SINGULAR CURVES, THETA FUNCTIONS & KP SOLUTIONS

§ 1. The KP equation and quasiperiodic solutions

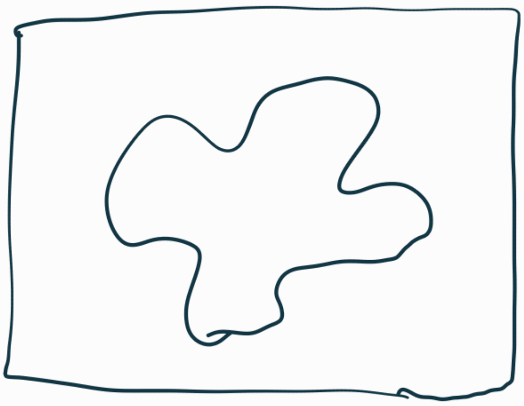
KP equation: $u = u(x,y,t)$

$$\frac{\partial}{\partial x}\left(4u_t - 6u\cdot u_x - u_{xxx}\right) = 3u_{yy}$$

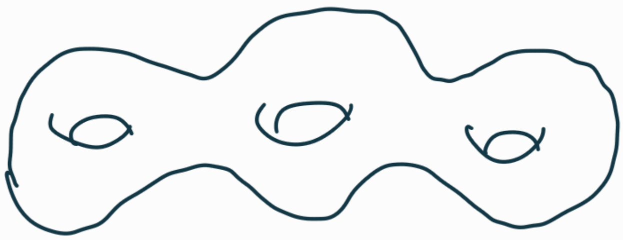
Describes the behavior of waves in shallow water.

This has possibly surprising connections to algebraic geometry and in particular algebraic curves. (KRICHEVER).

C = smooth projective complex algebraic curve of genus g = compact Riemann surface of genus g



PLANE QUARTIC
$\{x^4 + y^4 = 1\}$



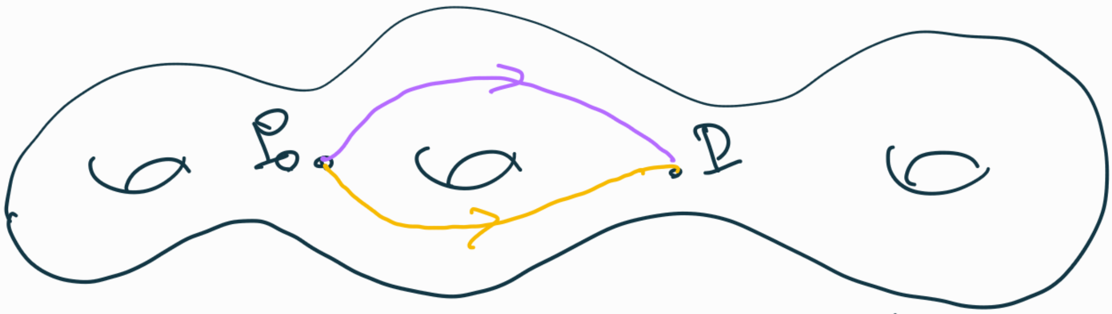
$\omega_1, \omega_2, \dots, \omega_g$ = basis of holomorphic differentials on C.

P_0 = base point on C.

We would like to integrate the differentials along C, so we would like to have a map:

$$
\begin{aligned}
a : C &\longrightarrow \mathbb{C}^g \\
P &\longmapsto \left(\int_{P_0}^{P} \omega_1, \int_{P_0}^{P} \omega_2, \dots, \int_{P_0}^{P} \omega_g \right)
\end{aligned}
$$

where $\int_{P_0}^{P}$ denotes the integral along a path from P_0 to P.



The integrals depend on the path chosen, and to resolve this ambiguity we want the integrals along all the cycles to be zero

So we define the ABEL-JACOBI map

$$a: C \longrightarrow \mathbb{C}^g / \Lambda_C = \left\{ \begin{array}{c} \text{integrals along} \\ \text{the cycles in} \\ H_1(C,\mathbb{Z}) \end{array} \right\}$$

$$P \longmapsto \left[\int_{P_0}^{P} \omega_1, \dots, \int_{P_0}^{P} \omega_g \right]$$

This complex torus $J(C) = \mathbb{C}^g / \Lambda_C$ is called the JACOBIAN VARIETY of the curve and it is actually a PROJECTIVE VARIETY.

$J(C)$ has a natural group structure by addition so we can extend this Abel-Jacobi map:

$$a: C^n \longrightarrow J(C)$$

$$(P_1, \dots, P_n) \longmapsto \left(\sum \int_{P_0}^{P_i} \omega_1, \sum \int_{P_0}^{P_i} \omega_2, \dots, \sum \int_{P_0}^{P_i} \omega_g \right)$$

<u>Thm</u>: [ABEL] It holds that

$a(P_1, \dots, P_n) = a(Q_1, \dots, Q_n) \iff P_1 + \dots + P_n \sim Q_1 + \dots + Q_n$ linearly equivalent.

Linear equivalence means that there exists a map $f: C \longrightarrow \mathbb{P}^1$ s.t.

$$f^{-1}(0) = P_1 + \dots + P_n$$
$$f^{-1}(\infty) = Q_1 + \dots + Q_n$$

Rmk: Observe that if the two divisors are linearly equivalent then the Abel map will send them to the same point: take $f : C \to \mathbb{P}^1$ and for any $t \in \mathbb{P}^1$ consider

$$\begin{aligned} \mathbb{P}^1 &\longrightarrow J(C) \\ t &\longmapsto \sum_{P \in f^{-1}(t)} \left(\int_{P_0}^{P} \omega_1, \dots, \int_{P_0}^{P} \omega_g \right) \end{aligned}$$

This is a holomorphic map from $\mathbb{P}^1$ to a complex torus and any such map is constant.

This describes the fiber of the Abel map, hence we can control the dimension of the image.

In particular

$$\begin{aligned} a : C^{g-1} &\longrightarrow J(C) \\ (P_1, \dots, P_{g-1}) &\longmapsto \sum_{i=1}^{g-1} \left(\int_{P_0}^{P_i} \omega_1, \dots, \int_{P_0}^{P_i} \omega_g \right) \end{aligned}$$

This has zero-dimensional fibers, so that the image has dimension $g-1$ in $J(C)$.

$$\boxed{a(C^{g-1}) = (4) \subseteq J(C) \quad \text{is called the THETA DIVISOR}}$$

And if we pull back to $\mathbb{C}^g \longrightarrow J(C)$ this gives an hypersurface in $\mathbb{C}^g$ which is described by one equation

$$\boxed{\Theta(z_1, \dots, z_g) = 0 \qquad \text{THETA FUNCTION}}$$

<u>Thm</u> [KRICHEVER]

There exist $U, V, W \in \mathbb{C}^g$, $c \in \mathbb{C}$ s.t.

$$\boxed{u(x,y,t) = 2\frac{\partial^2}{\partial x^2} \log \Theta(Ux + Vy + Wt) + c}$$

is a solution to the KP equation

<u>Rmk</u>: (1) Such solutions are called quasiperiodic.

(2) Everything here is <u>EXPLICITLY COMPUTABLE</u>:

$$C = \{ x^4 + y^4 - 1 = 0 \}$$

$$\omega_1 = \frac{1}{4y^3}\, dx \;,\quad \omega_2 = \frac{x}{4y^3}\, dx \;,\quad \omega_3 = \frac{y}{4y^3}\, dx$$

Then we can construct explicitly Θ, U, V, W, c.

In particular Θ can be expressed in terms of the Riemann Theta Function:

$$\Theta(z) = \sum_{n \in \mathbb{Z}^g} a_n \cdot \exp(2\pi i\, n^t z)$$

infinite linear combination of exponentials.

Can be computed: SAGE, MAPLE, JULIA, ..., MATLAB

(3) All the possible (U, V, W) form naturally a 3-fold. I studied together with BERND STURMFELS, TÜRKÜ ÇELIK and we named it the DUBROVIN THREEFOLD of the curve C.

(4) Krichever's proof uses the language of integrable systems. Recently we have given a different point of view together with JOHN LITTLE and TÜRKÜ ÇELIK: proof based on

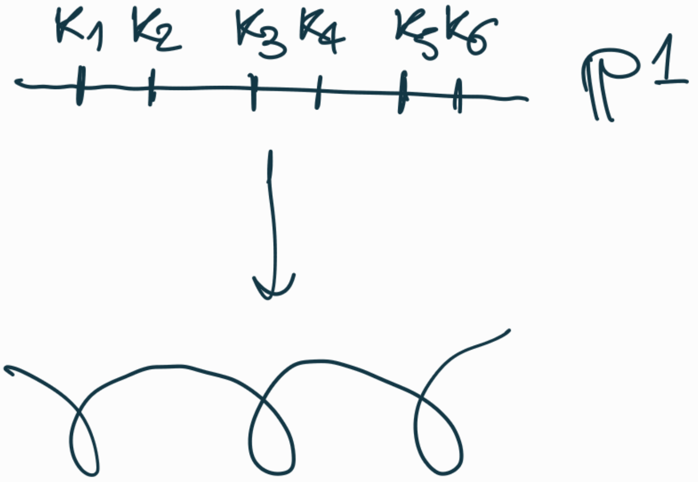
- SATO GRASSMANNIAN : integrable systems
- $\boxed{\text{ABEL'S THEOREM}}$: algebraic geometry

So this works whenever we have ABEL'S THEOREM at our disposal, also for $\boxed{\text{singular curves}}$.

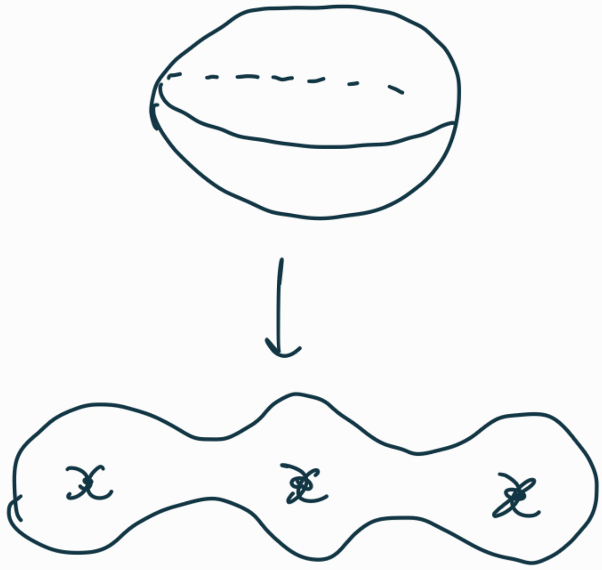
§2. RATIONAL NODAL CURVES

C = possibly reducible curve
all components are rational
all singularities are nodes

For example: an irreducible rational nodal curve C of (arithmetic) genus g is obtained from $\mathbb{P}^1$ by identifying g pairs of points



g nodes



g pinches

P_0 = smooth point on C

$\omega_1, \dots, \omega_g$ = basis of canonical differentials on C
(meromorphic differentials on $\mathbb{P}^1$ with certain properties)

$$\omega_i = \left(\frac{1}{x - K_{2i-1}} - \frac{1}{x - K_{2i}} \right) dx$$

We can integrate them

$$\int_{P_0}^{x} \omega_i = \log(x - K_{2i-1}) - \log(x - K_{2i})$$
$$= \log\left(\frac{x - K_{2i-1}}{x - K_{2i}} \right)$$

So we have another Abel map

$$a : C \longrightarrow \mathbb{C}^g$$

$$x \longmapsto \left(\log\left(\frac{x-k_1}{x-k_2}\right), \log\left(\frac{x-k_3}{x-k_4}\right), \dots \right)$$

These are not well defined: to fix this we pass to the GENERALIZED JACOBIAN of C which means in this case

$$\begin{tikzcd} & \boxed{\mathbb{C}^g} \arrow[dr, "\exp"] & \\ C \arrow[ur, dashed] \arrow[rr] & & (\mathbb{C}^*)^g = J(C) \end{tikzcd}$$

$$x \longmapsto \left(\left(\frac{x-k_1}{x-k_2}\right), \dots, \left(\frac{x-k_{2g-1}}{x-k_{2g}}\right) \right)$$

In this case the generalized Jacobian is an algebraic torus $(\mathbb{C}^*)^g$, we obtain again a theta divisor

$$\Theta = a(C^{g-1})$$
$$= \{ \theta(z_1, \dots, z_g) = 0 \}$$

It turns out that the theta function is a finite linear combination of exponentials

$$\theta = \sum a_c \cdot \exp(2\pi i \, c^t z) \qquad \text{FINITE}$$

Again we can obtain KP solutions

$$u = 2 \frac{\partial^2}{\partial x^2} \log \theta(Ux + Vy + Wt)$$

that are SOLITON SOLUTIONS.

This situation is studied by

- A., CLAUDIA FEVOLA, YELENA MANDELSTHAM, STURMFELS in the general case
- FEVOLA, MANDELSTAM : irreducible rational nodal curves

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§ 3. RATIONAL SOLUTIONS

C smooth : θ = infinite linear comb of exponentials

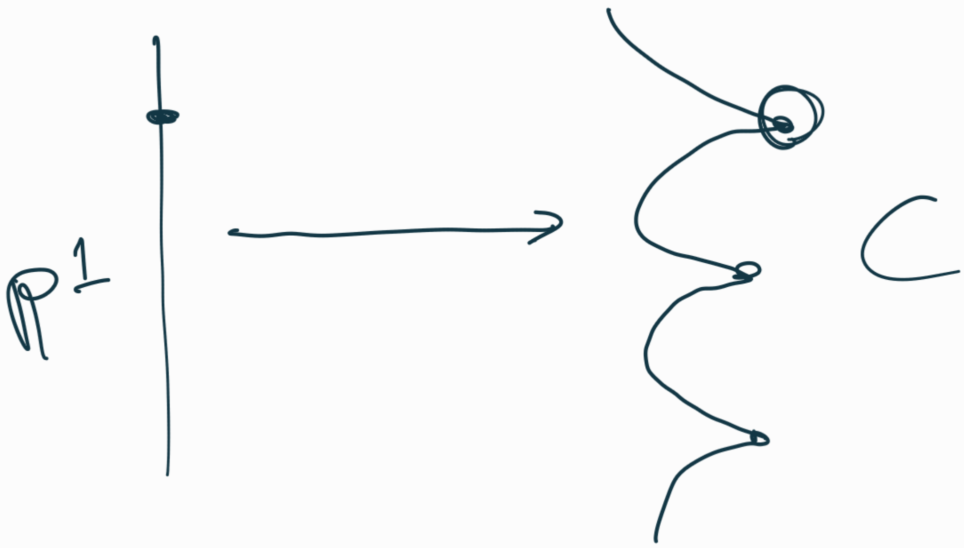
C rational nodal : θ = finite linear comb. of exponential

C more singular : θ = polynomial

Studied by LITTLE, A.-GECIK-STRUWE-STURMFELS

In a recent work with LITTLE and ÇELIK we completely classified the curves that give rise to a polynomial theta function

<u>Thm</u>: C has a polynomial theta function iff C is rational and all the singularities are unibranched (higher order cusps)



<u>Example</u>: Consider the curve C given by the image of

$$\nu : \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$
$$[u,t] \longmapsto [u^6, t^4u^2, t^5u, t^6]$$

Then C is rational and it has a unique unibranched singularity at $Q = [1,0,0,0]$

A basis of differentials is given by

$$\omega_1 = du \qquad \omega_2 = u\,du \qquad \omega_3 = u^2 du \qquad \omega_4 = u^6 du$$

$$\int_0^u \omega_1 = \boxed{u} \qquad \int_0^u \omega_2 = \boxed{\frac{u^2}{2}} \qquad \int_0^u \omega_3 = \boxed{\frac{u^3}{3}} \qquad \int_0^u \omega_4 = \boxed{\frac{u^7}{7}}$$

__Thm__: [A.-GECIK-LITTLE]

If C is an algebraic curve of genus g with polynomial theta function. Then this poly has degree at most $\frac{g(g+1)}{2}$ and we classify the curves attaining this bound.