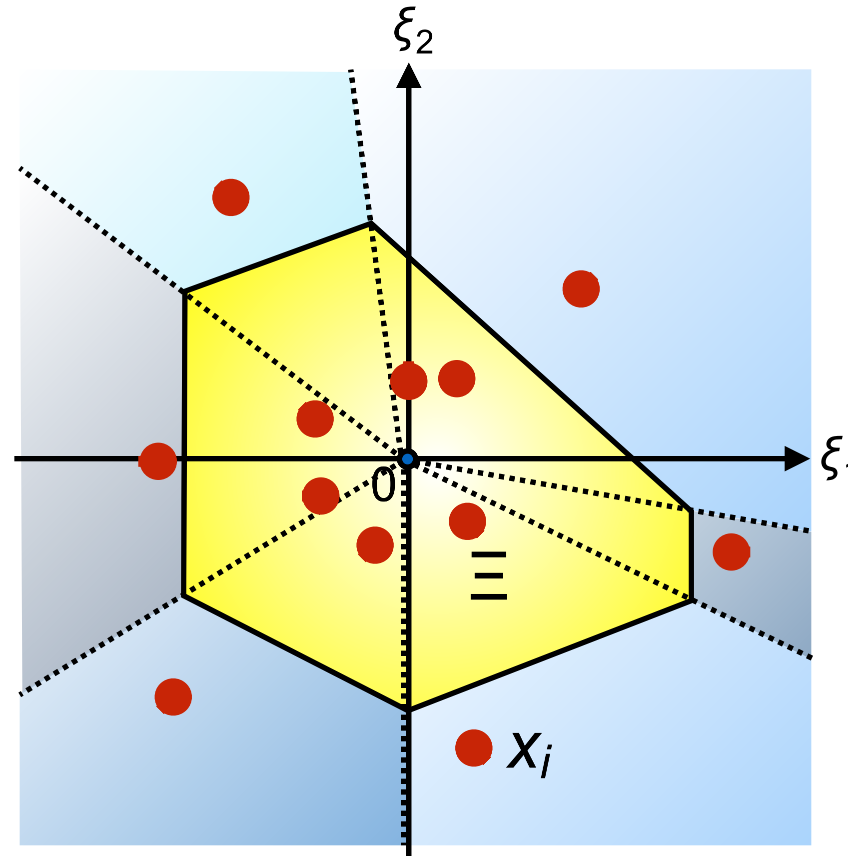


$$\mathbb{P}^* = \lambda_0 \cdot \mathbb{P}_0 + \sum_{i=1}^{k} \lambda_i \cdot \delta^{\alpha}_{[0,x_i]} \qquad m^* = \lambda_0 \cdot m_0 + \sum_{i=1}^{k} \lambda_i \cdot \delta_{x_i}$$

$$\lambda_0 \, \mathbb{E}_{m_0}\left[\begin{pmatrix} xx^\top & x \\ x^\top & 1 \end{pmatrix}\right] + \sum_{i=1}^{k} \lambda_i \begin{pmatrix} x_i x_i^\top & x_i \\ x_i^\top & 1 \end{pmatrix} = \begin{pmatrix} \frac{\alpha+2}{\alpha} S & \frac{\alpha+1}{\alpha}\mu \\ \frac{\alpha+1}{\alpha}\mu^\top & 1 \end{pmatrix}$$

Moment Conditions

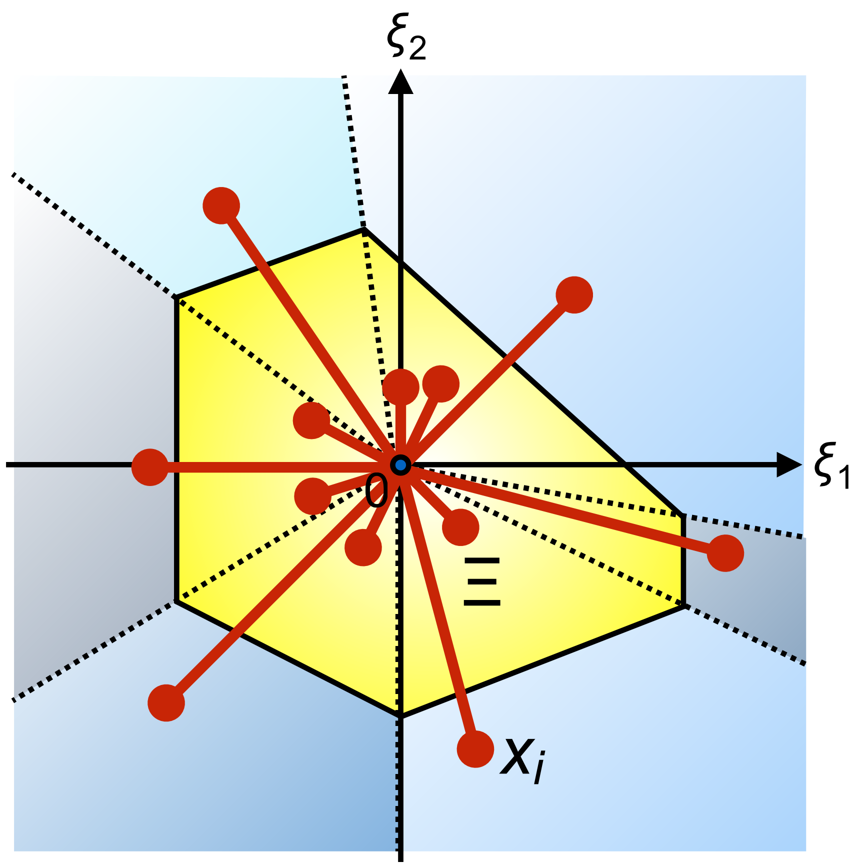


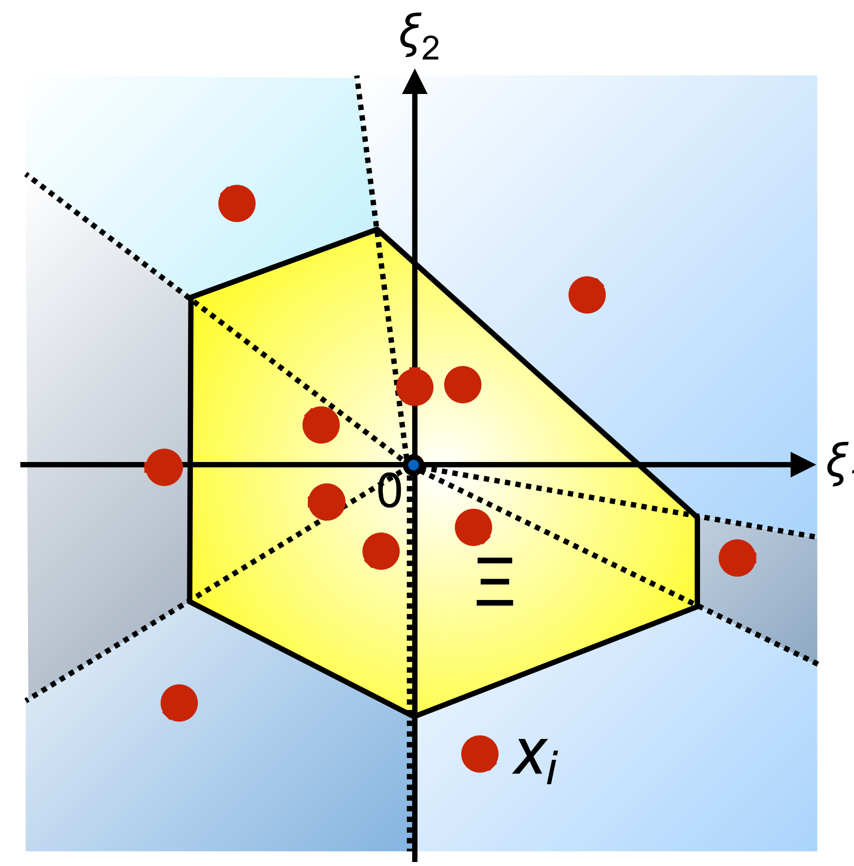


$$\mathbb{P}^* = \lambda_0 \cdot \mathbb{P}_0 + \sum_{i=1}^{k} \lambda_i \cdot \delta^{\alpha}_{[0,x_i]} \qquad m^* = \lambda_0 \cdot m_0 + \sum_{i=1}^{k} \lambda_i \cdot \delta_{x_i}$$

$$\sum_{i=1}^{k} \lambda_i \begin{pmatrix} x_i x_i^\top & x_i \\ x_i^\top & 1 \end{pmatrix} \preceq \begin{pmatrix} \frac{\alpha+2}{\alpha} S & \frac{\alpha+1}{\alpha}\mu \\ \frac{\alpha+1}{\alpha}\mu^\top & 1 \end{pmatrix}$$

Moment Conditions



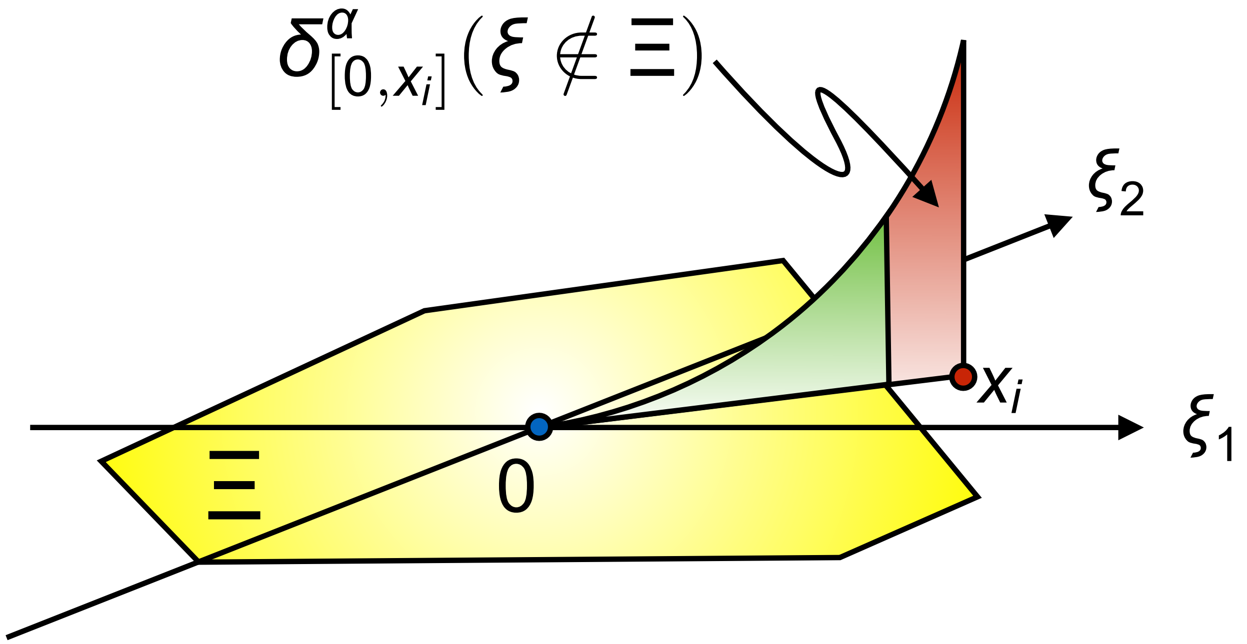


$$\mathbb{P}^* = \lambda_0 \cdot \mathbb{P}_0 + \sum_{i=1}^{k} \lambda_i \cdot \delta^{\alpha}_{[0,x_i]} \qquad m^* = \lambda_0 \cdot m_0 + \sum_{i=1}^{k} \lambda_i \cdot \delta_{x_i}$$

$$\sum_{i=1}^{k} \begin{pmatrix} Z_i & z_i \\ z_i^\top & \lambda_i \end{pmatrix} \preceq \begin{pmatrix} \frac{\alpha+2}{\alpha} S & \frac{\alpha+1}{\alpha}\mu \\ \frac{\alpha+1}{\alpha}\mu^\top & 1 \end{pmatrix}, \quad \begin{pmatrix} Z_i & z_i \\ z_i^\top & \lambda_i \end{pmatrix} \succeq 0 \quad \forall i$$

Objective Function

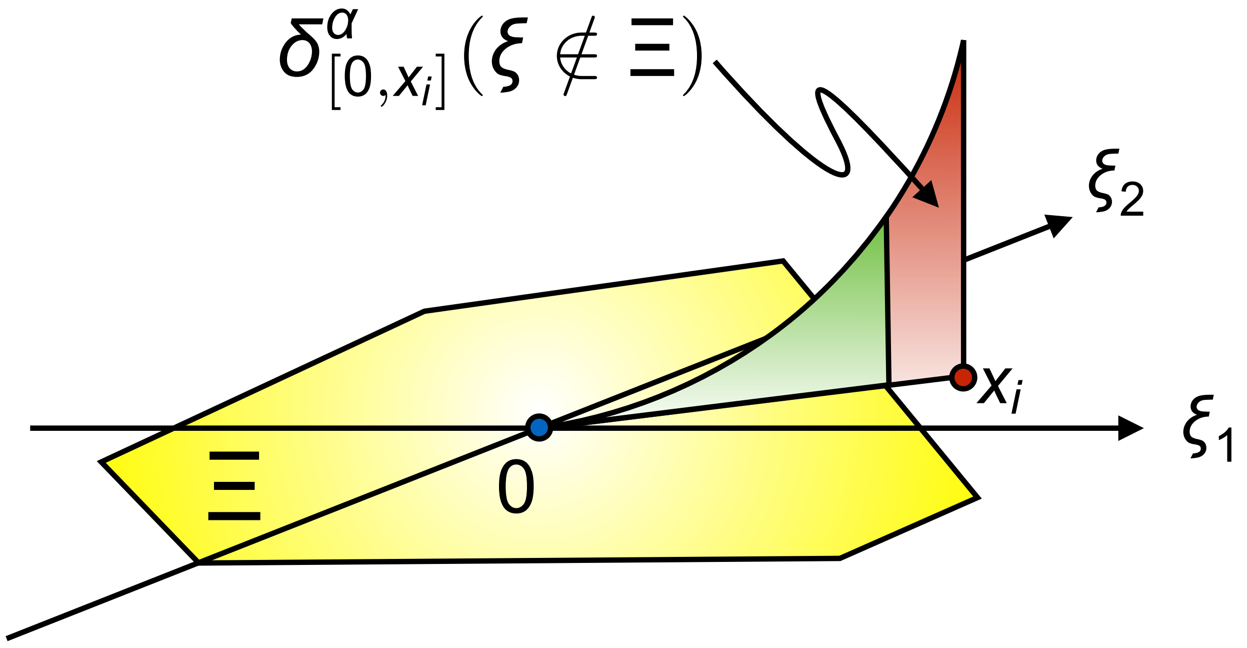
$$\mathbb{P}^* = \lambda_0 \cdot \mathbb{P}_0 + \sum_{i=1}^{k} \lambda_i \cdot \delta^{\alpha}_{[0,x_i]}$$



$$\mathbb{P}^*(\xi \notin \Xi) = \sum_{i=1}^{k} \lambda_i \cdot \delta^{\alpha}_{[0,x_i]}(\xi \notin \Xi)$$

Objective Function

$$\mathbb{P}^* = \lambda_0 \cdot \mathbb{P}_0 + \sum_{i=1}^{k} \lambda_i \cdot \delta^{\alpha}_{[0,x_i]}$$



$$\mathbb{P}^*(\xi \notin \Xi) \;=\; \max_{\tau} \quad \textstyle\sum_{i=1}^{k} \lambda_i - \tau_i$$

$$\text{s.t.} \quad \tau_i (a_i^\top z_i)^\alpha \geq \lambda_i^{\alpha+1} b_i^\alpha, \;\; \tau_i \geq 0$$

In Summary…

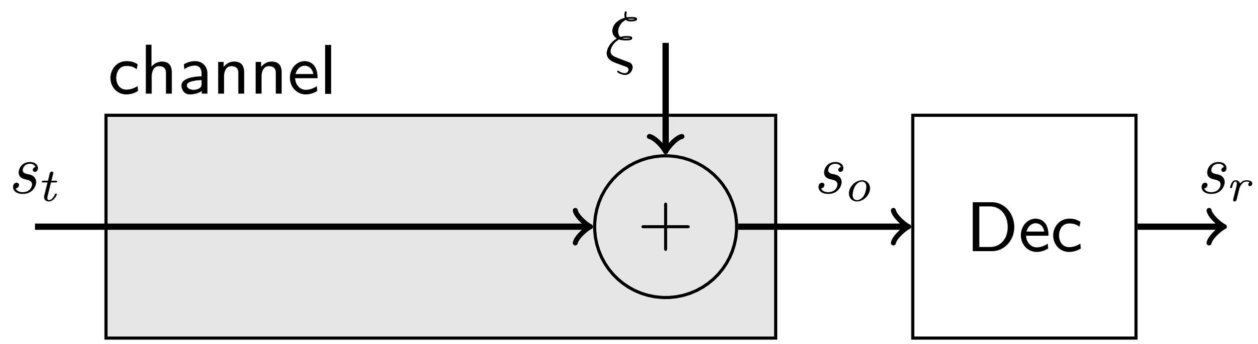
$$
\begin{aligned}
\max \quad & \textstyle\sum_{i=1}^{k} \lambda_i - \tau_i \\
\text{s.t.} \quad & a^\top z_i \geq 0, \;\; \tau_i \geq 0 \quad \forall i = 1, \ldots, k \\
& \tau_i (a_i^\top z_i)^\alpha \geq \lambda_i^{\alpha+1} b_i^\alpha \quad \forall i = 1, \ldots, k \\
& \textstyle\sum_{i=1}^{k} \begin{pmatrix} Z_i & z_i \\ z_i^\top & \lambda_i \end{pmatrix} \preceq \begin{pmatrix} \frac{n+2}{n} S & \frac{n+1}{n} \mu \\ \frac{n+1}{n} \mu^\top & 1 \end{pmatrix} \\
& \begin{pmatrix} Z_i & z_i \\ z_i^\top & \lambda_i \end{pmatrix} \succeq 0 \quad \forall i = 1, \ldots, k
\end{aligned}
$$

evaluation of $\mathbb{P}^*(\xi \notin \Xi)$

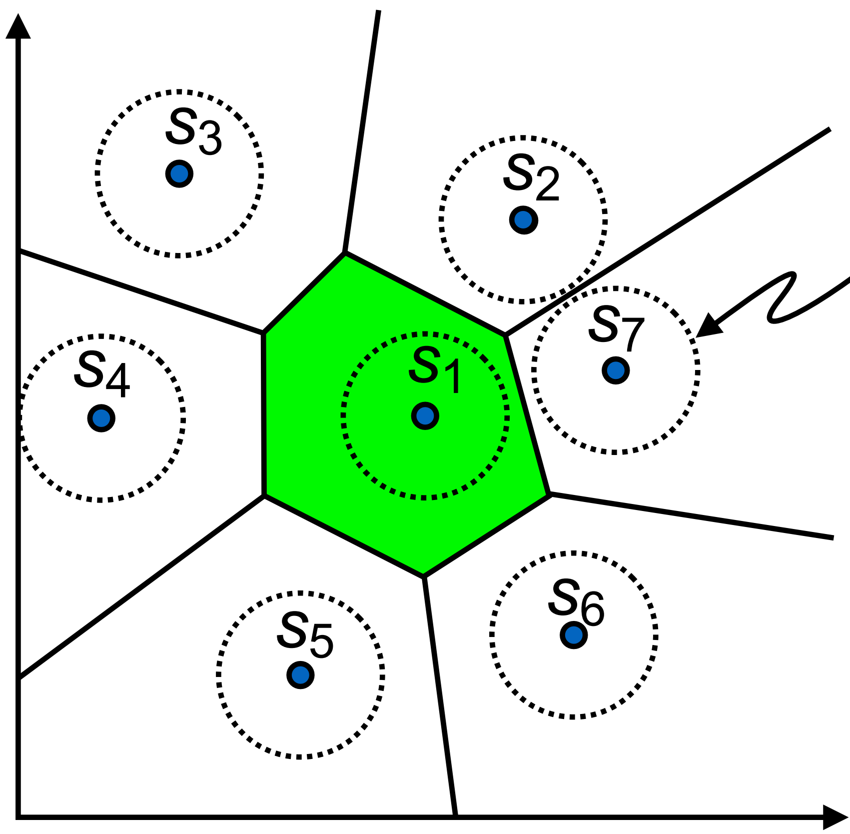
moment conditions for mixture distribution

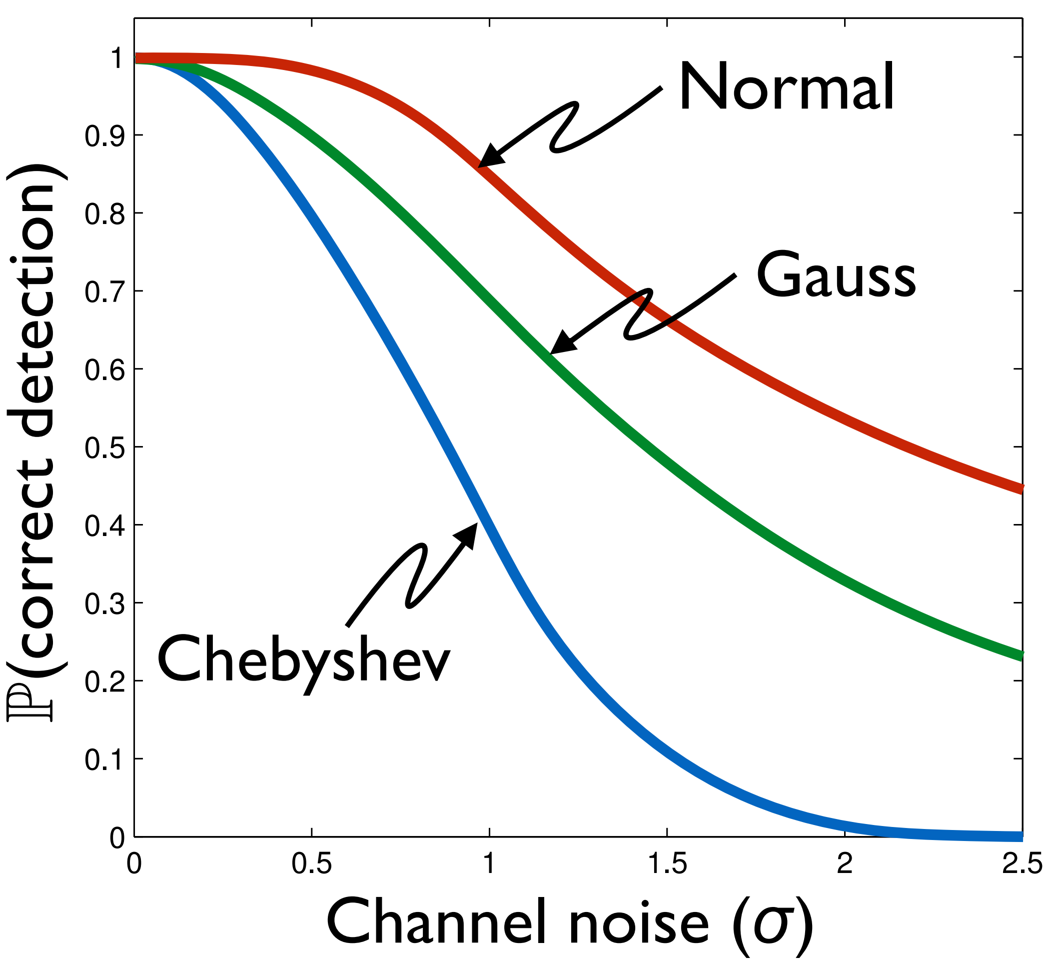
Application: Digital Communication Limits

Transmit symbols $s_1, s_2, \ldots, s_7$ over a noisy communication channel.



Minimum distance decoder





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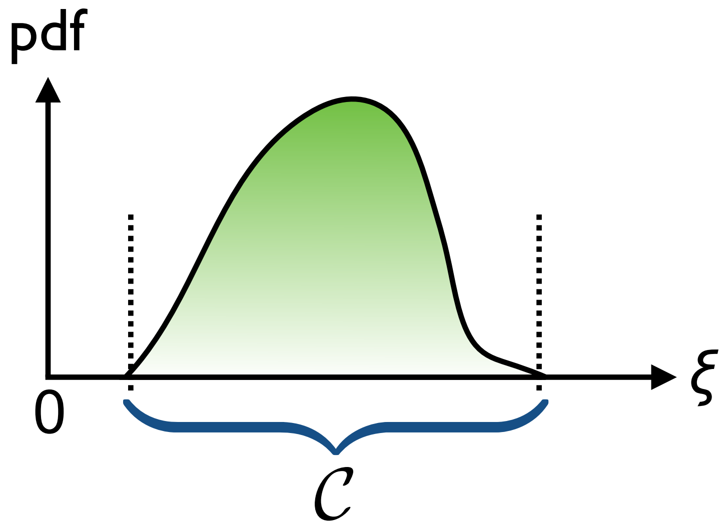
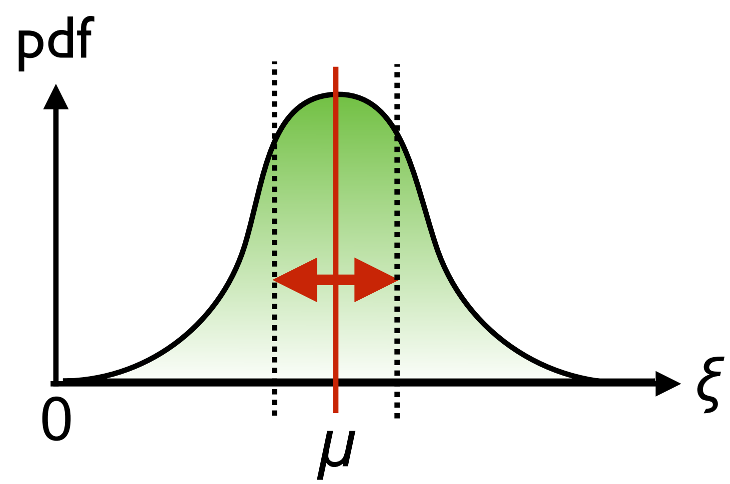
Tractable Extensions

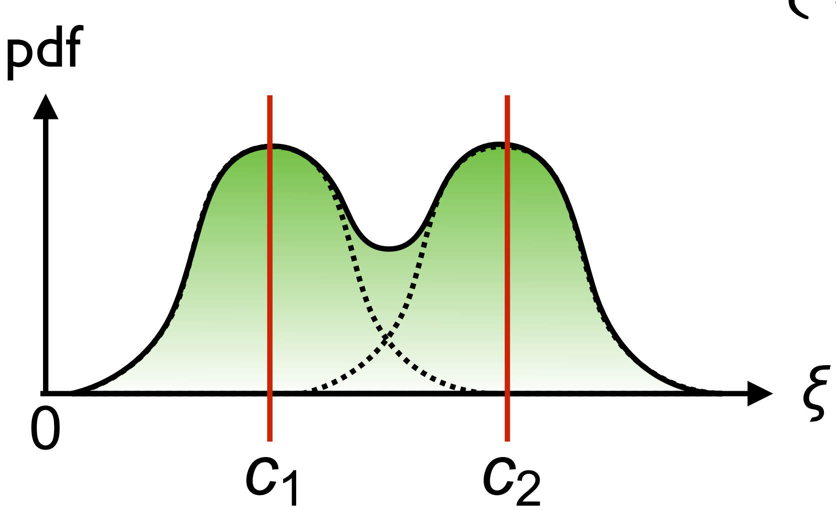
Support information: $\mathcal{P} = \left\{ \mathbb{P} \in \mathcal{P}_\alpha(\mu, S) : \mathbb{P}(\mathcal{C}) = 1 \right\}$



Moment ambiguity: $\mathcal{P} = \bigcup_{(\mu,S)\in\mathcal{M}} \mathcal{P}_\alpha(\mu, S)$



Multimodality: $\mathcal{P} = \left\{ \sum_m p_m \mathbb{P}_m : p \in \mathcal{U},\ \mathbb{P}_m \in \mathcal{P}_\alpha(c_m)\ \forall m \right\} \bigcap \mathcal{P}(\mu, S)$



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Bibliography

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