Non-parametric liquidity-adjusted VaR model: a stochastic programming approach

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Abstract

This paper proposes a Stochastic Programming (SP) approach for the calculation of the liquidity-adjusted Value-at-Risk (LVaR). The model presented in this paper offers an alternative to Almgren and Chriss's mean-variance approach (1999 and 2000). In this research, a two-stage stochastic programming model is developed with the intention of deriving the optimal trading strategies that respond dynamically to a given market situation. The sample paths approach is adopted for scenario generation. The scenarios are thus represented by a collection of simulated sample paths rather than the tree structure usually employed in stochastic programming. Consequently, the SP LVaR presented in this paper can be considered as a non-parametric approach, which is in contrast to Almgren and Chriss's parametric solution. Initially, a set of numerical experiments indicates that the LVaR figures are quite similar for both approaches when all the underlying financial assumptions are identical. Following this sanity check, a second set of numerical experiments shows how the randomness of the different types (i.e., bid and ask spread) can be easily incorporated into the problem due to the stochastic programming formulation and how optimal and adaptive trading strategies can be derived through a two-stage structure (i.e., a recourse problem). Hence, the results presented in this paper allow for the introduction of new dimensionalities into the computation of LVaR by incorporating different market conditions.

Developed over the last couple of decades, Value-at-Risk (VaR) models have been widely used as the main market risk management tool in the financial world [Jorion (2006)]. VaR estimates the likelihood of a portfolio loss caused by normal market movements over a given period of time. However, VaR fails to take into consideration the market liquidity impact. Its estimate is quite often based on mid-prices and the assumption that transactions do not affect market prices. Nevertheless, large trading blocks might impact prices, and trading activity is always costly. To overcome these problems, some researchers have proposed the calculation of liquidity adjusted VaR (LVaR) [Dowd (1998)]. Differing from the conventional VaR, LVaR takes both the size of the initial holding position and liquidity impact into account. The liquidity impact is commonly subcategorized into exogenous and endogenous illiquidity factors. The former is normally measured by the bid-ask spread, and the latter is expressed as the price movement caused by market transactions [Bangia et al. (1999)]. From this perspective, LVaR can be seen as a complementary tool for risk managers who need to estimate market risk exposure and are unwilling to disregard the liquidity impact.

Bangia et al. (1999) proposed a simple but practical solution that is directly derived from the conventional VaR model in which an illiquidity factor is expressed as the bid-ask spread. Although this approach avoids many complicated calculations, it fails to take into consideration endogenous illiquidity factors. Hence, liquidity risk and LVaR are underestimated. A more promising solution for LVaR estimation stems from the derivation of optimal trading strategies as suggested by Almgren and Chriss (1999 and 2000). In their model, Almgren and Chriss adopted the permanent and temporary market impact mechanisms from Holthausen et al.'s work (1987) and assumed linear functions for both of them. By externally setting a sales completion period, they derived an optimal trading strategy defined as the strategy with the minimum variance of transaction cost, or of shortfall, for a given level of expected transaction cost. Or inversely, a strategy that has the lowest level of expected transaction cost for a given level of variance. With the normal distribution and the mean and variance of transaction cost, LVaR can also be determined and minimized to derive optimal trading strategies. In this setting, LVaR can be understood as the p^{th} percentile possible loss that a trading position can encounter when liquidity effects are incorporated into the risk measure computation. Later on, Almgren (2003) extended this model by using a continuous-time approximation, and also introduced a non-linear and stochastic temporary market impact function. Another alternative is the liquidity discount approach presented by Jarrow and Subramanian (1997 and 2001). Similar to Almgren and Chriss's approach (1999 and 2000), the liquidity discount approach requires that the sales completion period be given as an exogenous factor. The optimal trading strategy is then derived by maximizing an investor's expected utility of consumption. Note that both approaches require externally setting a fixed horizon for liquidation. Aiming to overcome this problem, Hisata and Yamai (2000) extended Almgren and Chriss's approach by assuming a constant speed of sales and by using continuous approximation. They could derive a closed-form analytical solution for the optimal holding period. In this setting, the sales completion time thus becomes an endogenous variable. Yet, Hisata and Yamai's model relies on the strong assumption of a constant speed of sales.

Krokhmal and Uryasev (2006) argued that the solution offered by Almgren and Chriss and that of Jarrow and Subramanian were unable to dynamically respond to changes in market conditions. Therefore, they suggested a stochastic dynamic programming method and derived an optimal trading strategy by maximizing the expected stream of cash flows. Under their framework, the optimal trading strategy becomes highly dynamic as it can respond to market conditions at each time step. Another methodology that incorporates these dynamics into an optimal trading strategy is that of Bertsimas and Lo (1998). They applied a dynamic programming approach to the optimal liquidation problems. Analytical expressions of the dynamic optimal execution strategies are derived by minimizing the expected trading cost over a fixed time horizon.

In this paper, we present a new framework for the calculation of non-parametric LVaR by using stochastic programming (SP) techniques. Over the past few years, stochastic programming has grown into a mature methodology used to approach decision making problems in uncertain contexts. The main advantage of SP is its ability to better tackle optimization problems under conditions of uncertainty over time. Due to the fast development of computing power, it has been used to solve large scale optimization problems[1]. Therefore, we believe it is a promising methodology for LVaR modeling.

The SP approach presented in this paper is extended from Almgren and Chriss's framework (1999 and 2000). The sample path approach is adopted for scenario generation, rather than the scenario tree structure usually employed in SP. The scenario set is represented by a collection of simulated sample paths. Differing from Almgren and Chriss's parametric formulation of LVaR, we present a non-parametric formulation for LVaR. Both exogenous and endogenous illiquidity factors are taken into account. The former is measured by the bid-ask spread, and the latter is expressed by linear market impact functions, which are related to the quantity of sales. The model in this paper is built in a discrete-time manner, and the holding period is required to be determined externally. The permanent and temporary market impact mechanism proposed by Holthausen et al. (1987) is adopted to formulate the market impact, and both permanent and temporary market impacts are assumed as linear functions.

1 Gondzio and Grothey (2006) showed in their research that they could solve a quadratic financial planning problem exceeding 10^9 decision variables by applying a structure-exploiting parallel primal-dual interior-point solver.

Stochastic programming LVaR model

This paper proposes a SP approach to estimate the non-parametric LVaR, which is based on Almgren and Chriss's mean-variance approach (1999 and 2000). While their model has been shown to be an interesting methodology for the calculation of LVaR and has a huge potential in practice, the optimal trading strategies derived by their model fail to respond dynamically to the market situation, as they rely on a 'closed-form' or 'static' framework. For instance, if an increasing trend is observed in the market price, investors may decide to slow their liquidation process. If, on the other hand, unexpected market shocks occur, investors may decide to adjust their trading strategy and to speed up the completion of their sale. These market situations can be simulated and incorporated into scenarios. Clearly, any 'closed form' solution cannot deal with this type of uncertainty in such a dynamic manner. The LVaR formulation in Almgren and Chriss's model is based on the mean-variance framework; thus, it can be considered as a parametric approach. In contrast, the LVaR formulation presented in this paper is non-parametric and allows for the incorporation of various dynamics in the liquidation process. Thus, we propose a new framework for LVaR modeling.

Almgren and Chriss's mean-variance model

According to Almgren and Chriss's framework, a holding period T is required to be set externally. Then, this holding period is divided into N intervals of equal length ($\tau = T/N$). The trading strategy is defined as the quantity of shares sold in each time interval, which is denoted by a list of $n_1,\ldots, n_k,\ldots, n_N$, where n_k is the number of shares that the trader plans to sell in the k^{th} interval. Accordingly, the quantity of shares that the trader plans to hold at time $t_k = k\tau$ is denoted by x_k. Suppose a trader has a position X that needs to be liquidated before time T, then we have:

$$n_k = x_{k-1} - x_k,\ X = \sum_{k=1}^{N} n_k \quad \text{and} \quad x_k = X - \sum_{j=1}^{k} n_j = \sum_{j=k+1}^{N} n_j, \quad k = 0,\ldots, N.$$

Price dynamics in Almgren and Chriss's model are formulated as an arithmetic random walk as follows:

$$S_k = S_{k-1} + \sigma\xi_k\sqrt{\tau} + \mu\tau - \tau g\left(\frac{n_k}{\tau}\right) \tag{1}$$

where S_k is the equilibrium price after a sale, μ and σ are the drift and volatility of the asset price, respectively, and ξ_k is a random number that follows a standard normal distribution N (0, 1). The last term, $g(n_k/\tau)$, describes the permanent market impact from a sale. The actual sale price is calculated by subtracting the temporary impact, $h(n_k/\tau)$, from the equilibrium price:

$$\tilde{S}_k = S_k - h\left(\frac{n_k}{\tau}\right) \tag{2}$$

According to Almgren and Chriss's framework (1999 and 2000), both $g(n_k/t)$ and $h(n_k/\tau)$ are assumed to be linear functions:

$$g\left(\frac{n_k}{\tau}\right) = \gamma\cdot\frac{n_k}{\tau} \quad (3) \text{ and } \quad h\left(\frac{n_k}{\tau}\right) = \frac{1}{2}\varepsilon + \eta\cdot\frac{n_k}{\tau} \tag{4}$$

where γ and η are the permanent and temporary market impact coefficients[2], respectively, and ε denotes the bid-ask spread. They are all assumed to be constant.

Based on the previously presented equations, the formula for the actual sale price is derived as:

$$\tilde{S}_k = \underbrace{S_0 + \sigma\sum_{j=1}^{k}\xi_j\sqrt{\tau} + \mu k\tau}_{\text{I}} - \underbrace{\sum_{j=1}^{k}\gamma n_k}_{\text{II}} - \underbrace{\frac{1}{2}\varepsilon - \frac{\eta n_k}{\tau}}_{\text{III}} \tag{5}$$

As you can see from this formula, the actual sale price can be decomposed into three parts. Part I is the price random walk, which describes the price dynamics without any market impacts. Parts II and III are the price decline caused by the permanent and temporary market impact, respectively.

Then the total proceeds can be calculated by summing the sale values over the entire holding period:

$$\begin{aligned}\text{total proceed} &= \sum_{k=1}^{N} n_k\tilde{S}_k = XS_0 + \sigma\sum_{k=1}^{N} x_{k-1}\xi_k\sqrt{\tau} + \mu\sum_{k=1}^{N} x_{k-1}\tau - \gamma\sum_{k=1}^{N} n_k(X - x_k) - \frac{1}{2}\varepsilon X - \frac{\eta}{\tau}\sum_{k=1}^{N} n_k^2 \\ &= XS_0 + \sigma\sum_{k=1}^{N} x_{k-1}\xi_k\sqrt{\tau} + \mu\sum_{k=1}^{N} x_{k-1}\tau - \frac{1}{2}\gamma X^2 - \frac{1}{2}\varepsilon X - \left(\frac{\eta}{\tau} - \frac{1}{2}\gamma\right)\sum_{k=1}^{N} n_k^2 \qquad (6)\end{aligned}$$

Consequently, 'liquidation cost' (LC)[3] can be derived by subtracting the total actual sale proceeds from the trader's initial holding value, that is:

$$LC = XS_0 - \sum_{k=1}^{N} n_k\tilde{S}_k = -\sigma\sum_{k=1}^{N} x_{k-1}\xi_k\sqrt{\tau} - \mu\sum_{k=1}^{N} x_{k-1}\tau + \frac{1}{2}\gamma X^2 + \frac{1}{2}\varepsilon X + \left(\frac{\eta}{\tau} - \frac{1}{2}\gamma\right)\sum_{k=1}^{N} n_k^2 \quad (7).$$

Almgren and Chriss derive the formulae for the mean and variance of the liquidation cost as:

$$E[LC] = \frac{1}{2}\gamma X^2 - \mu\sum_{k=1}^{N}\tau x_{k-1} + \frac{1}{2}\varepsilon X + \left(\frac{\eta}{\tau} - \frac{1}{2}\gamma\right)\sum_{k=1}^{N} n_k^2 \tag{8}$$

$$\text{and } V[LC] = \sigma^2\sum_{k=1}^{N}\tau x_{k-1}^2 \tag{9}$$

Finally, they formulate the LVaR by using the parametric approach with the mean and variance of the LC as:

$$LVaR = E[LC] + \alpha_{cl}\sqrt{V[LC]} \tag{10}$$

where cl denotes the confidence level for the LVaR estimation, and α_{cl} is the corresponding percentile of the standard normal distribution. As expressed, LVaR measures a possible loss with a given position while taking into consideration both market risk conditions and liquidity effects.

2 For the estimation of temporary and permanent market impact coefficients, Almgren and Chriss did not propose a specific method. They assumed that: for the temporary market impact, trading each 1% of the daily volume incurs price depression of one bid-ask spread, and for the permanent market impact, trading 10% of the daily volume will have a significant impact on price, and incur price depression of one bid-ask spread. Since this paper focuses on the LVaR modeling, not the estimation of market impact coefficients, Almgren and Chriss's simple assumption is adopted for all the numerical experiments in this paper.

3 In Almgren and Chriss's paper, this 'cost' is referred as the transaction cost. However, the transaction cost is commonly known as the fees involved for participating in the market, such as the commission to the brokers. Therefore, in order to avoid any confusion, it is named 'liquidation cost' in this paper.

The optimal trading strategy could be derived by minimizing the LVaR. The mathematical programming formulation of this optimization problem is thus written as:

$$\begin{aligned} \min_{n_k} \quad & E[LC] + \alpha_{cl}\sqrt{V[LC]} \\ \text{s.t} \quad & x_k = \sum_{j=k+1}^{N} n_j \quad \forall k = 0, \ldots, N-1 \\ & X = \sum_{k=1}^{N} n_k \\ & n_1, \ldots, n_N \geq 0. \end{aligned}$$

Based on this brief introduction to Almgren and Chriss's mean-variance approach, we can thereon proceed to present the SP approach to LVaR modeling.

Stochastic programming transformation

In stochastic programming, uncertainty is modeled with scenarios that are generated by using available information to approximate future conditions. Before conducting the SP transformation, we need to briefly introduce the scenario generation technique used in this paper.

The liquidation process of investors' positions is a multi-period problem. The most commonly used technique is to model the evolution of stochastic parameters with multinomial scenario trees, as shown in Figure 1(a).

However, the use of scenario tree structures often leads to considerable computational difficulty, especially when dealing with large scale practical problems. In the scenario tree structure, the uncertainties are represented by the branches that are generated from each node. Increasing the number of branches per node can improve the quality of the approximation of the uncertainty. However, it causes an exponential growth in the number of nodes. Indeed, in order to approximate the future value of the uncertain parameters with a sufficient degree of accuracy, the resulting scenario tree could be of a huge size. It is commonly known as the "curse of dimensionality" [Bellman (1957)]. It is a significant obstacle for dynamic or stochastic optimization problems. An alternative method to overcome this problem is to simulate a collection of sample paths to reveal the future uncertainty as shown in Figure 1(b). Each simulated path represents a scenario. These sample paths can be generated by using Monte Carlo simulation, historical simulation, or bootstrapping. There have been several interesting papers regarding the application of the sample paths method in stochastic programming [Hibiki (2000), Krokhmal and Uryasev (2006)]. Using sample paths is advantageous because increasing the number of paths to achieve a better approximation causes the number of nodes to increase linearly rather than exponentially. This advantage is also present when the time period is increased. The number of nodes increases linearly with the sample paths method and exponentially with scenario tree structure.

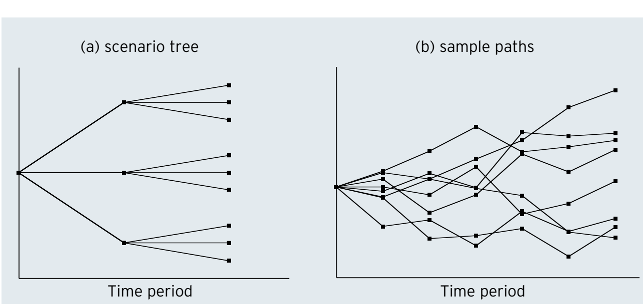

Figure 1 – Scenario generation

Let $\Im$ be a collection of sample paths

$$\Im = \{(C_0, C_{1,s}, C_{2,s}, \ldots, C_{k,s}, \ldots, C_{N,s}) \mid s = 1, \ldots, Sc\},$$

where $C_{k,s}$ represents the information about relevant parameters. In Almgren and Chriss's model (1999 and 2000), we should recall that the only randomness considered is market price. Hence, to set a point of comparison between their results and the results from the SP approach, we first assume the only randomness that is considered in the sample paths will come from the market price component, $\hat{S}_k$. Yet, this restrictive assumption can clearly be easily relaxed, and randomness can be added to other parameters, such as the bid-ask spread or the temporary and permanent market impact coefficients[4].

Under the SP framework, the trading strategy is no longer a vector but a two dimensional matrix

$$\text{strategy} = \begin{bmatrix} n_{1,1} & \cdots & n_{k,1} & \cdots & n_{N,1} \\ \vdots & \ddots & & & \vdots \\ n_{1,s} & & n_{k,s} & & n_{N,s} \\ \vdots & & & \ddots & \vdots \\ n_{1,Sc} & \cdots & n_{k,Sc} & \cdots & n_{N,Sc} \end{bmatrix}$$

first stage second stage

where $n_{k,s}$ is the quantity of shares sold in k^{th} interval on path s, s is the index of scenarios, and Sc is the number of scenarios. This is a two stage SP problem. $n_{1,s}$ (s = 1,...,Sc) are the first stage variables, and $n_{k,s}$ (k = 2,...,N and s = 1,...,Sc) are the second stage variables. Due to the nonanticipativity in the first stage, the first stage variables must be locked:

$$n_{1,s} = n_{1,s-1} \quad \forall\, s = 2, \ldots Sc.$$

For the actual sale price formulation, recall Equation (5). Taking into account the scenarios and replacing part I with $\hat{S}_{k,s}$ (i.e., the

4 An extension is presented below.

asset price without market impacts in k^{th} interval for each scenario), the actual sale price is reformulated as:

$$\tilde{S}_{k,s} = \hat{S}_{k,s} - \sum_{j=1}^{k} \gamma n_{j,s} - \frac{1}{2}\varepsilon - \frac{\eta n_{k,s}}{\tau} \quad (11)$$

As we now have the sale price formulation, the total sale proceeds corresponding to each scenario is naturally obtained by summing up the sale proceeds over the entire set of N intervals:

$$\begin{aligned} \text{total proceed} &= \sum_{k=1}^{N} n_{k,s}\tilde{S}_{k,s} = \sum_{k=1}^{N}\left(\hat{S}_{k,s}n_{k,s} - \frac{1}{2}\varepsilon n_{k,s} - n_{k,s}\sum_{j=1}^{k}\gamma n_{j,s} - \frac{\eta n_{k,s}^2}{\tau}\right). \\ &= \sum_{k=1}^{N}\hat{S}_{k,s}n_{k,s} - \frac{1}{2}\varepsilon X - \frac{1}{2}\gamma X^2 - \left(\frac{\eta}{\tau} - \frac{1}{2}\gamma\right)\sum_{k=1}^{N} n_{k,s}^2 \end{aligned} \quad (12)$$

Consequently, the liquidation cost under scenario s is derived by subtracting the corresponding total sale proceeds from the trader's initial holding value, that is:

$$LC_s = XS_0 - \sum_{k=1}^{N} n_{k,s}\tilde{S}_{k,s} = XS_0 + \frac{1}{2}\varepsilon X + \frac{1}{2}\gamma X^2 - \sum_{k=1}^{N}\hat{S}_{k,s}n_{k,s} + \left(\frac{\eta}{\tau} - \frac{1}{2}\gamma\right)\sum_{k=1}^{N} n_{k,s}^2 \quad (13)$$

The deterministic equivalent formulation of this SP problem with nonanticipativity constraints is:

$$\begin{aligned} \min_{n_{k,s}} \quad & \sum_{s=1}^{Sc} p_s \cdot LC_s \\ \text{s.t} \quad & X = \sum_{k=1}^{N} n_{k,s} \quad \forall s = 1,\ldots,Sc \\ & n_{1,s},\ldots,n_{N,s} \geq 0 \quad \forall s = 1,\ldots,Sc \\ & n_{1,s} = n_{1,s-1} \quad \forall s = 2,\ldots Sc \end{aligned}$$

where p_S is the probability of scenario s. Since the scenarios are obtained by the Monte Carlo simulation, they are thus equally probable with ps = 1 / Sc.

The resulting problem is a quadratic optimization one. The objective, the expected value of the liquidation cost, is a quadratic function, and all constraints are linear.

Non-parametric LVaR formulation

Depending on the set of assumptions, the calculation methodology, and their uses, two different types of VaRs usually exist, i.e., the parametric VaR and the non-parametric VaR. The same categorization obviously applies to LVaR. The LVaR estimated by Almgren and Chriss (1999 and 2000) is parametric as shown in Equation [10]. In this paper, we have to rely on a non-parametric formulation because it stems from the SP solution that we have adopted. More precisely, the calculation procedure is as follows:

1 Solve the stochastic optimization problem stated above and obtain the optimal trading strategy matrix.
2 Apply the optimal trading strategies to the corresponding scenario and calculate the liquidation cost for that specific scenario. That is to say, we substitute the optimal trading strategy matrix into the liquidation cost formula (Equation [13]), and calculate LC, which is a vector indexed by s.
3 Sort the vector LC, and find the value of the α^{th} percentile LC, i.e., the α%-LVaR. The most commonly used α value is 95 and 99.

Numerical experiments I

As previously mentioned, we first conducted a sanity check. This section details the numerical experiments for both the SP model and the Almgren and Chriss's mean-variance model with the restriction of randomness on one component only, i.e., the 'pure market price.'

JP Morgan's stock data was collected for the numerical experiments. The holding period, T, was set to be 5 days, and we selected the time interval to be 0.5 day. Thus, the total number of sales, N, was 10. The selection of the holding period and time interval was arbitrary.

For the price sample paths generation, the Monte Carlo simulation was applied. The stochastic evolution of the price was assumed to follow a geometric Brownian motion:

$$\hat{S}_k = \hat{S}_{k-1} \cdot \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)\tau + \xi_k \sigma\sqrt{\tau}\right) \quad (14)$$

Under Almgren and Chriss's mean-variance framework, market price was assumed to follow an arithmetic random walk because it is ultimately rather difficult to derive a closed-form solution based on an assumption of geometric Brownian motion. Yet, with Monte Carlo simulations, formulating the price evolution under different assumptions creates no issues related to the underlying distributions that could generate prices and returns. Since the geometric random walk is the most commonly used assumption for price stochastic processes, it is used in this paper even though differences between these two random walks are almost negligible over a short period of time.

10,000 sample paths were generated by using the Monte Carlo simulation. The simulated prices form a 10-by-10,000 matrix. The initial price is 37.72. These simulated sample paths are displayed in Figure 2.

Five different initial holdings were chosen for the numerical experiments with the aim of observing how the initial position affected the LVaR estimation. The LVaRs were calculated with the most commonly seen confidence levels of 95% and 99%. The results are summarized in Table 1.

The numerical results show that the LVaR ratios computed by the SP model are slightly lower than those computed by Almgren and

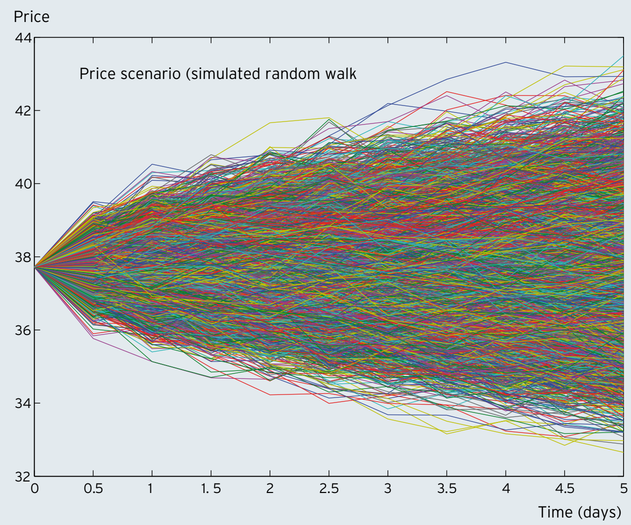

Figure 2 – Simulated price scenarios

Initial holding (shares)	1000000	500000	100000	50000	10000
Almgren and Chriss's Mean-Variance Model					
Parametric LVaR 95%	1.425E+06	6.186E+05	1.010E+05	4.845E+04	9.311E+03
Parametric LVaR per share	1.425	1.237	1.010	0.969	0.931
Parametric LVaR ratio	3.78%	3.28%	2.68%	2.57%	2.47%
Parametric LVaR 99%	1.863E+06	8.216E+05	1.388E+05	6.718E+04	1.304E+04
Parametric LVaR per share	1.863	1.643	1.388	1.344	1.304
Parametric LVaR ratio	4.94%	4.36%	3.68%	3.56%	3.46%
Stochastic Programming Model					
Non-parametric LVaR 95%	1.290E+06	5.399E+05	8.144E+04	3.832E+04	7.218E+03
Non-parametric LVaR per share	1.290	1.080	0.814	0.766	0.722
Non-parametric LVaR ratio	3.42%	2.86%	2.16%	2.03%	1.91%
Non-parametric LVaR 99%	1.811E+06	7.944E+05	1.342E+05	6.442E+04	1.249E+04
Non-parametric LVaR per share	1.811	1.589	1.342	1.288	1.249
Non-parametric LVaR ratio	4.80%	4.21%	3.56%	3.42%	3.31%

Table 1 – Numerical results summary
*LVaR per share = LVaR/Initial holding; LVaR ratio = LVaR per share/Initial price

Chriss's model at both the 95% and 99% confidence levels. Also, as expected, the numerical results show that the LVaR estimates increase when the initial holdings increase. This is true for both Almgren and Chriss's model and the SP model. As previously stated, as the initial holding becomes larger, the market impact becomes stronger when a trader liquidates his position. Consequently, the LVaR will increase as well. This is clearly a characteristic that distinguishes the LVaR from the traditional VaR.

SP LVaRs are lower than Almgren and Chriss's LVaRs because the SP model's optimal trading strategies can dynamically adapt to the market situation. This fits investors' actual trading behaviors in the market, as they will adjust their trading plans according to the market environment. Therefore, the SP model can provide more precise LVaR estimates due to the characteristic of the SP model's adaptive trading strategies.

Generalization of the stochastic programming LVaR model

A simple stochastic programming LVaR model that was transformed from Almgren and Chriss's mean-variance model was presented above in order to compare with other models discussed (i.e., both two LVaR approaches were used under the same setting). Let us now extend the analysis and show some of the advantages provided by the SP approach. Contrary to Almgren and Chriss's model that assumes that both the bid-ask spread and market impact coefficients are constants, we generalize the SP LVaR model by relaxing this assumption and treating these two components as random variables.

By incorporating randomness in the bid-ask spread and both the permanent and temporary market impact coefficients, the formula of the actual sale price needs to be rewritten as:

$$\tilde{S}_{k,s} = \hat{S}_{k,s} - \sum_{j=1}^{k} \gamma_{j,s} n_{j,s} - \frac{1}{2}\varepsilon_{k,s} - \frac{\eta_{k,s} n_{k,s}}{\tau} \quad (15)$$

For the formulation of $\varepsilon_{k,s}$, we employ a standardization process. Since bid-ask spreads tend to be proportional to asset prices, past observations may not accurately reflect the current variations. Bangia et al. (1999) suggested calculating a relative bid-ask spread that is equal to the bid-ask spread divided by the mid-price. By employing this calculation, the bid-ask spread is expressed as a proportion of the asset price; thus, the current bid-ask spread variation is sensitive to the current asset price rather than past observations. The relative bid-ask spread, as a normalizing device, can improve the accuracy of the bid-ask spread variation estimation. The bid-ask spread is thus formulated as:

$$\varepsilon_{k,s} = \hat{S}_{k,s} \bar{\varepsilon}_{k,s} \quad (16)$$

where $\bar{\varepsilon}_{k,s}$ is the relative bid-ask spread at time t_k on path s. Recall the sample path set

$$\Im = \{(C_0, C_{1,s}, C_{2,s}, \ldots, C_{k,s}, \ldots, C_{N,s}) \mid s = 1, \ldots, Sc\}.$$

By incorporating the randomness into the relative bid-ask spread and market impact coefficients, $C_{k,s}$ is extended to

$$C_{k,s} = (\hat{S}_{k,s}, \bar{\varepsilon}_{k,s}, \gamma_{k,s}, \eta_{k,s}).$$

In other words, in the generalized SP model, each node on the simulated sample paths contains information for the asset price, the relative bid-ask spread, and the permanent and temporary market impact coefficients.

As we now have the new sale price formulation, the formula for liquidation cost under scenario s (as shown in Equation [13]) is rewritten as:

$$LC_s = XS_0 - \sum_{k=1}^{N} n_{k,s}\tilde{S}_{k,s} = XS_0 - \sum_{k=1}^{N}\left(\hat{S}_{k,s}n_{k,s} - \frac{1}{2}\hat{S}_{k,s}\bar{\varepsilon}_{k,s}n_{k,s} - n_{k,s}\sum_{j=1}^{k}\gamma_{k,s}n_{j,s} - \frac{\eta_{k,s}n_{k,s}^2}{\tau}\right) \quad (17)$$

The deterministic equivalent formulation and the LVaR calculation procedure are the same as shown above. By generalizing the SP model, more parameters are incorporated in the sample path set, which leads to a more accurate approximation of future uncertainties.

Numerical experiments II

This section reports the numerical experiments for the generalized SP LVaR model presented above. We use the same dataset that was used for aforementioned numerical experiments. The holding period and time interval remain identical, i.e., 5 days and half a day, respectively.

For the sample paths' generation of the relative bid-ask spread, and the permanent and temporary market impacts, we assumed that they followed independent lognormal distributions and simulated each of them simply as a white noise:

$$\bar{\varepsilon}_{k,s} = \exp\left(\mu_{\bar{\varepsilon}} - \frac{1}{2}\sigma_{\bar{\varepsilon}}^2 + \xi_{k,s}\sigma_{\bar{\varepsilon}}\right) \quad (18)$$

$$\gamma_{k,s} = \exp\left(\mu_{\gamma} - \frac{1}{2}\sigma_{\gamma}^2 + \xi_{k,s}\sigma_{\gamma}\right) \quad (19)$$

$$\eta_{k,s} = \exp\left(\mu_{\eta} - \frac{1}{2}\sigma_{\eta}^2 + \xi_{k,s}\sigma_{\eta}\right) \quad (20)$$

where μ and σ are the means and standard deviations, respectively, of the three random variables (i.e., ε, γ and η).

Once again 10,000 sample paths were generated by using the Monte Carlo simulation for each parameter. The LVaRs at the 95% and 99% confidence levels were computed for the same five initial holding scenarios employed above. The results are summarized in Table 2.

Initial holding (shares)	1000000	500000	100000	50000	10000
Non-parametric LVaR 95%	1.084E+06	4.740E+05	7.800E+04	3.739E+04	7.119E+03
Non-parametric LVaR per share	1.084	0.948	0.780	0.748	0.712
Non-parametric LVaR ratio	2.87%	2.51%	2.07%	1.98%	1.89%
Non-parametric LVaR 99%	1.621E+06	7.389E+05	1.312E+05	6.403E+04	1.243E+04
Non-parametric LVaR per share	1.621	1.478	1.312	1.281	1.243
Non-parametric LVaR ratio	4.30%	3.92%	3.48%	3.40%	3.30%

Table 2 – Numerical results

In Figure 3, we can see that the LVaR ratios computed by the SP model with the incorporation of randomness into the bid-ask spread and the market impact coefficients are slightly lower than those computed by the SP model with the constant bid-ask spread and market impact coefficients. When the initial holding is small, incorporating these new random variables does not cause a significant change to the LVaR estimate. However, when the initial holding is large, the differences are substantial. For instance, when the initial holding is 1,000,000, incorporating randomness reduces the 95% LVaR ratio from 3.42% to 2.87% and the 99% LVaR ratio from 4.80% to 4.30%.

The main reason for these differences must lie in the way the optimal trading strategies that are derived by the SP model respond to the variation of the bid-ask spread and market impact coefficients. For example, if we assume that the bid-ask spread is constant, the

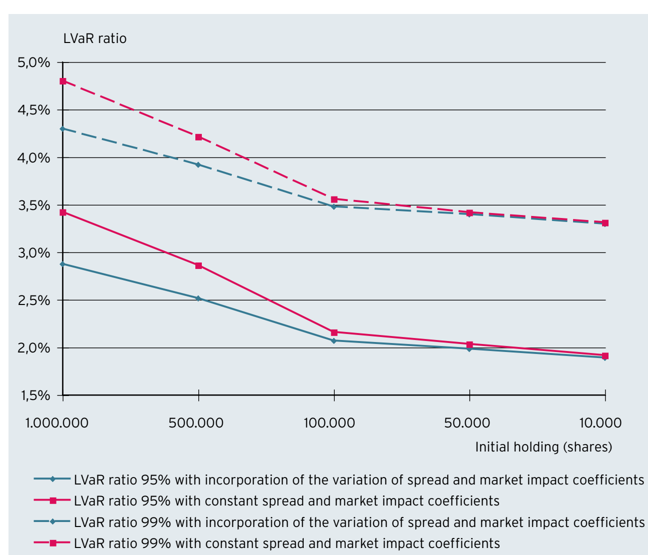

Figure 3: LVaR ratio comparison

loss caused by the spread in the whole liquidation process is ε X/2 for each scenario as shown in Equation (13) (ε is the mean value of the bid-ask spread). With the incorporation of randomness into the bid-ask spread, the optimal trading strategies are adjusted in accordance with its variation. When the spread is high, the optimal trading strategy may suggest selling less. On the contrary, when it is low, the optimal trading strategy may suggest selling more. Therefore, the average loss caused by the spread can be expected to be lower than ε X/2. Stated otherwise, the SP model's optimal trading strategies can take advantage of changes by acting in a flexible and timely manner. Also note that introducing the calculation of the relative bid-ask spread and the Monte Carlo simulation itself can cause certain differences. However, the effects are presumably small.

Finally, it is worthwhile mentioning that incorporating randomness into the bid-ask spread and market impact coefficients within the Almgren and Chriss's model will definitely enlarge the resulting LVaR estimates. Indeed, it would add new variance terms to the variance of the liquidation cost, since the variations of parameters are represented by their variances. This would lead to the increase of the LVaR estimates. The SP solution and its numerical experiments indicate that if uncertainty is handled well, it does not necessarily cause an increase in the LVaR estimates. It highlights the strength of the SP approach, which provides adaptive strategies (or 'recourse strategies'). Moreover, adding new random variables in the model does not increase the difficulty of the problem due to the non-parametric nature of the SP LVaR.

Conclusion

This paper presents a stochastic programming approach for LVaR modeling, which is extended from Almgren and Chriss's mean-variance approach. In contrast to their approach, the optimal trading strategies are derived by minimizing the expected liquidation cost. Thus, the SP strategies dynamically adapt to new market situations. This is the strength of SP in the context of decision making under uncertainty. Another contribution from this paper is the non-parametric formulation of the SP LVaR. It contrasts with the LVaR modeling methodologies that quite often rely on parametric approaches. Overall, the numerical results indicate that the two approaches are not identical. Indeed, the LVaRs computed using the SP model in this paper are lower than those computed by Almgren and Chriss's model. Yet, LVaR modeling still remains in its infancy, especially when using SP in this context.

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