Fluctuations for $\mathfrak{su}(2)$ from first principles

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AdS/CFT and Integrability
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Outline

Semiclassical quantisation

The zero-mode problem

Finite dimensions

Wentzel-Kramers-Brillouin-Maslov method

Finite-gap strings

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- *Aim: Give* ϕ_{cl} *a quantum interpretation.*

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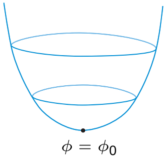

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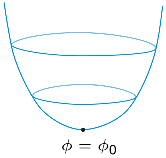

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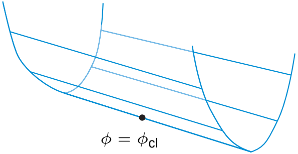

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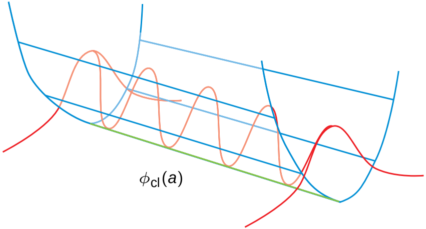

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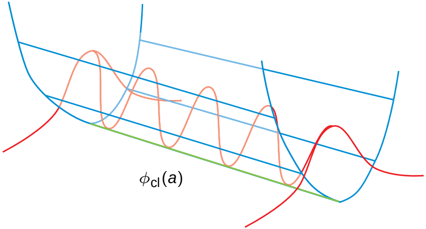

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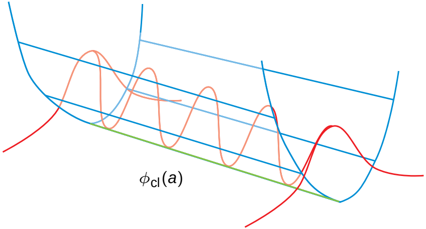

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Parameter a is called a collective coordinate.

zero-modes & broken symmetries

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$$(x_0, p_0, \ldots) \quad \xrightarrow{\phi_{\mathrm{cl}}} \quad (\phi, \pi) = (\phi_{\mathrm{cl}}(x, 0), \dot{\phi}_{\mathrm{cl}}(x, 0))$$

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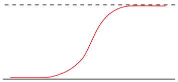

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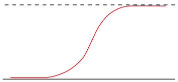

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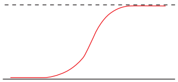

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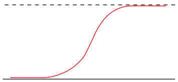

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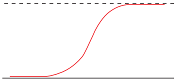

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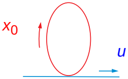

$$\hookrightarrow \quad (\phi_u(x+x_0,0), \dot{\phi}_u(x+x_0,0)) \in \mathcal{P}^\infty_{S-G}.$$

The Breather solution

2-parameter family of solutions

$$\phi_{\tau,v}(x,t) = \frac{4m}{\sqrt{\lambda}} \tan^{-1}\left\{ \frac{\sqrt{\left(\frac{\tau m}{2\pi}\right)^2 - 1}\,\sin\left[\frac{2\pi}{\tau}\frac{t-vx}{\sqrt{1-v^2}}\right]}{\cosh\left[\sqrt{\left(\frac{\tau m}{2\pi}\right)^2 - 1}\,\frac{2\pi}{\tau}\frac{x-vt}{\sqrt{1-v^2}}\right]} \right\}.$$

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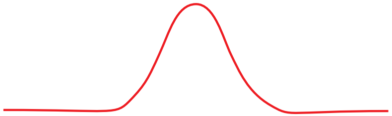

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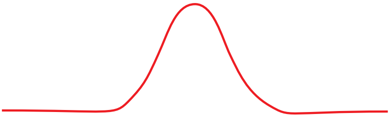

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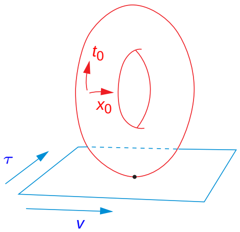

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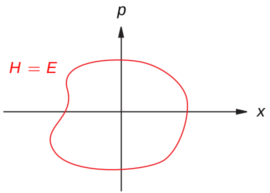

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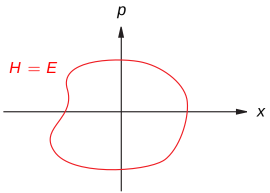

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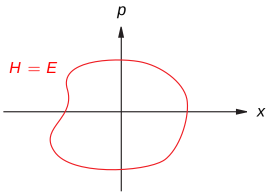

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WKB ansatz: $\psi_{WKB} = e^{iS(x)/\hbar}\rho(x) + O(\hbar)$

To order $O(1)$: $$H\left(x, \frac{\partial S(x)}{\partial x}\right) = E.$$

WKB method (continued)

Lemma

Let $\iota : H^{-1}(E) \hookrightarrow T^*X$ *and* $\alpha = pdx$*, then* $d\iota^*\alpha = 0$.

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Locally, $\iota^*\alpha = \frac{\partial S}{\partial x}dx = dS$ is exact. □

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Theorem (Bohr-Sommerfeld)

Single-valuedness of ψ_{WKB} on $H^{-1}(E)$ implies

$$\frac{1}{2\pi\hbar} \int_{H^{-1}(E)} \iota^*\alpha = N + \frac{1}{2} + O(\hbar).$$

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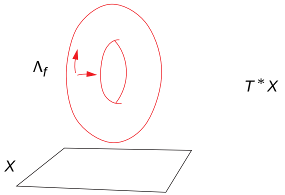

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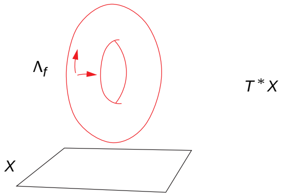

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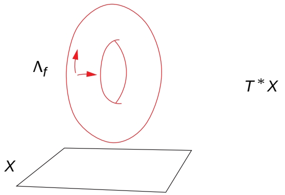

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Theorem (Einstein-Brillouin-Keller)

Single-valuedness of ψ_{WKB} on Λ_f implies

$$\frac{1}{2\pi\hbar}\int_\gamma \iota^*\alpha = N(\gamma) + \frac{\mu_\gamma}{2} + O(\hbar), \quad \forall\gamma \in H_1(\Lambda_f).$$

Degenerate torii

Recall,

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We will be interested in critical values of $\boldsymbol{F}$, *e.g.* $F_i = 0$ some i.

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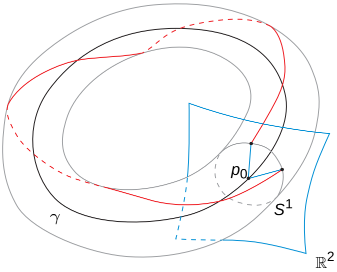

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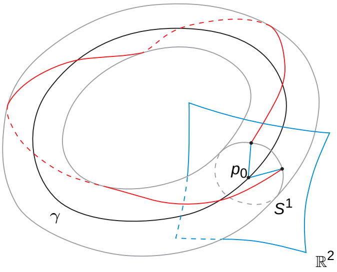

Quantise neighbouring 2-torus using EBK,

Theorem (Voros)

$$\frac{1}{2\pi\hbar}\int_{\gamma}\iota^{*}\alpha = N + \frac{\mu_{\gamma}}{2} + \left(n+\frac{1}{2}\right)\frac{\nu}{2\pi} + O(\hbar).$$

2-d harmonic oscillator

$$H = \frac{p_1^2}{2} + \frac{1}{2}\omega_1^2 x_1^2 + \frac{p_2^2}{2} + \frac{1}{2}\omega_2^2 x_2^2 = H_1 + H_2, \quad \omega_1 \neq \omega_2.$$

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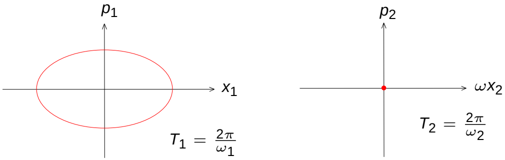

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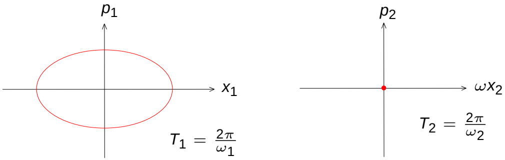
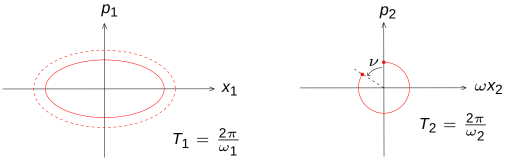


Perturb γ within $H^{-1}(E)$:



2-d harmonic oscillator (continued)

Voros $\Rightarrow \frac{1}{\hbar} I_1 = \left(n_1 + \frac{1}{2}\right) + \left(n_2 + \frac{1}{2}\right) \frac{\nu}{2\pi} + O(\hbar)$

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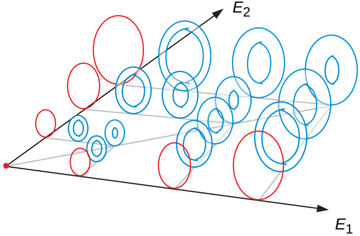

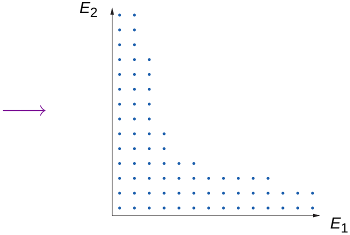

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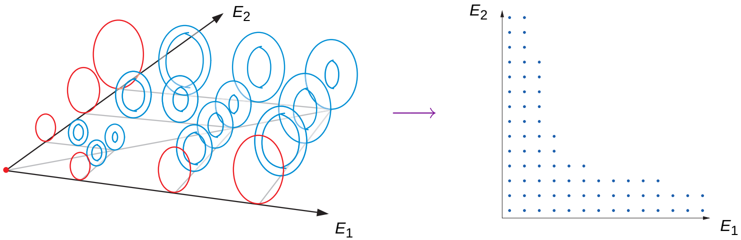

Remark

Could have applied EBK to 2-torii with $E_1 \neq 0, E_2 \neq 0$.

Quantising degenerate torii

Consider a family Λ_f of p-torii in n-dimensional system.

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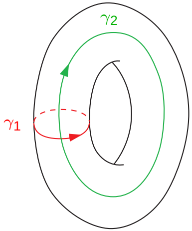
Theorem (Voros)

Let $k = 1, \ldots, p$ *and* $\gamma_k \in H_1(\Lambda_f)$ *then*

$$\frac{1}{2\pi\hbar}\int_{\gamma_k} \iota^*\alpha = \left(N_k + \frac{\mu_{\gamma_k}}{4}\right) + \sum_{\alpha=p+1}^{n}\left(n_\alpha + \frac{1}{2}\right)\frac{\nu_\alpha^k}{2\pi} + O(\hbar).$$



Outline

Semiclassical quantisation
- The zero-mode problem

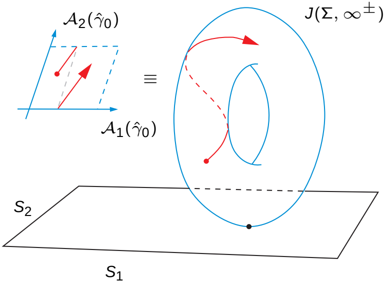
Finite dimensions
- Wentzel-Kramers-Brillouin-Maslov method

Finite-gap strings

Application to finite-gap strings

Recall,

$$\mathcal{M}_{\mathbb{C}}^{(2g+2)} \equiv$$



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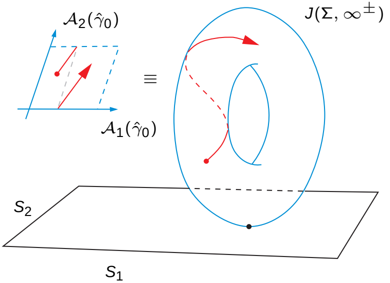

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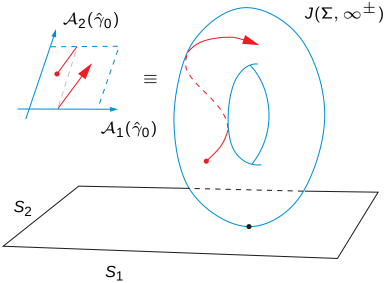

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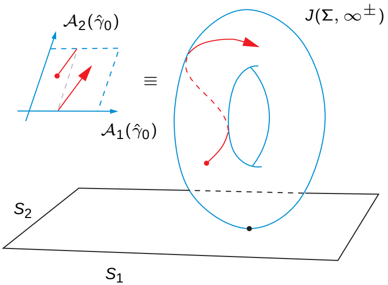

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Symplectic structure

Proposition

The pullback of the Poisson bracket on $\mathcal{P}^\infty$ by the finite-gap solution ι is (with $z = x + \frac{1}{x}$)

$$\hat{\omega}_{2g+2} = -\frac{\sqrt{\lambda}}{4\pi i}\sum_{i=1}^{g+1} dp(\hat{\gamma}_i)\wedge dz(\hat{\gamma}_i) = \sum_{i=1}^{g+1} d\left(-\frac{\sqrt{\lambda}}{4\pi i}p(\hat{\gamma}_i)dz(\hat{\gamma}_i)\right).$$

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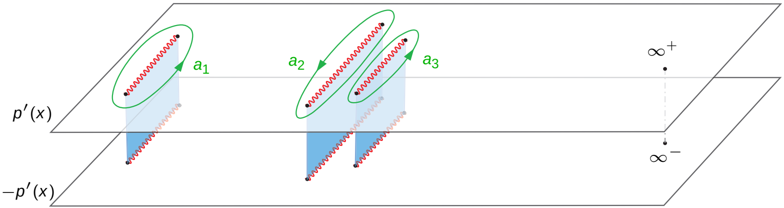

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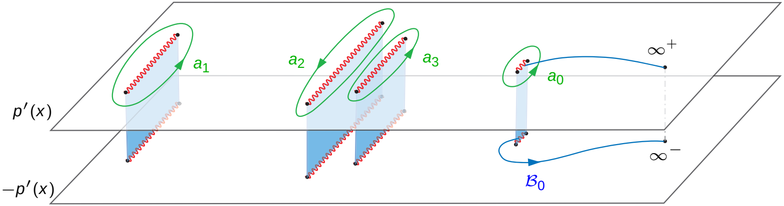

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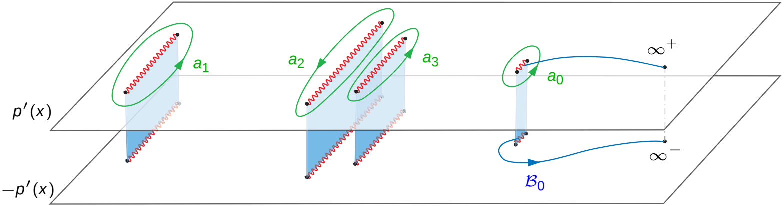

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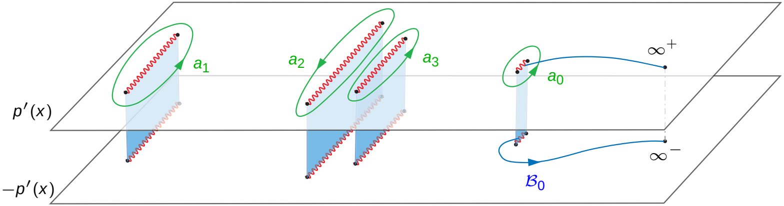

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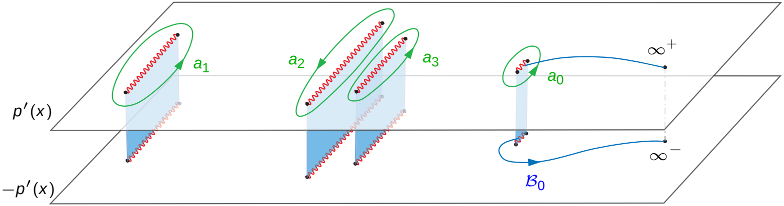

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Action spectrum

Apply Voros formula ($n \ll N$):

$$\frac{S_l}{\hbar} = N_l + \frac{1}{2} + \sum_{\alpha=g+2}^{\infty} \left(n_\alpha + \frac{1}{2} \right) \int_{\mathcal{B}_0} dq^{(l)} + O(\hbar).$$

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- *At $O(1)$:* $S_I \in \hbar\mathbb{Z}$.
- *At $O(\hbar)$:* S_I *receive* 1*-loop corrections.*

Energy spectrum

Now $[\hat{S}_I, \hat{S}_J] = O(\hbar^3)$ and classically $E = E_{\rm cl}[S_1, \ldots, S_{g+1}]$, hence semiclassical energy spectrum is just

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Proposition

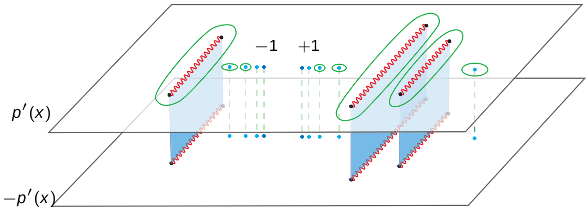
The energy spectrum can be rewritten, to order $O(\hbar)$ as

$$E = E_{cl}\left[\left(N_1 + \frac{1}{2}\right)\hbar, \ldots, \left(N_{g+1} + \frac{1}{2}\right)\hbar, \left(n_1 + \frac{1}{2}\right)\hbar, \ldots\right],$$

where E_{cl} is the classical energy of an infinite-gap solution.

Conclusion

Recall,



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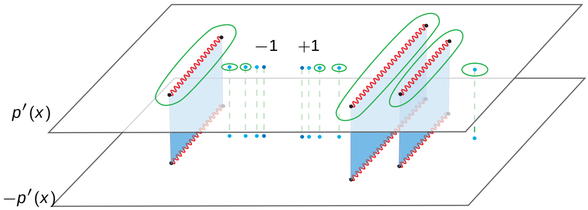

Semiclassical spectrum ⇔ energy of classical solution with

$$\int_\gamma \alpha \in \left(\frac{1}{2} + \mathbb{N}\right) \hbar$$

for contour γ around all cuts and all singular points.