Formal groups and elliptic functions

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References

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Recall classical definitions

An **elliptic function** is a meromorphic function in $\mathbb{C}$ that is doubly periodic:

$$f(z+2\omega)=f(z),\quad f(z+2\omega')=f(z),\quad \mathrm{Im}\frac{\omega'}{\omega}>0.$$

Properties

For any nonconstant elliptic functions $f(z)$, $g(z)$,
— there exists a polynomial $P(x_1,x_2)$, such that $P(f(z),g(z))\equiv 0$,
— any elliptic function $h(z)$ is a rational function of $f(z)$ and $f'(z)$.

The **Weierstrass function** $\wp(z;g_2,g_3)$
is the unique elliptic function with periods 2ω, $2\omega'$
and poles only in lattice points such that $\lim\limits_{z\to 0}\left(\wp(z)-\frac{1}{z^2}\right)=0$.
It defines the uniformization of an elliptic curve $\mathcal{V}$

$$\wp'(z)^2=4\wp(z)^3-g_2\wp(z)-g_3.$$

Relations between parameters and periods

The elliptic curve in standard Weierstrass form

$$\mathcal{V} = \{(x, y) | y^2 = 4x^3 - g_2 x - g_3\}.$$

The parameters ω, ω', η, η' are determined by the relations

$$2\omega = \oint_{\mathfrak{a}} \frac{dx}{y}, \quad 2\omega' = \oint_{\mathfrak{b}} \frac{dx}{y}, \quad 2\eta = -\oint_{\mathfrak{a}} \frac{xdx}{y}, \quad 2\eta' = -\oint_{\mathfrak{b}} \frac{xdx}{y},$$

where $\frac{dx}{y}$ and $\frac{xdx}{y}$ are a set of holomorphic differentials on $\mathcal{V}$, $\mathfrak{a}$ and $\mathfrak{b}$ are basis cycles on the curve such that $\eta\omega' - \omega\eta' = \frac{\pi i}{2}$. Vice versa,

$$g_2 = \sum_{(n,m)\neq(0,0)} \frac{60}{(2m\omega + 2n\omega')^4}, \quad g_3 = \sum_{(n,m)\neq(0,0)} \frac{140}{(2m\omega + 2n\omega')^6}.$$

Sigma function

The Weierstrass ζ-function is defined by

$$\zeta(z; g_2, g_3)' = -\wp(z; g_2, g_3), \qquad \lim_{z \to 0} (z\zeta(z)) = 1.$$

We have the relation

$$\eta = \zeta(\omega; g_2, g_3).$$

The Weierstrass σ-function is defined by

$$\big(\ln \sigma(z; g_2, g_3)\big)' = \zeta(z; g_2, g_3), \qquad \lim_{z \to 0} \left(\frac{\sigma(z)}{z}\right) = 1.$$

It is an entire odd quasiperiodic function
homogeneous with respect to the grading $\deg z = 1, \deg g_k = -2k$.
We have

$$\sigma(z; g_2, g_3) = z - \frac{g_2 z^5}{2 \cdot 5!} - \frac{6 g_3 z^7}{7!} - \frac{g_2^2 z^9}{4 \cdot 8!} - \frac{18 g_2 g_3 z^{11}}{11!} + (z^{13})$$

Elliptic function of level N

For a lattice L in $\mathbb{C}$ consider the elliptic function $g(x)$ with divisor $N \cdot 0 - N \cdot z$. We demand that $g(x) = x^N + \ldots$.

Set $f(x) = g(x)^{1/N}$ where $f(x) = x + \ldots$.

$$f(x) = \frac{\sigma(x)\sigma(z)}{\sigma(z-x)} \exp\left(\alpha x - \zeta(z)x\right).$$

Periodic properties

$$f(x + 2\omega_k) = f(x) \exp\left(2\eta_k z + 2\omega_k(\alpha - \zeta(z))\right).$$

For $\quad z = \frac{2n}{N}\omega_1 + \frac{2m}{N}\omega_2, \quad \alpha = -\frac{2n}{N}\eta_1 - \frac{2m}{N}\eta_2 + \zeta(z),$
we get elliptic functions of level N.

The function $f(x)$ is elliptic with respect to a sublattice L' of L.

Let the bundle $\mathbb{C}P(\xi) \to B$ with fiber $\mathbb{C}P(2)$ be the projectivization of a 3-dimensional complex vector bundle $\xi \to B$. A Hirzebruch genus $L_f : \Omega_U \to R$ is called $\mathbb{C}P(2)$-*multiplicative*, if we have $L_f[\mathbb{C}P(\xi)] = L_f[\mathbb{C}P(2)]L_f[B]$.

Theorem (V.M. Buchstaber, E.Yu. Bunkova 2014)

Let L_f be a $\mathbb{C}P(2)$-multiplicative genus.
If $L_f[\mathbb{C}P(2)] \neq 0$, then L_f is the two-parametric Todd genus, and

$$f(x) = \frac{e^{\alpha x} - e^{\beta x}}{\alpha e^{\alpha x} - \beta e^{\beta x}},$$

If $L_f[\mathbb{C}P(2)] = 0$, then L_f is level 3 elliptic genus and

$$f(x) = -\frac{2\wp(x) + \frac{a^2}{2}}{\wp'(x) - a\wp(x) + b - \frac{a^3}{4}}.$$

Here $g_2 = -\frac{1}{4}(8b - 3a^3)a$, $g_3 = \frac{1}{24}(8b^2 - 12a^3 b + 3a^6)$.

2-nd Hirzebruch functional equation

$$\frac{1}{f(x_1-x_2)f(x_1-x_3)}+\frac{1}{f(x_2-x_1)f(x_2-x_3)}+\frac{1}{f(x_3-x_1)f(x_3-x_2)}=C.$$

Solutions are formal series $f(x)$ such that $f(0)=0, f'(0)=1$.

If $C \neq 0$, then

$$f(x)=\frac{e^{\alpha x}-e^{\beta x}}{\alpha e^{\alpha x}-\beta e^{\beta x}}.$$

If $C=0$, then $f(x)$ is elliptic function of level 3 and

$$f(x)=-\frac{2\wp(x)+\frac{a^2}{2}}{\wp'(x)-a\wp(x)+b-\frac{a^3}{4}}.$$

Here $g_2=-\frac{1}{4}(8b-3a^3)a$, $g_3=\frac{1}{24}(8b^2-12a^3b+3a^6)$.

n-th Hirzebruch functional equation

$$\sum_{j=1}^{n+1} \prod_{i \neq j} \frac{1}{f(x_i - x_j)} = C$$

Solutions are formal series $f(x)$ such that $f(0) = 0, f'(0) = 1$.

The elliptic genus of level N is strictly multiplicative in fibre bundles with a manifold M_d with $c_1(M_d) \equiv 0 \mod N$ as fibre and a compact connected Lie group G of automorphisms of M_d as structure group.
(F. Hirzebruch comparing a paper by P.S. Landweber with a paper by S. Ochanine)

=> Elliptic function of level $n+1$ is a (two-parametric) solution to n-th Hirzebruch equation (with $C = 0$).

Let the bundle $\mathbb{C}P(\xi) \to B$ with fiber $\mathbb{C}P(2)$ be the projectivization of a 3-dimensional complex vector bundle $\xi \to B$. A Hirzebruch genus $L_f : \Omega_U \to R$ is called $\mathbb{C}P(2)$-*multiplicative*, if we have $L_f[\mathbb{C}P(\xi)] = L_f[\mathbb{C}P(2)]L_f[B]$.

Theorem (V.M. Buchstaber, E.Yu. Bunkova 2014)

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If $L_f[\mathbb{C}P(2)] = 0$, then L_f is level 3 elliptic genus and

$$f(x) = -\frac{2\wp(x) + \frac{a^2}{2}}{\wp'(x) - a\wp(x) + b - \frac{a^3}{4}}.$$

Here $g_2 = -\frac{1}{4}(8b - 3a^3)a$, $g_3 = \frac{1}{24}(8b^2 - 12a^3b + 3a^6)$.

Application to isogenies of elliptic curves

Theorem

Let Δ and J be the discriminant and J-invariant of an elliptic curve with generators $(2\omega_1, 2\omega_2)$.
Then the discriminant and J-invariant of elliptic curves with generators $(2\hat{\omega}_1 = 2\omega_1, 2\hat{\omega}_2 = \frac{2}{3}\omega_2)$ and $(2\hat{\omega}_1 = \frac{2}{3}\omega_1, 2\hat{\omega}_2 = 2\omega_2)$ can be expressed as follows:
Let us introduce formal parameters p, s such that

$$\Delta = -\frac{1}{27}p^3 s, \qquad J = -\frac{1}{64}\frac{(p-s)(p-9s)^3}{p^3 s},$$

then

$$\hat{\Delta} = -27 s^3 p, \qquad \hat{J} = -\frac{1}{64}\frac{(s-p)(s-9p)^3}{s^3 p}.$$

1-st Hirzebruch functional equation

$$\frac{1}{f(x_1 - x_2)} + \frac{1}{f(x_2 - x_1)} = C.$$

Elliptic function of level 2 is Jacobi's elliptic sine $sn(x)$.

$$sn'(x)^2 = 1 - 2\delta sn(x)^2 + \varepsilon sn(x)^4.$$

$sn(x)$ is the exponential of *universal* formal group of the form

$$F(u, v) = \frac{u^2 - v^2}{uB(v) - vB(u)}.$$

Thus $B(u)^2 = 1 - 2\delta u^2 + \varepsilon u^4$.

Formal group

Let R be a commutative ring with unity 1.
A commutative one-dimensional **formal group** over R is a formal series

$$F(u, v) = u + v + \sum_{i,j>0} a_{i,j} u^i v^j, \qquad a_{i,j} \in R,$$

with conditions

$$F(v, u) = F(u, v), \qquad F(u, F(v, w)) = F(F(u, v), w).$$

It's **exponential** $f(x) \in R \otimes \mathbb{Q}[[x]]$, $f(0) = 0, f'(0) = 1$, is determined by the addition law

$$f(x + y) = F(f(x), f(y)).$$

From the addition law

$$F(f(x), f(y)) = f(x+y)$$

we get the relations

$$\frac{\partial}{\partial v} F(u,v)\Big|_{v=0} = w(u)\,,$$

$$\frac{\partial^2}{\partial v^2} F(u,v)\Big|_{v=0} = w(u)(w'(u) - w_1)\,,$$

$$\frac{\partial^3}{\partial v^3} F(u,v)\Big|_{v=0} = w(u)^2 w''(u) - 2w(u)w_2 + \\ + w(u)(w'(u) - w_1)(w'(u) - 2w_1)\,.$$

1-st Hirzebruch functional equation

$$\frac{1}{f(x_1 - x_2)} + \frac{1}{f(x_2 - x_1)} = C.$$

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Thus $B(u)^2 = 1 - 2\delta u^2 + \varepsilon u^4$.

Theorem (V.M. Buchstaber, E.Yu. Bunkova 2015)

The exponential of the universal formal group of the form

$$F(u,v) = \frac{u^2C(v) - v^2C(u)}{uC(v)^2 - vC(u)^2}$$

is the elliptic function of level 3.

Krichever genus

$$\mathcal{F}_B(u,v) = \frac{u^2A(v) - v^2A(u)}{uB(v) - vB(u)}, \quad A(0) = B(0) = 1. \qquad (1)$$

Theorem (V.M. Buchstaber, 1990)

The exponential of the universal formal group of the form (1) is

$$f(x) = \frac{\exp(\alpha x)}{\Phi(x)} = \frac{\sigma(x)\sigma(z)}{\sigma(z-x)} e^{\alpha x - \zeta(z)x}.$$

Two-parametric Todd genus, elliptic genera of level 2 and 3

$$\frac{u^2A(v)-v^2A(u)}{uA(v)-vA(u)}, \qquad \frac{u^2-v^2}{uB(v)-vB(u)}, \quad \frac{u^2C(v)-v^2C(u)}{uC(v)^2-vC(u)^2},$$

$$\mathbb{Z}[\mu_1,\mu_2], \qquad \mathbb{Z}[\mu_2,\mu_4], \qquad \mathbb{Z}[\mu_1,\mu_3,\mu_6]/\{\mu_3^2=-\mu_6\},$$

$$\frac{e^{\alpha x}-e^{\beta x}}{\alpha e^{\alpha x}-\beta e^{\beta x}}, \qquad sn(x), \qquad \frac{\sigma(x)\sigma(z)}{\sigma(z-x)}\exp\left(\alpha x-\zeta(z)x\right),$$

$$\frac{-2(\wp(x)-\lambda)}{\wp'(x)-\mu_1\wp(x)+\mu_1\lambda}, \quad \frac{-2(\wp(x)-\frac{\delta}{3})}{\wp'(x)}, \quad \frac{-2(\wp(x)+\frac{a^2}{4})}{\wp'(x)-a\wp(x)+b-\frac{a^3}{4}}.$$

Tate elliptic formal group

General Weierstrass model of elliptic curve in Tate coordinates:

$$s = u^3 + \mu_1 us + \mu_2 u^2 s + \mu_3 s^2 + \mu_4 us^2 + \mu_6 s^3.$$

$$\mathcal{F}_T(u,v) = \big((u+v)(1-\mu_3 k-\mu_6 k^2)-uv(\mu_1+\mu_3 m+\mu_4 k+2\mu_6 mk)\big)\times$$
$$\times\ \frac{(1+\mu_2 m+\mu_4 m^2+\mu_6 m^3)}{(1+\mu_2 n+\mu_4 n^2+\mu_6 n^3)(1-\mu_3 k-\mu_6 k^2)^2}$$

$$m = \frac{s(u)-s(v)}{u-v},\ k = \frac{us(v)-vs(u)}{u-v},\ n = m+uv\frac{(1+\mu_2 m+\mu_4 m^2+\mu_6 m^3)}{(1-\mu_3 k-\mu_6 k^2)}.$$

$$\mathcal{F}_T(u,v) \in \mathbb{Z}[\mu_1,\mu_2,\mu_3,\mu_4,\mu_6][[u,v]].$$

$$f(x) = -2\frac{\wp(x)-\frac{1}{12}(\mu_1^2+4\mu_2)}{\wp'(x)-\mu_1\wp(x)+\frac{1}{12}\mu_1(\mu_1^2+4\mu_2)-\mu_3}.$$

Two-parametric Todd genus, elliptic genera of level 2 and 3

$$\frac{u^2A(v)-v^2A(u)}{uA(v)-vA(u)}, \qquad \frac{u^2-v^2}{uB(v)-vB(u)}, \qquad \frac{u^2C(v)-v^2C(u)}{uC(v)^2-vC(u)^2},$$

$$\mathbb{Z}[\mu_1,\mu_2], \qquad \mathbb{Z}[\mu_2,\mu_4], \qquad \mathbb{Z}[\mu_1,\mu_3,\mu_6]/\{\mu_3^2=-\mu_6\},$$

$$\frac{e^{\alpha x}-e^{\beta x}}{\alpha e^{\alpha x}-\beta e^{\beta x}}, \qquad sn(x), \qquad \frac{\sigma(x)\sigma(z)}{\sigma(z-x)}\exp\left(\alpha x-\zeta(z)x\right),$$

$$\frac{-2(\wp(x)-\lambda)}{\wp'(x)-\mu_1\wp(x)+\mu_1\lambda}, \qquad \frac{-2(\wp(x)-\frac{\delta}{3})}{\wp'(x)}, \qquad \frac{-2(\wp(x)+\frac{a^2}{4})}{\wp'(x)-a\wp(x)+b-\frac{a^3}{4}}.$$

Rings of coefficients

$$\mathcal{F}_B(u,v) = \frac{u^2A(v) - v^2A(u)}{uB(v) - vB(u)}, \qquad \mathbb{Z}[a_1, b_2, a_3, b_3, \ldots]/\mathcal{J}_a$$

$$\mathcal{F}_T(u,v) = \ldots, \qquad \mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6].$$

Tate and Buchstaber formal groups

$$\mathcal{F}_1(u,v) = \frac{u+v-\mu_1 uv}{1+\mu_2 uv}, \qquad \mathbb{Z}[\mu_1,\mu_2]$$

$$\mathcal{F}_2(u,v) = \frac{u^2-v^2}{u\sqrt{(1-\mu_2 v^2)^2-4\mu_4 v^4} - v\sqrt{(1-\mu_2 u^2)^2-4\mu_4 u^4}}, \quad \mathbb{Z}[\mu_2,\mu_4]$$

$$\mathcal{F}_3(u,v) = \frac{u^2(1-\mu_1 v-\mu_3 s(v)) - v^2(1-\mu_1 u-\mu_3 s(u))}{u(1-\mu_1 v-\mu_3 s(v))^2 - v(1-\mu_1 u-\mu_3 s(u))^2},$$

$$\mathbb{Z}[\mu_1,\mu_3,\mu_6]/\{3\mu_6 = -\mu_3^2\}$$

$$\mathcal{F}_4(u,v) = \frac{u^2(1+\mu_6 s(v)^2) - v^2(1+\mu_6 s(u)^2)}{u(1+\mu_2 v^2+\mu_6 s(v)^2) - v(1+\mu_2 u^2+\mu_6 s(u)^2)}, \quad \mathbb{F}_2[\mu_2,\mu_4,\mu_6]$$