Optimization of Truss Structures by Semidefinite Programming

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Outline

Introduction

Problem formulation

The optimization method

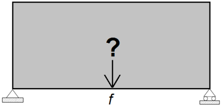
Techniques employed
- Exploiting the algebraic structures
- Member adding
- Warm-start strategy

Numerical examples

Conclusions and future works

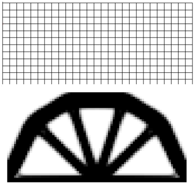
Structural optimization

- Consider the following design domain and loading conditions.

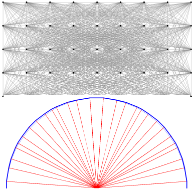


- The goal is to find the lightest structure that is able to carry the given set of loads.
- **Several approaches of structural optimization.**

Topology optimization (continuum*)

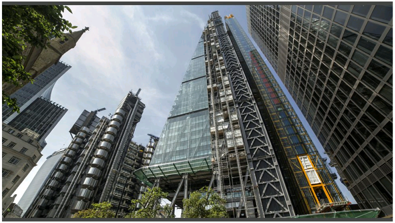


Topology optimization (truss)

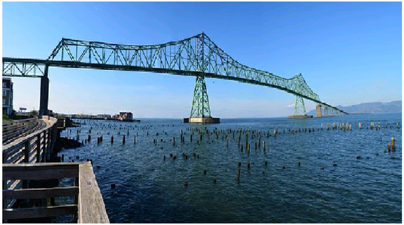


***O. Sigmund. A 99 line topology optimization code written in Matlab. Structural and Multidisciplinary Optimization, 21:120–127, 2001.**

Application

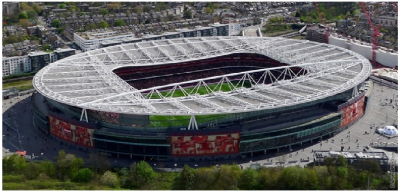


http://www.bbc.co.uk/programmes/p01rrnwc/p01rrbzl



https://en.wikipedia.org/wiki/Astoria%E2%80%93Megler_Bridge

https://en.wikipedia.org/wiki/London_King27s_Cross
_railway_station

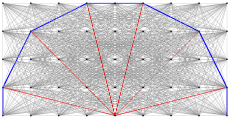


http://www.buildingtalk.com/wpcontent/uploads/arsenal-1.jpg

The underlying minimum weight problem

$$
\begin{aligned}
\underset{a, q_\ell, u_\ell}{\text{minimize}} \quad & l^T a \\
\text{subject to} \quad & \sum_i q_{\ell,i} \gamma_i = f_\ell, && \ell = 1, \cdots, n_L \\
& \frac{a_i E}{l_i} \gamma_i^T u_\ell = q_{\ell,i} && \ell = 1, \cdots, n_L, i = 1, \cdots, m \\
& -a\sigma^- \leq q_\ell \leq \sigma^+ a, && \ell = 1, \cdots, n_L \\
& a \geq 0
\end{aligned}
\tag{1}
$$

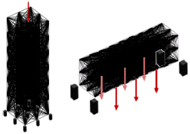
- n_L number of load cases,
- $l \in \mathbb{R}^n$ is a vector of bar lengths,
- $a \in \mathbb{R}^n$ is a vector of bar cross-sectional areas,
- $f_\ell \in \mathbb{R}^m$ is a vector of applied load forces,
- $q_\ell \in \mathbb{R}^n$ are axial forces in members,
- $\sigma^- > 0$ and $\sigma^+ > 0$ are the the material's yield stresses in compression and tension,
- E is Young's modulus.



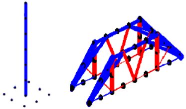
Bendsøe, M., Sigmund, O, Topology Optimization: Theory, Methods and Applications. Springer (2003)

Stability constraints

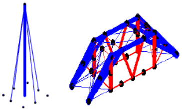
- Consider the following three dimensional problem.



(a) Design domains, bc, and loads.



(b) Without stability considerations.



(c) With stability considerations.

- **Without stability considerations:**
  - The optimal design (a slender of six bars in compression) needs some kind of support or bracing from orthogonal directions.
  - The optimal design for the bridge problem includes independent planar trusses. It lacks connectivity.
- **With stability considerations:**
  - The bar has bracing.
  - The independent planar trusses in the bridge are connected

The minimum weight problem with global stability constraints

$$
\begin{aligned}
\underset{a, q_\ell, u_\ell}{\text{minimize}} \quad & l^T a \\
\text{subject to} \quad & \sum_i q_{\ell,i}\gamma_i = f_\ell, && \ell = 1, \cdots, n_L \\
& \frac{a_i E}{l_i}\gamma_i^T u_\ell = q_{\ell,i} && \ell = 1, \cdots, n_L, i = 1, \cdots, m \\
& -a\sigma^- \leq q_\ell \leq \sigma^+ a, && \ell = 1, \cdots, n_L \\
& K(a) + \tau_\ell G(q_\ell) \succeq 0 && \ell = 1, \cdots, n_L \\
& a \geq 0
\end{aligned}
\tag{2}
$$

where the stiffness matrix K and geometry stiffness matrix G are given by

$$
K(a) = \sum_{j=1}^{m} a_j K_j, \text{ with } K_j = \frac{E_j}{l_j}\gamma_j\gamma_j^T, \text{ and } G(q) = \sum_{j=1}^{m} q_j G_j, \text{ with } G_j = \frac{1}{l_j}(\delta_j\delta_j^T + \eta_i\eta_j^T
$$

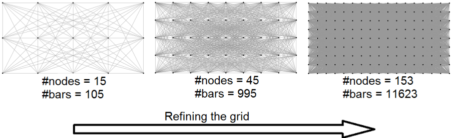
- The loading factor $\tau_\ell \geq 1$.
- $(\delta_j, \gamma_j, \eta_j)$ are mutually orthogonal. ($\eta = 0$ for 2D problems)

M. Kocvara. On the modelling and solving of the truss design problem with global stability constraints. Structural and Multidisciplinary Optimization, 23:189–203, 2002.

The minimum weight problem with global stability constraints

$$
\begin{aligned}
\underset{a, q_\ell, u_\ell}{\text{minimize}} \quad & l^T a \\
\text{subject to} \quad & \sum_i q_{\ell,i} \gamma_i = f_\ell, && \ell = 1, \cdots, n_L \\
& \frac{a_i E}{l_i} \gamma_i^T u_\ell = q_{\ell,i} && \ell = 1, \cdots, n_L, i = 1, \cdots, m \\
& -a\sigma^- \leq q_\ell \leq \sigma^+ a, && \ell = 1, \cdots, n_L \\
& K(a) + \tau_\ell G(q_\ell) \succeq 0 && \ell = 1, \cdots, n_L \\
& a \geq 0
\end{aligned}
\tag{2}
$$

The problem (2) is large-scale nonlinear non-convex semidefinite program.



For d number nodes, there are $m = \frac{d}{2}(d-1)$ potential member bars.

Simplification

- Ignore the kinematic compatibility constraints

$$\frac{a_i E}{l_i}\gamma_i^T u_\ell = q_{\ell,i},\ \ell = 1,\cdots,n_L,\ i = 1,\cdots,m.$$

- Hence, we solve the linear formulation

$$\begin{aligned} \underset{a,q_\ell}{\text{minimize}} \quad & l^T a \\ \text{subject to} \quad & \sum_i q_{\ell,i}\gamma_i = f_\ell, && \ell = 1,\cdots,n_L \\ & -a\sigma^- \leq q_\ell \leq \sigma^+ a, && \ell = 1,\cdots,n_L \\ & K(a) + \tau_\ell G(q_\ell) \succeq 0 && \ell = 1,\cdots,n_L \\ & a \geq 0. \end{aligned} \tag{3}$$

- We then measure the violation due to ignoring the kinematic compatibility constraints by solving the least-squares problem

$$\underset{u_\ell}{\text{minimize}} \quad \max_\ell \frac{1}{||q_\ell^*||^2}\sum_i \left(\frac{a_i^* E}{l_i}\gamma_i^T u_\ell - q_{\ell,i}^*\right)^2, \tag{4}$$

where a^* and q_ℓ^* are the solution of the relaxed problem (3).

Simplification

The (relaxation) SDP problem

$$
\begin{aligned}
\underset{a, q_\ell}{\text{minimize}} \quad & l^T a \\
\text{subject to} \quad & \sum_i q_{\ell,i} \gamma_i = f_\ell, \qquad \ell = 1, \cdots, n_L \\
& -a\sigma^- \leq q_\ell \leq \sigma^+ a, \quad \ell = 1, \cdots, n_L \\
& K(a) + \tau_\ell G(q_\ell) \succeq 0 \quad \ell = 1, \cdots, n_L \\
& a \geq 0
\end{aligned}
\tag{5}
$$

- can be efficiently solved.
- provides lower bounds to the nonlinear and non-convex formulation.
- its solution has (usually) small violation in the kinematic compatibility constraints for realistic input and reasonable value of τ_ℓ

Primal-Dual Interior Point Method

- Consider the following primal and dual semidefinite programs.

Primal

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & C \bullet X \\ \text{subject to} \quad & A_i \bullet X = b_i,\ i = 1, \dots, m \\ & X \succeq 0 \end{aligned}$$

Dual

$$\begin{aligned} \underset{y,S}{\text{maximize}} \quad & b^T y \\ \text{subject to} \quad & \sum_{i=1}^{m} y_i A_i + S = C \\ & S \succeq 0 \end{aligned}$$

where $C, A_i \in \mathbb{S}^{n\times n}$, $b, y \in \mathbb{R}^m$, and $U \bullet V = \sum_i \sum_j U_{ij} V_{ij}$ for $U, V \in \mathbb{R}^{n\times n}$.

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- The first-order optimality conditions are (solved for $\mu_k \to 0$)

$$\begin{aligned} \mathcal{A}X &= b \\ \mathcal{A}^* y + S &= C \\ X &= \mu S^{-1}. \end{aligned} \tag{6}$$

where $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m : \mathcal{A}X = (A_i \bullet X)_{i=1}^m$ and $\mathcal{A}^* : \mathbb{R}^m \to \mathbb{S}^n : \mathcal{A}^* y = \sum_{i=1}^m y_i A_i$

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Primal

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & C \bullet X \\ \text{subject to} \quad & A_i \bullet X = b_i, \; i = 1, ..., m \\ & X \succeq 0 \end{aligned}$$

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- Apply Newton method to solve (6).

$$\begin{bmatrix} 0 & \mathcal{A}^* & \mathcal{I} \\ \mathcal{A} & 0 & 0 \\ \mathcal{E} & 0 & \mathcal{F} \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta y \\ \Delta S \end{bmatrix} = \begin{bmatrix} \xi_d \\ \xi_p \\ \xi_c \end{bmatrix}.$$

where $\mathcal{E} = I \odot I, \mathcal{F} = X \odot S^{-1}$, and
$P \odot Q : \mathbb{S}^n \to \mathbb{S}^n : (P \odot Q)U = \frac{1}{2}(PUQ^T) + QUP^T)$

Primal-Dual Interior Point Method

- Consider the following primal and dual semidefinite programs.

Primal

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & C \bullet X \\ \text{subject to} \quad & A_i \bullet X = b_i,\ i = 1, \dots, m \\ & X \succeq 0 \end{aligned}$$

Dual

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where $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m : \mathcal{A}X = (A_i \bullet X)_{i=1}^m$ and $\mathcal{A}^* : \mathbb{R}^m \to \mathbb{S}^n : \mathcal{A}^* y = \sum_{i=1}^m y_i A_i$

- Usually solved for the reduced system **(normal equations)**

$$\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^* \Delta y = -\xi_p + \mathcal{A}\mathcal{E}^{-1}(\xi_d - \mathcal{F}\xi_c).$$

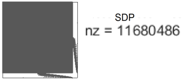
Primal-Dual Interior Point Method

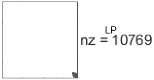
- The method obtains solution within modest number of iterations.
- Every iteration requires solving the linear system

$$\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^{*}\Delta y = -\xi_p + \mathcal{A}\mathcal{E}^{-1}(\xi_d - \mathcal{F}\xi_c).$$

- Forming the system requires $O(mn^3 + m^2n^2)$ arithmetic operations (straightforward expressions) (bottle-neck)
- Large storage requirement (bottle-neck). $\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^{*}$ is usually full matrix.

$$\begin{aligned} \underset{a, q_\ell, u_\ell}{\text{minimize}} \quad & l^T a \\ \text{subject to} \quad & Bq_\ell = f_\ell, & \ell = 1, \cdots, n_L \\ & -a\sigma^- \le q_\ell \le \sigma^+ a, & \ell = 1, \cdots, n_L \\ & K(a) + \tau_\ell G(q_\ell) \succeq 0 & \ell = 1, \cdots, n_L \\ & a \ge 0 \end{aligned} \qquad (7)$$





Exploiting algebraic structures

- The reduced system

$$\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^*\Delta y = -\xi_p + \mathcal{A}\mathcal{E}^{-1}(\xi_d - \mathcal{F}\xi_c).$$

for the truss problem is

$$\begin{bmatrix} A_{11} & A_{12}^T & 0 \\ A_{12} & A_{22} & \tilde{B}^T \\ 0 & \tilde{B} & 0 \end{bmatrix} \begin{bmatrix} \Delta a \\ \Delta q_\ell \\ \Delta \lambda_\ell \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}, \text{ where} \tag{8}$$

$$(A_{11})_{ij} = -\sum_\ell X_\ell K_i S_\ell^{-1} \bullet K_j + (D_{11})_{ij}$$

$$(A_{12})_{ij} = -X_\ell K_i S_\ell^{-1} \bullet G_j + (D_{12})_{ij}, \; (A_{22})_{ij} \quad = -X_\ell G_i S_\ell^{-1} \bullet G_j + (D_{22})_{ij}$$

$$K_j = \frac{E_j}{l_j}\gamma_j\gamma_j^T, \; G_j = \frac{1}{l_j}(\delta_j\delta_j^T + \eta_i\eta_j^T),$$

D_{kl} diagonal matrices, and $U \bullet V = \sum_i \sum_j U_{ij}V_{ij}$ for $U, V \in \mathbb{R}^{n\times n}$.

Exploiting algebraic structures

- $$\begin{bmatrix} A_{11} & A_{12}^T & 0 \\ A_{12} & A_{22} & \tilde{B}^T \\ 0 & \tilde{B} & 0 \end{bmatrix} \begin{bmatrix} \Delta a \\ \Delta q_\ell \\ \Delta \lambda_\ell \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}, \text{ where} \tag{8}$$

$$(A_{11})_{ij} = -\sum_\ell X_\ell K_i S_\ell^{-1} \bullet K_j + (D_{11})_{ij}$$

$$(A_{12})_{ij} = -X_\ell K_i S_\ell^{-1} \bullet G_j + (D_{12})_{ij}, \ (A_{22})_{ij} \quad = -X_\ell G_i S_\ell^{-1} \bullet G_j + (D_{22})_{ij}$$

$$K_j = \frac{E_j}{l_j} \gamma_j \gamma_j^T, \ G_j = \frac{1}{l_j}(\delta_j \delta_j^T + \eta_i \eta_j^T),$$

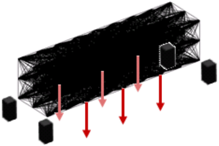
- We exploit the low rank property (and sparsity) of the K_i's and G_i's.

$$X_\ell K_i S_\ell^{-1} \bullet K_j = \frac{E^2}{l_i l_j} \gamma_j^T S_\ell^{-1} \gamma_i \gamma_i^T X_\ell \gamma_j, \tag{9}$$

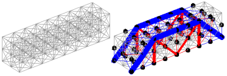
- The matrix in (8) can be computed in $\mathcal{O}(n^2 m)$ instead of $\mathcal{O}(nm^3 + n^2 m^2)$ arithmetic operations.

A. Ben-Tal and A. Nemirovski. Robust truss topology design via semidefinite programming. SIAM Journal on Optimization, 7(4):991–1016, 1997.

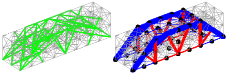
Member adding



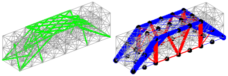
(a)



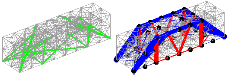
(b) Mem add iter.=1, 444 bars, vol= 0.05681m^3



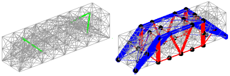
(c) Mem add iter.=2, 518 bars, vol= 0.05429m^3



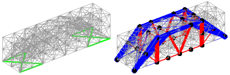
(d) Mem add iter.=3, 564 bars, vol= 0.05417m^3



(e) Mem add iter.=4, 588 bars, vol= 0.05414m^3



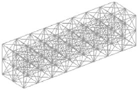
(f) Mem add iter.=5, 592 bars, vol= 0.05414m^3



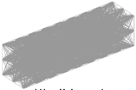
(g) Mem add iter.=6, 600 bars, vol= 0.05414m^3

Member adding

- Primal



(# inital bars)



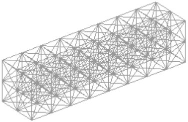
(# all bars)

$$\begin{aligned}
\underset{a,q}{\text{minimize}} \quad & \sum_{j\in\mathcal{K}_0} l_j a_j + \sum_{j\in\mathcal{K}_1} l_j a_j \\
\text{subject to} \quad & \sum_{j\in\mathcal{K}_0} \gamma_j q_{\ell,j} + \sum_{j\in\mathcal{K}_1} \gamma_j q_{\ell,j} = f_\ell, & \forall \ell \\
& -\sigma^- a_j \le q_{\ell,j} \le \sigma^+ a_j, & j \in \mathcal{K}_0, \forall \ell \\
& -\sigma^- a_j \le q_{\ell,j} \le \sigma^+ a_j, & j \in \mathcal{K}_1, \forall \ell \\
& \sum_{j\in\mathcal{K}_0} a_j K_j + \sum_{j\in\mathcal{K}_1} a_j K_j + \tau_\ell \sum_{j\in\mathcal{K}_0} q_{\ell,j} G_j + \tau_\ell \sum_{j\in\mathcal{K}_1} q_{\ell,j} G_j \succeq 0 & \forall \ell \\
& a_j \ge 0, j \in \mathcal{K}_0, a_j \ge 0, j \in \mathcal{K}_1, \forall \ell.
\end{aligned}$$

- Dual

$$\begin{aligned}
\max_{\lambda, X} \quad & \sum_\ell f_\ell^T \lambda_\ell \\
\text{s.t.} \quad & -\frac{l}{\sigma^-}(l_j - \sum_\ell K_j \bullet X_\ell) \le \sum_\ell (\gamma_j^T \lambda_\ell + \tau_\ell G_j \bullet X_\ell) \le \frac{1}{\sigma^+}(l_j - \sum_\ell K_j \bullet X_\ell), j \in \mathcal{K}_0, \forall \ell \\
& -\frac{l}{\sigma^-}(l_j - \sum_\ell K_j \bullet X_\ell) \le \sum_\ell (\gamma_j^T \lambda_\ell + \tau_\ell G_j \bullet X_\ell) \le \frac{1}{\sigma^+}(l_j - \sum_\ell K_j \bullet X_\ell), j \in \mathcal{K}_1, \forall \ell \\
& X_\ell \succeq 0, \ \forall \ell.
\end{aligned}$$

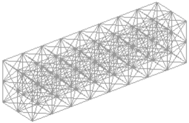
Member adding



- Set $a = 0$, $q = 0$.
- Primal

$$
\begin{aligned}
\underset{a,q}{\text{minimize}} \quad & \sum_{j\in\mathcal{K}_0} l_j a_j + \cancelto{0}{\sum_{j\in\mathcal{K}_1} l_j a_j} \\
\text{subject to} \quad & \sum_{j\in\mathcal{K}_0} \gamma_j q_{\ell,j} + \cancelto{0}{\sum_{j\in\mathcal{K}_1} \gamma_j q_{\ell,j}} = f_\ell, && \forall \ell \\
& -\sigma^- a_j \le q_{\ell,j} \le \sigma^+ a_j, && j \in \mathcal{K}_0, \forall \ell \\
& \xcancel{-\sigma^- a_j \le q_{\ell,j} \le \sigma^+ a_j,} && \xcancel{j \in \mathcal{K}_1, \forall \ell} \\
& \sum_{j\in\mathcal{K}_0} a_j K_j + \cancelto{0}{\sum_{j\in\mathcal{K}_1} a_j K_j} + \tau_\ell \sum_{j\in\mathcal{K}_0} q_{\ell,j} G_j + \tau_\ell \cancelto{0}{\sum_{j\in\mathcal{K}_1} q_{\ell,j} G_j} \succeq 0 && \forall \ell \\
& a_j \ge 0, j \in \mathcal{K}_0, \xcancel{a_j \ge 0, j \in \mathcal{K}_1, \forall \ell.}
\end{aligned}
$$

Member adding



- Dual

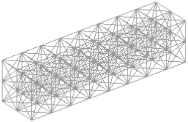
$$\max_{\lambda,X} \quad \sum_{\ell} f_\ell^T \lambda_\ell$$

$$\text{s.t.} \quad -\frac{I}{\sigma^-}(l_j - \sum_{\ell} K_j \bullet X_\ell) \leq \sum_{\ell}(\gamma_j^T \lambda_\ell + \tau_\ell G_j \bullet X_\ell) \leq \frac{1}{\sigma^+}(l_j - \sum_{\ell} K_j \bullet X_\ell), j \in \mathcal{K}_0, \forall \ell$$

$$-\frac{I}{\sigma^-}(l_j - \sum_{\ell} K_j \bullet X_\ell) \leq \sum_{\ell}(\gamma_j^T \lambda_\ell + \tau_\ell G_j \bullet X_\ell) \leq \frac{1}{\sigma^+}(l_j - \sum_{\ell} K_j \bullet X_\ell), j \in \mathcal{K}_1, \forall \ell$$

$$X_\ell \succeq 0, \ \forall \ell.$$

Member adding



Solve the primal restricted problem (RMP)

$$
\begin{aligned}
\underset{a,q}{\text{minimize}} \quad & \sum_{j\in\mathcal{K}_0} l_j a_j \\
\text{subject to} \quad & \sum_{j\in\mathcal{K}_0} \gamma_j q_{\ell,j} = f_\ell, && \forall \ell \\
& -\sigma^- a_j \le q_{\ell,j} \le \sigma^+ a_j, && j\in\mathcal{K}_0, \forall \ell \\
& \sum_{j\in\mathcal{K}_0} a_j K_j + \tau_\ell \sum_{j\in\mathcal{K}_0} q_j G_j \succeq 0, && \forall \ell \\
& a_j \ge 0, && j \in \mathcal{K}_0
\end{aligned}
$$

and the dual restricted problem (D-RMP)

$$
\max_{\lambda, X} \quad \sum_{\ell} f_\ell^T \lambda_\ell
$$

$$
\text{s.t.} \quad -\frac{l}{\sigma^-}\Big(l_j - \sum_\ell K_j \bullet X_\ell\Big) \le \sum_\ell \big(\gamma_j^T \lambda_\ell + \tau_\ell G_j \bullet X_\ell\big) \le \frac{1}{\sigma^+}\Big(l_j - \sum_\ell K_j \bullet X_\ell\Big), j \in \mathcal{K}_0
$$

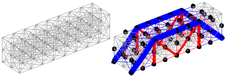
Member adding

- Generate the columns(matrices) as below.

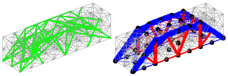
$$\mathcal{K} = \Big\{ j \in \{1, \cdots, m\} \backslash \mathcal{K}_0 | \sum_{\ell} (\gamma_j^T \lambda_\ell^* + \tau_\ell G_j \bullet X_\ell^*) < -\frac{I}{\sigma^-}(I_j - \sum_{\ell} K_j \bullet X_\ell^*) \text{ or } \sum_{\ell} (\gamma_j^T \lambda_\ell^* + \tau_\ell G_j \bullet X_\ell^*) > \frac{I}{\sigma^+}(I_j - \sum_{\ell} K_j \bullet X_\ell^*) \Big\} \quad (10)$$

where λ_ℓ^* and X_ℓ^* are solution of the D-RMPs.

- Filter, add, and the next problem instance.
- The sparsity of K_i's and G_i is exploited to generate the set $\mathcal{K}$



(a) Mem add iter.=1, 444 bars, vol= $0.05681 m^3$



(b) Mem add iter.=2, 518 bars, vol= $0.05429 m^3$

Figure

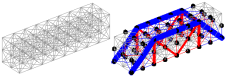
Member adding

- Generate the columns(matrices) as below.

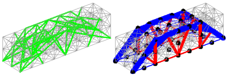
$$\mathcal{K} = \Big\{ j \in \{1, \cdots, m\} \backslash \mathcal{K}_0 | \sum_{\ell} (\gamma_j^T \lambda_\ell^* + \tau_\ell G_j \bullet X_\ell^*) < -\frac{l}{\sigma^-}(l_j - \sum_{\ell} K_j \bullet X_\ell^*) \text{ or } \sum_{\ell} (\gamma_j^T \lambda_\ell^* + \tau_\ell G_j \bullet X_\ell^*) > \frac{l}{\sigma^+}(l_j - \sum_{\ell} K_j \bullet X_\ell^*) \Big\} \quad (10)$$

where λ_ℓ^* and X_ℓ^* are solution of the D-RMPs.

- Filter, add, and the next problem instance.
- The sparsity of K_i's and G_i is exploited to generate the set $\mathcal{K}$



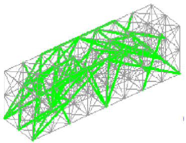
(a) Mem add iter.=1, 444 bars, vol= 0.05681m^3



(b) Mem add iter.=2, 518 bars, vol= 0.05429m^3

Figure

Member adding



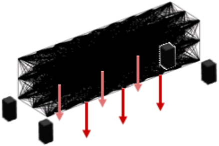
- the primal restricted problem (RMP)

$$\begin{aligned}
\underset{a,q}{\text{minimize}} \quad & \sum_{j\in\mathcal{K}_0} l_j a_j + \sum_{j\in\mathcal{K}} l_j a_j \\
\text{subject to} \quad & \sum_{j\in\mathcal{K}_0} \gamma_j q_{\ell,j} + \sum_{j\in\mathcal{K}} \gamma_j q_{\ell,j} = f_\ell, && \forall \ell \\
& -\sigma^- a_j \le q_{\ell,j} \le \sigma^+ a_j, && j \in \mathcal{K}_0, \forall \ell \\
& -\sigma^- a_j \le q_{\ell,j} \le \sigma^+ a_j, && j \in \mathcal{K}, \forall \ell \\
& \sum_{j\in\mathcal{K}_0} a_j K_j + \sum_{j\in\mathcal{K}} a_j K_j + \tau_\ell \sum_{j\in\mathcal{K}_0} q_{\ell,j} G_j + \tau_\ell \sum_{j\in\mathcal{K}} q_{\ell,j} G_j \succeq 0 && \forall \ell \\
& a_j \ge 0, j \in \mathcal{K}_0, a_j \ge 0, j \in \mathcal{K}, \forall \ell.
\end{aligned}$$

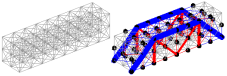
- the dual restricted problem (D-RMP)

$$\begin{aligned}
\max_{\lambda, X} \quad & \sum_\ell f_\ell^T \lambda_\ell \\
\text{s.t.} \quad & -\frac{l}{\sigma^-}(l_j - \sum_\ell K_j \bullet X_\ell) \le \sum_\ell (\gamma_j^T \lambda_\ell + \tau_\ell G_j \bullet X_\ell) \le \frac{1}{\sigma^+}(l_j - \sum_\ell K_j \bullet X_\ell), j \in \mathcal{K}_0, \forall \ell \\
& -\frac{l}{\sigma^-}(l_j - \sum_\ell K_j \bullet X_\ell) \le \sum_\ell (\gamma_j^T \lambda_\ell + \tau_\ell G_j \bullet X_\ell) \le \frac{1}{\sigma^+}(l_j - \sum_\ell K_j \bullet X_\ell), j \in \mathcal{K}, \forall \ell \\
& X_\ell \succeq 0, \ \forall \ell.
\end{aligned}$$

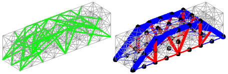
Warm-start strategy



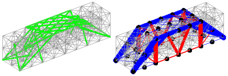
(a)



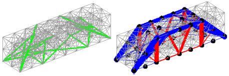
(b) Mem add iter.=1, 444 bars, vol= 0.05681m^3



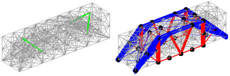
(c) Mem add iter.=2, 518 bars, vol= 0.05429m^3



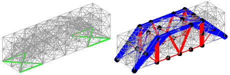
(d) Mem add iter.=3, 564 bars, vol= 0.05417m^3



(e) Mem add iter.=4, 588 bars, vol= 0.05414m^3

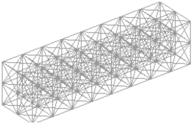


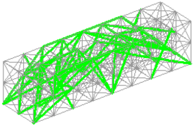
(f) Mem add iter.=5, 592 bars, vol= 0.05414m^3



(g) Mem add iter.=6, 600 bars, vol= 0.05414m^3

Warm-start strategy





- We extend the warm-start strategy

$$(a, q_\ell, S_\ell, s_\ell^+, s_\ell^-) \to (a, \bar{a}, q_\ell, \bar{q}_\ell, S_\ell, s_\ell^+, \bar{s}_\ell^+, s_\ell^-, \bar{s}_\ell^-)$$
$$(\lambda_\ell, X_\ell, x_\ell^+, x_\ell^-) \to (\lambda_\ell, X_\ell, x_\ell^+, \bar{x}_\ell^+, x_\ell^-, \bar{x}_\ell^-)$$

  The variables with the super-bar are vectors in $\mathbb{R}^k$, $k = |K|$

- Computing a warm-**start** point
  - Old vari**ables** $\leftarrow$ solution of the preceding save problem with loose tolerance, say $\epsilon_{opt} = 0.1$
  - New variables (those with super-bar)
    - $\bar{x}_{\ell,j}^+ = \max\{\bar{\gamma}_j^T \lambda_\ell + \tau_\ell \bar{G}_i \bullet X_\ell, \mu_0^{\frac{1}{2}}\}, \ \forall j \in K,$
    - $\bar{x}_{\ell,j}^- = \max\{-\bar{\gamma}_j^T \lambda_\ell - \tau_\ell \bar{G}_i \bullet X_\ell, \mu_0^{\frac{1}{2}}\}, \ \forall j \in K,$
    - $(\bar{x}_a)_j = \max\{|\bar{l}_j - \sigma^+ \sum_\ell (\bar{x^+}_\ell)_j - \sigma^- \sum_\ell (\bar{x^-}_\ell)_j - \bar{K}_i \bullet X_\ell|, \mu_0^{\frac{1}{2}}\}, \ \forall j \in$
    - $\bar{q}_\ell^+ = 0 \ \forall \ell \in \{1, ..., n_L\}$
    - $\bar{a}_j = \mu(\bar{x}_a^{-1})_j, \ \forall j \in K,$
    - $\bar{s}_\ell^+ = \sigma^+ \bar{a}, \ \forall \ell \in \{1, ..., n_L\}$
    - $\bar{s}_\ell^- = \sigma^- \bar{a}, \ \forall \ell \in \{1, ..., n_L\}$

Violation estimations

- **Primal infeasibility** $(\xi_{p_{1,\ell}}, \xi_{p_{2,\ell}}, \xi_{p_{3,\ell}}, \xi_{p_{4,\ell}})$

$$
\begin{aligned}
||\xi_{p_{1,\ell}}||_\infty &= ||f_\ell - \sum_i q_{\ell,i}\gamma_i - \sum_i \bar{q}_{\ell,i}\bar{\gamma}_i||_\infty = ||f_\ell - \sum_i q_{\ell,i}\gamma_i||_\infty = ||\xi^0_{p_1,\ell}||_\infty, \\
||\xi_{p_{2,\ell}}||_\infty &= ||\sigma^+\bar{a} - \bar{q}_\ell - \bar{s}^+_\ell||_\infty = 0, \\
||\xi_{p_{3,\ell}}||_\infty &= ||\sigma^-\bar{a} + \bar{q}_\ell - \bar{s}^-_\ell||_\infty = 0, \\
||\xi_{p_{4,\ell}}||_\infty &= ||-K(a) - \tau_\ell G(q_\ell) + S_\ell - \bar{K}(\bar{a}) - \tau_\ell \bar{G}(\bar{q}_\ell)||_\infty \le ||\xi^0_{p_{4,\ell}}||_\infty + \mu_0^{\frac{1}{2}} \sum \frac{E_i}{\bar{l}_i}
\end{aligned}
\tag{11}
$$

Violation estimations

- **Primal infeasibility** $(\xi_{p_{1,\ell}}, \xi_{p_{2,\ell}}, \xi_{p_{3,\ell}}, \xi_{p_{4,\ell}})$

$$\begin{aligned}
||\xi_{p_{1,\ell}}||_\infty &= ||f_\ell - \sum_i q_{\ell,i}\gamma_i - \sum_i \bar{q}_{\ell,i}\bar{\gamma}_i||_\infty = ||f_\ell - \sum_i q_{\ell,i}\gamma_i||_\infty = ||\xi^0_{p_1,\ell}||_\infty, \\
||\xi_{p_{2,\ell}}||_\infty &= ||\sigma^+\bar{a} - \bar{q}_\ell - \bar{s}^+_\ell||_\infty = 0, \\
||\xi_{p_{3,\ell}}||_\infty &= ||\sigma^-\bar{a} + \bar{q}_\ell - \bar{s}^-_\ell||_\infty = 0, \\
||\xi_{p_{4,\ell}}||_\infty &= ||-K(a) - \tau_\ell G(q_\ell) + S_\ell - \bar{K}(\bar{a}) - \tau_\ell \bar{G}(\bar{q}_\ell)||_\infty \le ||\xi^0_{p_{4,\ell}}||_\infty + \mu_0^{\frac{1}{2}} \sum \frac{E_i}{\bar{I}_i}
\end{aligned} \quad (11)$$

- **Dual infeasibility** $(\xi_{d_1}, \xi_{d_{2,\ell}})$

$$\begin{aligned}
||\xi_{d_1}||_\infty &= ||\sum_\ell (\sigma^+ x^+_\ell + \sigma^- x^-_\ell + \mathcal{K}X_\ell) + x_a - I||_\infty \\
&\le ||2(\bar{I} + \sum_\ell (\sigma_{max}(\varepsilon^-_\ell + \varepsilon^+_\ell) - \bar{\mathcal{K}}X_\ell)) + (4n_L + 1)\mu_0^{\frac{1}{2}} e||_\infty
\end{aligned}$$

$$||\bar{\xi}_{d_{2,\ell}}||_\infty = ||\bar{B}^T\lambda_\ell - \bar{x}^+_\ell + \bar{x}^-_\ell + \tau_\ell \bar{\mathcal{G}} X_\ell|| \le \mu_0^{\frac{1}{2}}.$$

Violation estimations

- **Primal infeasibility** $(\xi_{p_{1,\ell}}, \xi_{p_{2,\ell}}, \xi_{p_{3,\ell}}, \xi_{p_{4,\ell}})$

$$\begin{aligned}
||\xi_{p_{1,\ell}}||_\infty &= ||f_\ell - \sum_i q_{\ell,i}\gamma_i - \sum_i \bar{q}_{\ell,i}\bar{\gamma}_i||_\infty = ||f_\ell - \sum_i q_{\ell,i}\gamma_i||_\infty = ||\xi^0_{p_1,\ell}||_\infty, \\
||\xi_{p_{2,\ell}}||_\infty &= ||\sigma^+ \bar{a} - \bar{q}_\ell - \bar{s}^+_\ell||_\infty = 0, \\
||\xi_{p_{3,\ell}}||_\infty &= ||\sigma^- \bar{a} + \bar{q}_\ell - \bar{s}^-_\ell||_\infty = 0, \\
||\xi_{p_{4,\ell}}||_\infty &= ||-K(a) - \tau_\ell G(q_\ell) + S_\ell - \bar{K}(\bar{a}) - \tau_\ell \bar{G}(\bar{q}_\ell)||_\infty \le ||\xi^0_{p_{4,\ell}}||_\infty + \mu_0^{\frac{1}{2}} \sum \frac{E_i}{\bar{I}_i}
\end{aligned} \tag{11}$$

- **Dual infeasibility** $(\xi_{d_1}, \xi_{d_{2,\ell}})$

$$||\xi_{d_1}||_\infty \le ||2(\bar{I} + \sum_\ell (\sigma_{max}(\varepsilon^-_\ell + \varepsilon^+_\ell) - \bar{\mathcal{K}} X_\ell)) + (4n_L + 1)\mu_0^{\frac{1}{2}} e||_\infty$$

$$||\bar{\xi}_{d_{2,\ell}}||_\infty = ||\bar{B}^T \lambda_\ell - \bar{x}^+_\ell + \bar{x}^-_\ell + \tau_\ell \bar{\mathcal{G}} X_\ell|| \le \mu_0^{\frac{1}{2}}.$$

- **Centrality** $(\bar{a}, \bar{s}_a)$, $(\bar{X}_\ell, \bar{S}_\ell)$, $(\bar{x}^-_\ell, \bar{s}^-_\ell)$, $(\bar{x}^+_\ell, \bar{s}^+_\ell)$
  $(\bar{a}, \bar{s}_a)$, $(\bar{X}_\ell, \bar{S}_\ell)$ are μ_0 centered.

$$\frac{\sigma^+}{\sigma_{max} n_L \mu_0^{\frac{-1}{2}} (\max_\ell(\varepsilon^-_{\ell_j} + \varepsilon^+_{\ell_j}) + \bar{K}_i \bullet X_\ell) + 2n_L}\mu_0 \le (\bar{x}^+_\ell)_j (\bar{s}^+_\ell)_j \le \mu_0 \sigma^+ + \mu_0^{\frac{1}{2}} \sigma^+ (\varepsilon^-_{\ell_j} + \varepsilon^+_{\ell_j}), \ \forall j, \forall \ell.$$

$$\frac{\sigma^-}{\sigma_{max} n_L \mu_0^{\frac{-1}{2}} (\max_\ell(\varepsilon^-_{\ell_j} + \varepsilon^+_{\ell_j}) + \bar{K}_i \bullet X_\ell) + 2n_L}\mu_0 \le (\bar{x}^-_\ell)_j (\bar{s}^-_\ell)_j \le \mu_0 \sigma^- + \mu_0^{\frac{1}{2}} \sigma^- (\varepsilon^-_{\ell_j} + \varepsilon^+_{\ell_j}), \ \forall j, \forall \ell.$$

Example: Nonlinear Vs the relaxation

Nonlinear

$$
\begin{aligned}
\underset{a,q_\ell,u_\ell}{\text{minimize}} \quad & l^T a \\
\text{subject to} \quad & \sum_i q_{\ell,i}\gamma_i = f_\ell, && \forall \ell \\
& \frac{a_i E}{l_i}\gamma_i^T u_\ell = q_{\ell,i} && \forall \ell, \forall i \\
& -a\sigma^- \leq q_\ell \leq \sigma^+ a, && \forall \ell \\
& K(a) + \tau_\ell G(q_\ell) \succeq 0 && \forall \ell \\
& a \geq 0
\end{aligned}
$$

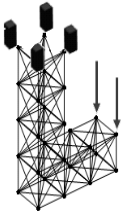
Relaxation

$$
\begin{aligned}
\underset{a,q_\ell,u_\ell}{\text{minimize}} \quad & l^T a \\
\text{subject to} \quad & \sum_i q_{\ell,i}\gamma_i = f_\ell, && \forall \ell \\
& -a\sigma^- \leq q_\ell \leq \sigma^+ a, && \forall \ell \\
& K(a) + \tau_\ell G(q_\ell) \succeq 0 && \forall \ell \\
& a \geq 0
\end{aligned}
$$

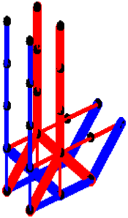
- Least-squares (LSQ) problem

$$
\underset{u_\ell}{\text{minimize}} \quad \max_\ell \frac{1}{||q_\ell^*||^2} \sum_i \Big(\frac{a_i^* E}{l_i}\gamma_i^T u_\ell - q_{\ell,i}^*\Big)^2, \tag{12}
$$

Example: Nonlinear Vs the relaxation

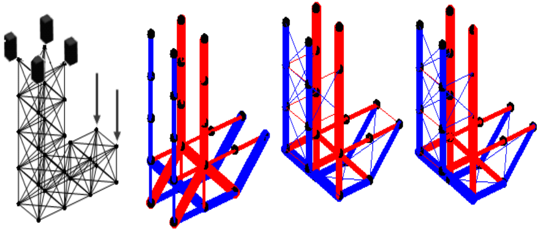


(a)



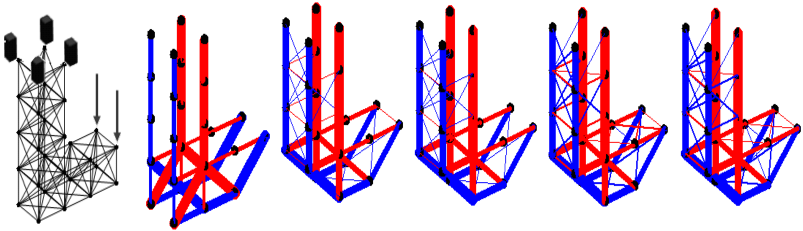
(b) $\tau_\ell = 0$

Example: Nonlinear Vs the relaxation



(a) (b) $\tau_\ell = 0$ (c) $\tau_\ell = 1$, Relaxation (d) $\tau_\ell = 1$ Nonlinear

Example: Nonlinear Vs the relaxation



(a) (b) $\tau_\ell = 0$ (c) $\tau_\ell = 1$, Relaxation (d) $\tau_\ell = 1$ Nonlinear (e) $\tau_\ell = 10$, Relaxation (f) $\tau_\ell = 10$, Nonlinear

τ	0	1	10
Volume (nonlinear SDP)	0.062	0.06222	0.06464
Volume (relaxed SDP)	-	0.06217	0.06433
Violation (LSQ problem)	-	4.96e-06	5.32e-4

M. Kocvara. On the modelling and solving of the truss design problem with global stability constraints. Structural and Multidisciplinary Optimization, 23:189–203, 2002.

Example: Nonlinear Vs the relaxation

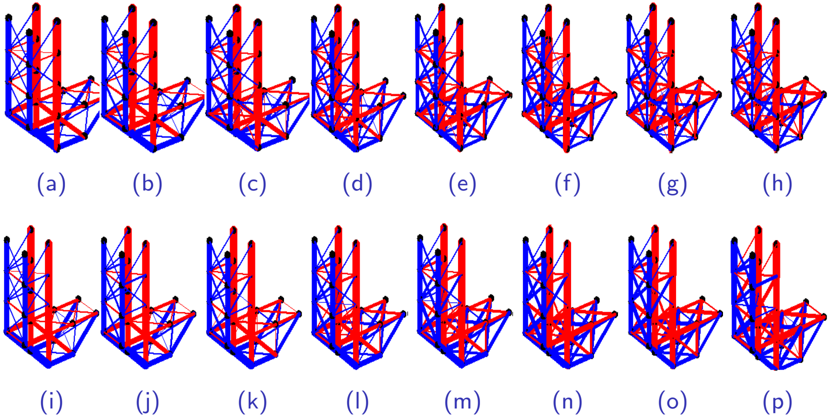
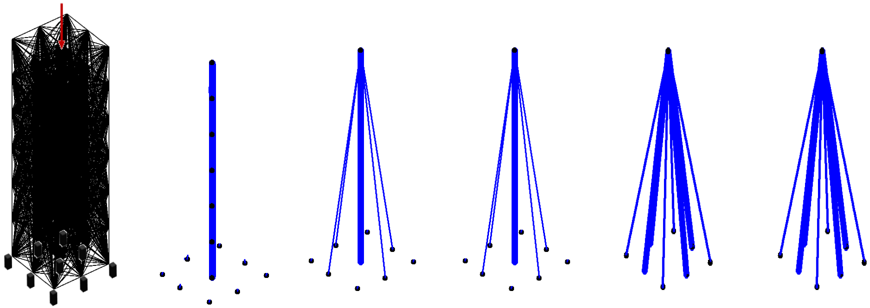


Figure: $\tau_\ell = 20, 30, 40, ..., 90$. (a)-(h) By solving the relaxation linear SDP. (i)-(p) By solving the nonlinear SDP.

τ	20	30	40	50	60	70	80	90
Volume (nonlinear SDP)	0.0677	0.0717	0.0772	0.0846	0.0933	0.1031	0.1139	0.1251
Volume (relaxed SDP)	0.0670	0.0703	0.0749	0.0805	0.0871	0.0947	0.1028	0.1117
Violation (LSQ problem)	0.0024	0.0066	0.0164	0.0306	0.0368	0.0459	0.0591	0.0702

Example: Nonlinear Vs the relaxation



(a) (b) $\tau_\ell = 0$ (c) $\tau_\ell = 1$, Relaxation (d) $\tau_\ell = 1$, Nonlinear (e) $\tau_\ell = 10$, Relaxation (f) $\tau_\ell = 10$, Nonlinear

τ	0	1	10
Volume (nonlinear SDP)	0.0300	0.0302	0.0320
Volume (relaxed SDP)	-	0.0301	0.0310
Violation (LSQ problem)	-	3.7e-6	5.1e-5

Example: Nonlinear Vs the relaxation

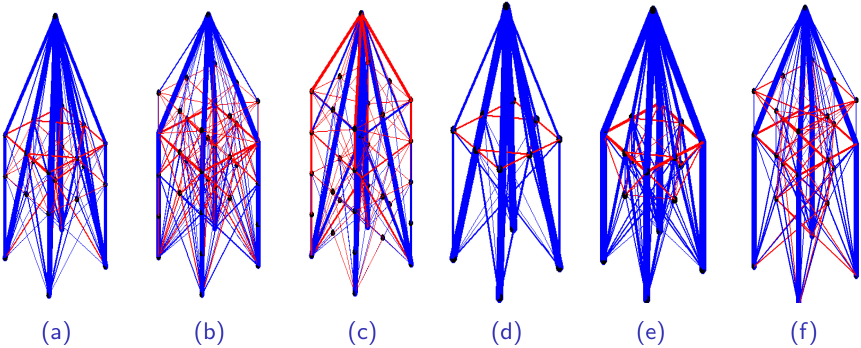
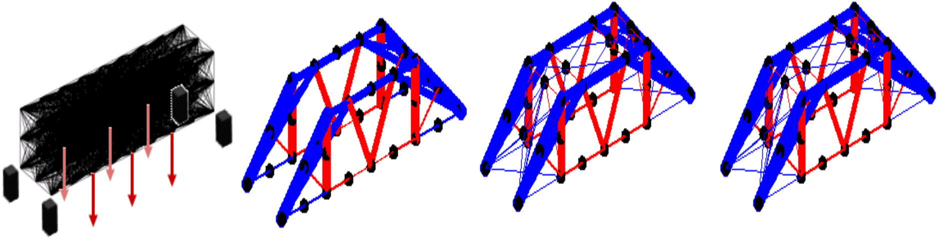


Figure: Optimal design with stability constraints for $\tau_\ell = 20, 30, 40$. (a)-(c) By solving the linear SDP relaxation. (d)-(f) By solving the nonlinear SDP.

τ	20	30	40
Volume (nonlinear SDP)	0.0370	0.0507	0.0663
Volume (relaxed SDP)	0.0358	0.0499	0.0642
Violation (LSQ problem)	0.0151	0.0510	0.5889

Example: Validating the member adding

	All at once	With member adding
Volume (m^3)	**0.05414**	**0.05414**
Final number of bars	**3240**	**600**
Mem. add. iter	1	6
Total CPU (Sec)	**145**	**28**

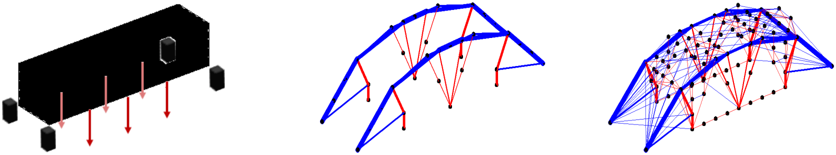


(a) Problem (b) Without stability (c) With stability, no mem. add. (d) With stability, mem. add.

The violation of the compatibility constraints by stable design is equal to $5.8336e-06$.

Example: Large-scale truss problems (SDP)

- #bars $=90,100$ ($n = 1,263$, $m = 180,200$ in standard SDP notation).
- The full-scale SDP requires at least 240GB memory.



(a) Problem (b) Without stability (c) With stability

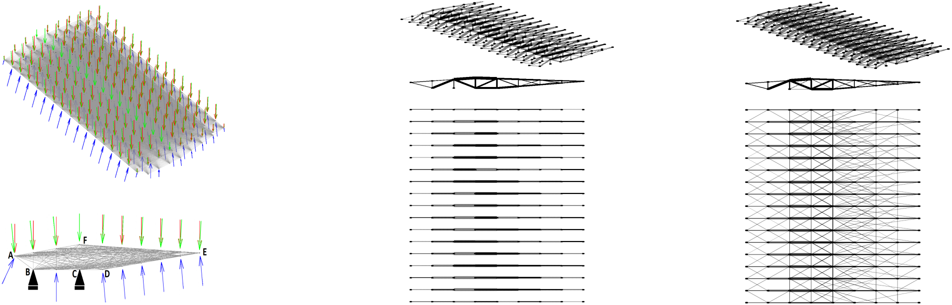
optimality tolerance = [1e-2,1e-2,1e-3,1e-4,1e-5,1e-5,...]

	without warm-start	with warm-start
Volume (m^3)	0.05147	0.05147
Mem. add. iter	7	7
IPM iter in each m. add.	19, 21, 23, 28, 33, 33, 32	19, 21, 23, 20, 23, 20, 18
Number of bars in each mem. add.	2904, 3922, 4584, 4808, 4976, 5064, 5078	2904, 3922, 4584, 4808, 4984, 5084, 5088
Total CPU	3638	2654

The violation of the compatibility constraints by stable design is equal to $52354e-06$.

Example: Stadium roof (multiple-load cases)

36856 members, 3-loads case, 2487 bars and 6 mem add iter needed. CPU=2238Sec.



(a) Problem (b) Without stability (c) With stability

Figure: f_1 (red), f_2 (blue), and f_3 (green). $A = (0, 0, 2.3)$, $B = (5, 0, 0)$, $C = (15, 0, 0)$, $D = (20, 0, 0)$, $E = (40, 0, 2.8)$, $F = (15, 0, 2.4.2)$. The roof is 80m the y-direction.

The violation of the compatibility constraints by unstable design is equal to 0.0011.
The violation of the compatibility constraints by stable design is equal to 0.3190.

Conclusions and future works

Conclusions

- Extended the member adding procedure to SDP.
- Developed and implemented a specialized primal-dual interior point method
  The method and its implementation:
  - exploits the structure of the problem.
  - uses warm-start strategy.

Future work

- Comparison to other SDP solvers.
- Look into the possibilities of using iterative methods for solving the linear systems.

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Thank you for your attention!