Computation and astrophysics of the N-body problem

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Course outline (tentative)

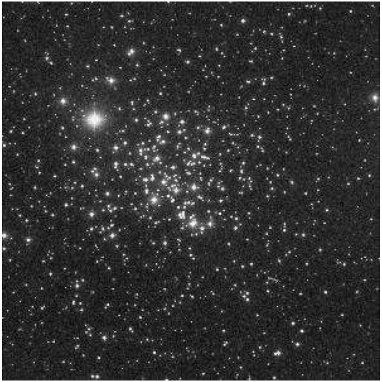
1. The *N*-body problem and its astrophysical settings. Initial conditions. Units. Introduction to a simple *N*-body code (NBODY1). Example - cold collapse. Virialisation and virial equilibrium.
2. Plummer's model. Movie of orbital motions. Crossing time scale. Core collapse - the movie. The relaxation time scale. The structure of the simplest direct-summation N-body code: constant, shared, time steps, Euler & Hermite integrators.
3. Quality control & error growth. Complexity. Acceleration of force computations with software & hardware. GPUs. Regularisation.
4. Refinements: external effects (tides), internal effects (stellar evolution, binary stellar evolution, collisions). The NBODY series of integrators. starlab. Example - cluster in a tidal field (NBODY6).

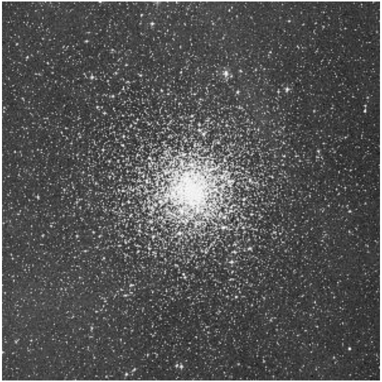
References

- Stellar Dynamics
  1. *Galactic Dynamics, 2e* James Binney and Scott Tremaine, Princeton University Press, 2008, 885 pp.
  2. *Dynamical Evolution of Globular Clusters* Lyman J. Spitzer Jr, Princeton UP, 1988, 196 pp.
  3. *The Gravitational Million Body Problem*, Douglas Heggie, Piet Hut; Cambridge UP, 2003
- *N*-body codes
  1. *Gravitational N-body simulations*, Sverre J. Aarseth Cambridge: CUP, 2003, 413pp
  2. *The Cambridge N-body Lectures*, S.J. Aarseth, C.A. Tout, R.A. Mardling, eds. Springer, LNP760, 2008, 402pp
  3. *Numerical Methods in Astrophysics: An Introduction*, P. Bodenheimer et al, CRC Press, 2006, 344pp
  4. http://www.ast.cam.ac.uk/~sverre/web/pages/nbody.htm
  5. https://github.com/nbodyx/Nbody6ppGPU
  6. http://www.ids.ias.edu/~starlab/ (starlab)
  7. http://amusecode.org/ (AMUSE)

Applications of *N*-Body Schemes in Astrophysics

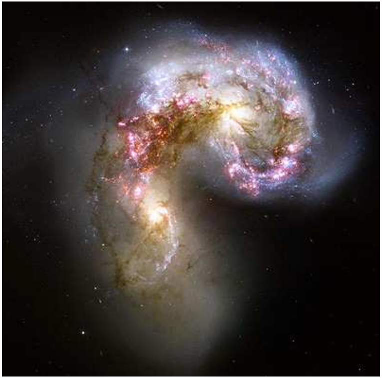
- the Galactic Centre (see http://www.astro.ucla.edu/~ghezgroup/gc/pictures/orbitsMovie.shtml)
- open clusters (M67 here)
- globular clusters (M4 here)





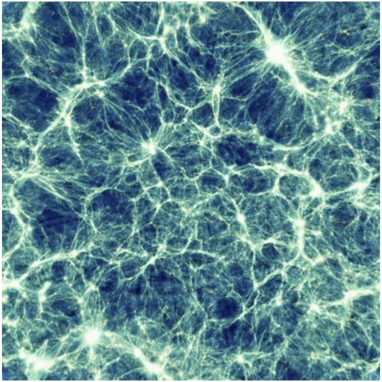
Applications of *N*-Body Schemes in Astrophysics (continued)

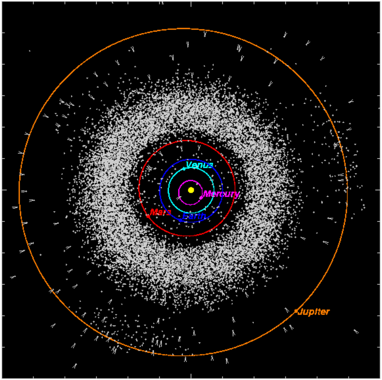
- young dense clusters - Magellanic Cloud clusters
- galaxy dynamics - the Antennae (requires specialised software because of large *N*)



Applications of *N*-Body Schemes in Astrophysics (continued)

- cosmic structure (requires specialised software)
- planetary systems (requires specialised software)





The *N*-body Problem

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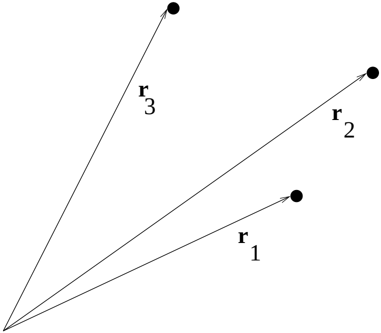
- *N* point masses (this approximation is good while separation of two stars is much greater than the sum of their radii)

The *N*-body Problem

- *N* point masses (this approximation is good while separation of two stars is much greater than the sum of their radii)
- classical gravitation and equations of motion (this approximation is good except close to horizon of black hole, or for close binaries emitting gravitational waves)

Equations of motion



Force on 1 due to 2 is in the direction of $\mathbf{r}_2 - \mathbf{r}_1$, i.e. the unit vector $\dfrac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}$, and has magnitude $\dfrac{Gm_1 m_2}{|\mathbf{r}_2 - \mathbf{r}_1|^2}$.

Equations of motion (continued)

Therefore force on 1 due to 2 is $\dfrac{Gm_1 m_2(\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3}$. Therefore total force on 1 is

$$m_1\ddot{\mathbf{r}}_1 = \sum_{j=1, j\neq 1}^{j=3} \frac{Gm_1 m_j(\mathbf{r}_j - \mathbf{r}_1)}{|\mathbf{r}_j - \mathbf{r}_1|^3}$$

In the N-body problem the equation of motion for body i is

$$\ddot{\mathbf{r}}_i = -G\sum_{j=1,\neq i}^{N} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

Softening

The equation of motion is

$$\ddot{\mathbf{r}}_i = -G \sum_{j=1,\neq i}^{N} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

There is a singularity if $|\mathbf{r}_i - \mathbf{r}_j| = 0$. To avoid this, the denominator is sometimes replaced by $(|\mathbf{r}_i - \mathbf{r}_j|^2 + \varepsilon^2)^{3/2}$, where ε is a small constant, the *softening parameter*. This approximation may be justifiable if close encounters between particles are unimportant – for example, in galaxy dynamics (with scaling by N). Not necessarily good for modelling star clusters.

Initial Conditions

These are $3N$ second-order ordinary differential equations, and hence require $6N$ initial conditions. Usually one uses the three cartesian components of position $\mathbf{r}_i$ and the three components of velocity $\dot{\mathbf{r}}_i$ of the N particles.

Example: **cold collapse**

The initial velocities are zero, and the initial positions are chosen randomly, with (in this example) a uniform spatial distribution in a sphere of some radius a.

How to Simulate Cold Collapse

1. Go to the web page
   http://www.ast.cam.ac.uk/∼sverre/web/pages/nbody.htm
2. Download nbody1.tar.Z
3. Uncompress it (“gunzip nbody1.tar.Z”)
4. Untar it (“tar xvf nbody1.tar”)
5. Go to the source subdirectory (“cd Real8”)
6. Edit the Makefile as follows
   - line 2: replace “f77” by your fortran compiler (e.g. “gfortran”)
   - after line 2, add the line “FC = gfortran” (or your compiler)
7. Make the code (“make”)
8. Go to the test subdirectory (“cd ../test”)
9. Make a copy of the file intest and edit to the following:
   1 **1.0**/25 1 200 1/0.01 **0.1** 10.0 2.0E-05 **0.0**/1 0 **0** 0 0 1 0 0 0 0 0 0 0 0 0/2.0 **1.0** 1.0/**0.0** 0.0 0.0 1.0 1.0

How to Simulate Cold Collapse (continued)

(It is best to give the file a new name, such as cc.in, and keep the old file intest. The changes are in boldface, and lines are here delimited by "/" to save space, but you should keep the format of intest.)

1. Run the code ("../Real8/nbody1 < cc.in")
2. Watch the numbers fly past until the job completes or you have to kill it ("Ctrl-c")[1]
3. To save the output, rerun it with redirection of output ("../Real8/nbody1 < cc.in > cc.out")

[1] If the run fails or fails to complete in a few minutes, kill it. Then edit cc.in to change the third number in line 2 of cc.in to some other positive integer, and try again.

Cold collapse: the output - p1

(Corresponds to slightly different input file.)

```
cc.out_p1        Fri Sep 14 14:14:10 2007        1



          N   NRAND    ETA    DELTAT    TCRIT    QE        EPS

         25    200    0.010     0.1     10.0   2.0E-05   0.0E+00

          OPTIONS       1   2   3   4   5   6   7   8   9  10  11  12  13  14  15

                        1   0   1   0   0   1   0   0   0   0   0   0   0   0   0

          SCALING:   SX  =  0.44  E = -5.72E-01  M(1) = 4.00E-02  M(N) = 4.00E-02  <M> = 4
.00E-02

          SCALING PARAMETERS:   R* = 1.00E+00  M* = 2.50E+01  V* = 3.28E-01  T* = 2.99E+00
```

Cold collapse: the output - p2

```
cc.out_p2        Fri Sep 14 14:14:29 2007        1

 T =    0.0  Q = 0.00  STEPS =       0  DE =  0.000000  E = -0.250000  TC =    0.0
 <R> =  2.00  RCM =  0.0000  VCM =  0.0000  AZ =   0.00000  T6 =   0  NRUN =  1

 T =    0.3  Q = 0.01  STEPS =      35  DE =  0.000000  E = -0.250000  TC =    0.1
 <R> =  1.99  RCM =  0.0000  VCM =  0.0000  AZ =   0.00000  T6 =   0  NRUN =  1

 T =    0.6  Q = 0.02  STEPS =      87  DE =  0.000000  E = -0.250000  TC =    0.2
 <R> =  1.95  RCM =  0.0000  VCM =  0.0000  AZ =   0.00000  T6 =   1  NRUN =  1

 T =    0.9  Q = 0.05  STEPS =     154  DE =  0.000000  E = -0.250000  TC =    0.3
 <R> =  1.89  RCM =  0.0000  VCM =  0.0000  AZ =   0.00000  T6 =   2  NRUN =  1

   BINARY     1  10  0.040  0.040  -0.1  0.3652    1.3  0.5156  1.54  0.970    0
   BINARY     6  15  0.040  0.040  -0.2  0.2525    2.2  0.3785  1.47  0.973    0
   BINARY     8   9  0.040  0.040  -0.2  0.2602    2.1  0.4063  1.94  0.989    0

 T =    1.1  Q = 0.11  STEPS =     258  DE =  0.000000  E = -0.250000  TC =    0.4
 <R> =  1.79  RCM =  0.0000  VCM =  0.0000  AZ =   0.00000  T6 =   3  NRUN =  1

   BINARY     6  15  0.040  0.040  -0.2  0.2509    2.2  0.2665  1.47  0.937    0
   BINARY     8   9  0.040  0.040  -0.2  0.2584    2.2  0.3032  1.89  0.983    0
   BINARY    14  22  0.040  0.040  -0.1  0.3191    1.6  0.4527  0.54  0.997    0
```

Cold collapse: the Movie

Initial conditions:

- All velocities are zero
- Particles are distributed uniformly in a sphere

Dynamic Equilibrium

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- The system “quickly” reaches a “steady state”
- The steady state is in “dynamic equilibrium”, i.e. there is no overal expansion or contraction of the system, or other bulk motion, even though all particles are in motion.
- A dynamic equilibrium is also a state of “virial equilbrium”, which can be analysed in terms of the Virial Theorem.

The Virial Theorem

Define the total kinetic energy T, the total potential energy V, the total energy E and the total "moment of inertia" I by

$$
\begin{aligned}
T &= \frac{1}{2}\sum_{i=1}^{N} m_i \mathbf{v}_i^2 \quad \text{(Kinetic Energy)} \\
V &= -\frac{G}{2}\sum_{i=1}^{N}\sum_{j=1,\neq i}^{N} \frac{m_j m_i}{|\mathbf{r}_i - \mathbf{r}_j|} \quad \text{(Potential Energy)} \\
E &= T + V \quad \text{(Total Energy)} \\
I &= \sum_{i=1}^{N} m_i |\mathbf{r}_i|^2 \quad \text{("Moment of Inertia").}
\end{aligned}
$$

Then from the equations of motion we deduce

$$\ddot{I} = 4T + 2V \quad \text{(Virial Theorem)}$$

$$E = \text{constant} \quad \text{(Energy Conservation)}$$

Proofs: Binney and Tremaine, Sec 7.2.1; this lecture, last pages

Virial Equilibrium

In dynamic equilibrium, I is approximately constant, and so $\ddot{I} \simeq 0$. From the Virial Theorem $\ddot{I} = 4T + 2V$ we deduce

$$2T + V \simeq 0.$$

Using total energy $E = T + V$ we deduce

$$\begin{aligned} T + E \simeq 0 \Rightarrow T &\simeq -E \\ \text{and similarly} \quad V &\simeq 2E. \end{aligned}$$

Mass, length and time scales

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- Total mass

$$M = \sum_{i=1}^{N} m_i$$

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- Characterise system size by "virial radius" R defined by
$$V = -\frac{GM^2}{2R}, \quad \text{where } M \text{ is total mass}$$

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- Characterise speeds by (mass weighted) mean square speed

$$v^2 = \frac{2T}{M}$$

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- Characterise speeds by (mass weighted) mean square speed

$$v^2 = \frac{2T}{M}$$

- Define time scale

$$t_{cr} = \frac{2R}{v} \quad \text{("Crossing time")}$$

Other useful expressions and definitions

In virial equilibrium $V = -2T$, and so $v^2 = \dfrac{GM}{2R}$.

The *virial ratio* is defined to be $Q = \dfrac{T}{|V|}$, and is 0.5 in virial equilibrium.

Another measure of the size of the system is the *half-mass radius*, the radius of a sphere containing the innermost half of the mass, measured with respect to the "centre" of the system.

> *More generally, a "lagrangian radius" is the radius of the sphere containing a given fixed fraction of the mass.*

Observational astronomers may prefer the *half-light radius*, the radius of a disk containing the innermost half of the light, since it can be "easily" measured.

Hénon Units (*aka N*-body units)

This is a conventional system of units in which

$$\begin{aligned} G &= 1 \\ M &= 1 \\ R &= 1 \end{aligned}$$

These are often used in *N*-body simulations.

Example

We have $v^2 = \dfrac{GM}{2R}$.

Suppose a star cluster has $M = 10^5 M_\odot, R = 5\text{pc}$. To convert a velocity from the *N*-body code to km/s, multiply by $\sqrt{\dfrac{GM}{R}}$, where G is expressed in the same units (i.e. km/s, $M_\odot$, pc), i.e. $G \simeq 0.0043$.

Hénon Units (continued)

We have

$$\begin{aligned} G &= 1 \\ M &= 1 \\ R &= 1 \end{aligned}$$

In these units:

- The characteristic speed $v^2 = \dfrac{GM}{2R} = \dfrac{1}{2}$
- The crossing time $t_{cr} = \dfrac{2R}{v} = 2\sqrt{2}$
- The total energy $E = -\dfrac{1}{2}Mv^2 = -\dfrac{1}{4}$

Significance of the crossing time

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- Time scale of approach to virial equilibrium

Significance of the crossing time

- Time scale of cold collapse
- Time scale of approach to virial equilibrium
- Time scale of orbital motions in virial equilibrium (Lecture 2)

The code NBODY1: input

The meaning of each input parameter is defined in the file define.f in the source subdirectory. For illustration we use the file for the cold collapse simulation:

```
1 0.5
25 1 200 1
0.01 0.1 10.0 2.0E-05 0.0
1 0 0 0 0 1 0 0 0 0 0 0 0 0
2.0 1.0 1.0
0.0 0.0 0.0 1.0 1.0
```

The code NBODY1: input

The meaning of each input parameter is defined in the file define.f in the source subdirectory. For illustration we use the file for the cold collapse simulation:

```
1 0.5
25 1 200 1
0.01 0.1 10.0 2.0E-05 0.0
1 0 0 0 0 1 0 0 0 0 0 0 0 0 0
2.0 1.0 1.0
0.0 0.0 0.0 1.0 1.0
```

Line 1: 1 0.5

1. KSTART Control index (1: new run; >1: restart; 3: new params).
   Comment: runs can be restarted following a crash or other accident
2. TCOMP Maximum computing time in minutes

The code NBODY1: input line 2

Line 2: 25 1 200 1

1. N Total particle number.
2. NFIX Output frequency of data save or binaries (option 3 & 6 – see below).
   Comment: There is a basic output interval. You need not output the data or information on binaries at all such output times.
3. NRAND Random number sequence skip.
   Comment: initialises the random number generator
4. NRUN Run identification index.

The code NBODY1: input line 3

Line 3: 0.01 0.1 10.0 2.0E-05 0.0

1. ETA Time-step parameter for total force polynomial.
   Comment: this controls the accuracy of the numerical solution of the equations of motion (see later)
2. DELTAT Output time interval in units of the crossing time.
   Comment: In N-body units the crossing time is $2\sqrt{2} \simeq 3$.
3. TCRIT Termination time in units of the crossing time.
4. QE Energy tolerance (stop if DE/E $>$ 5*QE & KZ(2) $\leq$ 1).
   Comment: the program stops if the relative change in energy exceeds the stated value and the appropriate option is chosen (see below)
5. EPS Softening parameter (square saved in EPS2).

The code NBODY1: input line 4

Line 4: 1 0 1 0 0 1 0 0 0 0 0 0 0 0 0
Note: This is the line of *options*. In general, a zero value indicates that the option is not selected

1. 1 COMMON save on unit 1 if TCOMP > CPU or if TIME > TCRIT.
   Comment: COMMON is a block of variables which are enough to restart the run, if desired.
2. 2 COMMON save on unit 2 at output (=1); restart if DE/E > 5*QE (=2).
   Comment: if this value is 1, the COMMON variables are saved every output time; if 2, and the accuracy of the run has deteriorated, the code attempts to repeat the most recent part of the run with higher accuracy (and carry on)

The code NBODY1: input line 4 (continued)

Line 4: 1 0 **1 0 0 1 0 0** 0 0 0 0 0 0 0

3 Basic data written to unit 3 at output time (frequency NFIX).
Comment: if this is positive, complete information on the particles is saved in a binary file OUT3 at each output time.

4 Initial conditions on unit 4 (=1: output; =2: input).
Comment: if this value is 1 the code dumps the initial conditions on a file called fort.4; if 2, the initial conditions are read in from this file (usually called fort.4)

5 Initial conditions (=0: uniform & isotropic; =1: Plummer).
Comment: we used 0. The Plummer model is explained in Lecture 2

6 Output of significant binaries.

7 Output of movie frames on unit 7.
Comment: you can experiment with this!

8 Generation of two subsystems (merger experiment).

The code NBODY1: input line 4 (continued)

Line 4: 1 0 1 0 0 1 0 0 **0 0 0 0 0** 0 0

9 Individual bodies printed at output time (MIN(5**KZ9,N)).
Comment: the value is referred to as KZ9, and can be used to control how many particles are listed

10 No scaling of initial conditions.
Comment: i.e. use units of the input file fort.4, and do not scale to *N*-body units

11 Modification of ETA by tolerance QE.
Comment: ETA controls the accuracy of the integration (see above), and this option lets the code attempt to adjust this by monitoring the relative change in energy.

12 Initial parameters for binary orbit.
Comment: if non-zero, the code reads the semi-major axis and eccentricity of a binary formed by the first two particles

13 Escaper removal (R > 2*RTIDE; RTIDE = 10*RSCALE).
Comment: here RSCALE is the virial radius.

The code NBODY1: input line 4 (continued)

Line 4: 1 0 1 0 0 1 0 0 0 0 0 0 **0 0**

14 Adjustment of coordinates & velocities to c.m. condition.
Comment: uses a barycentric coordinate system

15 Ignored

The code NBODY1: input line 5

Line 5: 2.0 1.0 1.0

1. ALPHAS Power-law index for initial mass function (routine DATA).
   Comment: the initial mass function of the stars is a power law, i.e. $f(m) \propto m^{-\alpha}$ in the range $BODYN < m < BODY1$ (see below)
2. BODY1 Maximum particle mass before scaling.
3. BODYN Minimum particle mass before scaling.
   Comment: choosing BODY1 = BODYN gives equal masses. The code interprets these as being in solar masses, and then scales to N-body units internally.

The code NBODY1: input line 6

Line 6: 0.0 0.0 0.0 1.0 1.0

1. Q Virial ratio (routine SCALE; Q = 0.5 for equilibrium).
2. VXROT XY-velocity scaling factor (> 0 for solid-body rotation).
   Comment: adds rotation about the *z*-axis
3. VZROT Z-velocity scaling factor (not used if VXROT = 0).
   Comment: if less than 1 the *z*-components of velocity are reduced and the systems tends to flatten
4. RBAR Virial radius in pc (for scaling to physical units).
   Comment: specifies the unit of length in parsecs
5. ZMBAR Mean mass in solar units.

Cold collapse: the output - p1

The first four lines repeat most of the input parameters

```
cc.out_p1        Fri Sep 14 14:14:10 2007        1


          N  NRAND    ETA   DELTAT   TCRIT   QE        EPS

         25    200   0.010     0.1    10.0   2.0E-05   0.0E+00

          OPTIONS       1   2   3   4   5   6   7   8   9  10  11  12  13  14  15

                        1   0   1   0   0   1   0   0   0   0   0   0   0   0   0

          SCALING:   SX  =   0.44  E = -5.72E-01  M(1) = 4.00E-02  M(N) = 4.00E-02  <M> = 4
.00E-02

          SCALING PARAMETERS:   R* = 1.00E+00  M* = 2.50E+01  V* = 3.28E-01  T* = 2.99E+00
```

Cold collapse: the output (continued)

line 5: SCALING: SX = 0.44 E = -5.72E-01 M(1) = 4.00E-02 M(N) = 4.00E-02 <M> = 4.00E-02

Comments:

1. E is the energy before scaling to *N*-body units
2. M(1),M(N) are the same as BODY1, BODYN (see above), but now in *N*-body units.
3. <M> is the mean mass (*N*-body units)

line 6: SCALING PARAMETERS: R* = 1.00E+00 M* = 2.50E+01 V* = 3.28E-01 T* = 2.99E+00

Comment: scaling factors for length (to parsecs), mass (to solar masses), velocities (to km/s), time (Myr, approximately)

Cold collapse: the output (continued)

The following gives an example of the output produced at each output time:

T = 0.9 Q = 0.05 STEPS = 154 DE = 0.000000 E = -0.250000 TC = 0.3

<R> = 1.89 RCM = 0.0000 VCM = 0.0000 AZ = 0.00000 T6 = 2 NRUN = 1

BINARY 1 10 0.040 0.040 -0.1 0.3652 1.3 0.5156 1.54 0.970 0

BINARY 6 15 0.040 0.040 -0.2 0.2525 2.2 0.3785 1.47 0.973 0

BINARY 8 9 0.040 0.040 -0.2 0.2602 2.1 0.4063 1.94 0.989 0

Line 1

1. Time in *N*-body units
2. Virial ratio
3. Number of integration steps taken (see below)
4. Change in energy (*N*-body units)
5. Total energy
6. Time in units of the crossing time

Cold collapse: the output (continued)

T = 0.9 Q = 0.05 STEPS = 154 DE = 0.000000 E = -0.250000 TC = 0.3

<R> = 1.89 RCM = 0.0000 VCM = 0.0000 AZ = 0.00000 T6 = 2 NRUN = 1

BINARY 1 10 0.040 0.040 -0.1 0.3652 1.3 0.5156 1.54 0.970 0
BINARY 6 15 0.040 0.040 -0.2 0.2525 2.2 0.3785 1.47 0.973 0
BINARY 8 9 0.040 0.040 -0.2 0.2602 2.1 0.4063 1.94 0.989 0

Line 2

1. Virial radius
2. Distance of centre of mass from origin
3. Velocity of centre of mass
4. Angular momentum about the *z*-axis
5. Time in Myr (approximately)
6. Run number (as in input)

Cold collapse: the output (continued)

T = 0.9 Q = 0.05 STEPS = 154 DE = 0.000000 E = -0.250000 TC = 0.3

<R> = 1.89 RCM = 0.0000 VCM = 0.0000 AZ = 0.00000 T6 = 2

NRUN = 1

BINARY **1 10 0.040 0.040 -0.1 0.365 1.3 0.5156 1.54 0.970 0**

Line 3 (one line for each binary)

1. (2 numbers) Names of the two components
2. (2 numbers) Masses of the components (*N*-body units)
3. Internal energy of the binary (per unit reduced mass); this excludes the energy of the centre of mass of the binary
4. Semi-major axis of the binary
5. Angular velocity (mean motion) of the binary
6. Separation of the components
7. Distance of the binary from the "centre" of the system
8. Eccentricity of the binary
9. Number of binary periods since $t = 0$

Extracting useful information with awk

The typical output:
T = 0.9 Q = 0.05 STEPS = 154 DE = 0.000000 E = -0.250000 TC = 0.3
<R> = 1.89 RCM = 0.0000 VCM = 0.0000 AZ = 0.00000 T6 = 2 NRUN = 1
BINARY 1 10 0.040 0.040 -0.1 0.3652 1.3 0.5156 1.54 0.970 0
(Note that there are only three lines in the output file, but these may be wrapped in the pdf you are reading.)
We can extract data with the linux command awk.
First, create a text file called (say) q.awk, containing the following line:

```
{if ($1==”T”) print $3,$6}
```

This means that we pick out only those lines of output beginning with the string “T”, and from such lines print out the third and sixth fields (fields being separated by spaces), i.e. the values of T, Q.

Extracting useful information with awk (continue)

First rerun the code but collect the output in a text file cc.out. (Again the run may have to be killed after a short time). The command is

../Real8/nbody1 < cc.in > cc.out

Then awk is run with

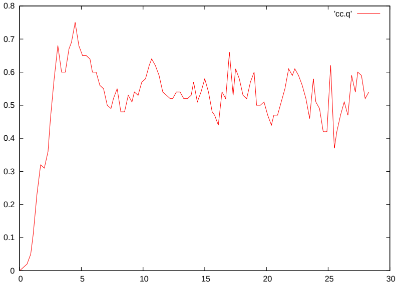
awk -f q.awk cc.out > cc.q

which means that awk is to pick up its instructions from the file q.awk, act on the file cc.out, and direct its output to a new file cc.q. Here is an example of cc.q (a table of T, Q values)

0.0 0.00
0.3 0.01
0.6 0.02
0.9 0.05
1.1 0.11
1.4 0.23

Plotting useful information with gnuplot

Start gnuplot (“gnuplot”), and then plot the file just produced:
gnuplot> plot ’cc.q’ w l
Note: “w l” means “with lines”; otherwise you just get points.
This produces a window with the following graph:



Plotting useful information with gnuplot (continued)

The plot can be improved and made permanent with a sequence of commands like this:

```
gnuplot> set xlab 'Time
gnuplot> set ylab 'Virial ratio
gnuplot> set ter post
gnuplot> set out 'q.ps
gnuplot> replot
```

Then you can print or view the postscript file q.ps

Proof of the Virial Theorem

"Moment of inertia"

$$I = \sum_{i=1}^{N} m_i (\mathbf{r}_i . \mathbf{r}_i)^2 .$$

Hence

$$\begin{aligned} \dot{I} &= \sum_{i=1}^{N} 2 m_i \mathbf{r}_i . \mathbf{v}_i \\ \ddot{I} &= \sum_{i=1}^{N} 2 m_i (\mathbf{v}_i . \mathbf{v}_i + \mathbf{r}_i . \dot{\mathbf{v}}_i). \end{aligned}$$

The first term gives 4 times the kinetic energy, i.e. $4T$. From the equations of motion (above), the second term gives

$$-G \sum_{i=1}^{N} 2 m_i \mathbf{r}_i . \sum_{j=1,\neq i}^{N} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} = -G \sum_{j=1}^{N} 2 m_j \mathbf{r}_j . \sum_{i=1,\neq j}^{N} m_i \frac{\mathbf{r}_j - \mathbf{r}_i}{|\mathbf{r}_j - \mathbf{r}_i|^3} \quad (1)$$

(swapping i and j).

Proof of the Virial Theorem (Contd)

Therefore the second term gives half their sum, i.e.

$$-G\sum_{i=1}^{N}\sum_{j=1,\neq i}^{N} m_i m_j(\mathbf{r}_i - \mathbf{r}_j).\frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} = -G\sum\sum\frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|},$$

the sum being over all distinct i, j between 1 and N. This is *twice* the potential energy, because each pair is counted twice.
Hence $\ddot{I} = 4T + 2V$.

Exercises from Lecture 1

1. Carry out the procedure of downloading NBODY1, running the cold collapse simulation, and plotting the time-dependence of the virial ratio.

Computation and astrophysics of the N-body problem

Outline of Lecture 1

1. Applications to star clusters
2. Equations of the *N*-body problem
3. Simulating cold collapse
4. The Virial Theorem; the crossing time; units and scaling
5. NBODY1: input, output, plotting

Plummer’s model

- In the context of star clusters, a “model” is a prescription for the distribution of the stars in phase space, i.e. the joint distribution of position and velocity

Plummer's model

- In the context of star clusters, a "model" is a prescription for the distribution of the stars in phase space, i.e. the joint distribution of position and velocity
- Plummer's model is a particular model of a system which is in virial (and dynamic) equilibrium
- Convenient analytical distributions
  - Density $\rho(r) = \dfrac{3M}{4\pi a^3\left(1 + r^2/a^2\right)^{5/2}}$
  - Potential $\phi(r) = -\dfrac{GM}{(1 + r^2/a^2)^{1/2}}$
  - Velocity distribution at radius r: $f(\mathbf{v}) \propto \left(-\phi(r) - \dfrac{1}{2}v^2\right)^{7/2}$, $v^2 < -2\phi$

In N-body units, $G = M = 1$ and $a = \dfrac{3\pi}{16}$.

Plummer's model in pictures

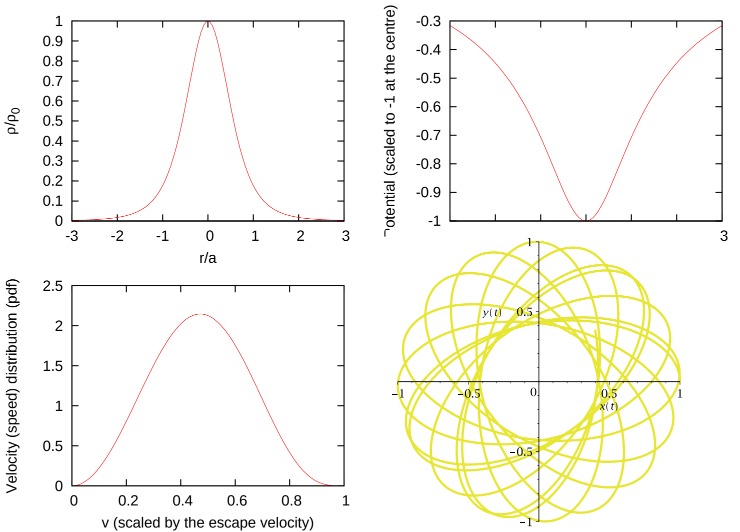


Figure : Density, potential, velocity distribution, an orbit

Models of stellar systems

- In the language of statistical physics, a system is specified by the distrbution of particles in phase space, which has coordinates $(\mathbf{r}, \mathbf{v})$.
- The distribution function is $f(\mathbf{r}, \mathbf{v}, t)$, i.e. the number of stars in a hypercube with sides $d\mathbf{r}, d\mathbf{v}$ is $f(\mathbf{r}, \mathbf{v}) d\mathbf{r} d\mathbf{v}$.
- Because particles are conserved, they obey a conservation equation

$$
\begin{aligned}
\frac{\partial f}{\partial t} + \nabla_{\mathbf{r}}.(f\dot{\mathbf{r}}) + \nabla_{\mathbf{v}}.(f\dot{\mathbf{v}}) &= 0 \\
\Rightarrow \frac{\partial f}{\partial t} + \nabla_{\mathbf{r}}.(f\mathbf{v}) - \nabla_{\mathbf{v}}.(f\nabla_{\mathbf{r}}\phi(\mathbf{r}, t)) &= 0 \\
\Rightarrow \frac{\partial f}{\partial t} + \mathbf{v}.\nabla_{\mathbf{r}} f - \nabla_{\mathbf{r}}\phi(\mathbf{r}, t).\nabla_{\mathbf{v}} f &= 0
\end{aligned}
$$

This is the *collisionless Boltzmann equation*

Solving the CBE $\partial f/\partial t + \mathbf{v}.\nabla_{\mathbf{r}} f - \nabla_{\mathbf{r}}\phi(\mathbf{r}, t).\nabla_{\mathbf{v}} f = 0$

- Example: suppose that the potential $\phi(\mathbf{r})$ is time-independent, and that f is some function of the particle energy $E = v^2/2 + \phi(\mathbf{r})$, i.e. $f(\mathbf{r}, \mathbf{v}) = F(E)$.
- Then

$$\begin{aligned}
\partial f/\partial t &= 0 \\
\mathbf{v}.\nabla_{\mathbf{r}} f &= F'(E)\mathbf{v}.\nabla_{\mathbf{r}}\phi(\mathbf{r}) \\
\nabla_{\mathbf{r}}\phi(\mathbf{r}, t).\nabla_{\mathbf{v}} f &= F'(E)\nabla_{\mathbf{r}}\phi(\mathbf{r}).\mathbf{v}
\end{aligned}$$

- Hence this f satisfies the CBE with $\partial f/\partial t = 0$, i.e. it is an equilibrium (stationary) solution.
- In general, f is a solution if it is expressed as a function of constants of the motion (e.g. E if the potential is independent of t, and angular momentum if the potential is spherically symmetric.) This roughly is *Jeans' Theorem*.

Self-consistent models

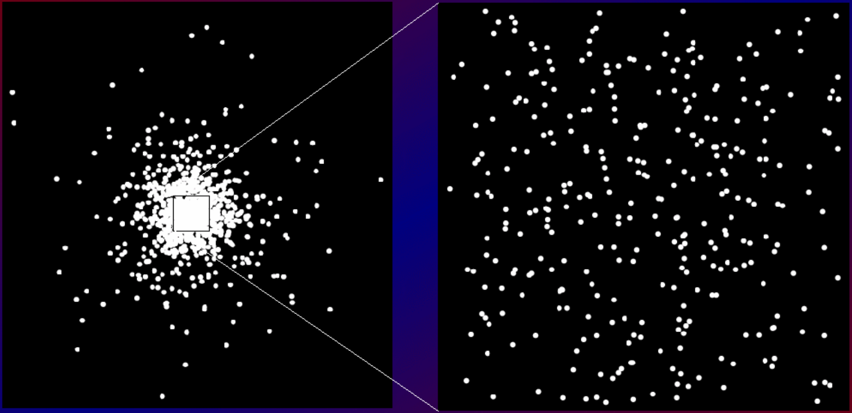
- Plummer's model is given by $F(E) = C|E|^{7/2}$ when $E < 0$, 0 otherwise, where C is constant.
- In terms of $f(\mathbf{r}, \mathbf{v}) = F(E)$, the space density is given by $\rho(\mathbf{r}) = m \int F(E) d^3\mathbf{v}$, where m is the mass of one star.
- For Plummer's model

$$\rho(\mathbf{r}) = 4\pi m C \int_0^{\sqrt{-2\phi}} (-\phi - v^2/2)^{7/2} v^2 dv, = C'(-\phi)^5, \qquad (2)$$

  where C' is another constant.
- ρ and ϕ are also related by Poisson's equation $\nabla^2\phi(\mathbf{r}) = 4\pi G\rho(\mathbf{r})$
- If we assume that ϕ is spherically symmetric (depending only on r), for Plummer's model this is $\phi''(r) + 2\phi'(r)/r = 4\pi G C'(-\phi(r))^5$.
- Solution: $\phi(r) = C''\left(1 + \dfrac{r^2}{a^2}\right)^{-1/2}$ for suitable constants C'', a.

YITP

Core collapse: Two simulations



Entire system
13 seconds to t = 3.3

Central area
13 seconds to t = 330

The initial conditions (Plummer's model)

Lessons from the simulations

Evolution on two time scales:

Lessons from the simulations

Evolution on two time scales:

- orbital motions (crossing time scale; see Lecture 1)
- much slower evolution of the statistical distribution: evolution of the central density
  - This is the statistical result of numerous “close” encounters between pairs of particles
  - This process is called “two-body relaxation”, or “collisional relaxation”
  - It acts on a time scale called the “relaxation time”

Other observations:

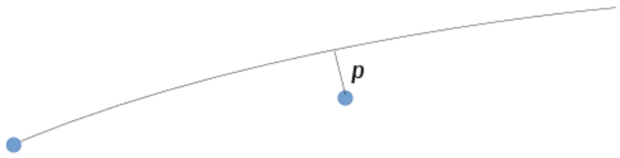
- Plummer’s model is in dynamical equilibrium
- The slow increase in the central density is called “core collapse”
- *Note: the core is the name given to the region in the centre where the density is nearly uniform*

(Two-body) relaxation time

- Local definition $t_r = \dfrac{0.065v^3}{\rho m G^2 \ln \gamma N}$ where
  - v is the velocity dispersion (root mean square velocity)
  - ρ is the space (mass-)density
  - m is the particle mass
  - γ is a constant (about 0.11 for equal masses)
  - N is number of particles
- Global definition: half-mass relaxation time
  $t_{rh} = 0.138\dfrac{N^{1/2}r_h^{3/2}}{m^{1/2}G^{1/2}\ln(\gamma N)}$, where
  - r_h is the *half-mass radius* (containing the innermost half of the system; see Lecture 1); comparable with the virial radius
  - Recall (lecture 1) $\langle v^2 \rangle = \dfrac{GNm}{2R}$
  - Mean density inside r_h is $0.5Nm/(4\pi r_h^3/3)$

"Derivation" of the relaxation time

- Consider a distant encounter between two stars at distance p (impact parameter, which is approximately the distance of closest approach in a distant encounter). Let v be relative speed.



- Acceleration at closest approach is Gm/p^2, duration $\sim p/v$
- Hence change of velocity has magnitude $\Delta v \sim Gm/(pv)$.
- After many encounters, total *transverse* change of velocity is nearly zero. Therefore we measure the strength of two-body encounters by summing $(\Delta v)^2$.

"Derivation" of the relaxation time (contd)

- $\Delta v \sim Gm/(pv)$.
- Let n by the number-density of stars.
- In time t the number of encounters within distance p is of order $n.\pi p^2.vt$

- The number between p and $p + dp$ is of order $n.\pi 2p.dp.vt$
- Hence the sum of $(\Delta v)^2$ over this time is

$$\begin{aligned} \sum(\Delta v)^2 &\sim \int nvt\left(\frac{Gm}{pv}\right)^2 pdp \\ &= \frac{ntG^2m^2}{v}\int\frac{dp}{p} \end{aligned}$$

“Derivation” of the relaxation time (contd)

- $$\sum(\Delta v)^2 \sim \frac{ntG^2m^2}{v}\int\frac{dp}{p} = \frac{ntG^2m^2}{v}[\ln p]_{p_{min}}^{p_{max}}$$

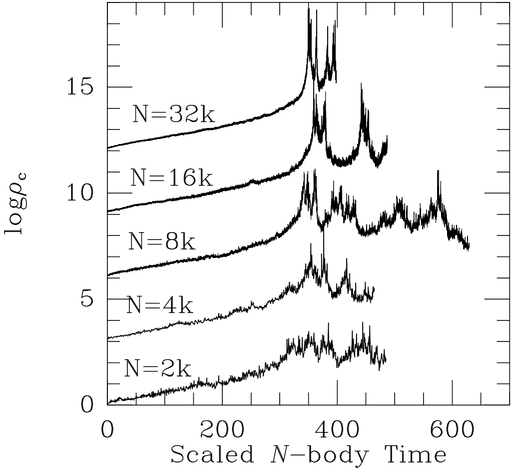
  - For p_{max} we take the radius of the cluster, say the virial radius R
  - For p_{min} we take the value at which the assumption of small deflection breaks down, i.e. $\Delta v \sim v$, or $p \sim Gm/v^2$.
  - Then $[\ln p]_{p_{min}}^{p_{max}} \sim \ln(Rv^2/(Gm))$
  - By the virial theorem, $2T + V \simeq 0$. Therefore $Nm\langle v^2\rangle - GN^2m^2/(2R) \simeq 0$, and so $R\langle v^2\rangle/(Gm) \simeq N/2$
- The total effect of encounters becomes important when $\sum(\Delta v)^2 \simeq v^2$, i.e. at a time $t \sim v^3/(G^2m^2n\ln N)$.

Dynamical friction

- The assumption that the average of $\Delta \mathbf{v}$ is negligible is only true for the component orthogonal to the space velocity of the star.
- The longitudinal component (parallel to the velocity **v** of the star) is of the form $\sum \Delta v_{\|} \sim -vt/t_r$.
- Because the effect is in the opposite direction to **v** it is called "dynamical *friction*"
- The statistical effect of encounters is a balance between the "dissipative" effect of dynamical friction and the "heating" effect described by $\sum (\Delta v)^2$.
- These effects drive the velocities towards a Maxwellian distribution
- Because high-speed stars escape, this distribution is never achieved.

Significance of the relaxation time

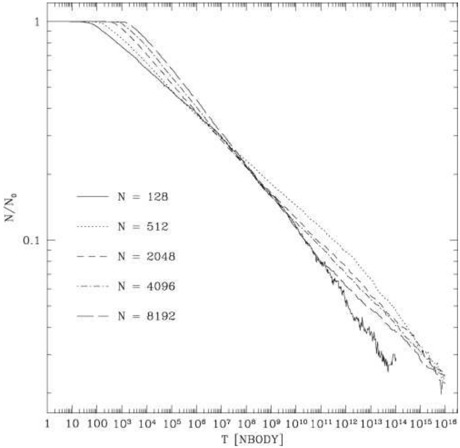
- Time scale of core collapse - here is central density v. time



Source: Makino 1996

Significance of the relaxation time (cont)

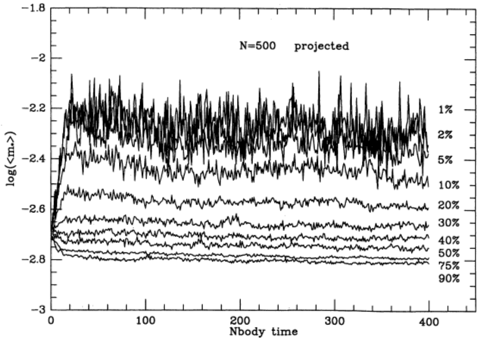
- Time scale of escape (actually several/many t_r)



Source: Baumgardt+ 2002

Significance of the relaxation time (cont)

- Time scale of *mass segregation* (if there is a distribution of masses, the heavier particles sink to the centre on a time scale which is a fraction of t_r)



Mean mass in Lagrangian shells against time (*Source: Giersz+ 1996*)

Why does core collapse happen?

- Relaxation acts like collisions in a gas: it causes “conduction” of energy from warmer regions (where the typical stellar speeds are large, i.e. in the core) to cooler (where the speeds are smaller, i.e. in the outer “halo”)
- Stars in the core are “cooled”, which makes them fall nearer to the centre, and speed up; stars in the halo are “heated”, which makes them move further from the centre, and slow down.
- This new “temperature” profile accentuates the temperature gradient, and the collapse of the core. There is an accelerating “gravothermal runaway” (or “gravothermal catastrophe”).
- The time scale of core collapse is the relaxation time scale.

The negative specific heat of self-gravitating systems

- Virial theorem $\ddot{I} = 4T + 2V$
- Virial equilibrium: $4T + 2V \simeq 0$
- Energy $E = T + V$, and so $2T + 2E \simeq 0$
- Initial state $T_0, E_0 \simeq -T_0$
- Suppose system suddenly loses a little kinetic energy.
  - New state has $E_1 = T_1 + V_1 \simeq T_1 + V_0 < T_0 + V_0 = E_0$.
  - New state virialises with kinetic energy $T_1' = -E_1 > -E_0 = T_0$
  - Thus, although kinetic energy was removed, the new kinetic energy is larger than the old kinetic energy
  - Strictly, the virial theorem does not apply (without modification) when applied to *part* of a system
- Example: an artificial satellite, subject to air drag, speeds up as it falls onto a lower orbit.

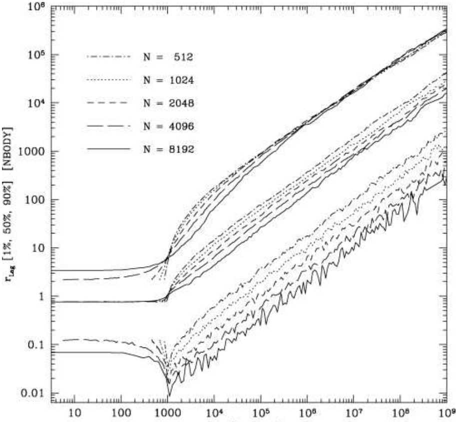
The Two Types of Stellar Dynamics

- *Collisional* stellar dynamics deals with phenomena on time scales of a few t_{rh} (open and globular star clusters; some galactic nuclei; core collapse; mass segregation)
- *Collisionless* stellar dynamics deals with phenomena on time scales much less than t_{rh} (spiral structure, galaxy collisions (!), virialisation)
- In the context of the phrase “Collisional stellar dynamics” the “collisions” are gravitational two-body encounters.

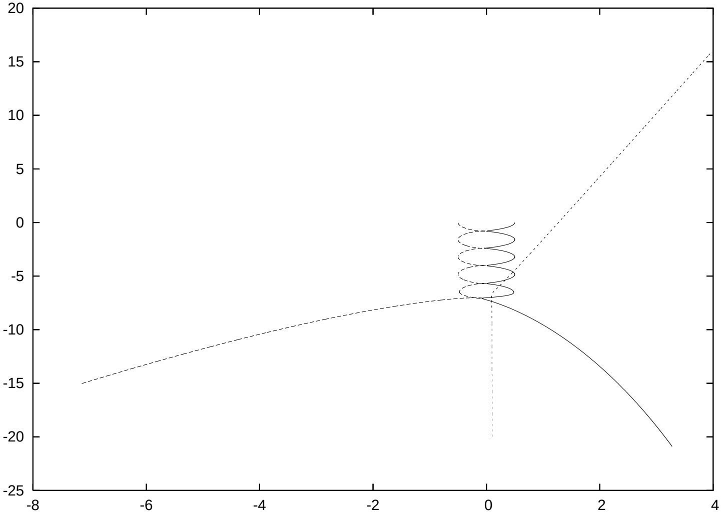
Post-collapse evolution

Depends on boundary conditions:

- "isolated" system: binaries form in the core, liberating energy, which expands the system on the time scale t_{rh} (by a feedback mechanism). As r_h expands, t_{rh} increases. The system *very* slowly loses mass



Formation of Binaries - an example



The dynamics of binary stars

- The dynamics of a binary star is a *two-body problem*, whose solution is well known ("Kepler's problem")
- A binary has two kinds of dynamical degrees of freedom
  - *internal*, i.e. the relative motion of the two stars (relative speed v), and their gravitational interaction (separation r)
  - *external*, i.e. the motion of the barycentre (centre of mass) of the two stars (speed V)
- The energy of a binary consisting of masses m_1, m_2 is

$$\begin{aligned} E &= \frac{1}{2}(m_1+m_2)V^2 + \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}v^2 - \frac{Gm_1 m_2}{r} \\ &= \frac{1}{2}(m_1+m_2)V^2 - \frac{Gm_1 m_2}{2a}, \end{aligned}$$

  where a is the semi-major axis of the Kepler orbit
- An encounter of a binary with a single star is a *three-body problem*, and has no known general solution (example)

Three-body interactions

- If the three stars come to comparable distances, they exchange energy
- On average, the exchange of energy can be understood as an approach to equipartition
  - If $v \gg V$ then the internal degrees of freedom tend to lose kinetic energy in the encounter. Therefore the energy of the binary becomes more negative, and a decreases.
  - If $v \ll V$ then the internal degrees of freedom tend to gain kinetic energy, and a increases.

Hard and soft binaries

- Reminder: v is the relative speed of the components of a binary, V is its speed relative to an incoming third star.
- Now $v^2 \sim Gm/a$ (e.g. for a circular binary with equal masses m), and $V^2 \sim \sigma^2$, the rms speed of stars in the cluster which the binary encounters.
- The condition $v^2 \sim V^2$ is thus roughly the condition $Gm/a \sim \sigma^2$, or $a \sim Gm/\sigma^2$.
- Thus if $a \ll Gm/\sigma^2$ then $v^2 \gg V^2$, and a tends to decrease; and if $a \gg Gm/\sigma^2$, a tends to increase.
- Binaries with $a \ll Gm/\sigma^2$ are called (very) hard, and tend to become harder; binaries with $a \gg Gm/\sigma^2$ are called (very) soft, and tend to become softer.

The formation rate of binaries

- Only hard binaries are of interest, i.e. $a < Gm/\sigma^2$.
- Consider a system of density n stars per unit volume
- In time t, one star moves typical distance σt
- It encounters another star within a distance a if $n.\pi a^2.\sigma t \sim 1$, i.e. $t \sim 1/(\pi n \sigma a^2)$

The formation rate of binaries (contd)

- The probability that there is a third star within a distance $\sim a$ at this time is of order na^3, and so the rate of formation of a hard binary is

$$\begin{aligned}\dot{n}_b &\sim (n/t)na^3 \text{ per unit volume}\\ &\sim n^3\sigma a^5\\ &\sim \frac{G^5m^5n^3}{\sigma^9}\end{aligned}$$

  where we have set $a \sim Gm/\sigma^2$ (threshold between hard and soft binaries)
- Because of the n-dependence, binary formation is concentrated in the core
- In core collapse $n, \sigma \to \infty$ and the volume tends to zero. But n wins.

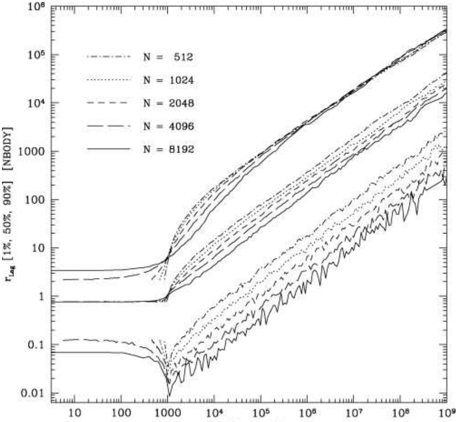
Energy generation by binaries

- When a binary forms in a three-body encounter, the energy of the three stars long before the encounter is their kinetic energy T_0. (Their potential energy is small when they are far apart.)
- After the encounter, there is a binary with negative energy (in its barycentric frame).
- By energy conservation, the kinetic energy of the two products of the encounter (the barycentre of the binary, and the single star) must exceed T_0.
- Encounters with an existing hard binary tend to have the same effect, i.e. an increase in the kinetic energy of the barycentre of the binary and the single star.
- Binary formation and evolution are an energy-generating mechanism, like nuclear reactions in a star.
- In this way, binaries can halt core collapse.

Post-collapse evolution (again)

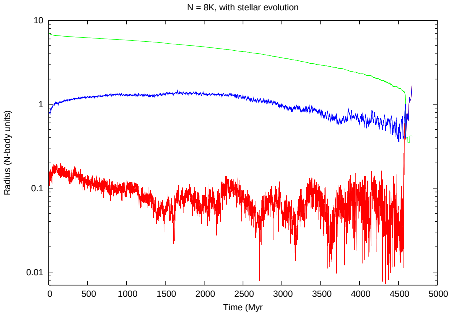
Depends on boundary conditions:

- "isolated" system: binaries form in the core, liberating energy, which expands the system on the time scale t_{rh} (by a feedback mechanism). As r_h expands, t_{rh} increases. The system *very* slowly loses mass



Post-collapse evolution (continued)

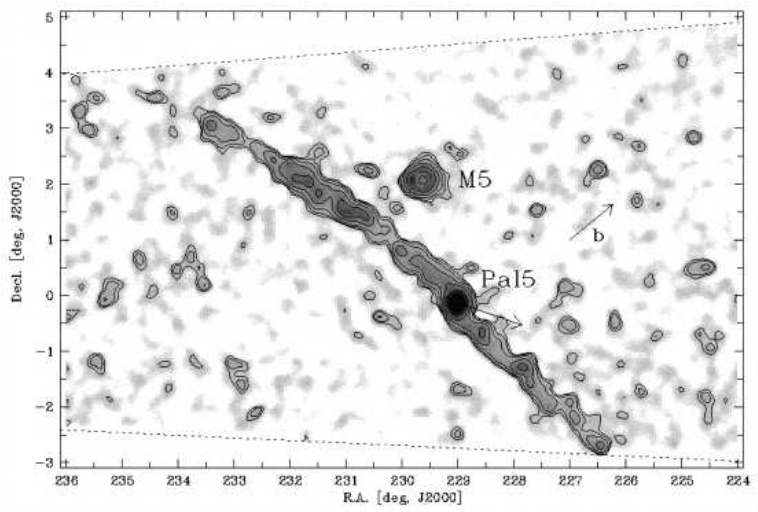
- "tidally limited" systems: stars escape (roughly speaking) at a tidal radius r_t, where external forces become dominant (see Lecture 4). Mass is lost on time scale t_{rh}; r_t contracts, r_h contracts (eventually). System dissolves in few t_{rh}.



Plotted (from the top): r_t, r_h and the 1% Lagrangian radius (see Lecture 1)

Post-collapse evolution (continued)

In fact stars escape along "tidal tails". Here is the example of Pal 5:



Exercises for Lecture 2

1. In N-body units, for a system in virial equilibrium, show that the mean square speed is $1/2$, the virial radius is 1, and the crossing time is $t_{cr} = 2\sqrt{2}$.
2. The following awk script extracts the time and the virial radius from the output of NBODY1:

   ```
   {if ($1==”T”) t = $3}
   {if ($1==”<R>”) print t,$3}
   ```

   Plot the virial radius for a cold-collapse simulation (Lecture 1), and try to understand it. Repeat for larger N.