Singular Curves and Giant Magnons

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AdS/CFT and Integrability
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Outline

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The Hofman-Maldacena limit

Definition (HM limit)

$J_1 \to \infty, \; E \to \infty, \; E - J_1 = \textit{fixed}, \; \lambda = \textit{fixed}, \; J_2 = \textit{fixed}.$

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Gauge theory: $\mathrm{tr}\,(Z^{J_1} W^{J_2})$ becomes infinitely long. Relax trace and consider

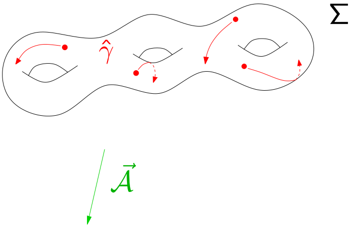
$$\mathcal{O}_{\mathfrak{p}} = \sum_{l} e^{i\mathfrak{p}l} (\ldots ZZZW^{J_2}ZZZ\ldots).$$

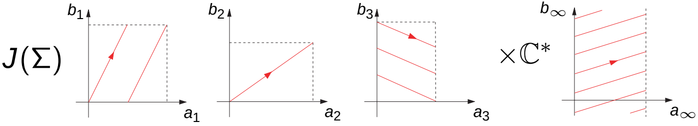
Outline

Finite-Gap picture

The story so far,

$$\left(\begin{array}{c} \text{finite-gap solution to} \\ d * j = 0 \\ dj - j \wedge j = 0 \\ \frac{1}{2}\operatorname{tr} j_{\pm}^2 = -\kappa^2 \end{array} \right) \quad \overset{1:1}{\longleftrightarrow}$$





Real curves

Recall $\Omega(x,\sigma,\tau) = P\overleftarrow{\exp}\int_{[\gamma(\sigma,\tau)]} \frac{1}{1-x^2}(j - x * j)$, hence
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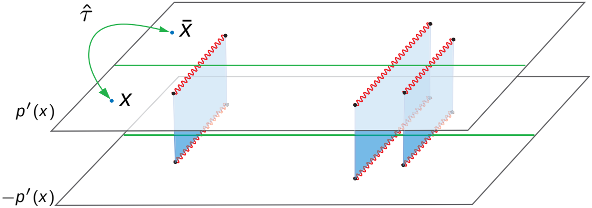

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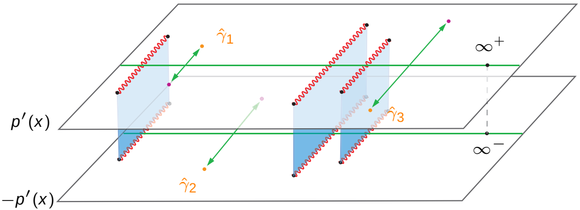

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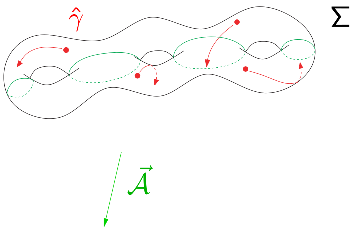

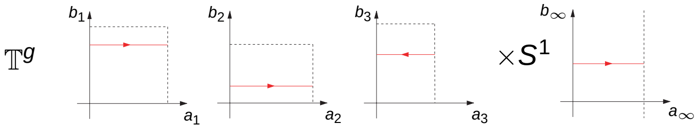

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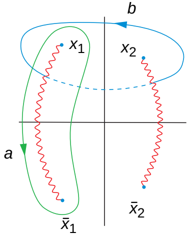

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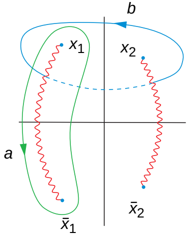

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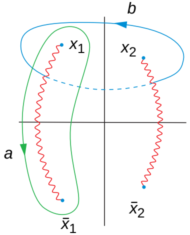

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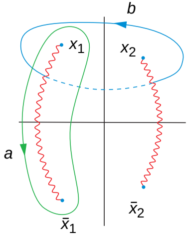

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σ-periodicity: $\frac{2K(k)\sqrt{1-v^2}}{\kappa'} = \frac{2\pi}{n}$, $\kappa' \equiv \kappa \frac{|x_1 - \bar{x}_2|}{\sqrt{y_+ y_-}}$, $v \equiv \frac{y_+ - y_-}{y_+ + y_-}$.

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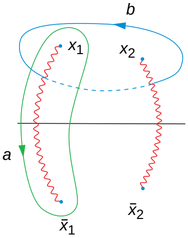

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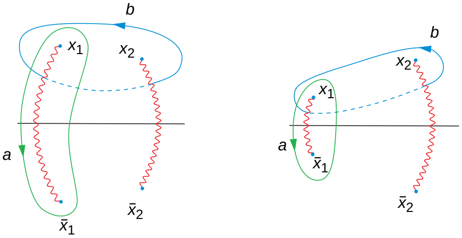

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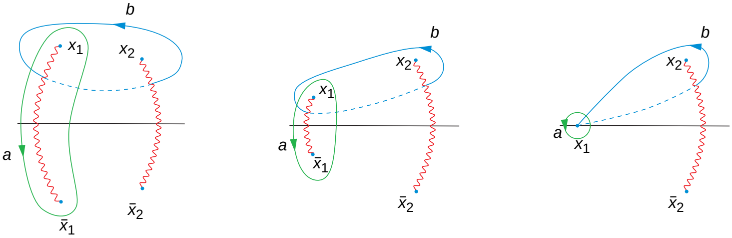

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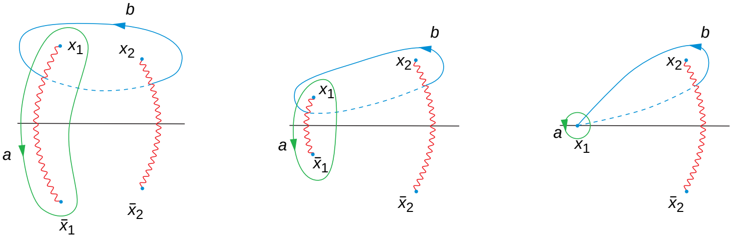

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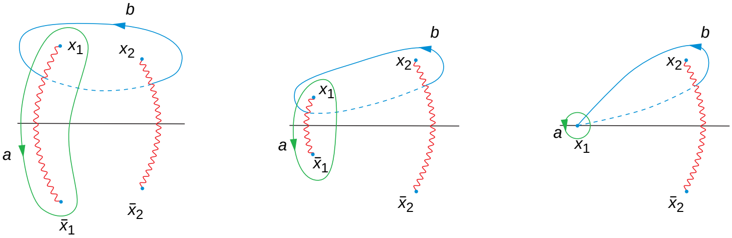

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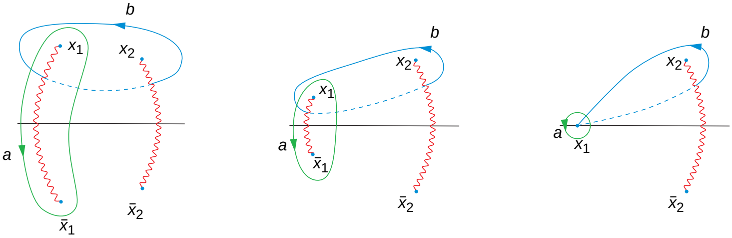

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- ... *semiclassical quantisation.*

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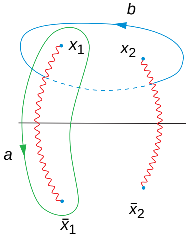

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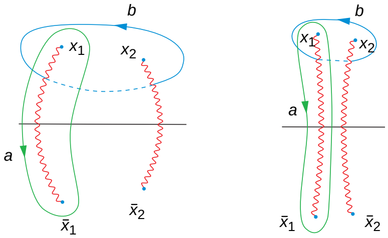

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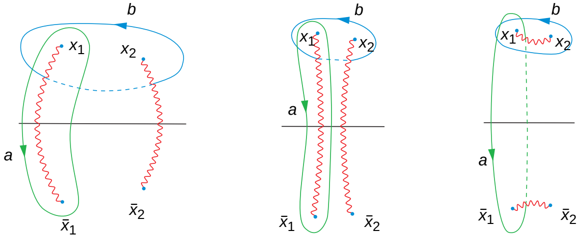

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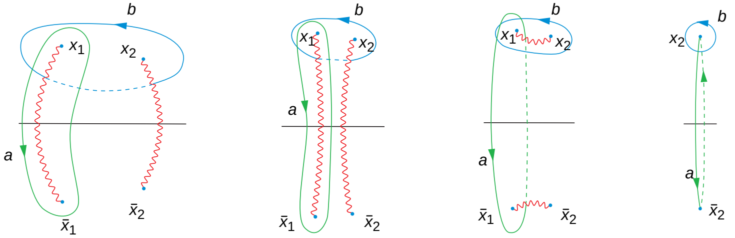

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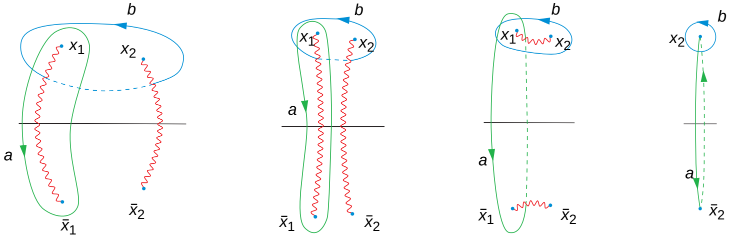

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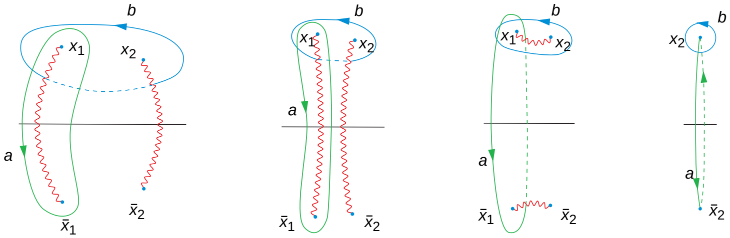

In particular $E = \sqrt{\lambda}\kappa \to \infty$ but

$$\left\{ \begin{array}{c} E - J_1 \\ J_2 \end{array} \right\} \to n\frac{\sqrt{\lambda}}{4\pi} \left| \left(x_2 \mp \frac{1}{x_2} \right) - \left(\bar{x}_2 \mp \frac{1}{\bar{x}_2} \right) \right| < \infty.$$

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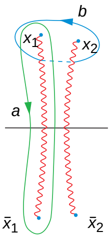

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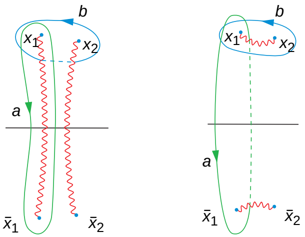

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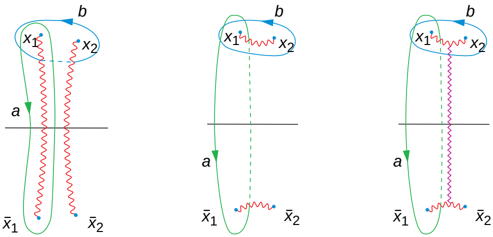

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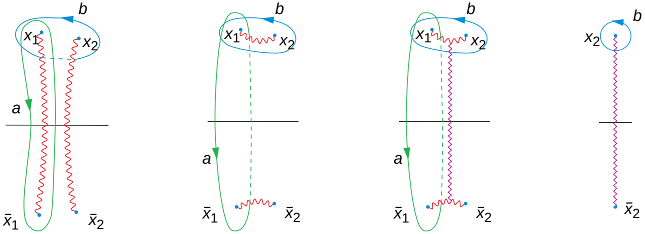

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where $\Theta_\mu(\cdot, k), Z_\mu(\cdot, k)$ are Jacobi Θ- and ζ-functions, and

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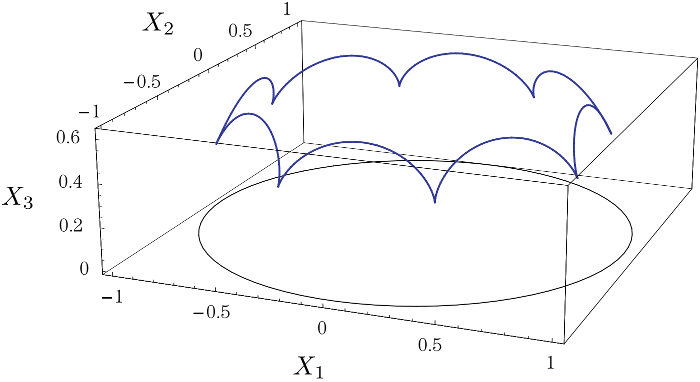
Definition

Focus on a 'single hop' region $\sigma \in [\frac{-\pi}{n}, \frac{\pi}{n}] \Leftrightarrow \tilde{X} \in [-K, K]$.

Profile



Soliton limit ($k \to 1$)

Recall that motion
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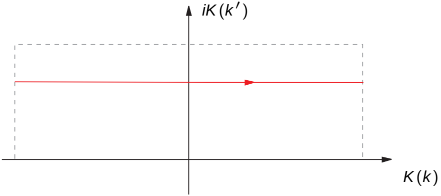

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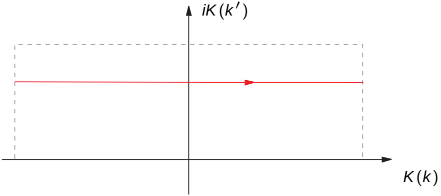

Proposition

In the limit $k \to 1$

$$Z_1 = \frac{\cos(\tilde{\rho}_-)}{cosh(\tilde{X})} \exp\left(iv_+\tilde{T} + i\varphi_1^0\right),$$

$$Z_2 = \frac{sinh(\tilde{X} - i\tilde{\rho}_-)}{cosh(\tilde{X})} \exp\left(i\tan(\tilde{\rho}_-)\tilde{X} + iv_-\tilde{T} + i\varphi_2^0\right).$$

which is known as the dyonic giant magnon on S^3.

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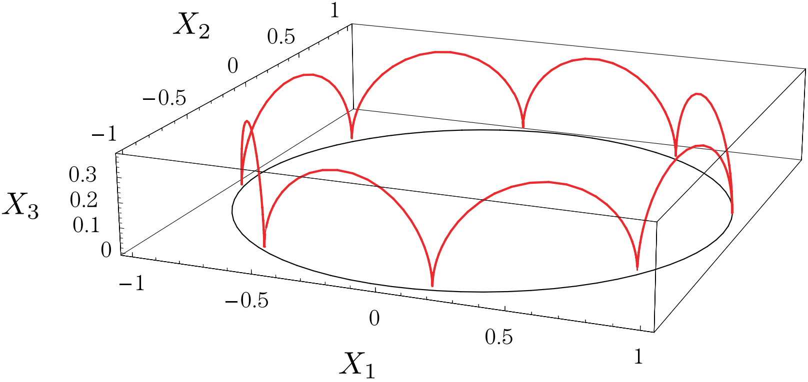

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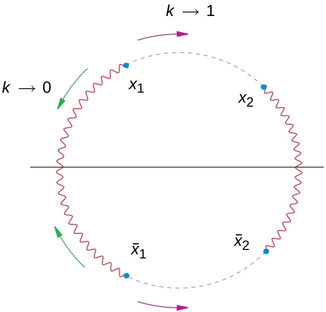
Conclusion

- Study of elliptic case $\Rightarrow \exists$ two interesting singular limits



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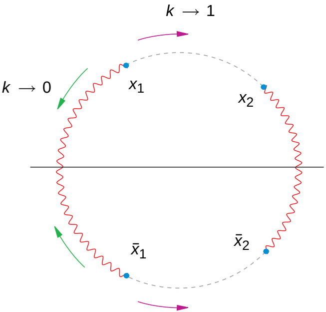

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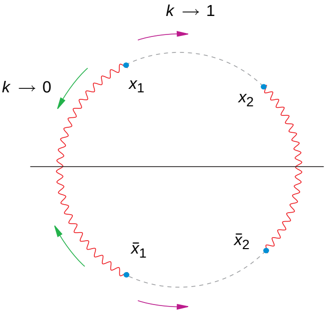

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- Partial degenerations?