The Secret Life of Graphs

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Outline

References

References:

- R. Bacher, P. de la Harpe, T. Nagnibeda, "The lattice of integral flows and the lattice of integral cuts on a finite graph" (1997).
- M. Baker and S. Norine, "Riemann-Roch and Abel-Jacobi theory on a finite graph" (2007).

Graphs

By a graph G, we mean a connected, finite, undirected multigraph without loop edges.

Divisors

- The group $\mathrm{Div}(G)$ of divisors on G is the free abelian group on $V(G)$.
- We write elements of $\mathrm{Div}(G)$ as formal sums

$$D = \sum_{v \in V(G)} a_v(v)$$

with $a_v \in \mathbb{Z}$.
- A divisor D is effective if $a_v \geq 0$ for all v.
- The degree of $D = \sum a_v(v)$ is $\deg(D) = \sum a_v$.
- We set

$$\mathrm{Div}^0(G) = \{D \in \mathrm{Div}(G) \ : \ \deg(D) = 0\}.$$

Rational functions and principal divisors

- The group of rational functions on G is

$$\mathcal{M}(G) = \{\text{functions } f : V(G) \to \mathbb{Z}\}.$$

- The Laplacian operator $\Delta : \mathcal{M}(G) \to \mathrm{Div}^0(G)$ is defined by

$$\Delta f = \sum_{v \in V(G)} \left(\sum_{e = vw} (f(v) - f(w)) \right) (v).$$

- The group of principal divisors on G is the subgroup

$$\mathrm{Prin}(G) = \{\Delta f \ : \ f \in \mathcal{M}(G)\}$$

of $\mathrm{Div}^0(G)$.

Linear equivalence

- Divisors $D, D' \in \mathrm{Div}(G)$ are linearly equivalent, written $D \sim D'$, if $D - D'$ is principal.
- If we think of D as an assignment of pounds to each vertex, then $D \sim D'$ iff one can get from D to D' by a sequence of "legal moves" of the following type:
  1. A vertex v lends one pound across each edge adjacent to v.
  2. A vertex v borrows one pound along each edge adjacent to v.

The Jacobian

- The Jacobian (or Picard group) of G is

$$\mathrm{Jac}(G) = \mathrm{Div}^0(G)/\,\mathrm{Prin}(G).$$

- This is a finite abelian group whose cardinality is the number of spanning trees in G (Kirchhoff's Matrix-Tree Theorem).

Riemann-Roch for graphs

Serguei Norine and I proved a Riemann-Roch theorem for finite graphs, which was quickly extended by Gathmann–Kerber and Mikhalkin–Zharkov to a Riemann-Roch theorem for metric graphs / tropical curves.

The Riemann-Roch theorem

- The canonical divisor on G is

$$K_G = \sum_{v \in V(G)} (\deg(v) - 2)\,(v).$$

Its degree is $2g - 2$, where $g = \dim_{\mathbb{R}} H_1(G, \mathbb{R})$ is the genus of G.

- The complete linear system $|D|$ of a divisor D is

$$|D| = \{E \in \mathrm{Div}(G) \ : \ E \geq 0, \ E \sim D\}.$$

- Define $h^0(D)$ by the formula

$$h^0(D) = \min\{\deg(E) \ : \ E \geq 0, |D - E| = \emptyset\}.$$

Theorem ("Riemann-Roch for graphs", B.–Norine)

For every $D \in \mathrm{Div}(G)$, we have

$$h^0(D) - h^0(K_G - D) = \deg(D) + 1 - g.$$

A consequence of Riemann-Roch

Corollary (B.-Norine)

If the total amount of money on the graph is at least g, then one can get every vertex out of debt by a sequence of legal moves.

Outline

References

References:

- G. Mikhalkin and I. Zharkov, "Tropical curves, their Jacobians, and Theta functions" (2008).
- A. Gathmann and M. Kerber, "A Riemann-Roch theorem in tropical geometry" (2008).

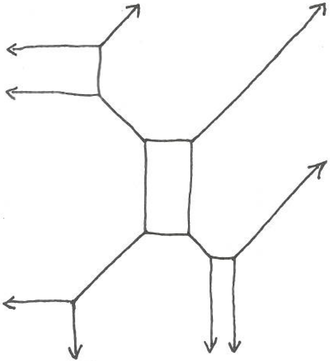
Tropical geometry

- Let K be an algebraically closed field which is complete with respect to a (non-trivial) non-archimedean valuation val.
- Examples: $K = \mathbb{C}_p$ or $K =$ the Puiseux series field $\mathbb{C}\{T\}$.
- If X is a d-dimensional irreducible algebraic subvariety of the torus $(K^*)^n$, then

$$\mathrm{Trop}(X) = \overline{\mathrm{val}(X)} \subseteq \mathbb{R}^n$$

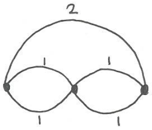
  is a connected polyhedral complex of pure dimension d.

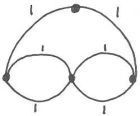
A tropical cubic curve in $\mathbb{R}^2$



Metric graphs

- A weighted graph is a graph G together with an assignment of a "length" $\ell(e) > 0$ to each edge $e \in E(G)$.
- A (compact) metric graph Γ is just the "geometric realization" of a weighted graph: it is obtained from a weighted graph G by identifying each edge e with a line segment of length $\ell(e)$. In particular, Γ is a compact metric space.
- A weighted graph G whose geometric realization is Γ will be called a model for Γ.





Abstract tropical curves

Following Mikhalkin, an abstract tropical curve is just a "metric graph with a finite number of unbounded ends".

Convention

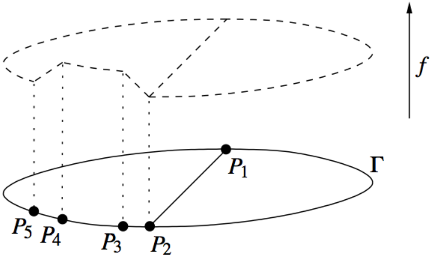
We will ignore the unbounded ends and use the terms "tropical curve" and "metric graph" interchangeably.

Divisors

For a tropical curve Γ, we make the following definitions:

- $\mathrm{Div}(\Gamma)$ is the free abelian group on Γ.
- $\mathcal{M}(\Gamma)$ consists of all continuous piecewise affine functions $f : \Gamma \to \mathbb{R}$ with integer slopes.
- The Laplacian operator $\Delta : \mathcal{M}(\Gamma) \to \mathrm{Div}^0(\Gamma)$ is defined by $-\Delta f = \sum_{p \in \Gamma} \sigma_p(f)(p)$, where $\sigma_p(f)$ is the sum of the slopes of f in all tangent directions emanating from p.
- $\mathrm{Prin}(\Gamma) = \{\Delta f \ : \ f \in \mathcal{M}(\Gamma)\}$.
- $\mathrm{Jac}(\Gamma) = \mathrm{Div}^0(\Gamma) / \mathrm{Prin}(\Gamma)$.

The divisor of a tropical rational function



Tropical Abel theorem

Let $\Omega^1(\Gamma)$ be the g-dimensional real vector space of harmonic 1-forms on Γ.

One can identify $\Omega^1(\Gamma)$ with $H^1(\Gamma, \mathbb{R})$ and think of a harmonic 1-form ω as a real-valued flow on Γ (net amount into each vertex equals net amount out).

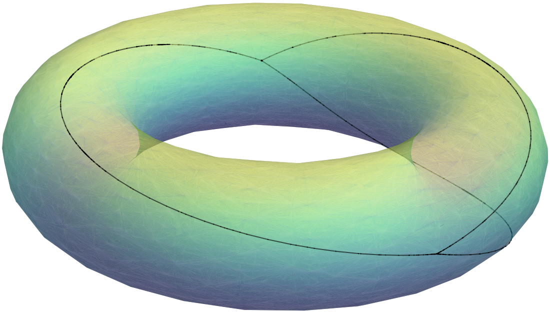
Theorem ("Tropical Abel Theorem", Mikhalkin–Zharkov)

There is a canonical isomorphism

$$\mathrm{Div}^0(\Gamma)/\,\mathrm{Prin}(\Gamma) \cong \mathrm{Hom}(\Omega^1(\Gamma), \mathbb{R})/H_1(\Gamma, \mathbb{Z}).$$

In particular, $\mathrm{Jac}(\Gamma)$ *is a real torus of dimension* g.

A tropical curve of genus 2 in its Jacobian



Tropical Riemann-Roch

As before, define $K_\Gamma = \sum_{p \in \Gamma} (\deg(p) - 2)\,(p)$ and for $D \in \mathrm{Div}(\Gamma)$, define

$$|D| = \{E \in \mathrm{Div}(\Gamma) \ : \ E \geq 0, \ E \sim D\}$$
$$h^0(D) = \min\{\deg(E) \ : \ E \geq 0, |D - E| = \emptyset\}.$$

Theorem ("Riemann-Roch for tropical curves", Gathmann–Kerber, Mikhalkin–Zharkov)

For every $D \in \mathrm{Div}(\Gamma)$, we have

$$h^0(D) - h^0(K_\Gamma - D) = \deg(D) + 1 - g.$$

Outline

References

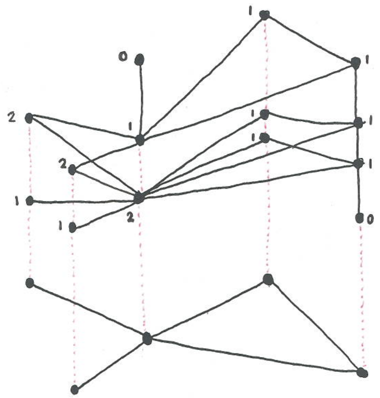
- H. Urakawa, "A discrete analogue of the harmonic morphism and Green kernel comparison theorems" (2000).
- M. Baker and S. Norine, "Harmonic morphisms and hyperelliptic graphs" (2009).

Harmonic morphisms

A continuous map $\phi : \Gamma \to \Gamma'$ between metric graphs is called a harmonic morphism if:

- ϕ is piecewise affine with integer slopes.
- If $p \in \Gamma$ and $f : \Gamma' \to \mathbb{R}$ is harmonic at $\phi(p)$, then $f \circ \phi$ is harmonic at p.

A degree 3 harmonic morphism



Multiplicities

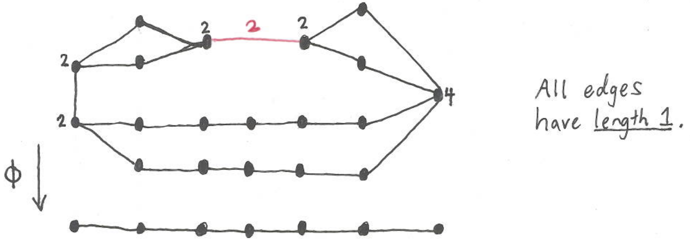
If $\phi : \Gamma \to \Gamma'$ is a harmonic morphism of metric graphs, one can define a local multiplicity (or ramification index) $m_\phi(x)$ at every $x \in \Gamma$.

The quantity

$$\deg(\phi) := \sum_{\phi(x)=x'} m_\phi(x)$$

is the same for every $x' \in \Gamma'$, and is called the degree of ϕ.

A degree 4 harmonic morphism



The tropical Riemann-Hurwitz formula

As with algebraic curves, one can use the multiplicities $m_\phi(x)$ to define functorial pushforward and pullback maps on divisors, harmonic 1-forms, Jacobians, etc.

Proposition (Tropical Riemann-Hurwitz formula)

Assume (for simplicity) that $\phi : \Gamma \to \Gamma'$ is a harmonic morphism of metric graphs having finite fibers. Then

$$K_\Gamma = \phi^* K_{\Gamma'} + \sum_{x \in \Gamma} \left(2m_\phi(x) - 2 - \sum_{e \ni x} (m_\phi(e) - 1) \right) (x).$$

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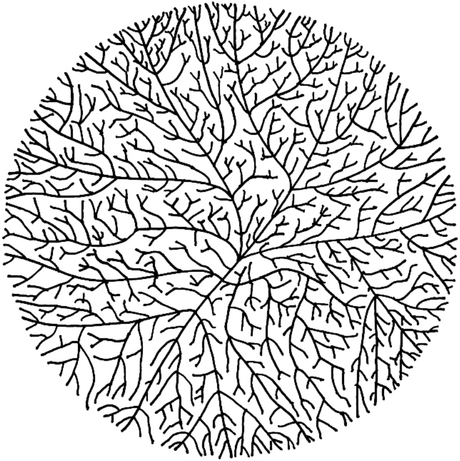
Outline

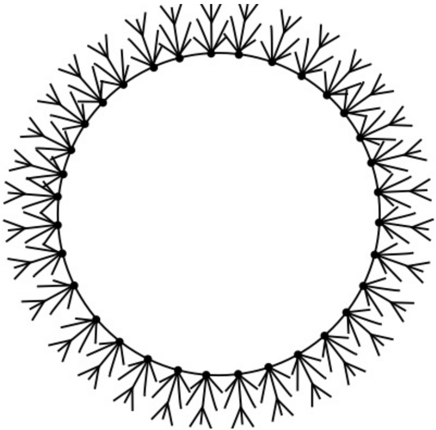
Crash course on Berkovich analytic spaces

- Let k be a complete and algebraically closed non-Archimedean valued field.
- If V is an irreducible algebraic variety over k, the Berkovich analytic space associated to V is a path-connected, locally compact Hausdorff space V^{an} containing $V(k)$ as a dense subspace.
- The construction $V \rightsquigarrow V^{\mathrm{an}}$ is functorial.
- For an open affine subscheme $U = \mathrm{Spec}(A)$ of V, U^{an} is the space of all bounded multiplicative seminorms on A extending the given absolute value on k (endowed with the topology of pointwise convergence).

Example: The Berkovich projective line



Example: A Berkovich elliptic curve



Example: A Berkovich K3 surface

Potential theory and dynamics on the Berkovich projective line

- Rumely and I showed that potential theory on trees could be used to do non-Archimedean potential theory on $(\mathbb{P}^1)^{\mathrm{an}}$, with results that closely parallel the classical theory of harmonic and subharmonic functions on $\mathbb{P}^1(\mathbb{C})$ (Poisson formula, Harnack's principle, Poincaré-Lelong formula, Frostman's theorem,...)
- In particular, the Laplacian on metric graphs can be used to define a Laplacian operator on $(\mathbb{P}^1)^{\mathrm{an}}$. If one thinks of a metric graph as a resistive electrical network, non-Archimedean potential theory is intimately related to Kirchhoff's laws.

A sample application

Potential theory on the Berkovich projective line has found many applications in recent years. For example:

Theorem (B.-DeMarco, 2011)

Let $a, b \in \mathbb{C}$ with $a \neq \pm b$. Then the set of $c \in \mathbb{C}$ such that both a and b have finite orbit under $z^2 + c$ is finite.

Intersection theory as non-Archimedean potential theory

- OK, so what about curves of higher genus?
- Amaury Thuillier, a student of Chambert-Loir, developed (independently and at the same time) non-Archimedean potential theory for arbitrary Berkovich curves.
- He applies this theory to give a symmetrical version of Arakelov intersection theory for curves in which one is doing potential theory at all places.
- Slogan: Intersection theory is non-Archimedean potential theory.
- Remark: The first person to develop this slogan (albeit without Berkovich spaces) was Ernst Kani.

The non-Archimedean Poincaré-Lelong formula

Theorem (Thuillier)

Let f be a rational function on a curve X over a complete non-Archimedean field k. Then

$$\Delta \log |f| = \delta_{\mathrm{div}(f)}.$$

Reinterpretation in the language of tropical geometry

- For a finite metric subgraph Γ of X^{an} containing the skeleton, let $\mathrm{Trop}(f)$ denote the restriction of $\log|f|$ to Γ. This is a piecewise-linear function with integer slopes, i.e., a "tropical rational function" on Γ.
- For $D \in \mathrm{Div}(X)$, let $\mathrm{trop}(D)$ denote the retraction of D to Γ. This is a "divisor" on Γ.
- If F is a tropical rational function on Γ, define the associated principal divisor on Γ to be the Laplacian of F, i.e.,

$$\mathrm{div}(F) := \sum_{p \in \Gamma} \Delta_p(F)(p),$$

  where $\Delta_p(F)$ is the sum of the incoming slopes of F at p.
- Thuillier's Poincare-Lelong formula is equivalent to the statement that for every such Γ, we have

$$\mathrm{div}(\mathrm{Trop}(f)) = \mathrm{trop}(\mathrm{div}(f)).$$

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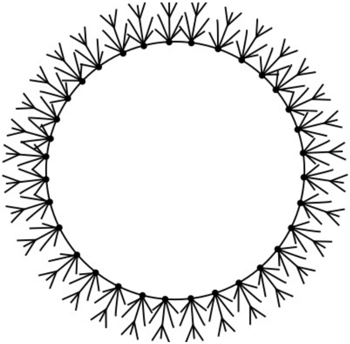
  where $\Delta_p(F)$ is the sum of the incoming slopes of F at p.
- Thuillier's Poincare-Lelong formula is equivalent to the statement that for every such Γ, we have

$$\mathrm{div}(\mathrm{Trop}(f)) = \mathrm{trop}(\mathrm{div}(f)).$$

Example: retraction of divisors



Semicontinuity

Shortly after establishing the Riemann-Roch theorem, I noticed that the combinatorial rank $r(D)$ has the following semicontinuity property:

Lemma (B.)

Let X be an algebraic curve over a complete non-Archimedean field k. For every finite metric subgraph Γ of X^{an} containing the skeleton,

$$r_\Gamma(\mathrm{trop}(D)) \geq r_X(D).$$

I began to wonder whether this result might have some applications to classical algebraic geometry...

Outline

Brill-Noether theory

We begin with the following result, known as the Brill-Noether theorem (due to Griffiths–Harris and others):

Theorem (Griffiths-Harris, Eisenbud-Harris, Lazarsfeld,. . .)

Given nonnegative integers g, r, d, define $\rho := g - (r+1)(g-d+r)$. If X is a nonsingular projective curve of genus g, define $W^r_d(X)$ to be the variety parametrizing line bundles $\mathcal{L}$ of degree d on X with $h^0(\mathcal{L}) \geq r+1$. Then for a general nonsingular projective curve X of genus g, $W^r_d(X)$ has dimension ρ if $\rho \geq 0$, and is empty if $\rho < 0$.

The proof, which Joe Harris explained in his class, uses a brilliant idea that goes back to Castelnuovo:
Since $\dim W^r_d(X)$ is upper semicontinuous on $\overline{\mathcal{M}}_g$, to show the statement for a general smooth curve of genus g it suffices to prove it for a single stable curve of genus g.

A rational backbone with g elliptic tails

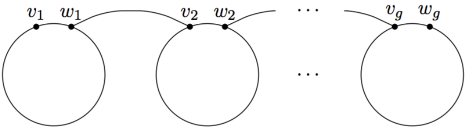
A tropical approach to degenerating linear series

- In 2008 I conjectured a tropical analogue of the Brill-Noether theorem. The conjecture was motivated by extensive computational evidence from my summer REU student Adam Tart.
- I also proved that this purely combinatorial conjecture would imply the classical Brill-Noether Theorem.
- My conjecture was proved by Cools, Draisma, Payne, and Robeva in a very explicit way:

Theorem (CDPR, 2012)

If $\rho := g - (r+1)(g-d+r) < 0$, then for the metric graph Γ consisting of a chain of g loops with general edge lengths, there is no divisor of degree d and rank at least r on Γ.

A chain of g loops



Other applications of the combinatorics of chains of loops

Generic chains of loops have also been used to prove the following:

- (Jensen–Payne, 2015) [Maximal Rank Conjecture for Quadrics] If X is a general curve of genus g and $\mathcal{L}$ is a general line bundle of degree d and rank r on X, then the natural map $\mathrm{Sym}^2 H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}^{\otimes 2})$ has maximal rank, i.e., it is either injective or surjective.
- (Pflueger, 2016) [Brill–Noether theory for k-gonal curves] For $r > 0$ and $g - d + r > 1$, a general smooth projective k-gonal curve X of genus g has $\dim W^r_d(X) = \rho$ if and only if $g - k \leq d - 2r$.

Applications to number theory

Linear series on metric graphs also play a key role in the proofs of the following two results in number theory:

Theorem (Katz–Zureick-Brown, 2013)

Let X be a curve of genus g over $\mathbb{Q}$ and suppose that the Mordell–Weil rank r of $J(\mathbb{Q})$ is less than g. Then for every prime $p > 2r + 2$, we have

$$\#X(\mathbb{Q}) \leq \#\mathfrak{X}^{\mathrm{sm}}(\mathbb{F}_p) + 2r,$$

where $\mathfrak{X}$ denotes the minimal proper regular model of X over $\mathbb{Z}_p$.

Theorem (Katz–Rabinoff–Zureick-Brown, 2015)

There is an explicit bound $M(g) = 76g^2 - 82g + 22$ such that if $X/\mathbb{Q}$ is a curve of genus g with Mordell-Weil rank at most $g - 3$, then $\#X(\mathbb{Q}) \leq M(g)$.