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DEL PEZZO SURFACES WITH INFINITE AUTOMORPHISM GROUPS

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Abstract. We classify del Pezzo surfaces with Du Val singularities that have infinite automorphism groups, and describe the connected components of their automorphisms groups.

Throughout this paper, we always assume that all varieties are projective and defined over an algebraically closed field $\Bbbk$ of characteristic 0.

1. Introduction

Automorphism groups of smooth del Pezzo surfaces are well-studied (see, for example, [11, 12]). In particular, if X is a smooth del Pezzo surface, then $\mathrm{Aut}(X)$ is infinite if and only if X is toric. Moreover, if X is a smooth toric del Pezzo surface, then $\mathrm{Aut}^0(X)$ can be described as follows:

K_X^2	$\mathrm{Aut}^0(X)$	equation & total space	
6	$\mathbb{G}_{\mathrm{m}}^2$	$u_0v_0w_0 = u_1v_1w_1$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
7	$\mathbb{B}_2 \times \mathbb{B}_2$		
8	$\mathbb{G}_{\mathrm{a}}^2 \rtimes \mathrm{GL}_2(\Bbbk)$	$u_0v_0 = u_1v_1$	$\mathbb{P}^2 \times \mathbb{P}^1$
8	$\mathrm{PGL}_2(\Bbbk) \times \mathrm{PGL}_2(\Bbbk)$	—	$\mathbb{P}^1 \times \mathbb{P}^1$
9	$\mathrm{PGL}_3(\Bbbk)$	—	$\mathbb{P}^2$

where $\mathbb{G}_{\mathrm{a}}$ is a one-dimensional unipotent additive group, $\mathbb{G}_{\mathrm{m}}$ is a one-dimensional algebraic torus, and $\mathbb{B}_2$ is the Borel subgroup of $\mathrm{PGL}_2(\Bbbk)$. In this paper, we prove similar result for del Pezzo surfaces with at worst Du Val singularities. For short, we call such surfaces *Du Val del Pezzo surfaces*. Our main result is the following.

Main Theorem. *Let X be a Du Val del Pezzo surface. Then the group* $\mathrm{Aut}(X)$ *is infinite if and only if X is described in Big Table in Section* 8.

Everywhere below the number n^0 refers to the corresponding surface in Big Table in Section 8. As a consequence of our classification we have the following

Corollary 1.1. *Let X be a Du Val del Pezzo surface. Then the group* $\mathrm{Aut}(X)$ *is not reductive if and only if X is one of the* 23 *surfaces* 7^o, 14^o, 15^o, 18^o, 24^o, 25^o, 26^o, 27^o, 28^o, 31^o, 36^o, 37^o, 38^o, 39^o, 42^o, 43^o, 44^o, 45^o, 46^o, 48^o, 49^o, 50^o, 51^o.

Thus, the surfaces listed in this corollary are not K-polystable [1, 21], which is known (see [24]).

Corollary 1.2. *Let X be a Du Val del Pezzo surface.*

(i) *If $K_X^2 = 1$ and* $\mathrm{Aut}(X)$ *is infinite, then $\uprho(X) = 1$.*
(ii) *If $K_X^2 > 1$ and $\uprho(X) = 1$, then* $\mathrm{Aut}(X)$ *is infinite.*
(iii) *If $K_X^2 \geqslant 6$ or $K_X^2 = 5$ and X is singular, then* $\mathrm{Aut}(X)$ *is infinite.*

Many particular parts of our classification have been previously studied from different perspectives. For examples, the Du Val del Pezzo surfaces admitting an effective action of the group $\mathbb{G}_{\mathrm{a}}^2$ and $\mathbb{G}_{\mathrm{a}} \rtimes \mathbb{G}_{\mathrm{m}}$ have been classified in [9, 10]. The classification of toric Du Val del Pezzo surfaces is well-known for specialists (see e.g. [25]). Du Val del Pezzo surfaces that admit a faithful action of the group $\mathbb{G}_m$ have been studied in [2, 15, 16] in terms of their Cox rings. Moreover, when we were finishing the final version of this paper, we were informed that Main Theorem has been independently proven in [20] using completely different approach, which also works in positive characteristic.

Note that the complete classification of *all* Du Val del Pezzo surfaces have been known for a long time [13, 11]. The basic problem is that its is very huge and to choose surfaces with infinite automorphism group typically takes a lot of efforts.

Remark 1.3. Almost all surfaces in Big Table are explicitly given by their defining equations, since they are not always uniquely determined by their degree and singularities. For example, the cubic surface in $\mathbb{P}^3$ given by

$$x_3x_0^2 + x_1^3 + x_2^3 + x_0x_1x_2 = 0$$

has one singular point of type D_4, its automorphism group is finite, and it is not isomorphic to the cubic surface 22^o, which has the same singularity. Similarly, the quartic surface in the weighted projective space $\mathbb{P}(1,1,1,2)$ that is given by the equation

$$y_2^2 = y_1^3y_1'' + y_1'^4 + y_1'^2y_1^2$$

is a del Pezzo surface of degree 2 that has one singular point of type E_6. It is not isomorphic to the del Pezzo surface 11^o, which has the same degree and the same singularity. There are more examples like this: the surface 1^o and the sextic surface in $\mathbb{P}(1,1,2,3)$ given by

$$y_3^2 = y_2^3 + y_1'y_1^5 + y_2^2y_1^2$$

are the only del Pezzo surfaces of degree 1 with singular point of type E_8. They are not isomorphic. In fact, the latter surface is the only Du Val del Pezzo surface whose class group is $\mathbb{Z}$ that does not appear in our Big Table (see Remark B.5).

Let us briefly describe the structure of this paper. In Section 2, we present several basic facts about Du Val del Pezzo surfaces which are used in the proof of Main Theorem. In Section 3, we prove Theorem 3.8, which together with Main Theorem imply

Corollary 1.4. *Let X be a Du Val del Pezzo surface with $K_X^2 \geqslant 3$, and let $\tau(X)$ be its Fano–Weil index. Suppose that $\tau(X) > 1$. Then* $\mathrm{Aut}(X)$ *is infinite.*

In Section 4, we prove Main Theorem for del Pezzo surfaces of degree $\geqslant 4$. Then, in Sections 5, 6, 7, we prove Main Theorem for del Pezzo surfaces of degree 1, 2, 3, respectively. In Section 8, we present Big Table. In Appendix A, we describe lines on del Pezzo surfaces that appear in Big Table together with the dual graphs of the curves with negative self-intersection numbers on their minimal resolutions. Finally, in Appendix B, we will recall classification of Du Val del Pezzo surfaces whose Weil divisor class group is cyclic, and present an alternative proof of Main Theorem for them.

Notations. Throughout this paper, we will use the following notation:

- $\boldsymbol{\mu}_n$ is a cyclic subgroup of order n.
- $\mathbb{G}_a$ is a one-dimensional unipotent additive group.
- $\mathbb{G}_m$ is a one-dimensional algebraic torus.
- $\mathbb{B}_n$ is a Borel subgroup of $\mathrm{PGL}_n(\Bbbk)$.
- $\mathbb{U}_n$ is a maximal unipotent subgroup of $\mathrm{PGL}_n(\Bbbk)$.
- $\mathbb{G}_a \rtimes_{(n)} \mathbb{G}_m$ is a semidirect product $\mathbb{G}_a$ and $\mathbb{G}_m$ such that $\mathbb{G}_m$ acts on $\mathbb{G}_a$ as $\mathbf{x} \mapsto t^n\mathbf{x}$. This group is isomorphic to the following group:

$$\left\{ \begin{pmatrix} t^r & 0 \\ a & t^s \end{pmatrix} \in \mathrm{GL}_2(\Bbbk) \;\middle|\; t \in \Bbbk^* \text{ and } a \in \Bbbk \right\},$$

where $n = s - r$. Indeed, the required isomorphism follows from

$$\begin{pmatrix} t^r & 0 \\ 0 & t^s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} t^r & 0 \\ 0 & t^s \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ at^{s-r} & 1 \end{pmatrix}.$$

Observe that $\mathbb{G}_{\mathrm{a}} \rtimes_{(0)} \mathbb{G}_{\mathrm{m}} = \mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}}$, $\mathbb{G}_{\mathrm{a}} \rtimes_{(1)} \mathbb{G}_{\mathrm{m}} = \mathbb{B}_2$ and

$$\mathbb{G}_{\mathrm{a}} \rtimes_{(n)} \mathbb{G}_{\mathrm{m}} \cong \mathbb{G}_{\mathrm{a}} \rtimes_{(-n)} \mathbb{G}_{\mathrm{m}}.$$

Therefore, we will always assume that $n \geqslant 0$. If $n > 0$, the center of $\mathbb{G}_{\mathrm{a}} \rtimes_{(n)} \mathbb{G}_{\mathrm{m}}$ is $\boldsymbol{\mu}_n$. This implies that $\mathbb{G}_{\mathrm{a}} \rtimes_{(n_1)} \mathbb{G}_{\mathrm{m}} \cong \mathbb{G}_{\mathrm{a}} \rtimes_{(n_2)} \mathbb{G}_{\mathrm{m}} \iff n_1 = \pm n_2$.

- $\mathbb{F}_n$ is the Hirzebruch surface.
- $\mathbb{P}(a_1, \ldots, a_n)$ is the weighted projective space.
- For a weighted projective space $\mathbb{P}(a_0, a_1, \ldots, a_n)$, we denote by $y_{a_0}, y_{a_1}, \ldots, y_{a_n}$ the coordinates on it of weights $a_0, a_1, \ldots, a_n$, respectively.
- For a variety X, we denote by $\mathrm{Sing}(X)$ the set of its singular points.
- For a variety X, we denote by $\rho(X)$ the rank of the Weil divisor class group $\mathrm{Cl}(X)$.
- For a variety X and its (possibly reducible) reduced subvariety $Y \subseteq X$, $\mathrm{Aut}(X, Y)$ denotes the group consisting of all automorphisms in $\mathrm{Aut}(X)$ that maps Y into itself.
- For a surface X with Du Val singularities, $\mathrm{Type}(X)$ denotes the type of its singularities. If $\mathrm{Type}(X) = \mathrm{D}_4 2\mathrm{A}_1$, then $\mathrm{Sing}(X)$ consists of a point of type D_4, and 2 points of type A_1.
- For a Du Val del Pezzo surface X, $\tau(X)$ denotes its Fano–Weil index, which is defined as follows:

$$\tau(X) = \max\Big\{ t \in \mathbb{Z} \ \Big| \ -K_X \sim tA, \ \text{where } A \text{ is a Weil divisor on } X \Big\}.$$

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2. Del Pezzo surfaces with Du Val singularities

Let X be a Du Val del Pezzo surface with $d := K_X^2$. Then d is known as the degree of the surface X. Let $\mu\colon \widetilde{X} \to X$ be the minimal resolution of singularities. Then

$$K_{\widetilde{X}} \sim \mu^* K_X,$$

so that $\widetilde{X}$ is a *weak del Pezzo surface*, that is, the anticanonical divisor $-K_{\widetilde{X}}$ is nef and big. By the Noether formula $d = 10 - \rho(X) \leqslant 9$ and by the genus formula every irreducible curve on $\widetilde{X}$ with negative self-intersection number is either (-1) or (-2)-curve. Moreover, one of the following holds (see [4, 17]):

(i) $K_X^2 = 9$ and $\widetilde{X} \cong X \cong \mathbb{P}^2$;
(ii) $K_X^2 = 8$ and $\widetilde{X} \cong X \cong \mathbb{F}_1$;
(iii) $K_X^2 = 8$ and $\widetilde{X} \cong X \cong \mathbb{P}^1 \times \mathbb{P}^1$;
(iv) $K_X^2 = 8$, $\widetilde{X} \cong \mathbb{F}_2$ and X is a quadric cone in $\mathbb{P}^3$;
(v) $K_X^2 \leqslant 7$ and there exists a $\mathrm{Aut}^0(X)$-equivariant diagram

(2.1)
$$\begin{array}{ccc} & \widetilde{X} & \\ {}^{\varphi}\swarrow & & \searrow^{\mu} \\ \mathbb{P}^2 & & X. \end{array}$$

where φ is a suitable contraction of (-1)-curves.

Moreover, it follows from the Kawamata–Viehweg vanishing and the exponential exact sequence that the group $\mathrm{Pic}(X)$ is torsion free.

Corollary 2.2. *Let G be a connected algebraic subgroup in $\mathrm{Aut}(X)$. Suppose that $d = K_X^2 \leqslant 7$. Then G is isomorphic to a subgroup of the following group:*

$$\left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix} \in \mathrm{GL}_3(\Bbbk) \;\middle|\; a_{ij} \in \Bbbk, a_{11} \neq 0, a_{22} \neq 0 \right\} \cong \mathbb{U}_3 \rtimes \mathbb{G}_{\mathrm{m}}^2.$$

In particular, the group G is solvable. If G is reductive and non-trivial, then $G \cong \mathbb{G}_{\mathrm{m}}$ or $G \cong \mathbb{G}_{\mathrm{m}}^2$. Similarly, if G is unipotent, then $G \cong \mathbb{G}_{\mathrm{a}}$ or $G \cong \mathbb{G}_{\mathrm{a}}^2$ or $G \cong \mathbb{U}_3$.

Proof. This follows from the fact that the diagram (2.1) is G-equivariant. □

Example 2.3 ([7, Proposition 8.1])**.** Suppose that $d = 7$ and X is singular. The surface X is unique. The morphism φ is the blow up of two points, $\widetilde{X}$ contains unique (-2)-curve, and $\mathrm{Type}(X) = \mathrm{A}_1$. The surface X contains two (-1)-curves. The dual graph of curves with negative self-intersection numbers has the form

$$\circ \text{——} \bullet \text{——} \bullet$$

where $\bullet$ denotes a (-1)-curve, and $\circ$ denotes the (-2)-curve. Using (2.1), we see that

$$\mathrm{Aut}^0(X) \cong \mathrm{Aut}(\mathbb{P}^2, \ell, P) \cong \mathbb{B}_3,$$

where ℓ is a line on $\mathbb{P}^2$, and P is a point in ℓ. Note that X is the surface 48^o.

The type $\mathrm{Type}(X)$ does not always determine the dual graph of curves with negative self-intersection numbers in $\widetilde{X}$. However, this graph is always determined by the type $\mathrm{Type}(X)$ and the number of (-1)-curves in $\widetilde{X}$. In the following, we denote by $\#(X)$ the number of (-1)-curves in the surface $\widetilde{X}$.

Example 2.4. If $d = 6$ and $\mathrm{Type}(X) = \mathrm{A}_1$, then X is one of the surfaces 45^o or 46^o in Big Table. If X is the surface 45^o, then $\#(X) = 3$. On the other hand, if X is the surface 46^o, then $\#(X) = 4$. The dual graph of curves with negative self-intersection numbers in $\widetilde{X}$ is given in Appendix A.

Using the Riemann–Roch formula and Kawamata–Viehweg vanishing, we get $\dim |-K_X| = d$. Let $\Phi\colon X \dashrightarrow \mathbb{P}^d$ be the rational map given by $|-K_X|$. The linear system $|-K_X|$ does not have fixed components, and it contains a smooth elliptic curve (see [8]). Using this fact, one can prove

Theorem 2.5 ([8, 17])**.** *The following assertions hold:*

- (i) *if $d \geqslant 2$, then $|-K_X|$ is base point free, so that Φ is a morphism;*
- (ii) *if $d \geqslant 3$, then $-K_X$ is very ample, so that Φ is an embedding;*
- (iii) *if $d = 3$, then $\Phi(X)$ is a cubic surface in $\mathbb{P}^3$;*
- (iv) *if $d \geqslant 4$, then $\Phi(X)$ is an intersection of quadrics in $\mathbb{P}^d$;*
- (v) *if $d = 2$, then Φ is a double cover that is branched over a possibly reducible quartic curve, so that X is a hypersurface in $\mathbb{P}(1,1,1,2)$ of degree 4;*
- (vi) *if $d = 1$, then $|-K_X|$ is an elliptic pencil, its base locus consists of one point $O \notin \mathrm{Sing}(X)$, and every curve in $|-K_X|$ is irreducible and smooth at O;*
- (vii) *if $d = 1$, then $|-2K_X|$ defines a double cover $X \to \mathbb{P}(1,1,2)$ branched over a sextic curve, so that X is a hypersurface in $\mathbb{P}(1,1,2,3)$ of degree 6.*

The number of (-1)-curves in $\widetilde{X}$ is finite.

Definition 2.6. An irreducible curve $L \subset X$ is a *line* if $L = \mu(\widetilde{L})$ for a (-1)-curve $\widetilde{L} \subset \widetilde{X}$.

If $d \geqslant 3$, then lines in X are usual (projective) lines in $\Phi(X) \subset \mathbb{P}^d$. Conversely, if $d \geqslant 3$, then lines in $\Phi(X)$ are lines in the sense of Definition 2.6. Moreover, if $d = 2$, then lines in X are smooth rational curve. Furthermore, if $d = 1$ and L is a line in X, then

- either L is singular curve in $|-K_X|$ such that $\mathrm{Sing}(L) \subset \mathrm{Sing}(X)$,
- or L is a smooth rational curve that does not contain the base point of the pencil $|-K_X|$.

Note that $\#(X)$ is the number of lines in X. Then $\#(X) > 0$ unless X is $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}(1,1,2)$.

Lemma 2.7. *Assume that $d \geqslant 3$. Let P be a point in X, and let $\#(X,P)$ be the number of lines in X passing through P.*

(i) *If $P \in X$ is smooth, then*

$$\#(X,P) \leqslant \begin{cases} 3 & \text{if } d = 3, \\ 2 & \text{if } d \geqslant 4. \end{cases}$$

(ii) *If $P \in X$ is singular, then*

$$\#(X,P) \leqslant \begin{cases} 6 & \text{if } d = 3, \\ 4 & \text{if } d = 4, \\ 3 & \text{if } d \geqslant 5. \end{cases}$$

Proof. Let $L_1, \ldots, L_r$ be all the lines on X passing through P, and let $\mathbb{T}_{P,X} \subset \mathbb{P}^d$ be the embedded tangents space to X at the point P. Then

$$\bigcup_{i=1}^{r} L_i \subseteq X \cap \mathbb{T}_{P,X}.$$

If $P \in X$ is a smooth point, then $\dim \mathbb{T}_{P,X} = 2$, so that $r \leqslant d$. Moreover, if $d \geqslant 4$, then $r \leqslant 2$, because X is an intersection of quadrics in this case. Thus, we may assume that P is a singular point of the surface X. Then $\dim \mathbb{T}_{P,X} = 3$, since P is a Du Val singular point of the surface X. Hence, if $d \geqslant 4$, then $r \leqslant 4$, because X is an intersection of quadrics in this case.

Suppose that $d = 3$. We may assume that $P = (0:0:0:1)$. Then X is given in $\mathbb{P}^3$ by

$$x_3 q_2(x_0, x_1, x_2) + q_3(x_0, x_1, x_2) = 0,$$

where q_2 and q_3 are homogeneous forms of degree 2 and 3, respectively. Then the (set-theoretic) union of the lines $L_1, \ldots, L_r$ is given by the system of equations $q_2 = q_3 = 0$, so that $r \leqslant 6$.

To complete the proof, we may assume that $d \geqslant 5$. We only consider the case $d = 5$, since the proof is similar in the remaining cases. Let us show that $r \leqslant 3$. To do this, suppose that $r \geqslant 4$. Let us seek for a contradiction.

Let Q be a point in X that is not contained in any line in X (it exists since $\#(X)$ is finite). Keeping in mind that the Zariski tangent space of the surface X at the point P is three-dimensional, we conclude that there exists a hyperplane H in $\mathbb{P}^5$ that contains the lines L_1, L_2, L_3, L_4 and the point Q. Then

$$H\big|_X = C + \sum_{i=1}^{4} L_i,$$

where C is a curve in X that passes through Q. Counting degrees, we see that $\deg(C) \leqslant 1$, so that C is a line, which contradicts the choice of the point Q. □

Lemma 2.8. *Suppose that $d \leqslant 7$. For any singular point of X there is a line passing through it.*

Proof. The required assertion follows from the existence of the diagram (2.1). □

Since the Du Val singularities are $\mathbb{Q}$-factorial, $\rho(X)$ is equal to the rank of the Weil divisor class group $\mathrm{Cl}(X)$.

Lemma 2.9. *Suppose that $d \leqslant 7$. Then the following assertions hold.*

(i) *The group $\mathrm{Cl}(X)$ is generated by the classes of lines in X.*

(ii) *Let $\mathrm{Cl}(X)_{\mathrm{tors}} \subset \mathrm{Cl}(X)$ be the torsion subgroup and let n be the order of the group $\mathrm{Cl}(X)_{\mathrm{tors}}$. There is a Galois abelian cover $\pi\colon Y \to X$ of degree n which is étale outside of $\mathrm{Sing}(X)$, where Y is a Du Val del Pezzo surface such that*

$$K_Y^2 = d\,n,$$

so that $n \leqslant \frac{9}{d}$.

(iii) *If $\rho(X) = 1$ and X contains two distinct lines L and L', then $L \not\sim L'$ and $L \sim_{\mathbb{Q}} L'$.*

(iv) *Every extremal ray of the Mori cone $\overline{\mathrm{NE}}(X)$ is generated by the class of a line.*

(v) *For every effective divisor $D \in \mathrm{Cl}(X)$, there are $a_0, a_1, \ldots, a_r \in \mathbb{Z}_{\geqslant 0}$ such that*

$$D \sim a_0(-K_X) + \sum_{i=1}^{r} a_i L_i$$

where $L_1, \ldots, L_r$ are lines in X, $r = \#(X)$, and $a_0 = 0$ if $d \neq 1$.

Proof. The assertion (ii) follows from a well-known construction, see e.g. [27, § 3.6].

To prove the assertion (iii), observe that

$$L \sim_{\mathbb{Q}} L',$$

because the numerical and $\mathbb{Q}$-linear equivalences on the surface X coincide. But $L \not\sim L'$, because otherwise X would contain a pencil of lines, which contradicts $\#(X) < \infty$.

The assertion (iv) follows from Lemma 2.12 below.

Let us prove the assertion (v). Let $\widetilde{D}$ be the proper transform on $\widetilde{X}$ of the divisor D. To prove (v), it is sufficient to show that the divisor $\widetilde{D}$ is rationally equivalent to a convex integral linear combination of (-1) and (-2)-curves (and $-K_{\widetilde{X}}$ if $d = 1$).

We may assume that $\widetilde{D}$ is an irreducible curve.

Let us use induction on $\dim|\widetilde{D}|$. If $\dim|\widetilde{D}| = 0$, then $\widetilde{D}$ is either a (-1) or (-2)-curve by the Riemann-Roch formula and Kawamata-Viehweg vanishing. This is the base of induction.

Suppose that $\dim|\widetilde{D}| \geqslant 1$ and the required assertion holds for any effective divisor $\widetilde{D}'$ on $\widetilde{X}$ such that $\dim|\widetilde{D}'| < \dim|\widetilde{D}|$. Observe that $\widetilde{D}$ is nef. Thus, if $\widetilde{D}^2 = 0$, then $|\widetilde{D}|$ is base point free and gives a conic bundle $\widetilde{X} \to \mathbb{P}^1$, which must have at least one reducible fiber, because $\rho(\widetilde{X}) \geqslant 3$. Hence, if $\widetilde{D}^2 = 0$, then we can proceed by induction. Thus, we may assume that $\widetilde{D}^2 \geqslant 1$.

If $\widetilde{D}$ is not ample, then $\widetilde{X}$ contains an irreducible curve $\widetilde{C}$ such that $\widetilde{D} \cdot \widetilde{C} = 0$, which implies that $\widetilde{C}^2 < 0$ by the Hodge index theorem, so that $\widetilde{C}^2 = -1$ or $\widetilde{C}^2 = -2$, which gives

$$\dim|\widetilde{D} - \widetilde{C}| \geqslant \dim|\widetilde{D}| - 1 \geqslant 0.$$

Hence, if $\widetilde{D}$ is not ample, then there a exists an effective divisor $\widetilde{D}'$ such that $\widetilde{D} \sim \widetilde{D}' + \widetilde{C}$, so that we can proceed by induction. Therefore, we may assume that $\widetilde{D}$ is ample.

Suppose that $\widetilde{D} \sim -K_{\widetilde{X}}$ and $K_{\widetilde{X}}^2 \geqslant 2$. Then for any (-1)-curve $\widetilde{C}$ on $\widetilde{X}$ we have

$$\dim|\widetilde{D} - \widetilde{C}| \geqslant \dim|\widetilde{D}| - 2 \geqslant 0,$$

so that there is an effective divisor $\widetilde{D}'$ such that $\widetilde{D} \sim \widetilde{D}' + \widetilde{C}$, and we can proceed by induction.

Finally, we assume that $\widetilde{D}$ is ample and $\widetilde{D} \not\sim -K_{\widetilde{X}}$. There is $a \in \mathbb{N}$ such that $\widetilde{D} + aK_{\widetilde{X}}$ is nef but not ample, because the Mori cone of the surface $\widetilde{X}$ is generated by (-1)-curves and (-2)-curves.

Now using the Riemann-Roch formula and Kawamata–Viehweg vanishing, we see that the linear system $|\widetilde{D} + aK_{\widetilde{X}}|$ contains a divisor $\widetilde{D}'$, so that

$$\widetilde{D} \sim \widetilde{D}' - aK_{\widetilde{X}},$$

where $\widetilde{D}'$ and $-K_{\widetilde{X}}$ are both decomposable in the required form. □

Corollary 2.10. *One has $\#(X) \geqslant \rho(X)$. Moreover, if $\#(X) = \rho(X)$, then $\mathrm{Cl}(X)$ is torsion free, and every line in X generates an extremal ray of the Mori cone $\overline{\mathrm{NE}}(X)$.*

Corollary 2.11. *Suppose that $d \leqslant 7$, and X admits a faithful $\mathbb{G}_a^2$-action. Then $\#(X) = \rho(X)$. Moreover, the complement to the open orbit coincides with the union of lines.*

Proof. Let U be the open $\mathbb{G}_a^2$-orbit, let $\widetilde{U} = \mu^{-1}(U)$, let $\overline{U} = \varphi(\widetilde{U})$, let $B = X \setminus U$, let $\widetilde{B} = \widetilde{X} \setminus \widetilde{U}$, and let $\overline{B} = \mathbb{P}^2 \setminus \overline{U}$. Then

$$U \cong \widetilde{U} \cong \overline{U} \cong \mathbb{A}^2,$$

so that the curve $\overline{B}$ must be a line. Then $\widetilde{B}$ has $\rho(\widetilde{X})$ components, and B has $\rho(X)$ components. Since all (-1)-curves on $\widetilde{X}$ are contained in $\widetilde{B}$, we see that all the lines in X are contained in B. This gives $\#(X) \leqslant \rho(X)$. But $\#(X) \geqslant \rho(X)$ by Corollary 2.10. □

Observe that a line L on the surface X generates an extremal ray of $\overline{\mathrm{NE}}(X) \iff L^2 \leqslant 0$.

Lemma 2.12 ([23, Proposition 1.2],[26, § 7.1])**.** *Let V be a surface that has Du Val singularities, and let $\psi\colon V \to Y$ be an extremal Mori contraction. Then one of the following holds:*

(i) *either ψ is a weighted blow up of a smooth point in Y with weights $(1, n)$, the exceptional curve E is smooth and rational, one has $E^2 = -\frac{1}{n}$, and $E \cap \mathrm{Sing}(V)$ consists of one point which is of type A_{n-1} on V;*

(ii) *or ψ is a conic bundle, one has $-K_V \cdot F = 2$ and $F_{\mathrm{red}} \cong \mathbb{P}^1$ for any its scheme fiber F, and if F is not reduced, then one of the following three cases holds:*
   - *$F \cap \mathrm{Sing}(V)$ consists of two singular points of type A_1;*
   - *$F \cap \mathrm{Sing}(V)$ consists of one singular point of type A_3;*
   - *$F \cap \mathrm{Sing}(V)$ consists of one singular point of type D_n, where $n \geqslant 4$.*

In the case (i)*, we say that ψ is a* $(1, n)$-contraction.

Applying this lemma to our Du Val del Pezzo surface X, we get

Corollary 2.13. *Let E be an irreducible curve on X such that $E^2 < 0$. Then E is a line on X, and E is an exceptional divisor of a $(1, n)$-contraction for some $n \geqslant 1$.*

Corollary 2.14. *Suppose there exists a birational morphism $\psi\colon X \to Y$ that is a $(1, n)$-contraction, and let E be the exceptional curve of the morphism ψ. Then*

- *the point $\psi(E)$ is a smooth point of the surface Y;*
- *Y is a Du Val del Pezzo surface, $K_Y^2 = d + n$ and $\rho(Y) = \rho(X) - 1$;*
- *the point $\psi(E)$ is not contained in a line in Y.*

Corollary 2.15. *Let $\psi\colon X \to Y$ be a contraction of a proper face of the cone $\overline{\mathrm{NE}}(X)$. Then*

- *either the morphism ψ is birational, Y is a Du Val del Pezzo surface, and ψ contracts a disjoint union of lines on the surface X,*
- *or the morphism ψ is a conic bundle and $Y \cong \mathbb{P}^1$.*

If a del Pezzo surface X is smooth and $\rho(X) \geqslant 2$, then X always admits a conic bundle contraction. However, this is not always the case if X has Du Val singularities.

Lemma 2.16. *Let X be a Du Val del Pezzo surface of degree d with $\rho(X) \geqslant 2$.*

(i) *Assume that* $d=3$. *Then there exists a conic bundle structure* $\psi\colon X\to\mathbb{P}^1$ *if and only if* X *contains a line* L *that is contained in* $X\setminus\mathrm{Sing}(X)$.

(ii) *Assume that* $d=4$. *Then there exists a conic bundle structure* $\psi\colon X\to\mathbb{P}^1$ *if and only if there is a double cover* $\pi\colon X\to\mathbb{P}^1\times\mathbb{P}^1$ *branched over a curve of degree* $(2,2)$.

Proof. If $d=3$, then X is a cubic surface in $\mathbb{P}^3$, so that every conic bundle $\psi\colon X\to\mathbb{P}^1$ is given by the linear projection from some line in X that does not contain singular points of the surface X, so that X admits a conic bundle contraction if and only if such a line exists.

Assume that $d=4$. If there is a double cover $X\to\mathbb{P}^1\times\mathbb{P}^1$ branched over a curve of degree $(2,2)$, then composing it with a projection to one of the factors, we obtain the required conic bundle. Thus, we may assume that there exists a conic bundle $\psi\colon X\to\mathbb{P}^1$. Let C be its general fiber. Then $|-K_X-C|$ is base point free and gives another conic bundle $\psi'\colon X\to\mathbb{P}^1$. Let $\pi=\psi\times\psi'$. Then $\pi\colon X\to\mathbb{P}^1\times\mathbb{P}^1$ is the required double cover. ☐

3. The Fano–Weil index of Du Val del Pezzo surfaces

Recall from Section 1 that $\tau(X)$ is the Fano–Weil index of a del Pezzo surface X.

Lemma 3.1. *Let* X *be a Du Val del Pezzo surface with* $\rho(X)=1$ *and* $K_X^2\geqslant 3$. *Then* $\tau(X)=K_X^2$.

Proof. Let $d:=K_X^2$ and $D:=K_X+dL$, where L is a line on X. If $D\sim 0$, then we are done. Thus, we may assume that $D\not\sim 0$. Since $D\sim_{\mathbb{Q}}0$, the divisor D is a non-trivial torsion in $\mathrm{Cl}(X)$. Let n be its order. Then

$$2\leqslant n\leqslant\frac{9}{d}$$

by Lemma 2.9, so that either $d=3$ or $d=4$.

Suppose that $d=4$. Then either $\mathrm{Type}(X)=\mathrm{A}_3 2\mathrm{A}_1$ or $\mathrm{Type}(X)=\mathrm{D}_5$ by [7, Proposition 6.1]. In the former case, we see that $4L$ is a Cartier divisor, so that $D\sim 0$, since $\mathrm{Pic}(X)$ is torsion free. In the latter case, L is the unique line in X by [7, Proposition 6.1], so that $D\sim 0$ by Lemma 2.9. Thus, in both cases we obtain a contradiction with our assumption that $D\not\sim 0$.

Thus, we see that $d=3$. Then either $n=2$ or $n=3$. If $n=2$, we have

$$K_X+3(L+D)\sim 4K_X+12L\sim 4(K_X+3D)\sim 0,$$

so that $\tau(X)=3$. If $n=3$, then $X\cong\mathbb{P}^2/\boldsymbol{\mu}_3$ by Lemma 2.9, which implies that $\mathrm{Type}(X)=3\mathrm{A}_2$. In this case, the divisor $3L$ is Cartier, which gives $D\sim 0$, because $\mathrm{Pic}(X)$ is torsion free. ☐

The number $\tau(X)$ divides the degree d of the del Pezzo surface X, so that $\tau(X)=1$ if $d=1$. If $d\geqslant 2$, then the Fano–Weil index $\tau(X)$ is closely related to the following notion:

Definition 3.2. A del Pezzo surface X is said to be *weakly minimal* if X does not contain lines that are contained in the smooth locus of the surface X.

Remark 3.3. If X is a weakly minimal Du Val del Pezzo surface, and $\psi\colon X\to Y$ is a birational contraction, then Y is also a weakly minimal Du Val del Pezzo surface by Corollary 2.14.

Now, we prove the following result.

Proposition 3.4. *Let* X *be a Du Val del Pezzo surface and let* $d:=K_X^2$.

(i) *If* $\tau(X)=d$, *then* $d\leqslant 6$ *and* X *is a hypersurface in* $\mathbb{P}(1,2,3,d)$ *of degree* 6 *given by*

$$y_3^2+y_2^3+\lambda_1y_1^4y_2+\lambda_2y_1^6+y_d\phi(y_1,y_2,y_d)=0,$$

where ϕ *is a polynomial of degree* $6-d$, *and* λ_1 *and* $\lambda_2\in\Bbbk$ *such that* $4\lambda_1^3+27\lambda_2^2\neq 0$,

(ii) *If* $\tau(X)=\frac{d}{2}$*, then* X *is a hypersurface in* $\mathbb{P}(1,1,2,\frac{d}{2})$ *of degree* 4 *given by*

$$y_2^2+y_1y_1'(y_1-y_1')(y_1-\lambda y_1')+x_3\phi(y_1,y_1',x_3)=0,$$

where ϕ *is a polynomial of degree* $4-\frac{d}{2}$*, and* $\lambda\in\Bbbk$ *such that* $\lambda\neq 0$ *and* $\lambda\neq 1$*.*

(iii) *If* $d=6$ *and* $\tau(X)=2$*, then* X *is a hypersurface in* $\mathbb{P}(1,1,1,2)$ *of degree* 3 *given by*

$$\psi(y_1,y_1',y_1'')+y_2\phi(y_1,y_1',y_1'')=0,$$

where ψ *and* ϕ *are polynomials of degree* 3 *and* 1*, respectively.*

Proof. Let us only prove the assertion (i), since the assertions (ii) and (iii) can be proved similarly. Let C be a general curve in $|-K_X|$. Then C is a smooth elliptic curve (see, for example, [8]). Suppose that $\tau(X)=d$. Then $-K_X\sim dA$, where A is a Weil divisor on X. Consider the natural homomorphism of graded algebras

$$\Phi:\mathrm{R}(X,A):=\bigoplus_{n\geqslant 0}H^0\Big(X,\mathscr{O}_X\big(nA\big)\Big)\longrightarrow\bigoplus_{n\geqslant 0}H^0\Big(C,\mathscr{O}_C\big(nA\big)\Big)=:\mathrm{R}(C,A)$$

By the Kawamata–Viehweg vanishing it is surjective. Note that $\mathscr{O}_C(A)$ is a line bundle of degree 1. It is well-known that $\mathrm{R}(C,A)$ is generated by 3 elements $\bar{y}_1$, $\bar{y}_2$, $\bar{y}_3$ with $\deg\bar{y}_i=i$ such that

$$\bar{y}_3^2+\bar{y}_2^3+\lambda_1\bar{y}_1^4\bar{y}_2+\lambda_2\bar{y}_1^6=0$$

for some λ_1 and λ_2 in $\Bbbk$ such that $4\lambda_1^3+27\lambda_2^2\neq 0$. The kernel of Φ is generated by a homogeneous element y_d of degree d. Take arbitrary elements y_1, y_2 and y_3 in $\mathrm{R}(X,A)$ such that $\Phi(y_i)=\bar{y}_i$. Then $\mathrm{R}(X,A)$ is generated by y_1, y_2, y_3 and y_d. This gives us an embedding

$$X\cong\mathrm{Proj}\Big(\mathrm{R}\big(X,A\big)\Big)\hookrightarrow\mathrm{Proj}\Big(\Bbbk\big[y_1,y_2,y_3,y_d\big]\Big)\cong\mathbb{P}(1,2,3,d)$$

whose image is given by an equation of the required form. □

Remark 3.5. The embedding of the surface X described in Proposition 3.4 is almost canonical. It only depends on the choice of the divisor class $A\in\mathrm{Cl}(X)$ such that $-K_X\sim\tau(X)A$, which is uniquely defined modulo $\tau(X)$-torsion. Thus, this embedding is $\mathrm{Aut}^0(X)$-equivariant.

Using Proposition 3.4, we can describe many del Pezzo surfaces:

Example 3.6. Suppose that $d=3$, $\tau(X)=3$, and $\mathrm{Type}(X)=2\mathrm{A}_2$. Using Proposition 3.4, we see that X is a hypersurface in $\mathbb{P}(1,2,3,3)$ of degree 3 that is given by

$$y_3y_3'=y_2(y_2-y_1^2)(y_2-\lambda y_1^2),$$

where $\lambda\in\Bbbk\setminus\{0,\,1\}$.

Example 3.7. Suppose that $d=4$, $\tau(X)=2$, and $\mathrm{Type}(X)=2\mathrm{A}_1$. Using Proposition 3.4, we see that X is a hypersurface in $\mathbb{P}(1,1,2,2)$ of degree 4 that is given by

$$y_2y_2'=y_1y_1'(y_1'-y_1)(y_1'-\lambda y_1),$$

where $\lambda\in\Bbbk\setminus\{0,\,1\}$. This surface is known as *the Iskovskikh surface* (see [19]).

Similarly, we can use Proposition 3.4 to prove the following result:

Theorem 3.8. *Let* X *be a Du Val del Pezzo surface, let* $d:=K_X^2$*. Suppose that* $d\geqslant 3$*,* $\tau(X)>1$*, and the surface* X *is singular. Then* X *is a hypersurface in a weighted projective space* $\mathbb{P}$ *such that one of the following possibilities holds:*

№	d	ρ	Type	τ	$\mathbb{P}$	equation of X

23^o		3	$2\mathrm{A}_2$			see Example 3.6
21^o	3	2	$2\mathrm{A}_2\mathrm{A}_1$	3	$\mathbb{P}(1,2,3,3)$	$y_3y_3' = y_2^2(y_2 + y_1^2)$
16^o		1	$3\mathrm{A}_2$			$y_3y_3' = y_2^3$
18^o		2	A_5			$y_3^2 = y_2^3 + y_1^6 + y_1y_2y_3'$
15^o		1	$\mathrm{A}_5\mathrm{A}_1$			$y_3^2 = y_2^3 + y_3'y_1y_2$
14^o		1	E_6			$y_3^2 = y_2^3 + y_3'y_1^3$
28^o		2	$\mathrm{A}_3\mathrm{A}_1$			$y_3^2 = y_1^6 + y_2y_4$
25^o	4	1	$\mathrm{A}_32\mathrm{A}_1$	4	$\mathbb{P}(1,2,3,4)$	$y_3^2 = y_2y_4$
24^o		1	D_5			$y_3^2 = y_2^3 + y_1^2y_4$
35^o		4	$2\mathrm{A}_1$			see Example 3.7
34^o		3	$3\mathrm{A}_1$		$\mathbb{P}(1,1,2,2)$	$y_2y_2' = y_1^2y_1'(y_1' + y_1)$
30^o		2	$4\mathrm{A}_1$			$y_2y_2' = y_1^2y_1'^2$
29^o	4	2	$\mathrm{A}_22\mathrm{A}_1$	2		$y_2y_2' = y_1^3y_1'$
31^o		3	A_3			$y_2^2 = y_2'y_1y_1' + y_1^4 + y_1'^4$
26^o		2	D_4			$y_2^2 = y_2'y_1^2 + y_1'^4$
36^o	5	1	A_4	5	$\mathbb{P}(1,2,3,5)$	$y_3^2 + y_2^3 + y_1y_5 = 0$
42^o		1	$\mathrm{A}_2\mathrm{A}_1$	6	$\mathbb{P}(1,2,3)$	—
45^o	6	3	A_1	2	$\mathbb{P}(1,1,1,2)$	$y_1''y_2 = y_1y_1'(y_1 - y_1')$
44^o		2	$2\mathrm{A}_1$			$y_1''y_2 = y_1^2y_1'$
43^o		2	A_2	3	$\mathbb{P}(1,1,2,3)$	$y_1y_3 = y_2^2 - y_1'^4$
50^o	8	1	A_1	4	$\mathbb{P}(1,1,2)$	—

Proof. The required assertion follows from Proposition 3.4. Let us show this in the case $d = 3$. Suppose that $d = 3$ and $\tau(X) > 1$. Observe that $\tau(X)$ must divide d. Thus, we have $\tau(X) = 3$. By Proposition 3.4, X is a surface in $\mathbb{P}(1,2,3,3)$ given by

$$y_3^2 + y_2^3 + \lambda_1 y_1^4 y_2 + \lambda_2 y_1^6 + \lambda_3 x_3^2 + \lambda_4 x_3 y_1 y_2 + \lambda_5 x_3 y_1^3 = 0$$

for some λ_1, λ_2, λ_3, λ_4 and λ_5 in $\Bbbk$. If $\lambda_3 \neq 0$, then completing the square we reduce this equation to the defining equation of one the surfaces 16^o, 21^o or 23^o,. Thus, we may assume that $\lambda_3 = 0$.

If $\lambda_4 \neq 0$, then we can use a coordinate change $x_3 \mapsto \alpha x_3 + \beta y_1^3$ and $y_2 \mapsto \gamma y_2 + \delta y_1^2$ for appropriate α, β, γ and δ in $\Bbbk$ to reduce our equation to

$$y_3^2 + y_2^3 + \lambda_2 y_1^6 + x_3 y_1 y_2 = 0.$$

If $\lambda_2 = 0$, this equation defines the surface 15^o. On the other hand, if $\lambda_2 \neq 0$, we can scale the coordinates to get $\lambda_2 = 1$, so that we obtain the defining equation of the surface 18^o.

We may assume that $\lambda_3 = \lambda_4 = 0$. If $\lambda_5 = 0$, then X has a non-Du Val singularity at $(0:0:0:1)$, so that $\lambda_5 \neq 0$. Then we reduce our equation to the defining equation of the surface 14^o. □

Remark 3.9. Using Theorem 3.8, we can easily obtain the anticanonical embedding of the surface $X \hookrightarrow \mathbb{P}^d$. For instance, if $d = 3$ and $\tau(X) = 3$, the map $\mathbb{P}(1,2,3,3) \dashrightarrow \mathbb{P}^3$ given by

$$\big(y_1 : y_2 : y_3 : y_3'\big) \longmapsto \big(y_1^3 : y_1 y_2 : y_3 : y_3'\big)$$

defines an embedding $X \hookrightarrow \mathbb{P}^3$, so that X is a cubic in $\mathbb{P}^3$ given by

23^o: $x_0x_2x_3 = x_1(x_1 - x_0)(x_1 - \lambda x_0)$, where $\lambda \in \Bbbk \setminus \{0, 1\}$;
21^o: $x_0x_2x_3 = x_1^3 + x_0x_1^2$;
16^o: $x_0x_2x_3 = x_1^3$;
18^o: $x_0x_2^2 = x_1^3 + x_0^3 + x_0x_3x_1$;
15^o: $x_0x_2^2 = x_1^3 + x_0x_3x_1$;
14^o: $x_0x_2^2 = x_1^3 + x_3x_0^2$.

If X is not weakly minimal, then $\tau(X) = 1$. In particular, if the del Pezzo surface X is smooth, then $\tau(X) = 1$ unless $X \cong \mathbb{P}^2$ or $X \cong \mathbb{P}^1 \times \mathbb{P}^1$. However, if X is weakly minimal and $d \geqslant 2$, we cannot immediately conclude that $\tau(X) > 1$. Let us present two examples.

Example 3.10. Let X be a quintic del Pezzo surface with $\rho(X) = 2$ admitting a conic bundle contraction $\psi_1 \colon X \to \mathbb{P}^1$. It is easy to see from Lemma 2.12 that $\mathrm{Type}(X) = \mathrm{A}_3$ and the second extremal contraction is a birational $(1, 4)$-contraction $\psi_2 \colon X \to \mathbb{P}^2$. Then X is weakly-minimal. We have an $\mathrm{Aut}(X)$-equivariant morphism $\psi = (\psi_1, \psi_2) \colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ that is finite and birational onto its image, which is given by

$$\phi\big(v_0, v_1, u_0, u_1, u_2\big) = 0,$$

where ϕ is a bihomogeneous polynomial such that its degree with respect to v_0, v_1 equals 1, and its degree with respect to u_0, u_1, u_2 equals 2, since ψ_2 is birational, ψ_1 is a conic bundle, $\rho(X) = 2$. Let P be the singular point of the surface X, and let F be the fiber of ψ_1 that passes through P. We may assume that $\psi(P) = (1 : 0; 0 : 1 : 0)$. Since F is a multiple fiber of the conic bundle ψ_1, we may assume that F is given by $u_2^2 = 0$. Then

$$\phi = u_2^2 v_0 + q(u_0, u_1, u_2)v_1,$$

where q is a quadratic form of rank 3. Changing coordinates, we may assume that $q = u_0^2 + u_1u_2$, so that X is the surface 37^o. Let $\tau = \tau(X)$, and let A be a Weil divisor on X such that $-K_X \sim \tau A$. Then $5 = K_X^2 = -\tau K_X \cdot A$ and $2 = -K_X \cdot C = \tau A \cdot C$, where C is a general fiber of the conic bundle ψ_1. Since $K_X \cdot A$ and $A \cdot C$ are integers, we have $\tau = 1$.

Example 3.11. Let X be a Du Val cubic surface in $\mathbb{P}^3$ with $\mathrm{Type}(X) = \mathrm{A}_4\mathrm{A}_1$. Then $\rho(X) = 2$, and it follows from [6] that X is unique up to isomorphism and can be given by the equation

$$x_0x_2x_3 + x_0^2x_1 + x_1^2x_3 = 0,$$

so that X is the surfaces 19^o. Observe that X contains exactly four lines:

$$L_1 = \{x_0 = x_1 = 0\}, \quad L_2 = \{x_1 = x_3 = 0\}, \quad L_3 = \{x_1 = x_2 = 0\}, \quad L_4 = \{x_0 = x_3 = 0\}$$

and $-K_X \sim L_1 + L_2 + L_3 \sim 2L_1 + L_4 \sim 4L_1 - L_2$. Then $\mathrm{Cl}(X) = \mathbb{Z}[L_1] \oplus \mathbb{Z}[L_2]$ and $\tau(X) = 1$.

Thus, the surfaces 19^o or 37^o are weakly minimal and their Fano–Weil index is 1. On the other hand, we have the following result:

Theorem 3.12. *Let X be a Du Val del Pezzo surface and let $d := K_X^2$. Suppose X is weakly minimal, $d \geqslant 3$, and $\tau(X) = 1$. Then X is one of the surfaces 19^o or 37^o.*

Proof. Observe that $\rho(X) \geqslant 2$ by Lemma 3.1. First, let us consider the case where $\rho(X) = 2$. In this case, the Mori cone $\overline{\mathrm{NE}}(X)$ is generated by two lines L_1 and L_2 such that $L_1 \cap L_2 \neq \varnothing$. Without loss of generality, we may assume that $L_1^2 \geqslant L_2^2$. Then

$$-K_X \sim_{\mathbb{Q}} \alpha_1 L_1 + \alpha_2 L_2$$

for some $\alpha_1 \in \mathbb{Q}_{>0}$ and $\alpha_2 \in \mathbb{Q}_{>0}$. Since $-K_X \cdot L_1 = -K_X \cdot L_2 = 1$ and $K_X^2 = d$, we get

$$\begin{cases} \alpha_1 + \alpha_2 = d, \\ \alpha_1 L_1^2 + \alpha_2 L_1 \cdot L_2 = 1, \\ \alpha_1 L_1 \cdot L_2 + \alpha_2 L_2^2 = 1. \end{cases} \tag{3.13}$$

Let $\psi_1 \colon X \to Y_1$ and $\psi_2 \colon X \to Y_2$ be the contractions of the extremal rays that are generated by the lines L_1 and L_2, respectively.

Assume that ψ_1 is a conic bundle. Since X is weakly minimal, by Lemma 2.16(i) we have $d \geqslant 4$. In particular, we see that the anticanonical model of the surface X is an intersection of quadrics. Let C_1 be a general fiber of ψ_1. Then $C_1 \cdot L_2 = 1$ and $C_1 \sim 2L_1$. Hence, $L_1 \cdot L_2 = \frac{1}{2}$ and $L_1^2 = 0$. Then (3.13) gives $\alpha_2 = 2$, $\alpha_1 = d - 2$ and $L_2^2 = 1 - \frac{d}{4}$. To proceed, we may assume that $d \leqslant 6$. If $d = 5$, then $L_2^2 = -\frac{1}{4}$ and ψ_2 is an $(1,4)$-contraction by Lemma 2.12, so that X is the surface 37^o. If $d = 6$, then $\alpha_1 = 4$, $L_2^2 = -\frac{1}{2}$ and ψ_2 is an $(1,2)$-contraction. Then $4L_1 + 2L_2$ is a Cartier divisor, so that

$$-K_X \sim 4L_1 + 2L_2 = 2(2L_1 + L_2),$$

which is impossible, since $\tau(X) = 1$. Finally, if $d = 4$, then $L_2^2 = 0$ and ψ_2 is also a conic bundle. Since $2L_1 + 2L_2$ is Cartier, we have $-K_X \sim 2L_1 + 2L_2$ and so $\tau(X) > 1$, a contradiction.

Thus, we may assume that both ψ_1 and ψ_2 are birational.

Each line L_1 and L_2 contains exactly one singular point of the surface X by Lemma 2.12. Let P_1 be the singular point contained in L_1, and let P_2 be the singular point contained in L_2. By Lemma 2.12, the points P_1 and P_2 are singular points of types A_{n_1} and A_{n_2} for some positive integers n_1 and n_2. Then

$$-\frac{1}{n_1 + 1} = L_1^2 \geqslant L_2^2 = -\frac{1}{n_2 + 1}$$

by Lemma 2.12, so that $n_1 \geqslant n_2 \geqslant 1$.

Suppose that $P_1 \neq P_2$. Then $L_1 \cap L_2$ is a smooth point of the surface X, so that $L_1 \cdot L_2 = 1$. Then (3.13) gives

$$\begin{cases} \alpha_1 + \alpha_2 = d, \\ -\alpha_1 + \alpha_2(n_1 + 1) = n_1 + 1, \\ \alpha_1(n_2 + 1) - \alpha_2 = n_2 + 1. \end{cases}$$

Note also that $d + n_1 + n_2 \leqslant 8$, since $\rho(\widetilde{X}) = 10 - d$. Eliminating α_1 and α_2, we get

$$d(n_1 n_2 + n_2 + n_1) = 2n_1 n_2 + 3n_1 + 3n_2 + 4$$

This give us the following solutions:

- $d = 4$, $n_1 = n_2 = 1$, $\alpha_1 = \alpha_2 = 2$, $-K_X \sim_{\mathbb{Q}} 2(L_1 + L_2)$,
- $d = 3$, $n_1 = n_2 = 2$, $\alpha_1 = \alpha_2 = 3/2$, $-K_X \sim_{\mathbb{Q}} \frac{3}{2}(L_1 + L_2)$,
- $d = 3$, $n_1 = 4$, $n_2 = 1$, $\mathrm{Type}(X) = \mathrm{A}_4\mathrm{A}_1$, $\alpha_1 = \frac{5}{3}$, $\alpha_2 = \frac{4}{3}$, $-K_X \sim_{\mathbb{Q}} \frac{1}{3}(5L_1 + 4L_2)$.

If $d = 4$, then $2(L_1 + L_2)$ is a Cartier divisor, which gives

$$-K_X \sim 2(L_1 + L_2),$$

which is impossible. If $d = 3$ and $n_1 = n_2 = 2$, then X is a cubic surface in $\mathbb{P}^3$, so that X contains a line L such that L passes through P_1 and P_2 and

$$-K_X \sim L_1 + L_2 + L$$

which gives $-K_X \sim_{\mathbb{Q}} 3L$, because $-K_X \sim_{\mathbb{Q}} \frac{3}{2}(L_1 + L_2)$. In this case, the divisor $3L$ is Cartier, so that $-K_X \sim 3L$, which contradicts $\tau(X) = 1$. Thus, we conclude that $d = 3$, $n_1 = 4$ and $n_2 = 1$. Then X is the surface 19^o. Hence, to proceed, we may assume that $P_1 = P_2$.

Let $n = n_1 = n_2$. Then $L_1^2 = L_2^2 = -\frac{1}{n+1}$. Moreover, we have $L_1 \cdot L_2 = \frac{k}{n+1}$ for some $k \in \mathbb{Z}_{>0}$. Then (3.13) gives $\alpha_1 = \alpha_2 = \frac{d}{2}$ and $2(n+1) = d(k-1)$, so that

$$-K_X \sim_{\mathbb{Q}} \tfrac{d}{2}\big(L_1 + L_2\big).$$

Note also that $d+n \leqslant 8$, since $\mathsf{\rho}(\widetilde{X}) = 10-d$. Then $d \neq 5$ and $d \neq 7$, because $2(n+1) = d(k-1)$. Likewise, if $d = 6$, then $n = 2$, so that $-K_X \sim_{\mathbb{Q}} 3(L_1+L_2)$ and $\mathrm{Cl}(X)$ is torsion free by Lemma 2.9, which gives $-K_X \sim 3(L_1 + L_2)$, which contradicts $\tau(X) = 1$. Hence, either $d = 3$ or $d = 4$.

Suppose that $d = 4$. Then $n = 3$, because $2(n + 1) = d(k - 1)$ and $d + n \leqslant 8$. Since $\mathsf{\rho}(X) = 2$, we see that X has a singular point of type A_1. On the other hand, we have $-K_X \sim_{\mathbb{Q}} 2(L_1 + L_2)$. Since $\tau(X) = 1$, we have $-K_X \not\sim 2(L_1 + L_2)$, so that $K_X + 2(L_1 + L_2)$ is a non-trivial torsion element in $\mathrm{Cl}(X)$. Now, applying Lemma 2.9, we obtain a double cover $\pi\colon Y \to X$ that is étale outside of the point $L_1 \cap L_2$. Then Y is a del Pezzo surface of degree 8 such that it contains two singular points of type A_1, which is absurd. This shows that $d \neq 4$.

Therefore, we see that $d = 3$. Since $2(n + 1) = d(k - 1)$ and $d + n \leqslant 8$, we have $n = 2$ or $n = 5$. Since X is a cubic surface in $\mathbb{P}^3$, it contains a line L such that

$$-K_X \sim L_1 + L_2 + L,$$

so that $L \sim_{\mathbb{Q}} \frac{1}{2}(L_1 + L_2)$, because $-K_X \sim_{\mathbb{Q}} \frac{3}{2}(L_1 + L_2)$. This gives $-K_X \sim_{\mathbb{Q}} 3L$. But $-K_X \not\sim 3L$. In particular, we have $n \neq 2$, because $3L$ is a Cartier divisor if $n = 2$. We conclude that $n = 5$. Now, using Lemma 2.9, we conclude that there is a finite Galois cover $\pi\colon Y \to X$ of degree $r \geqslant 2$, which is étale outside of the point $L_1 \cap L_2$. Here, r be the order of the torsion divisor $K_X + 3L$. By construction, the surface Y is a del Pezzo surface of degree rd, so that either $r = 2$ or $r = 3$. If $r = 3$, then $Y \cong \mathbb{P}^2$, which is impossible, since $\mathsf{\rho}(Y) \geqslant \mathsf{\rho}(X) = 2$. Thus, we have $r = 2$. Then

$$-K_X \sim 3L - (K_X + 3L) \sim 3L - (K_X + 3L) + 4(K_X + 3L) \sim 3L + 3(K_X + 3L) \sim 3(K_X + 4L),$$

which is impossible, since $\tau(X) = 1$. The obtained contradiction shows that $\mathsf{\rho}(X) \neq 2$.

We see that $\mathsf{\rho}(X) \geqslant 3$ and X is singular. Then $d \leqslant 6$.

Let $\psi\colon X \to Y$ be an extremal Mori contraction. Since $\mathsf{\rho}(X) \geqslant 3$, the morphism ψ is birational, so that ψ is a $(1,m)$-contraction of a line $L \subset X$ by Lemma 2.12. Then Y is a weakly minimal Du Val del Pezzo surface such that $K_Y^2 = d + m$ with $\mathsf{\rho}(Y) = \mathsf{\rho}(X) - 1 \geqslant 2$ (see Remark 3.3). In particular, we have $d + m \neq 7$, because a Du Val del Pezzo surface of degree 7 is not weakly minimal (see Example 2.3). Note that $m \geqslant 2$, since X is weakly minimal.

Consider the case $d = 6$. Using the Noether formula, we see that $\mathsf{\rho}(X) = 3$ and $\mathrm{Type}(X) = \mathrm{A}_1$. Then $m = 2$, $K_Y^2 = 8$, $\mathsf{\rho}(Y) = 2$, so that $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then $K_X \sim \psi^*K_Y + 2L$ is divisible by 2, which is a contradiction. Thus, we have $d \neq 6$.

Consider the case $d = 5$. Since $d + m \neq 7$, we have $m > 2$. Then $\mathsf{\rho}(X) = 3$ and $\mathrm{Type}(X) = \mathrm{A}_2$ by the Noether formula, so that X is not weakly minimal by [7, Proposition 8.5]. This contradicts our assumption.

Consider the case $d = 4$. Then $m \neq 3$, since $d + m \neq 7$. If $m > 3$, it follows from the Noether formula that $m = 4$, $\mathsf{\rho}(X) = 3$ and $\mathrm{Type}(X) = \mathrm{A}_3$, so that $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$, which gives

$$-K_X \sim 2\big(\widetilde{F}_1 + \widetilde{F}_2\big),$$

where $\widetilde{F}_1$ and $\widetilde{F}_2$ are proper transforms on X of the curves in Y of bi-degree $(1,0)$ and $(0,1)$ that contains $\psi(L)$, respectively. This contradicts our assumption $\tau(X) = 1$. So, we see that all extremal contractions on X are birational $(1,2)$-contractions. Then $\tau(Y) > 1$, since we already dealt with sextic del Pezzo surfaces. By Theorem 3.8, we see that Y is one of the surfaces 45^o, 44^o, 43^o. If Y is the surface 43^o, then it has a birational $(1,3)$-contraction, so that X also has a birational

(1, 3)-contraction by Corollary 2.14, which is a contradiction. Then Y is one of the surfaces 45^o or 44^o. By Lemma 2.9(i), we have

$$-K_Y \sim 2\sum_{i=1}^{s} a_i M_i$$

for some lines $M_1, \ldots, M_s$ on the surface Y and some integers $a_1, \ldots, a_s$. Since $\psi(L)$ is a smooth point of Y that does not lie on a line by Corollary 2.14, we obtain

$$-K_X \sim 2\sum_{i=1}^{s} a_i \widetilde{M}_i - 2L \sim 2\Big(\sum_{i=1}^{s} a_i \widetilde{M}_i - L\Big),$$

where $\widetilde{M}_i$ is a proper transform on X of the line M_i. This contradicts our assumption $\tau(X) = 1$.

Finally, we consider the case $d = 3$. Then $m \neq 4$, since $d + m \neq 7$. Moreover, since X is weakly minimal, there exists no dominant morphisms from X to a curve by Lemma 2.16(i), and the same holds for Y. Using this, we conclude that $m \neq 5$. Thus, we have the following possibilities:

- either $K_Y^2 = 5$ and $m = 2$,
- or $K_Y^2 = 6$ and $m = 3$.

Moreover, if $K_Y^2 = 5$, then Y is not the surface 37^o, because del Pezzo surface 37^o admits a dominant morphism to $\mathbb{P}^1$ (see Example 3.10). Therefore, we conclude that $\tau(Y) > 1$, because we already dealt with weakly minimal Du Val del Pezzo surfaces of degree 5 and 6.

Now, using Theorem 3.8, we see that $K_Y^2 \neq 5$, because $\rho(Y) > 1$. Therefore, we have $K_Y^2 = 6$. Then Y is the surface 43^o, 44^o or 45^o again by Theorem 3.8. If Y is the surface 44^o, then

$$\frac{y_1}{y_1''} = \frac{y_1'^2}{y_2}$$

on the surface Y, so that the map $Y \dashrightarrow \mathbb{P}^1$ given by

$$(y_1 : y_1' : y_1'' : y_2) \longmapsto (y_1 : y_1'') = (y_1'^2 : y_2)$$

is a morphism, which is a contradiction. Similarly, we obtain a contradiction when Y is the surface 45^o, because the map $Y \dashrightarrow \mathbb{P}^1$ given by

$$(y_1 : y_1' : y_1'' : y_2) \longmapsto (y_1 : y_1') = (y_1''(y_1' + y_1'') : y_2)$$

is a morphism in this case. Thus, we see that Y is the surface 43^o. Then X is a cubic surface such that $\mathrm{Type}(X) = 2\mathrm{A}_2$. Now, using [6], we conclude that X is one of the surfaces 23^o, so that $\tau(X) = 3$ by Theorem 3.8, which contradicts our assumption. ☐

4. The proof of Main Theorem: higher degree cases

Let X be a Du Val del Pezzo surface of degree d whose automorphism group $\mathrm{Aut}(X)$ is infinite. If $d \geqslant 8$, then X is either $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{F}_1$ or $\mathbb{P}(1, 1, 2)$. In each of this case, the corresponding automorphism group is well-known and listed in Big Table (cases 53^o, 52^o, 51^o, 50^o respectively).

If $d \leqslant 7$, we have an $\mathrm{Aut}^0(X)$-equivariant diagram (2.1). If $d = 7$, then X is one of the del Pezzo surfaces 48^o and 49^o, and the morphism φ in (2.1) is a blow up of two (possibly infinitely near) points. From this we obtain the following

Lemma 4.1. *Let X be a Du Val del Pezzo surface of degree 7, and let U be the complement in the surface X to the union of all lines. Then Main Theorem holds for X, the subset U is the open orbit of the group $\mathrm{Aut}^0(X)$, and $U \cong \mathbb{A}^2$.*

All del Pezzo surfaces of degree 6 have infinite automorphism groups, so that all of them appear in our Big Table. These are the del Pezzo surfaces 42^o, 43^o, 44^o, 45^o, 46^o and 47^o. Going through these six cases one by one, we obtain

Lemma 4.2. *Let X be a Du Val del Pezzo surface of degree 6, and let U be the complement in the surface X to the union of all lines. Then Main Theorem holds for X, the subset U is the open orbit of the group $\mathrm{Aut}^0(X)$, and $U \cong \mathbb{A}^2$ in the cases 42^o, 43^o, 44^o, and 45^o.*

Proof. If X is the surface 47^o, then φ in (2.1) is a blow up of three distinct non-collinear points, so that X is toric and $\mathrm{Aut}^0(X) \cong \mathbb{G}_{\mathrm{m}}^2$.

Likewise, if X is the surface 45^o, then φ is the blow up of three distinct collinear points in $\mathbb{P}^2$. Using this, it is not hard to see that $\mathrm{Aut}^0(X) \cong \mathbb{G}_{\mathrm{a}}^2 \rtimes \mathbb{G}_{\mathrm{m}}$ in this case.

For infinitely near points we use the notation of [11].

If X is the surface 42^o, then the morphism φ is the blow up of three infinitely near collinear points $P_1 \prec P_2 \prec P_3$ in the plane $\mathbb{P}^2$ in the notations of [11], which implies that $\mathrm{Aut}^0(X) \cong \mathbb{B}_3$.

Similarly, if X is the surface 43^o, then φ is the blow up of three infinitely near non-collinear points $P_1 \prec P_2 \prec P_3$, which implies that $\mathrm{Aut}^0(X) \cong \mathbb{U}_3 \rtimes \mathbb{G}_{\mathrm{m}}$.

If X is the surface 44^o, then φ is the blow up of three collinear points P_1, P_2 and P_3 such that the points P_1 and P_2 are distinct and $P_3 \succ P_1$, which implies that $\mathrm{Aut}^0(X) \cong \mathbb{B}_2 \times \mathbb{B}_2$.

Finally, if X is the surface 46^o, then φ is the blow up of three non-collinear points P_1, P_2 and P_3, so that P_1 and P_2 are distinct, but $P_3 \succ P_1$. Hence, in this case, we have $\mathrm{Aut}^0(X) \cong \mathbb{B}_2 \times \mathbb{G}_{\mathrm{m}}$.

The last assertions follow from Corollary 2.11 in the cases 42^o, 43^o, 44^o, and 45^o, and it follows from Lemma 4.1 in the cases 46^o and 47^o. ☐

Similarly, all singular del Pezzo surfaces of degree 5 also have infinite automorphism groups. These are the surfaces 36^o, 37^o, 38^o, 39^o, 40^o and 41^o in Big Table.

Lemma 4.3. *Let X be a Du Val del Pezzo surface of degree 5, and let U be the complement in the surface X to the union of all lines. Then Main Theorem holds for X, the subset U is the open orbit of the group $\mathrm{Aut}^0(X)$ in the cases 36^o, 37^o, 38^o, 39^o, 40^o, and $U \cong \mathbb{A}^2$ in the cases 36^o, 37^o.*

Proof. If X is weakly minimal and $\tau(X) > 1$, then X is the surface 36^o by Theorems 3.12 and 3.8. Then the group $\mathrm{Aut}^0(X)$ consists of the transformations that send the point $(y_1 : y_2 : y_3 : y_5)$ to

$$\Big(y_1 : t^2y_2+ay_1^2 : t^3y_3+by_1^3+cy_1y_2 : t^6y_5-(a^3+b^2)y_1^5-(3a^2t^2+2bc)y_1^3y_2-2bt^3y_1^2y_3-(3at^4+c^2)y_1y_2^2-2ct^3y_2y_3\Big),$$

where $t \in \Bbbk^*$ and $a, b, c \in \Bbbk$. This gives $\mathrm{Aut}^0(X) \cong \mathbb{U}^3 \rtimes \mathbb{G}_{\mathrm{m}}$ (cf. Corollary 2.2, see Corollary B.8).

If X is weakly minimal and $\tau(X) = 1$, then X is the del Pezzo surface 37^o by Theorem 3.12, and its $\mathrm{Aut}^0(X)$-equivariant embedding $X \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2$ is described in Example 3.10. In this case, the group $\mathrm{Aut}(X)$ contains a two-dimensional unipotent subgroup

$$(v_0 : v_1;\ u_0 : u_1 : u_2) \longmapsto \big(v_0 - (a_1^2 + a_2)v_1 : v_1;\ u_0 + a_1u_2 : u_1 - 2a_1u_0 + a_2u_2 : u_2\big)$$

and a one-dimensional torus

$$(v_0 : v_1;\ u_0 : u_1 : u_2) \longmapsto (v_0 : t^{-2}v_1;\ tu_0 : t^2u_1 : u_2),$$

where $a_1, a_2 \in \Bbbk$ and $t \in \Bbbk^*$. This implies that $\mathrm{Aut}(X) \cong \mathbb{G}_{\mathrm{a}}^2 \rtimes \mathbb{G}_{\mathrm{m}}$ as required.

We may assume that X is not weakly minimal. Then there is a birational morphism $\psi\colon X \to Y$ such that Y is one of the surfaces 42^o, 43^o, 44^o, 45^o, and ψ is a blow up of a smooth point $P \in Y$. By Corollary 2.14 and Lemma 4.2, the point P is contained in the open orbit of the group $\mathrm{Aut}^0(Y)$. Since $\mathrm{Aut}^0(X)$ is the connected component of the stabilizer in $\mathrm{Aut}^0(Y)$ of the point P, this gives the required description of the group $\mathrm{Aut}^0(X)$ in Big Table.

The last assertions follow from Corollary 2.11 in the cases 36^o, 37^o, and it follows from Lemma 4.2 in the cases 38^o, 39^o, 40^o. ☐

Let us conclude this section by proving Main Theorem for Du Val del Pezzo surfaces of degree 4.

Proposition 4.4. *Main Theorem holds for Du Val del Pezzo surfaces of degree 4.*

Proof. Let X be Du Val del Pezzo surface of degree 4 such that the group $\mathrm{Aut}(X)$ is infinite. Suppose that X is not weakly minimal. Then there exists a birational morphism $\psi\colon X \to Y$ such that Y is a singular quintic Du Val del Pezzo surface, and ψ is a blow up of a smooth point $P \in Y$, which is not contained in a line by Corollary 2.14. Observe that the group $\mathrm{Aut}^0(X)$ is the connected component of the stabilizer in $\mathrm{Aut}^0(Y)$ of the point P. Using Lemma 4.3, we see that Y must be one of the surfaces 36^o, 37^o, 38^o, and P must be contained in the open orbit of the group $\mathrm{Aut}^0(Y)$. This implies that X is one of the surfaces 27^o, 32^o, 33^o, respectively. Now, it is not hard to check that $\mathrm{Aut}^0(X) \cong \mathbb{G}_\mathrm{m}$ in the cases 32^o and 33^o, and $\mathrm{Aut}^0(X) \cong \mathbb{B}_2$ in the case 27^o.

Hence, we may assume that the surface X is weakly minimal. Then $\tau(X) > 1$ by Theorem 3.12. Using Theorem 3.8, we see that X is one of the surfaces 24^o, 25^o, 26^o, 28^o, 29^o, 30^o, 31^o, 34^o, 35^o. Let us show that $\mathrm{Aut}(X)$ is infinite, and $\mathrm{Aut}^0(X)$ is described in Big Table.

Let X be the surface 24^o. Then X is embedded into $\mathbb{P}(1,2,3,4)$ as a hypersurface that is given by the equation $y_3^2 = y_2^3 + y_1^2 y_4$. This embedding is $\mathrm{Aut}^0(X)$-equivariant by Remark 3.5. Observe that $\mathrm{Aut}^0(X)$ contains transformations

$$(y_1 : y_2 : y_3 : y_4) \longmapsto \big(y_1 : y_2 + a y_1^2 : y_3 + b y_1^3 : y_4 - (a^3 - b^2) y_1^4 - 3a^2 y_1^2 y_2 + 2b y_1 y_3 - 3a y_2^2\big),$$

where $a \in \Bbbk$ and $b \in \Bbbk$. These transformations generates a subgroup in $\mathrm{Aut}^0(X)$ isomorphic to $\mathbb{G}_\mathrm{a}^2$. Moreover, the surface X also admits an action of a one-dimensional torus which acts diagonally:

$$(y_1 : y_2 : y_3 : y_4) \longmapsto (t^2 y_1 : t^2 y_2 : t^3 y_3 : t^2 y_4),$$

where $t \in \mathbb{G}_\mathrm{m}$. The described transformations generate a subgroup that is isomorphic to $\mathbb{G}_\mathrm{a}^2 \rtimes \mathbb{G}_\mathrm{m}$. Since X has a singularity of type D_5, it is not toric, so that $\mathrm{Aut}^0(X) \cong \mathbb{G}_\mathrm{a}^2 \rtimes \mathbb{G}_\mathrm{m}$ (cf. Corollary B.8).

Now we suppose that X is the surface 25^o. Then $\mathrm{Aut}(X)$ contains transformations

$$\gamma(a) : (y_1 : y_2 : y_3 : y_4) \longmapsto \big(y_1 : y_2 : y_2 + a y_1 y_2 : y_4 + 2a y_1 x_2 + a^2 y_1^2 y_2\big)$$

for every $a \in \Bbbk$. These transformations generates a proper subgroup in $\mathrm{Aut}^0(X)$ isomorphic to $\mathbb{G}_\mathrm{a}$. Moreover, the surface X also contains transformations

$$\delta(t_1, t_2) : (y_1 : y_2 : y_3 : y_4) \longmapsto \big(y_1 : t_1 t_2^2 y_2 : t_1 t_2 y_3 : t_1 y_4\big)$$

for every $t_1 \in \Bbbk^*$ and $t_2 \in \Bbbk^*$. They generates a subgroup isomorphic to $\mathbb{G}_\mathrm{m}^2$. Observe that

$$\gamma(a) \circ \delta(t_1, t_2) = \delta(t_1, t_2) \circ \gamma_{a t_2}.$$

Therefore, all described transformations generate a subgroup in $\mathrm{Aut}^0(X)$ isomorphic to $\mathbb{B}_2 \times \mathbb{G}_\mathrm{m}$. Then $\mathrm{Aut}^0(X) \cong \mathbb{B}_2 \times \mathbb{G}_\mathrm{m}$, because X does not admit an effective $\mathbb{G}_\mathrm{a}^2$-action by Corollary 2.11.

Now we suppose that 28^o. Then the group $\mathrm{Aut}(X)$ contains transformations

$$(y_1 : y_2 : y_3 : y_4) \longmapsto \big(t y_1 : t^3 y_2 : t^3 y_3 + a t^3 y_1 y_2 : t^3 y_4 + 2a t^3 y_1 y3 + a^2 t^3 y_1^2 y_2\big)$$

for any $a \in \Bbbk$ and $t \in \Bbbk^*$. These transformations generate a subgroup in $\mathrm{Aut}^0(X)$ isomorphic to $\mathbb{B}_2$. Since all three lines on X pass through one point, X is not toric. Hence, $\mathrm{rk}\,\mathrm{Aut}^0(X) = 1$. The surface X does not admit an effective $\mathbb{G}_\mathrm{a}^2$-action by Corollary 2.11, so that $\dim \mathrm{Aut}^0(X) = 2$, which implies $\mathrm{Aut}^0(X) \cong \mathbb{B}_2$.

Let X be one of the surfaces 29^o or 30^o. Then $\#(X) = 4$. Let L_1, L_2, L_3, L_4 be the lines in X. Recall that X is an intersection of two quadrics in $\mathbb{P}^4$. We have

$$L_1 + L_2 + L_3 + L_4 \sim -K_X,$$

so that $L_1 + L_2 + L_3 + L_4$ is cut out by a hyperplane $H \subset \mathbb{P}^4$. On the other hand, this curve form a combinatorial cycle. Thus, if X admits an effective $\mathbb{G}_\mathrm{a}$-action, then this action is trivial on each line among L_1, L_2, L_3 and L_4, so that it is trivial on H, which implies that the closure of any one-dimensional $\mathbb{G}_\mathrm{a}$-orbit is a line. The latter is impossible, since X contains finitely many lines. Therefore, we conclude that the surface X does not admit an effective action of the group $\mathbb{G}_\mathrm{a}$. On the other hand, the equations of X are binomial. This implies that X admits a diagonal action of two-dimensional torus. Hence, $\mathrm{Aut}^0(X) \cong \mathbb{G}_\mathrm{m}^2$.

Let X be one of the surfaces 34^o or 35^o. Then X contains two lines L_1 and L_2 such that the intersection $L_1 \cap L_2$ is a smooth point of X, and there exist an $\mathrm{Aut}^0(X)$-equivariant diagram

$$\begin{array}{ccccc} & & \widehat{X} & & \\ & \overset{\alpha}{\swarrow} & & \overset{\beta}{\searrow} & \\ X & & & & Y \end{array}$$

where α is a blow up of the point $L_1 \cap L_2$, and β is the birational contraction of the proper transforms of the lines L_1 and L_2. Then we have the following possibilities:

- if X is the surface 34^o, then Y is a cubic surface such that $\mathrm{Type}(Y) = \mathrm{A}_4\mathrm{A}_1$;
- if X is the surface 35^o, then Y is a cubic surface such that $\mathrm{Type}(Y) = 2\mathrm{A}_2$.

Hence, it follows from Corollary 7.4, that Y does not admit an effective action of the group $\mathbb{G}_\mathrm{a}$. Since X contains three lines passing through one point, it is not toric. One the other hand, it is easy to see that X admits an effective diagonal action of a one-dimensional torus.

To complete the proof, we may assume that X is one of the surfaces 26^o or 31^o. By Lemma 2.16(ii), there is a double cover $\pi\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ branched over a curve B of degree $(2,2)$. By construction, this double cover is $\mathrm{Aut}^0(X)$-equivariant, and the curve B is $\mathrm{Aut}^0(X)$-invariant. Therefore, there exists an exact sequence of groups

$$1 \longrightarrow \boldsymbol{\mu}_2 \longrightarrow \mathrm{Aut}(X) \longrightarrow \mathrm{Aut}\big(\mathbb{P}^1 \times \mathbb{P}^1, B\big).$$

If X is the surface 26^o, then B is a union of irreducible smooth curves of degrees $(1,1)$, $(1,0)$, $(0,1)$, which intersect in one point, which implies that

$$\mathrm{Aut}\big(\mathbb{P}^1 \times \mathbb{P}^1, B\big) \cong \mathbb{B}_2 \rtimes \boldsymbol{\mu}_2.$$

This can be shown by taking linear projection $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ from the singular point $\mathrm{Sing}(B)$, where we consider $\mathbb{P}^1 \times \mathbb{P}^1$ as a quadric in $\mathbb{P}^3$. Thus, in this case, we have $\mathrm{Aut}^0(X) \cong \mathbb{G}_\mathrm{a} \rtimes_{(2)} \mathbb{G}_\mathrm{m}$, since $\mathrm{Aut}(X)$ contains a subgroup isomorphic to $\mathbb{G}_\mathrm{a} \rtimes_{(2)} \mathbb{G}_\mathrm{m}$ generated by transformations

$$(y_1 : y_1' : y_2 : y_2') \longmapsto \big(y_1 : ty_1' : t^2y_2 + at^2y_1^2 : t^4y_2' + 2at^4y_2 + a^2t^4y_1^2\big),$$

where $a \in \Bbbk$ and $t \in \Bbbk^*$. Similarly, if X is the surface 31^o, then

$$\mathrm{Aut}\big(\mathbb{P}^1 \times \mathbb{P}^1, B\big) \cong \mathbb{G}_\mathrm{a} \rtimes \boldsymbol{\mu}_2,$$

because B is a union of two irreducible smooth curves of degree $(1,1)$, which intersect in one point. In this case, we have $\mathrm{Aut}^0(X) \cong \mathbb{G}_\mathrm{a}$, since the $\mathbb{G}_\mathrm{a}$-action lifts from $\mathbb{P}^1 \times \mathbb{P}^1$ to the surface X. □

Corollary 4.5. *Main Theorem holds del Pezzo surfaces of degree $\geqslant 4$.*

5. Del Pezzo surfaces of degree 1

In this section, we prove Main Theorem for del Pezzo surfaces of degree 1.

Proposition 5.1. *Main Theorem holds for del Pezzo surfaces of degree* 1.

We start with

Lemma 5.2. *Let X be a Du Val del Pezzo surface of degree* 1*. Then X does not admit effective actions of the group $\mathbb{G}_\mathrm{a}$.*

Proof. Suppose that X admit an effective $\mathbb{G}_{\mathrm{a}}$-action. Let $\Phi\colon X \dashrightarrow \mathbb{P}^1$ be the anticanonical map. It is $\mathrm{Aut}(X)$-equivariant, and all its fibers are reduced irreducible curves of arithmetic genus 1. Since $\mathbb{G}_{\mathrm{a}}$ cannot effectively act on a smooth elliptic curve, we conclude that $\mathbb{G}_{\mathrm{a}}$ acts non-trivially on the base of Φ. Thus, there is exactly one $\mathbb{G}_{\mathrm{a}}$-invariant fiber, say C. Any fiber of Φ different from C is a smooth elliptic curve. Thus we have

$$\rho(X) + 2 = \chi(X) = \chi(C) - 1,$$

so that $\chi(C) = \rho(X) + 3 \geqslant 4$. But $\chi(C) \leqslant 2$, because C is a curve of arithmetic genus 1. □

Our next step in proving Main Theorem for del Pezzo surfaces of degree 1 is the following

Lemma 5.3. *If X is one of the surfaces 1^o, 2^o, 3^o, 4^o, then* $\mathrm{Aut}^0(X) \cong \mathbb{G}_{\mathrm{m}}$.

Proof. By Lemma 5.2 these surfaces do not admit a $\mathbb{G}_{\mathrm{a}}$-action and they are not toric because their singularities are not cyclic quotient. On the other hand, it is easy to see that each of these surfaces admits a $\mathbb{G}_{\mathrm{m}}$-action. □

We also need the following easy local fact.

Lemma 5.4. *Let $(X \ni P)$ be a Du Val singularity defined over $\mathbb{C}$ that contains a reduced irreducible curve C such that C is a Cartier divisor on X, and the singularity $(C \ni P)$ is a simple cusp.*

(i) *If $(X \ni P)$ is of type A_n, then $n \leqslant 2$.*

(ii) *If $(X \ni P)$ is of type D_n with $n \geqslant 5$, then some small analytic neighborhood of $(X \ni P)$ can be given by the equation*

$$x^2 + y^2 z + z^{n-1} = 0,$$

so that C is cut out by $z = y + \phi(x, y, z)$, where $\mathrm{mult}_0(\phi) \geqslant 2$.

Proof. Let us prove the assertion (i). In a neighborhood of the point P, the curve C is cut out by a hypersurface, say H. Thus, we have $C = X \cap H$ in $\mathbb{C}^3$. Since the multiplicity of the curve C at P equals 2, the hypersurface H is smooth at P. Therefore, we may assume that H is given by $z = 0$, and C is given by

$$\begin{cases} x^2 + y^3 = 0, \\ z = 0. \end{cases}$$

Hence, the equation of the surface X is

$$x^2 + y^3 + z\phi(x, y, z) = 0.$$

Since $P \in X$ is a point of type A_n, the rank of the quadratic part of this equation is at least 2. Then $\phi(x, y, z)$ contains a linear term. This implies that $P \in X$ is of type A_2 or A_3.

Now, let us prove the assertion (ii). We may assume that $P \in X$ is given in $\mathbb{C}^3$ by the equation

$$x^2 + y^2 z + z^{n-1} = 0.$$

As above, we have $C = X \cap H$, where H is a hypersurface that is smooth at P. Then the equation of the hypersurface H must contain a linear term. Moreover, one can see that this equation must be of the form $z = y + \phi(x, y, z)$, which implies (ii). □

Corollary 5.5. *Let X be a surface admitting an effective $\mathbb{G}_{\mathrm{m}}$-action, let C be a $\mathbb{G}_{\mathrm{m}}$-invariant reduced irreducible curve in X that is a Cartier divisor on X, and let P be its singular point. Suppose that X has Du Val singularity of type D_n at P, and C has a simple cusp at P. Then $n = 4$.*

Proof. Suppose that $n > 4$. There exists a $\mathbb{G}_{\mathrm{m}}$-equavariant embedding of the germ $P \in X$ to $\mathbb{C}^3$. Let us choose $\mathbb{G}_{\mathrm{m}}$-semi-invariant coordinates in $\mathbb{C}^3$. By Lemma 5.4, we see that $C = X \cap H$, where the equation of H has the form $z = y + \phi(x, y, z)$. But this equation cannot be $\mathbb{G}_{\mathrm{m}}$-semi-invariant, which is a contradiction. □

Now, we are ready to prove Proposition 5.1.

Proof of Proposition 5.1. Let X be a Du Val del Pezzo surface of degree 1. Then X does not admit an effective $\mathbb{G}_a$-action by Lemma 5.2. Thus, the group $\mathrm{Aut}(X)$ is infinite if and only if X admits an effective action of the group $\mathbb{G}_m$.

Suppose that X admits an effective $\mathbb{G}_m$-action. By Lemma 5.3, to complete the proof, it is enough to show that X is one of the surfaces 1^o, 2^o, 3^o, 4^o.

Let $\Phi\colon X \dashrightarrow \mathbb{P}^1$ be the anticanonical map. It is $\mathbb{G}_m$-equivariant, and all its fibers are reduced irreducible curves of arithmetic genus 1. Since $\mathbb{G}_m$ cannot effectively act on a smooth elliptic curve, we conclude that $\mathbb{G}_m$ acts non-trivially on the base of Φ. There are exactly two $\mathbb{G}_m$-invariant fibers. Denote them by C_1 and C_2. Any fiber of Φ different from C_1 and C_2 is a smooth elliptic curve. Thus we have

$$\rho(X) + 2 = \chi(X) = \chi(C_1) + \chi(C_2) - 1,$$

so that $\chi(C_1)+\chi(C_2) = \rho(X)+3 \geqslant 4$. Since $\chi(C_i) \leqslant 2$, we have $\rho(X) = 1$ and $\chi(C_1) = \chi(C_2) = 2$. This means in particular that C_1 and C_2 are cuspidal curves of arithmetic genus 1.

Let P_i be the singular point of C_i. Then

$$\varnothing \neq \mathrm{Sing}(X) \subset \big\{P_1,\, P_2\big\}.$$

The singularity of the surface X at the point P_1 is of type A_{n_1}, D_{n_1} or E_{n_1} for some $n_1 \geqslant 1$. Likewise, if X is singular at P_2, then P_2 is a singular point of type A_{n_2}, D_{n_2} or E_{n_2} for some $n_2 \geqslant 0$, where $n_2 = 0$ simply means that the point P_1 is the only singular point of the del Pezzo surface X. Without loss of generality, we may assume that $n_1 \geqslant n_2$. Since $\rho(X) = 1$, $n_1 + n_2 = 8$ and $n_1 \geqslant 4$. Now, using Corollary 5.5, we obtain the following possibilities for $\mathrm{Type}(X)$: E_8, $\mathrm{E}_7\mathrm{A}_1$, $\mathrm{E}_6\mathrm{A}_2$, $2\mathrm{D}_4$. But X is uniquely determined by $\mathrm{Type}(X)$ and the fact that $|-K_X|$ has two singular curves that are both cuspidal. This follows from [30, Theorem 1.2] and [30, Table 4.1], see also Remark B.5. Thus, we conclude that X is one of the surfaces 1^o, 2^o, 3^o, 4^o (see e.g. [5, Satz 2.11]). ☐

Let us conclude this section by an observation that the surfaces 2^o, 3^o, 4^o can be obtained as finite quotients of other surfaces in Big Table. The surface 2^o is the quotient of the surface 11^o by $\boldsymbol{\mu}_2$. The surface 3^o is the quotient of the cubic surface 22^o by $\boldsymbol{\mu}_3$, and 4^o is the quotient of a special member of the family 35^o by $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$. In all the cases the action of the group is free outside the singular locus. This observation can be used to obtain the description of the surfaces 2^o, 3^o, 4^o. To show this one can look at the exact sequence

$$0 \longrightarrow \mathrm{Pic}(X) \xrightarrow{\alpha} \mathrm{Cl}(X) \xrightarrow{\beta} \bigoplus_{P\in X} \mathrm{Cl}(X,P),$$

where $\mathrm{Cl}(X,P)$ is the local Weil divisor class group of the point $P \in X$. The map α is a primitive embedding. Hence, we have $\mathrm{Cl}(X) = \mathrm{Pic}(X) \oplus \mathrm{Cl}(X)_{\mathrm{tors}}$. By Corollary B.6, we have

$$\mathrm{Cl}(X)_{\mathrm{tors}} \neq 0.$$

By Lemma 2.9(ii), the group $\mathrm{Cl}(X)_{\mathrm{tors}}$ defines a Galois abelian cover $\pi\colon X' \to X$ which is étale outside of the locus $\mathrm{Sing}(X)$ and whose degree is $|\,\mathrm{Cl}(X)_{\mathrm{tors}}|$. Using the local description of such covers (see [27, 5]), we see that $\mathrm{Type}(X') = \mathrm{E}_6$, D_4, $2\mathrm{A}_1$ in the cases 2^o, 3^o, 4^o, respectively.

6. Del Pezzo surfaces of degree 2

In this section, we prove Main Theorem for del Pezzo surfaces of degree 2. To do this, we need one (probably known) result about singular cubic and quartic curves (cf. [18, 29]).

Proposition 6.1. *Let C be a reduced cubic or quartic curve in $\mathbb{P}^2$ such that $\mathrm{Aut}(\mathbb{P}^2, C)$ is infinite. Then the curve C and the group $\mathrm{Aut}^0(\mathbb{P}^2, C)$ are given in the following table:*

Equation of the curve C up to the action of $\mathrm{PGL}_3(\Bbbk)$	$\mathrm{Aut}^0(\mathbb{P}^2, C)$
$x_0x_1(x_0+x_1)=0$	$\mathbb{G}_a^2 \rtimes \mathbb{G}_m$
$x_0x_1x_2=0$	$\mathbb{G}_m^2$
$x_0(x_0x_2+x_1^2)=0$	$\mathbb{B}_2$
$x_0^2x_2+x_1^3=0$	$\mathbb{G}_m$
$x_1(x_0x_2+x_1^2)=0$	$\mathbb{G}_m$
$x_0x_1(x_0-x_1)(x_0-\lambda x_1)=0$ for $\lambda \in \Bbbk \setminus \{0,1\}$	$\mathbb{B}_2$
$x_0(x_0^2x_2+x_1^3)=0$	$\mathbb{G}_m$
$x_0x_1(x_0x_2+x_1^2)=0$	$\mathbb{G}_m$
$(x_0x_2+x_1^2)^2-x_0^4=0$	$\mathbb{G}_a$
$x_2(x_0^2x_2+x_1^3)=0$	$\mathbb{G}_m$
$x_0x_1x_2(x_1+x_2)=0$	$\mathbb{G}_m$
$x_0x_1(x_0x_1+x_2^2)=0$	$\mathbb{G}_m$
$x_0^3x_2+x_1^4=0$	$\mathbb{G}_m$
$x_1(x_0^2x_2+x_1^3)=0$	$\mathbb{G}_m$
$(x_2^2+x_0x_1)(x_2^2+\lambda x_0x_1)=0$ for $\lambda \in \Bbbk \setminus \{0,1\}$	$\mathbb{G}_m$

Proof. If C is one of the curves in the table, we can explicitly describe $\mathrm{Aut}^0(\mathbb{P}^2, C)$ by finding all elements in $\mathrm{Aut}(\mathbb{P}^2) \cong \mathrm{PGL}_3(\Bbbk)$ that leaves every irreducible component of the curve C invariant. For example, if C is given by $x_0x_1(x_0+x_1)=0$, then $\mathrm{Aut}^0(\mathbb{P}^2, C)$ consists of the transformations

$$(x_0 : x_1 : x_2) \longmapsto \big(tx_0 : tx_1 : x_2 + ax_0 + bx_1\big)$$

for any $t \in \Bbbk^*$, $a \in \Bbbk$ and $b \in \Bbbk$. Thus, in this case, we have $\mathrm{Aut}^0(\mathbb{P}^2, C) \cong \mathbb{G}_a^2 \rtimes \mathbb{G}_m$ as required.

Similarly, if C is given by $x_0(x_0x_2+x_1^2)=0$, then $\mathrm{Aut}^0(\mathbb{P}^2, C)$ consists of the transformations

$$(x_0 : x_1 : x_2) \longmapsto \Big(t^2x_0 : tx_1 + ax_0 : x_2 - \frac{2a}{t}x_1 - \frac{a^2}{t^2}x_0\Big)$$

for any $t \in \Bbbk^*$ and $a \in \Bbbk$, so that $\mathrm{Aut}^0(\mathbb{P}^2, C) \cong \mathbb{G}_a \rtimes_{(1)} \mathbb{G}_m$. Likewise, if C is the cubic $x_0^2x_2+x_1^3=0$, then $\mathrm{Aut}^0(\mathbb{P}^2, C)$ consists of the transformations $(x_0 : x_1 : x_2) \longmapsto (t^3x_0 : t^2x_1 : x_2)$, where $t \in \Bbbk^*$. Thus, in this case, we have $\mathrm{Aut}^0(\mathbb{P}^2, C) \cong \mathbb{G}_m$.

The computations are very similar in all remaining cases. For instance, if C is the quartic curve that is given by $(x_0x_2+x_1^2)^2-x_0^4=0$, then $\mathrm{Aut}(\mathbb{P}^2, C)$ consists of the transformations

$$(x_0 : x_1 : x_2) \longmapsto \Big(\zeta^2x_0 : \zeta x_1 - \frac{\zeta a}{2}x_0 : x_2 + ax_1 - \frac{a^2}{4}x_0\Big)$$

where $\zeta \in \{\pm 1, \pm i\}$ and $a \in \Bbbk$, which implies that $\mathrm{Aut}(\mathbb{P}^2, C) \cong \mathbb{G}_a \rtimes \boldsymbol{\mu}_4$, so that $\mathrm{Aut}^0(\mathbb{P}^2, C) \cong \mathbb{G}_a$. We leave the computations in the remaining cases to the reader.

Therefore, to complete the proof, we must show that C is one of the curves listed in the table. If C is the cubic curve, then C must be singular. On the other hand, all singular cubic curves are already listed in the table except for the nodal one that is given by

$$x_2(x_0^2+x_1^2)+x_1^2=0.$$

However, if C is this curve, then $\mathrm{Aut}(\mathbb{P}^2, C)$ is finite. Therefore, we may assume that $\deg(C) = 4$. Then we may have the following five cases:

(i) the curve C is irreducible;
(ii) $C = C_1 + C_2$, where C_1 is a line and C_2 is an irreducible cubic;
(iii) $C = C_1 + C_2$, where C_1 and C_2 are irreducible conics;
(iv) $C = C_1 + C_2 + C_3$, where C_1 and C_2 are lines, and C_2 are irreducible conic;
(v) $C = C_1 + C_2 + C_3 + C_3$, where C_1, C_2, C_3 and C_4 are lines.

Moreover, by our assumption, the group $\mathrm{Aut}(\mathbb{P}^2, C)$ contains a subgroup isomorphic to either $\mathbb{G}_\mathrm{a}$ or $\mathbb{G}_\mathrm{m}$ (or both). We deal with these two (slightly overlapping) possibilities separately.

Suppose that $\mathrm{Aut}(\mathbb{P}^2, C) \supset \mathbb{G}_\mathrm{a}$. Since each irreducible component is $\mathbb{G}_\mathrm{a}$-invariant, we conclude that the case (ii) is impossible. Likewise, if C is irreducible, then C has a $\mathbb{G}_\mathrm{a}$-open orbit $U \cong \mathbb{A}^1$. Hence the curve C must be rational and its normalization morphism must be a homeomorphism, and the complement $C \setminus U$ is a single point, say P. The projection $C \dashrightarrow \mathbb{P}^1$ from P must be $\mathbb{G}_\mathrm{a}$-equivariant, so it is an isomorphism on U. This implies that P is a triple point and there is exactly one line $L \subset \mathbb{P}^2$ such that $C \cap L = P$. We may assume that $P = (0 : 0 : 1)$ and L is given by $x_0 = 0$. Then the equation of C has the form $x_0^3 x_2 + x_1^4 = 0$. But then $\mathrm{Aut}^0(\mathbb{P}^2, C) \cong \mathbb{G}_\mathrm{m}$, which is a contradiction. Hence, we conclude that case (i) is also impossible.

If we are in case (iii), then the $\mathbb{G}_\mathrm{a}$-action of each irreducible conic C_1 and C_2 is effective, so that the intersection $C_1 \cap C_2$ consists of one point. In this case, in appropriate projective coordinates, the curve C is given by

$$\big(x_1x_2 + x_0^2 + x_1^2\big)\big(x_1x_2 + x_0^2 - x_1^2\big) = 0,$$

so that C is listed in the table as required.

If we are in case (iv) or case (v), then the closure of any one-dimensional $\mathbb{G}_\mathrm{a}$-orbit is a line in the pencil generated by C_1 and C_2, which implies that C is a union of four lines passing through one point. Hence, we are in case (v), and the curve C can be given by

$$x_0x_1(x_0 - x_1)(x_0 - \lambda x_1) = 0$$

for some $\lambda \in \Bbbk \setminus \{0, 1\}$, so that C is in the table as well.

To complete the proof, we may assume that $\mathrm{Aut}(\mathbb{P}^2, C) \supset \mathbb{G}_\mathrm{m}$. If C is irreducible, then it can be given as the closure of the image of the map $t \mapsto (1 : t : t^4)$, so that C is the curve $x_0^3x_2 - x_1^4 = 0$ in the table. Hence, we may assume that C is reducible, i.e. we are not in case (i).

Suppose that $\mathbb{G}_\mathrm{m}$ acts trivially on some irreducible component of the curve C. This component must be a line, so that we are in one of the cases (ii), (iv) or (v). Without loss of generality, we may assume that $\mathbb{G}_\mathrm{m}$ acts trivially on the line C_1. Then there exists a $\mathbb{G}_\mathrm{m}$-fixed point $O \in \mathbb{P}^2 \setminus C_1$, so that the closure of any $\mathbb{G}_\mathrm{m}$-orbit in $\mathbb{P}^2$ is a line connecting O and a point in C_1. This implies that we are in case (v), and the lines C_2, C_3, C_4 all pass through O, so that C is given by

$$x_0x_1x_2(x_1 + x_2) = 0$$

in appropriate projective coordinates. This curve is in the table. Therefore, to complete the proof, we may assume that $\mathbb{G}_\mathrm{m}$ acts effectively on each irreducible component of the curve C.

If we are in case (v), then each line among C_1, C_2, C_3 and C_4 meet the union of the remaining lines in at most two points. This implies that all these four lines must pass through one point. Such curve is in the table and we already met it earlier in the proof. Hence, case (v) is done.

Suppose that we are in case (iii), so that $C = C_1 + C_2$, where both C_1 and C_2 are irreducible conics. Then the intersection $C_1 \cap C_2$ consists of at most two points. Moreover, the intersection cannot consists of one point, since otherwise we would have $\mathrm{Aut}(\mathbb{P}^2, C) \cong \mathbb{G}_\mathrm{a}$. Hence, we see that the intersection $C_1 \cap C_2$ consists of exactly two points. Then the curve C can be given by

$$(x_2^2 + x_0x_1)(x_2^2 + \lambda x_0x_1) = 0$$

for some $\lambda \in \Bbbk \setminus \{0,1\}$. This curve is also in the table. Thus, case (iii) is also done.

Now we suppose that we are in case (iv). Then C_1 and C_2 are lines, and C_3 is a conic. Then

$$\#\Big(C_3 \cap \big(C_1 \cup C_2\big)\Big) \leqslant 2,$$

so that at least one of the lines C_1 and C_2 must be tangent to C_3. If only one of them is tangent, then C can be given by $x_0x_1(x_0x_2 + x_1^2) = 0$. Similarly, if both lines are tangent to the conic C_3, then C can be given by $x_0x_1(x_0x_1 + x_2^2) = 0$. In both subcases, the curve C is in the table.

Finally, we suppose that we are in case (ii). Then C_2 is a cuspidal cubic curve. Now, choosing appropriate coordinates on $\mathbb{P}^2$, we may assume that C_2 is given by

$$x_0^2x_2 - x_1^3 = 0,$$

and the $\mathbb{G}_\mathrm{m}$-action on $\mathbb{P}^2$ is described earlier in the proof. Then the $\mathbb{G}_\mathrm{m}$-action on C_2 has exactly two fixed points: the points $(0:0:1)$ and $(1:0:0)$. If the line C_1 passes through both of them, then the curve C is given by

$$x_0x_1(x_0x_2 + x_1^2) = 0.$$

Similarly, if $(1:0:0) \in C_1$ and $(0:0:1) \notin C_1$, then the curve C is given by $x_2(x_0^2x_2 + x_1^3) = 0$. Vice versa, if $(1:0:0) \notin C_1$ and $(0:0:1) \in C_1$, then the curve C is given by $x_0(x_0^2x_2 + x_1^3) = 0$. In every subcase, we see that the quartic curve C is listed in the table as required. ☐

Using Proposition 6.1, we immediately obtain

Corollary 6.2. *Main Theorem holds for del Pezzo surfaces of degree* 2.

Proof. Let X be a Du Val del Pezzo surface of degree 2. Then X is a hypersurface in $\mathbb{P}(1,1,1,2)$ that is given by

$$w^2 = \phi_4(x_0, x_1, x_2),$$

where $\phi_4(x_0,x_1,x_2)$ is a homogeneous polynomial of degree 4. The natural projection to $\mathbb{P}^2$ gives a double cover $\pi\colon X \to \mathbb{P}^2$. Let B be the branch curve of this double cover. Then B is the quartic curve in $\mathbb{P}^2$ that is given by $\phi_4(x_0,x_1,x_2) = 0$.

Since the double cover π is $\mathrm{Aut}(X)$-equivariant, it gives a homomorphism $\mathrm{Aut}^0(X) \to \mathrm{Aut}^0(\mathbb{P}^2, B)$, whose kernel is either trivial or isomorphic to $\boldsymbol{\mu}_2$. Thus, the curve B must be one of the quartic curves listed in the table in Proposition 6.1 except for the quartic curve consisting of four lines passing through one point, because X has Du Val singularities. Now, going through the equations listed in the table in Proposition 6.1, we obtain all possibilities for the polynomial $\phi_4(x_0,x_1,x_2)$. This shows that if $\mathrm{Aut}(X)$ is infinite, then

- either $\mathrm{Aut}(X) \cong \mathbb{G}_\mathrm{a}$ and X is the surface 7^o,
- or $\mathrm{Aut}(X) \cong \mathbb{G}_\mathrm{m}$ and X is one of the surfaces 5^o, 6^o, 8^o, 9^o, 10^o, 11^o, 12^o, 13^o.

Vice versa, if X is the surface 7^o, then the group $\mathbb{G}_\mathrm{a}$ acts on X as follows:

$$\big(x_0 : x_1 : x_2 : w\big) \longmapsto \big(x_0 + tx_1 : x_1 : x_2 - 2tx_0 - t^2x_1 : w\big),$$

where $t \in \mathbb{G}_\mathrm{a}$. Thus, in this case, we have $\mathrm{Aut}(X) \cong \mathbb{G}_\mathrm{a}$. Similarly, if X is one of the del Pezzo surfaces 5^o, 6^o, 8^o, 9^o, 10^o, 11^o, 12^o, 13^o, then X admits an effective action of the group $\mathbb{G}_\mathrm{m}$, so that $\mathrm{Aut}(X) \cong \mathbb{G}_\mathrm{m}$ as listed in Big Table. ☐

7. Cubic surfaces

Now, we prove Main Theorem for del Pezzo surfaces of degree 3, which easily follows from [28]. Nevertheless, we prefer to give an independent proof here.

Lemma 7.1. *Let X be a Du Val cubic surface in* $\mathbb{P}^3$.

(i) *If X contains three lines L_1, L_2, L_3 that meet each other at three distinct points (a triangle), then X does not admit an effective action of the group $\mathbb{G}_a$.*
(ii) *If X is toric, then $\rho(X) = 1$, $\#(X) = 3$, and the toric boundary is composed of three lines forming a triangle.*

Proof. If X contains a triangle and admits an effective $\mathbb{G}_a$-action, then the $\mathbb{G}_a$-action is trivial on the triangle, so that this action is trivial on the hyperplane in $\mathbb{P}^3$ that passes through the triangle, which implies that the closure of any $\mathbb{G}_a$-orbit in X is contained in a line. The latter is impossible, since X contains finitely many lines. This proves (i)

To prove (ii), suppose that the surface X is toric. Let $D = D_1 + \cdots + D_r$ be the toric boundary. Then $r = \rho(X) + 2$. Since every line on X is torus-invariant and $D \sim -K_X$, we have

$$3 = \sum_{i=1}^{r} (-K_X) \cdot D_i \geqslant r.$$

Therefore, we conclude that $\rho(X) = 1$, $r = 3$ and $-K_X \cdot D_1 = -K_X \cdot D_2 = -K_X \cdot D_3 = 1$. Moreover, the lines D_1, D_2, D_3 form a triangle, because the pair (X, D) has log canonical singularities. □

Now, we are ready to prove

Proposition 7.2. *Main Theorem holds for weakly minimal cubic surfaces.*

Proof. Let X be a weakly minimal Du Val cubic surface. If $\tau(X) = 1$, then Theorem 3.12 implies that X is the surfaces 19^o, and its basic properties are described in Example 3.11. In this case, the surface X admits an algebraic torus action

$$(x_0, x_1, x_2, x_3) \longmapsto (x_0, tx_1, t^2x_2, t^{-1}x_3).$$

Since X contains three lines passing through one point, it is not toric. Since X contains a triangle, it does not admit an unipotent group action by Lemma 7.1, so that $\mathrm{Aut}^0(X) \cong \mathbb{G}_m$ as required.

Thus, to complete the proof, we may assume that $\tau(X) > 1$. By Theorem 3.8, we have only the following possibilities: 14^o, 15^o, 16^o, 18^o, 21^o, 23^o.

Consider the cases 16^o, 21^o, 23^o. From the equations in Theorem 3.8, we see that X contains a triangle that is cut out by $y_1y_2 = 0$. Then by Lemma 7.1(i), we conclude that the unipotent radical of $\mathrm{Aut}^0(X)$ is trivial. The surface 16^o is a toric cubic surface because its equation is binomial, and the surfaces 21^o and 23^o are not toric by Lemma 7.1(ii). Therefore, if X is the surface 16^o, then $\mathrm{Aut}^0(X) \cong \mathbb{G}_m^2$. Similarly, if X is one of the surfaces 21^o or 23^o, then we have $\mathrm{Aut}^0(X) \cong \mathbb{G}_m$, because X admits a diagonal effective action of the group $\mathbb{G}_m$.

Now, we suppose that X is the surface 18^o. Then $\mathrm{Type}(X) = \mathrm{A}_5$, and it follows from Theorem 3.8 that X is a hypersurface in $\mathbb{P}(1, 2, 3, 3)$ that is given by

$$y_3^2 = y_2^3 + y_1^6 + y_1y_2y_3'.$$

Let L_1, L_2, L_3 be the curves $y_1 = y_3^2 - y_2^3 = 0$, $y_2 = y_3 - y_1^3 = 0$, $y_2 = y_3 + y_1^3 = 0$, respectively. Then L_1, L_2 and L_3 are lines meeting at one point. If X admits an effective action of the group $\mathbb{G}_m$, then L_3 contains a $\mathbb{G}_m$-fixed point $P \notin \mathrm{Sing}(X)$, and there exists a $\mathbb{G}_m$-equivariant diagram

$$X \xleftarrow{\ \alpha\ } \widehat{X} \xrightarrow{\ \beta\ } Y$$

where α is the blow up of the point P, the morphism β is the birational contraction of the proper transform of the line L_3, and Y is a singular del Pezzo surface of degree 2 such that $\mathrm{Type}(X) = \mathrm{A}_6$. The latter contradicts Corollary 6.2, so that X does not admit an effective action of the group $\mathbb{G}_m$.

Then $\mathrm{Aut}^0(X)$ is unipotent. By Corollary 2.11, the surface X does not admit an effective $\mathbb{G}_a^2$-action. Then $\dim \mathrm{Aut}^0(X) \leqslant 1$. On the other hand, the group $\mathrm{Aut}(X)$ contains transformations

$$\big(y_1 : y_2 : y_3 : y_3'\big) \longmapsto \big(y_1 : y_2 : y_3 + ay_1y_2 : y_3' + 2ay_3 + a^2y_1y_2\big),$$

where $a \in \Bbbk$. They generate a group isomorphic to $\mathbb{G}_a$. Then $\mathrm{Aut}^0(X) \cong \mathbb{G}_a$ by Corollary 2.2.

Let X be the surface 15^o. As in the previous case, the surface X is a sextic hypersurface in the weighted projective space $\mathbb{P}(1,2,3,3)$. But now the surface X is given by $y_3^2 = y_2^3 + y_3'y_1y_2$. Observe that the group $\mathrm{Aut}(X)$ contains transformations

$$(y_1 : y_2 : y_3 : y_3') \longmapsto \big(y_1 : t^2y_2 : t^3y_3 + at^3y_1y_2 : t^4y_3' + 2at^4y_3 + a^2t^4y_1y_2\big)$$

for any $a \in \Bbbk$ and $t \in \Bbbk^*$. They generate a group isomorphic to $\mathbb{B}_2$. This implies that $\mathrm{Aut}^0(X) \cong \mathbb{B}_2$, because X is not toric by Lemma 7.1(ii), and X admits no effective $\mathbb{G}_a^2$-action by Corollary 2.11.

Finally, if X is the surface 14^o, then it follows from Theorem 3.8 that X is a hypersurface in $\mathbb{P}(1,2,3,3)$ that is given by $y_3^2 = y_2^3 + y_3'y_1^3$. Using this, one can show that the group $\mathrm{Aut}^0(X)$ consists of transformations

$$\big(y_1 : y_2 : y_3 : y_3'\big) \longmapsto \big(y_1 : t^2y_2 : t^3y_3 + ay_1^3 : t^6y_3' + a^2y_1^3 + 2at^3y_3\big),$$

where $a \in \Bbbk$ and $t \in \Bbbk^*$. Thus, in this case, we have $\mathrm{Aut}^0(X) \cong \mathbb{G}_a \rtimes_{(3)} \mathbb{G}_m$, which also follows from Corollary B.8. ☐

To complete the proof of Main Theorem for Du Val cubic surfaces, we need

Lemma 7.3. *Let X be a non-weakly minimal Du Val cubic surface such that $\mathrm{Aut}^0(X)$ is infinite. Then $\mathrm{Aut}^0(X) \cong \mathbb{G}_m$ and X is one of the surfaces 17^o, 20^o or 22^o.*

Proof. The surface X contains a line L such that $L \subset X \setminus \mathrm{Sing}(X)$. By Lemma 2.16(i) there is a conic bundle $\psi \colon X \to \mathbb{P}^1$ such that L is its double section. If X admits an effective $\mathbb{G}_a$-action, then the group $\mathbb{G}_a$ fixes the ramification points of the double cover $L \to \mathbb{P}^1$ induced by ψ, so that the group $\mathbb{G}_a$ acts trivially on L, which implies that it also acts trivially on the fibers of the conic bundle ψ, so that the $\mathbb{G}_a$-action on X is trivial, which is a contradiction. Hence, we conclude that the group $\mathrm{Aut}^0(X)$ contains no unipotent subgroups. Then $\mathrm{Aut}^0(X)$ must be a torus, which implies that $\mathrm{Aut}^0(X) \cong \mathbb{G}_m$, because X is not toric by Lemma7.1(ii).

Let $\psi' \colon X \to X'$ be the contraction of the line L. Then X' is a quartic Du Val del Pezzo surface such that $\rho(X') = \rho(X) - 1$ and $\mathrm{Type}(X) = \mathrm{Type}(X')$. Note that the group $\mathrm{Aut}(X')$ is infinite, and $\mathrm{Aut}^0(X)$ is the stabilizer in $\mathrm{Aut}^0(X')$ of the point $\psi'(L)$. Let U' be the complement in X' to the union of all lines. Then $\psi'(L) \in U'$ by Corollary 2.14.

Suppose that $\rho(X) = 2$. Then ψ is an extremal contraction. By Lemma 2.12(ii), the singular points of the surface X can be of types D_4, D_5, A_3, and A_1, where A_1 appears even number of times. We have two possibilities: $\mathrm{Type}(X) = \mathrm{D}_5$ and $\mathrm{A}_3 2\mathrm{A}_1$, so that X' is one of the surfaces 24^o or 25^o. In both cases, the subset U' is the open $\mathrm{Aut}^0(X')$-orbit (cf. Remark B.4), which immediately implies that $\mathrm{Aut}^0(X) \cong \mathbb{G}_m$ and X is one of the surface 17^o or 20^o.

Now, we assume that $\rho(X) > 2$. If $\mathrm{Type}(X) = \mathrm{D}_4$, then X' is the surface 26^o. Arguing as above, we see that $\mathrm{Aut}^0(X) \cong \mathbb{G}_m$ and X is the surface 22^o. Thus, to complete the proof, we may assume that $\rho(X) > 2$ and all the singularities of X are of type A_n.

We claim that the action of $\mathrm{Aut}^0(X)$ on L is trivial. Indeed, suppose that this is not the case. Let us seek for a contradiction. Let P_1 and P_2 be the ramification points of the double cover $L \to \mathbb{P}^1$, let F_1 and F_2 be the fibers of the conic bundle ψ passing through the points P_1 and P_2, respectively. Then P_1 and P_2 are fixed by $\mathrm{Aut}^0(X)$, and these are all $\mathrm{Aut}^0(X)$-fixed points on L, so that all fibers of the conic bundle ψ other than F_1 and F_2 are smooth. Since $\rho(X) > 2$, there exists at least one reducible fiber. Thus, we may assume that F_1 is reducible. Then

$$F_1 = F_1' + F_1'',$$

where F_1' and F_1'' are lines. Then $P_1 = F_1' \cap F_1'' \cap L$ and the surface X is smooth along F_1, which implies that $\mathrm{Sing}(X) \subset F_2$ and F_2 is irreducible (but multiple). In particular, we have $\rho(X) = 3$. On the other hand, using Lemma 2.12(ii), we see that either $\mathrm{Type}(X) = 2\mathrm{A}_1$ or $\mathrm{Type}(X) = \mathrm{A}_3$. This contradicts the Noether formula. Therefore, the action of $\mathrm{Aut}^0(X)$ on the line L is trivial, so that the action of the group $\mathrm{Aut}^0(X)$ on the base of the conic bundle ψ is trivial as well.

Let M' be a line in X' (it does exist since $\rho(X') > 1$), and let M be its proper transform on X. Then $\psi'(L) \notin M'$ by Corollary 2.14, so that M is a line on X, which is disjoined from the line L. Then M is an $\mathrm{Aut}^0(X)$-invariant curve, which is not contained in the fibers of the conic bundle ψ, since ψ is given by the projection from L. Therefore, if C is a general fiber of ψ, then C contains at least three $\mathrm{Aut}^0(X)$-fixed points $C \cap (L \cup M)$, so that the $\mathrm{Aut}^0(X)$-action on C must be trivial. This implies that the action of $\mathrm{Aut}^0(X)$ on X is also trivial, which is a contradiction. □

Combining Proposition 7.2 and Lemma 7.3, we obtain

Corollary 7.4. *Main Theorem holds for del Pezzo surfaces of degree* 3.

Thus Main Theorem holds for Du Val del Pezzo surface of degrees 1, 2, 3, 4, 5, 6. This follows from Proposition 5.1 and Corollaries 4.5, 6.2, 7.4. This completes the proof of Main Theorem.

8. Big Table

Let X be a Du Val del Pezzo surface such that $\mathrm{Aut}(X)$ is infinite. Then the type $\mathrm{Type}(X)$, the degree K_X^2, the Picard rank $\rho(X)$, the number of lines $\#(X)$, the Fano–Weil index $\tau(X)$, the group $\mathrm{Aut}^0(X)$, and the equation of the surface X are given below. The column No indicates a del Pezzo surface from which X can be obtained by blowing up a smooth point that does not lie on a line.

	K_X^2	$\rho(X)$	$\#(X)$	$\mathrm{Type}(X)$	$\tau(X)$	No	$\mathrm{Aut}^0(X)$	equation & total space	
1^o	1	1	1	E_8	1	–	$\mathbb{G}_\mathrm{m}$	$y_3^2 = y_2^3 + y_1' y_1^5$	$\mathbb{P}(1,1,2,3)$
2^o	1	1	3	$\mathrm{E}_7\mathrm{A}_1$	1	–	$\mathbb{G}_\mathrm{m}$	$y_3^2 = y_1^3 y_1' y_2 + y_2^3$	$\mathbb{P}(1,1,2,3)$
3^o	1	1	4	$\mathrm{E}_6\mathrm{A}_2$	1	–	$\mathbb{G}_\mathrm{m}$	$y_3^2 = y_2^3 + y_1'^2 y_1^4$	$\mathbb{P}(1,1,2,3)$
4^o	1	1	5	$2\mathrm{D}_4$	1	–	$\mathbb{G}_\mathrm{m}$	$y_3^2 = y_2(y_2 + y_1 y_1')(y_2 + \lambda y_1 y_1')$ for $\lambda \in \Bbbk \setminus \{0,1\}$	$\mathbb{P}(1,1,2,3)$
5^o	2	1	1	E_7	2	–	$\mathbb{G}_\mathrm{m}$	$y_2^2 = y_1(y_1^2 y_1'' + y_1'^3)$	$\mathbb{P}(1,1,1,2)$
6^o	2	1	2	$\mathrm{D}_6\mathrm{A}_1$	2	–	$\mathbb{G}_\mathrm{m}$	$y_2^2 = y_1 y_1'(y_1 y_1'' + y_1'^2)$	$\mathbb{P}(1,1,1,2)$
7^o	2	1	2	A_7	1	–	$\mathbb{G}_\mathrm{a}$	$y_2^2 = (y_1 y_1'' + y_1'^2)^2 - y_1^4$	$\mathbb{P}(1,1,1,2)$
8^o	2	1	3	$\mathrm{A}_5\mathrm{A}_2$	2	–	$\mathbb{G}_\mathrm{m}$	$y_2^2 = y_1''(y_1^2 y_1'' + y_1'^3)$	$\mathbb{P}(1,1,1,2)$
9^o	2	1	4	$\mathrm{D}_4 3\mathrm{A}_1$	2	–	$\mathbb{G}_\mathrm{m}$	$y_2^2 = y_1 y_1' y_1''(y_1' + y_1'')$	$\mathbb{P}(1,1,1,2)$
10^o	2	1	4	$2\mathrm{A}_3\mathrm{A}_1$	2	–	$\mathbb{G}_\mathrm{m}$	$y_2^2 = y_1 y_1'(y_1 y_1' + y_1''^2)$	$\mathbb{P}(1,1,1,2)$
11^o	2	2	4	E_6	1	14^o	$\mathbb{G}_\mathrm{m}$	$y_2^2 = y_1^3 y_1'' + y_1'^4$	$\mathbb{P}(1,1,1,2)$
12^o	2	2	5	$\mathrm{D}_5\mathrm{A}_1$	2	–	$\mathbb{G}_\mathrm{m}$	$y_2^2 = y_1'(y_1^2 y_1'' + y_1'^3)$	$\mathbb{P}(1,1,1,2)$
13^o	2	2	6	$2\mathrm{A}_3$	1	–	$\mathbb{G}_\mathrm{m}$	$y_2^2 = (y_1''^2 + y_1 y_1')(y_1''^2 + \lambda y_1 y_1')$ for $\lambda \in \Bbbk \setminus \{0,1\}$	$\mathbb{P}(1,1,1,2)$
14^o	3	1	1	E_6	3	–	$\mathbb{G}_\mathrm{a} \rtimes_{(3)} \mathbb{G}_\mathrm{m}$	$x_0 x_2^2 = x_1^3 + x_3 x_0^2$	$\mathbb{P}^3$
15^o	3	1	2	$\mathrm{A}_5\mathrm{A}_1$	3	–	$\mathbb{B}_2$	$x_0 x_2^2 = x_1^3 + x_0 x_3 x_1$	$\mathbb{P}^3$
16^o	3	1	3	$3\mathrm{A}_2$	3	–	$\mathbb{G}_\mathrm{m}^2$	$x_0 x_2 x_3 = x_1^3$	$\mathbb{P}^3$
17^o	3	2	3	D_5	1	24^o	$\mathbb{G}_\mathrm{m}$	$x_0^2 x_3 = x_2(x_0 x_2 - x_1^2)$	$\mathbb{P}^3$
18^o	3	2	3	A_5	3	–	$\mathbb{G}_\mathrm{a}$	$x_0 x_2^2 = x_1^3 + x_0^3 + x_0 x_3 x_1$	$\mathbb{P}^3$
19^o	3	2	4	$\mathrm{A}_4\mathrm{A}_1$	1	–	$\mathbb{G}_\mathrm{m}$	$x_3(x_0 x_2 - x_1^2) = x_0^2 x_1$	$\mathbb{P}^3$
20^o	3	2	5	$\mathrm{A}_3 2\mathrm{A}_1$	1	25^o	$\mathbb{G}_\mathrm{m}$	$x_3(x_0 x_2 - x_1^2) = x_0 x_1^2$	$\mathbb{P}^3$
21^o	3	2	5	$2\mathrm{A}_2\mathrm{A}_1$	3	–	$\mathbb{G}_\mathrm{m}$	$x_0 x_2 x_3 = x_1^3 + x_0 x_1^2$	$\mathbb{P}^3$
22^o	3	3	6	D_4	1	26^o	$\mathbb{G}_\mathrm{m}$	$x_0^2 x_3 = x_1 x_2(x_1 + x_2)$	$\mathbb{P}^3$
23^o	3	3	7	$2\mathrm{A}_2$	3	–	$\mathbb{G}_\mathrm{m}$	$x_0 x_2 x_3 = x_1(x_1 - x_0)(x_1 - \lambda x_0)$ for $\lambda \in \Bbbk \setminus \{0,1\}$	$\mathbb{P}^3$
24^o	4	1	1	D_5	4	–	$\mathbb{G}_\mathrm{a}^2 \rtimes \mathbb{G}_\mathrm{m}$	$y_3^2 = y_2^3 + y_1^2 y_4$	$\mathbb{P}(1,2,3,4)$
25^o	4	1	2	$\mathrm{A}_3 2\mathrm{A}_1$	4	–	$\mathbb{B}_2 \times \mathbb{G}_\mathrm{m}$	$y_3^2 = y_2 y_4$	$\mathbb{P}(1,2,3,4)$

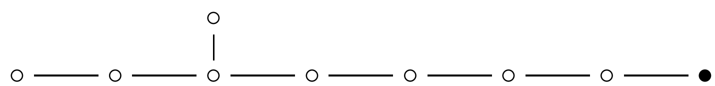
	K_X^2	$\rho(X)$	$\#(X)$	Type(X)	$\tau(X)$	No	$\mathrm{Aut}^0(X)$	equation & total space	
26^o	4	2	2	D_4	2	–	$\mathbb{G}_\mathrm{a} \rtimes_{(2)} \mathbb{G}_\mathrm{m}$	$y_2^2 = y_2' y_1^2 + y_1'^4$	$\mathbb{P}(1,1,2,2)$
27^o	4	2	3	A_4	1	36^o	$\mathbb{B}_2$	$x_0x_1 - x_2x_3 = x_0x_4 + x_1x_2 + x_3^2 = 0$	$\mathbb{P}^4$
28^o	4	2	3	$\mathrm{A}_3\mathrm{A}_1$	4	–	$\mathbb{B}_2$	$y_3^2 = y_1^6 + y_2y_4$	$\mathbb{P}(1,2,3,4)$
29^o	4	2	4	$\mathrm{A}_2 2\mathrm{A}_1$	2	–	$\mathbb{G}_\mathrm{m}^2$	$y_2y_2' = y_1^3 y_1'$	$\mathbb{P}(1,1,2,2)$
30^o	4	2	4	$4\mathrm{A}_1$	2	–	$\mathbb{G}_\mathrm{m}^2$	$y_2y_2' = y_1^2 y_1'^2$	$\mathbb{P}(1,1,2,2)$
31^o	4	3	4	A_3	2	–	$\mathbb{G}_\mathrm{a}$	$y_2^2 = y_2' y_1 y_1' + y_1^4 + y_1'^4$	$\mathbb{P}(1,1,2,2)$
32^o	4	3	5	A_3	1	37^o	$\mathbb{G}_\mathrm{m}$	$x_0x_1 - x_2x_3 = x_0x_3 + x_2x_4 + x_1x_3 = 0$	$\mathbb{P}^4$
33^o	4	3	6	$\mathrm{A}_2\mathrm{A}_1$	1	38^o	$\mathbb{G}_\mathrm{m}$	$x_0x_1 - x_2x_3 = x_1x_2 + x_2x_4 + x_3x_4 = 0$	$\mathbb{P}^4$
34^o	4	3	6	$3\mathrm{A}_1$	2	–	$\mathbb{G}_\mathrm{m}$	$y_2y_2' = y_1^2 y_1'(y_1' + y_1)$	$\mathbb{P}(1,1,2,2)$
35^o	4	4	8	$2\mathrm{A}_1$	2	–	$\mathbb{G}_\mathrm{m}$	$y_2y_2' = y_1y_1'(y_1' - y_1)(y_1' - \lambda y_1)$ for $\lambda \in \Bbbk \setminus \{0,1\}$	$\mathbb{P}(1,1,2,2)$
36^o	5	1	1	A_4	5	–	$\mathbb{U}_3 \rtimes \mathbb{G}_\mathrm{m}$	$y_3^2 + y_2^3 + y_1y_5 = 0$	$\mathbb{P}(1,2,3,5)$
37^o	5	2	2	A_3	1	–	$\mathbb{G}_\mathrm{a}^2 \rtimes \mathbb{G}_\mathrm{m}$	$u_2^2v_0 + (u_0^2 + u_1u_2)v_1 = 0$	$\mathbb{P}^2 \times \mathbb{P}^1$
38^o	5	2	3	$\mathrm{A}_2\mathrm{A}_1$	1	42^o	$\mathbb{B}_2 \times \mathbb{G}_\mathrm{m}$		
39^o	5	3	4	A_2	1	43^o	$\mathbb{B}_2$	$u_0u_1v_0 + (u_1^2 + u_0u_2)v_1 = 0$	$\mathbb{P}^2 \times \mathbb{P}^1$
40^o	5	3	5	$2\mathrm{A}_1$	1	44^o	$\mathbb{G}_\mathrm{m}^2$	$u_0^2v_0 + u_1u_2v_1 = 0$	$\mathbb{P}^2 \times \mathbb{P}^1$
41^o	5	4	7	A_1	1	45^o,46^o	$\mathbb{G}_\mathrm{m}$	$u_0u_1v_0 + (u_0 + u_1)u_2v_1 = 0$	$\mathbb{P}^2 \times \mathbb{P}^1$
42^o	6	1	1	$\mathrm{A}_2\mathrm{A}_1$	6	–	$\mathbb{B}_3$	—	$\mathbb{P}(1,2,3)$
43^o	6	2	2	A_2	3	–	$\mathbb{U}_3 \rtimes \mathbb{G}_\mathrm{m}$	$y_1y_3 = y_2^2 + y_1'^4$	$\mathbb{P}(1,1,2,3)$
44^o	6	2	2	$2\mathrm{A}_1$	2	–	$\mathbb{B}_2 \times \mathbb{B}_2$	$y_1y_2 = y_1'^2 y_1''$	$\mathbb{P}(1,1,1,2)$
45^o	6	3	3	A_1	2	–	$\mathbb{G}_\mathrm{a}^2 \rtimes \mathbb{G}_\mathrm{m}$	$y_1y_2 = y_1'y_1''(y_1' + y_1'')$	$\mathbb{P}(1,1,1,2)$
46^o	6	3	4	A_1	1	48^o	$\mathbb{B}_2 \times \mathbb{G}_\mathrm{m}$	$u_0v_0 + u_1v_1 + u_2v_2 = 0,\ u_0v_1 + u_1v_2 = 0$	$\mathbb{P}^2 \times \mathbb{P}^2$
47^o	6	4	6	smooth	1	49^o	$\mathbb{G}_\mathrm{m}^2$	$u_0v_0w_0 = u_1v_1w_1$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
48^o	7	2	2	A_1	1	50^o	$\mathbb{B}_3$		
49^o	7	3	3	smooth	1	51^o,52^o	$\mathbb{B}_2 \times \mathbb{B}_2$		
50^o	8	1	0	A_1	4	–	$\mathbb{G}_\mathrm{a}^3 \rtimes (\mathrm{GL}_2(\Bbbk)/\boldsymbol{\mu}_2)$	—	$\mathbb{P}(1,1,2)$
51^o	8	2	1	smooth	1	53^o	$\mathbb{G}_\mathrm{a}^2 \rtimes \mathrm{GL}_2(\Bbbk)$	$u_0v_0 = u_1v_1$	$\mathbb{P}^2 \times \mathbb{P}^1$
52^o	8	2	0	smooth	2	–	$\mathrm{PGL}_2(\Bbbk) \times \mathrm{PGL}_2(\Bbbk)$	—	$\mathbb{P}^1 \times \mathbb{P}^1$
53^o	9	1	0	smooth	3	–	$\mathrm{PGL}_3(\Bbbk)$	—	$\mathbb{P}^2$

Appendix A. Lines and dual graphs

In this appendix, we present equations of the lines on del Pezzo surfaces that appear in Big Table. We also present the dual graphs of all the curves with negative self-intersection numbers on their minimal resolutions. As in the paper [7], we will denote a (-1)-curve by $\bullet$, and we will denote a (-2)-curve by $\circ$. We will exclude surfaces 48^o (see Example 2.3), 49^o, 50^o, 51^o, 52^o, 53^o.

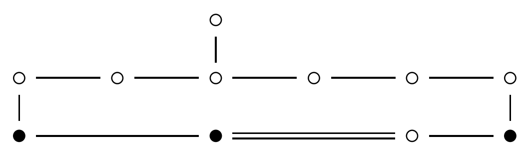
Let X be a del Pezzo surface of degree d in Big Table. Take the equation of X from Big Table. The lines on X and the dual graphs of all the curves with negative self-intersection numbers on its minimal resolution can be described as follows.

(1^o) One has $d = 1$ and $\mathrm{Type}(X) = \mathrm{E}_8$. The dual graph is



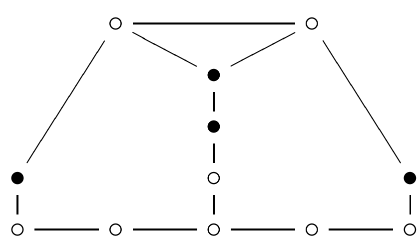
The surface X is weakly-minimal. The only line on X is $x_3^2 - x_2^3 = x_0 = 0$.

(2^o) One has $d = 1$ and $\mathrm{Type}(X) = \mathrm{E}_7\mathrm{A}_1$. The dual graph is



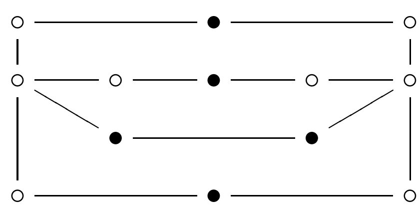
The surface X is weakly-minimal. The lines are cut out by $x_0 = 0$, $x_1 = 0$ and $x_3 = x_2 = 0$.

(3^o) One has $d = 1$ and $\mathrm{Type}(X) = \mathrm{E}_6\mathrm{A}_2$. The dual graph is



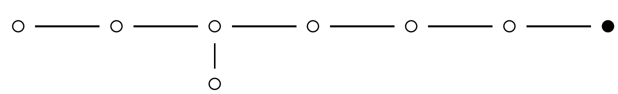
The surface X is weakly-minimal. The lines are cut out by $x_0 = 0$, $x_1 = 0$ and $x_2 = 0$.

(4^o) One has $d = 1$ and $\mathrm{Type}(X) = 2\mathrm{D}_4$. The dual graph is



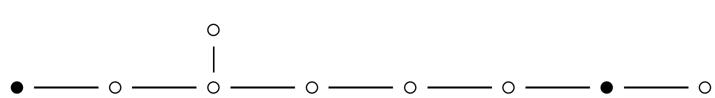
The surface X is weakly-minimal. The lines are cut out by $x_0 = 0$, $x_1 = 0$ and $x_3 = 0$.

(5^o) One has $d = 2$ and $\mathrm{Type}(X) = \mathrm{E}_7$. The dual graph is



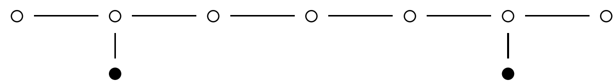
The surface X is weakly-minimal, and the line is $x_0 = x_3 = 0$.

(6^o) One has $d = 2$ and $\mathrm{Type}(X) = \mathrm{D}_6\mathrm{A}_1$. The dual graph is



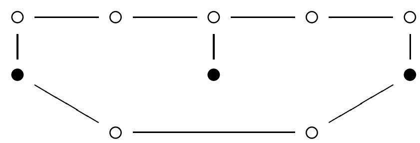
The surface X is weakly-minimal. The lines are $x_0 = x_3 = 0$ and $x_1 = x_3 = 0$.

(7^o) One has $d = 2$ and $\mathrm{Type}(X) = \mathrm{A}_7$. The dual graph is



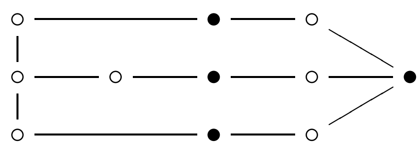
The surface X is weakly-minimal, and the lines are $x_1 = x_3 \pm x_0^2 = 0$.

(8^o) One has $d = 2$ and $\mathrm{Type}(X) = \mathrm{A}_5\mathrm{A}_2$. The dual graph is



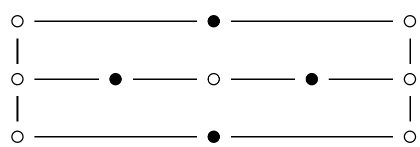
The surface X is weakly-minimal. The lines are $x_1 = x_3 \pm x_0x_2 = 0$ and $x_2 = x_3 = 0$.

(9^o) One has $d = 2$ and $\mathrm{Type}(X) = \mathrm{D}_4 3\mathrm{A}_1$. The dual graph is



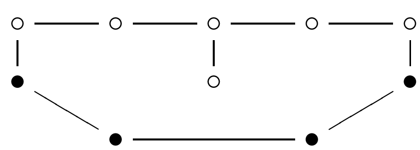
The surface X is weakly-minimal. The lines are cut out by $x_3 = 0$.

(10^o) One has $d = 2$ and $\mathrm{Type}(X) = 2\mathrm{A}_3\mathrm{A}_1$. The dual graph is



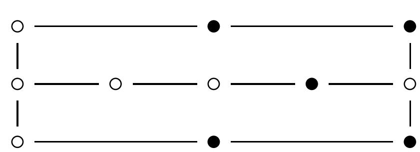
This surface is weakly-minimal. The lines are $x_0 = x_3 = 0$, $x_1 = x_3 = 0$, $x_2 = x_3 \pm x_0x_1 = 0$.

(11^o) One has $d = 2$ and $\mathrm{Type}(X) = \mathrm{E}_6$. The dual graph is



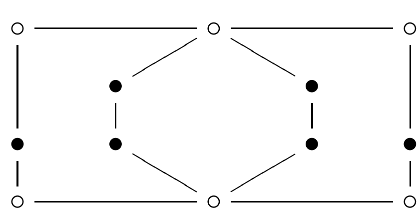
This surface is not weakly-minimal. The lines are $x_0 = x_3 \pm x_1^2 = 0$ and $x_2 = x_3 \pm x_1^2 = 0$.

(12^o) One has $d = 2$ and $\mathrm{Type}(X) = \mathrm{D}_5\mathrm{A}_1$. The dual graph is



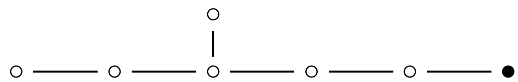
The surface X is weakly-minimal. The lines are cut out by $x_0 = 0$, $x_2 = 0$ and $x_1 = x_3 = 0$.

(13^o) One has $d = 2$ and $\mathrm{Type}(X) = 2\mathrm{A}_3$. The dual graph is



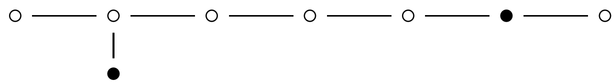
The surface X is weakly-minimal. The lines are cut out by $x_0 = 0$, $x_1 = 0$ and $x_2 = 0$.

(14^o) One has $d = 3$ and Type$(X) = \mathrm{E}_6$. The dual graph is



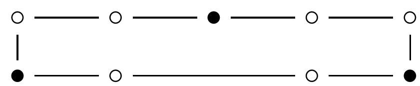
The surface X is weakly-minimal. The line is $x_0 = x_1 = 0$.

(15^o) One has $d = 3$ and Type$(X) = \mathrm{A}_5\mathrm{A}_1$. The dual graph is



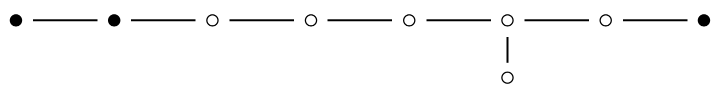
The surface X is weakly-minimal. The lines are $x_0 = x_1 = 0$ and $x_1 = x_2 = 0$.

(16^o) One has $d = 3$ and Type$(X) = 3\mathrm{A}_2$. The dual graph is



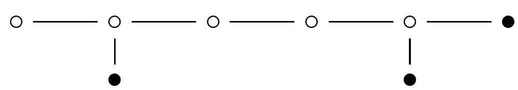
The surface X is weakly-minimal. The lines are $x_0 = x_3 = 0$, $x_1 = x_3 = 0$ and $x_2 = x_3 = 0$.

(17^o) One has $d = 3$ and Type$(X) = \mathrm{D}_5$. The dual graph is



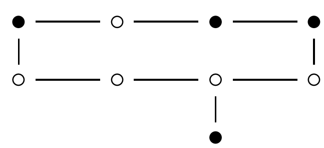
The surface X is weakly-minimal. The lines are $x_0 = x_1 = 0$, $x_0 = x_2 = 0$ and $x_2 = x_3 = 0$.

(18^o) One has $d = 3$ and Type$(X) = \mathrm{A}_5$. The dual graph is



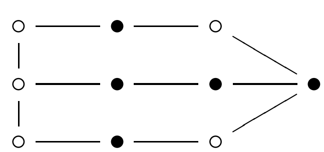
The surface X is weakly-minimal. The lines are cut out by $x_1 = 0$.

(19^o) One has $d = 3$ and Type$(X) = \mathrm{A}_4\mathrm{A}_1$. The dual graph is



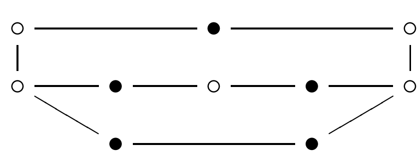
The surface X is weakly-minimal. The lines are cut out by $x_0 = 0$ and $x_1 = 0$.

(20^o) One has $d = 3$ and Type$(X) = \mathrm{A}_3 2\mathrm{A}_1$. The dual graph is



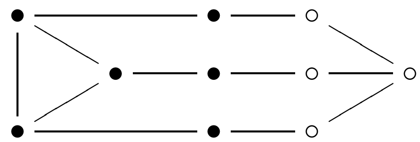
Then X is not weakly-minimal. The lines are cut out by $x_0 = 0$, $x_1 = 0$ and $x_2 = 0$.

(21^o) One has $d = 3$ and Type$(X) = 2\mathrm{A}_2\mathrm{A}_1$. The dual graph is



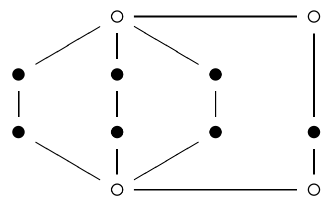
The surface X is weakly-minimal. The lines are cut out by $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$.

(22^o) One has $d = 3$ and $\mathrm{Type}(X) = \mathrm{D}_4$. The dual graph is



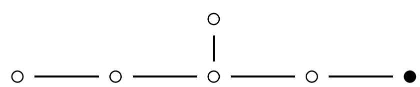
Then X is not weakly-minimal. The lines on X are cut by $x_0 = 0$ and $x_3 = 0$.

(23^o) One has $d = 3$ and $\mathrm{Type}(X) = 2\mathrm{A}_2$. The dual graph is



The surface X is weakly-minimal. The lines are cut out by $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$.

(24^o) One has $d = 4$ and $\mathrm{Type}(X) = \mathrm{D}_5$. The dual graph is

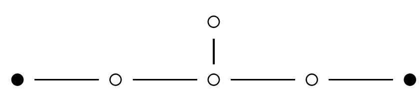


The surface X is weakly-minimal. The line is $x_2^2 - x_1^3 = x_0 = 0$.

(25^o) One has $d = 4$ and $\mathrm{Type}(X) = \mathrm{A}_3 2\mathrm{A}_1$. The dual graph is

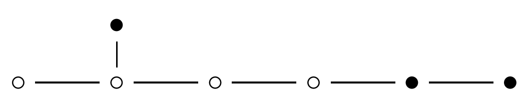
The surface X is weakly-minimal. The lines are $x_1 = x_2 = 0$ and $x_2^2 - x_1^3 = x_0 = 0$.

(26^o) One has $d = 4$ and $\mathrm{Type}(X) = \mathrm{D}_4$. The dual graph is



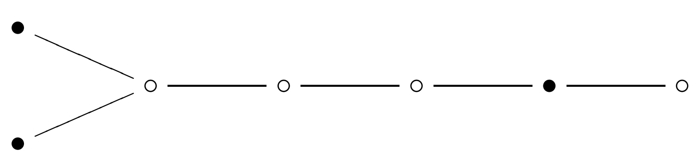
The surface X is weakly-minimal, and the lines are $x_2 \pm x_1^2 = x_0 = 0$.

(27^o) One has $d = 4$ and $\mathrm{Type}(X) = \mathrm{A}_4$. The dual graph is



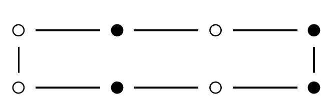
Then X is not weakly-minimal. The lines are cut out by $x_0 = 0$ and $x_1 = 0$.

(28^o) One has $d = 4$ and $\mathrm{Type}(X) = \mathrm{A}_3\mathrm{A}_1$. The dual graph is



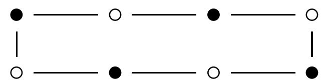
The surface X is weakly-minimal. The lines are $x_2^2 - x_1x_3 = x_0 = 0$ and $x_1 = x_2 \pm x_0^3 = 0$.

(29^o) One has $d = 4$ and $\mathrm{Type}(X) = \mathrm{A}_2 2\mathrm{A}_1$. The dual graph is



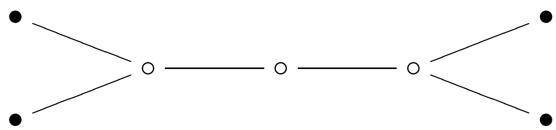
The surface X is weakly-minimal. The lines are cut out by $x_2 = 0$ and $x_3 = 0$.

(30^o) One has $d = 4$ and Type$(X) = 4\mathrm{A}_1$. The dual graph is



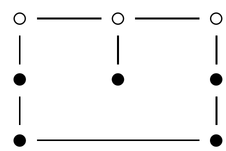
The surface X is weakly-minimal. The lines in X are cut by $x_4 = 0$.

(31^o) One has $d = 4$ and Type$(X) = \mathrm{A}_3$ and $\#(X) = 4$. The dual graph is



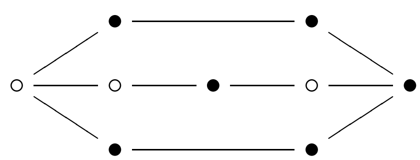
The surface X is weakly minimal. The lines are $x_2 \pm x_1^2 = x_0 = 0$ and $x_2 \pm x_0^2 = x_1 = 0$.

(32^o) One has $d = 4$ and Type$(X) = \mathrm{A}_3$ and $\#(X) = 5$. The dual graph is



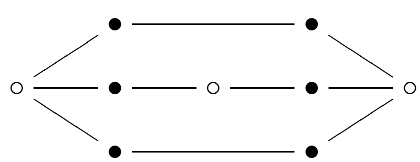
Then X is not weakly-minimal. The lines are cut out by $x_0 = 0$ and $x_1 = 0$.

(33^o) One has $d = 4$ and Type$(X) = \mathrm{A}_2\mathrm{A}_1$. The dual graph is



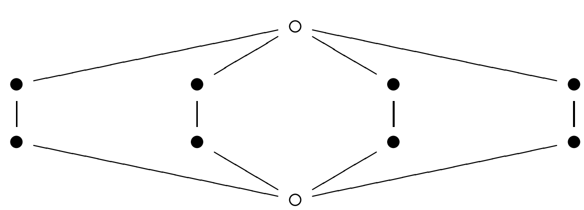
Then X is not weakly-minimal. The lines are cut out by $x_1 = 0$ and $x_0 = 0$.

(34^o) One has $d = 4$ and Type$(X) = 3\mathrm{A}_1$. The dual graph is



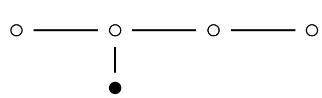
The surface X is weakly-minimal. The lines are cut out by $x_2 = 0$ and $x_3 = 0$.

(35^o) One has $d = 4$ and Type$(X) = 2\mathrm{A}_1$ and $\#(X) = 8$. The dual graph is



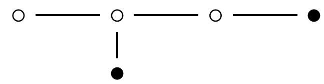
The surface X is weakly-minimal. The lines are cut out by $x_2 = 0$ and $x_3 = 0$.

(36^o) One has $d = 5$ and Type$(X) = \mathrm{A}_4$. The dual graph is



The surface X is weakly minimal. The line is given by $x_2^2 + x_1^3 = x_0 = 0$.

(37^o) One has $d = 5$ and $\mathrm{Type}(X) = \mathrm{A}_3$. The dual graph is



Then X is weakly minimal. The lines are $x_0 = x_2 = 0$ and $y_1 = x_2 = 0$.

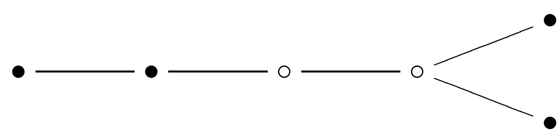
(38^o) One has $d = 5$ and $\mathrm{Type}(X) = \mathrm{A}_2\mathrm{A}_1$. The dual graph is

Then surface X is not weakly minimal. It is given in $\mathbb{P}^5$ by

$$\begin{cases} x_0x_2 = x_1x_5, \\ x_0x_3 = x_5^2, \\ x_1x_3 = x_2x_5, \\ x_1x_4 = x_5^2, \\ x_2x_4 = x_3x_5. \end{cases}$$

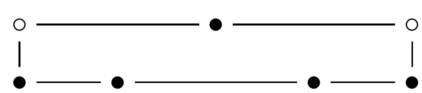
The lines on X are cut out by $x_5 = 0$.

(39^o) One has $d = 5$ and $\mathrm{Type}(X) = \mathrm{A}_2$. The dual graph is



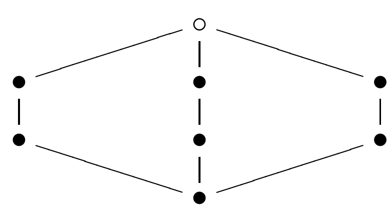
Then X is not weakly minimal. The lines are cut out by $x_0 = 0$ and $x_1 = 0$.

(40^o) One has $d = 5$ and $\mathrm{Type}(X) = 2\mathrm{A}_1$. The dual graph is



Then X is not weakly minimal. The lines are cut out $x_0 = 0$, $x_1 = 0$ and $x_2 = 0$.

(41^o) One has $d = 5$ and $\mathrm{Type}(X) = \mathrm{A}_1$. The dual graph is

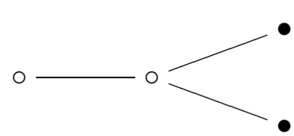


Then X is not weakly minimal. The lines are cut out by $x_0 = 0$, $x_1 = 0$, $y_0 = 0$, $y_1 = 0$.

(42^o) One has $d = 6$ and $\mathrm{Type}(X) = \mathrm{A}_2\mathrm{A}_1$. The dual graph is

Then X is weakly minimal. The line is $x_0 = 0$.

(43^o) One has $d = 6$ and $\mathrm{Type}(X) = \mathrm{A}_2$. The dual graph is

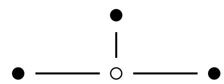


The surface X is weakly minimal. The lines are $x_0 = x_1 = 0$ and $x_0 = x_2 = 0$.

(44^o) One has $d = 6$ and $\mathrm{Type}(X) = 2\mathrm{A}_1$. The dual graph is

The surface X is weakly minimal. The lines are cut out by $x_2 = 0$.

(45^o) One has $d = 6$ and $\mathrm{Type}(X) = \mathrm{A}_1$. The dual graph is

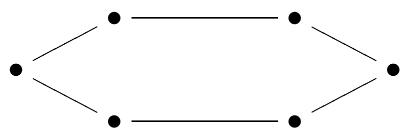


The surface X is weakly minimal. The lines are cut out by $x_2 = 0$.

(46^o) One has $d = 6$ and $\mathrm{Type}(X) = \mathrm{A}_1$ and $\#(X) = 4$. The dual graph is

The surface X is not weakly minimal. The lines are cut out by $x_0 = 0$.

(47^o) One has $d = 6$ and X is smooth. The dual graph is



Then X is not weakly minimal. The lines are cut out by $x_0 = 0$, $y_0 = 0$ and $z_0 = 0$.

Appendix B. Del Pezzo surfaces of homology type of $\mathbb{P}^2$

In this section, we recall classification of Du Val del Pezzo surfaces whose Weil divisor class group is cyclic [22, 14, 30]. By Lemma 2.9(iii), each such surface except $\mathbb{P}^2$ and $\mathbb{P}(1,1,2)$ contains a unique line that generated its class group.

Proposition B.1. *Let X be a Du Val del Pezzo surface such that $\mathrm{Cl}(X) \cong \mathbb{Z}$, and let $d := K_X^2$. Then $d \neq 7$. If $d \leqslant 6$, then there is an embedding $X \hookrightarrow \mathbb{P}(1,2,3,d)$ such that X is given by*

$$\phi(y_1, y_2, y_3, y_d) = 0,$$

where ϕ is a homogeneous polynomial of weighted degree 6. If $2 \leqslant d \leqslant 6$, then

$$\phi = y_3^2 + y_2^3 + y_1^{6-d} y_d,$$

so that X is uniquely determined by its degree. If $d = 1$, there are exactly two possibilities:

$$\phi = y_3^2 + y_2^3 + y_1^5 y_d, \tag{B.2}$$

$$\phi = y_3^2 + y_2^3 + y_1^5 y_d + y_1^2 y_2^2, \tag{B.3}$$

which give us two non isomorphic surfaces. The only line $L \subset X$ is cut out by $y_1 = 0$.

Proof. The proof is similar to the proof of Theorem 3.8. □

Thus, if $d = 9$, 8, 6, then $X \cong \mathbb{P}^2$, $\mathbb{P}(1,1,2)$, $\mathbb{P}(1,2,3)$, respectively.

Remark B.4. In the notation and assumptions of Proposition B.1, one can show that $X \setminus L \cong \mathbb{A}^2$. Moreover, there is cell decomposition $X = \mathbb{A}^2 \cup \mathbb{A}^1 \cup \mathbb{A}^0$. In particular, we have

$$H_q(X, \mathbb{Z}) \cong H_q(\mathbb{P}^2, \mathbb{Z})$$

for all q (cf. [3]). In the case $d = 1$, we can say even more: $H^*(X, \mathbb{Z}) \cong H^*(\mathbb{P}^2, \mathbb{Z})$ as rings.

Remark B.5. Suppose that X is a Du Val del Pezzo surface of degree 1 such that $\mathrm{Cl}(X) \cong \mathbb{Z}$. By Proposition B.1, X is a hypersurface in $\mathbb{P}(1,1,2,3)$ that is given

(i) either by $y_3^2 + y_2^3 + y_1^5 y_1' = 0$,
(ii) or by $y_3^2 + y_2^3 + y_1^5 y_1' + y_1^2 y_2^2 = 0$.

These possibilities are distinguished by the collection of singular curves in the pencil $|-K_X|$. Indeed, in the first case, the pencil $|-K_X|$ contains two singular curves $y_1 = 0$ and $y_1' = 0$, which are both cuspidal. In the second case, it has three singular curves $y_1 = 0$, $y_1' = 0$ and $4y_1 + 27y_1' = 0$. One of them is cuspidal, and the remaining two curves are nodal.

Corollary B.6. *Let X be a Du Val del Pezzo surface, let $d := K_X^2$. If $\mathrm{Cl}(X) \cong \mathbb{Z}$, then $\mathrm{Type}(X)$ is*

$$\mathrm{E}_8, \quad \mathrm{E}_7, \quad \mathrm{E}_6, \quad \mathrm{D}_5, \quad \mathrm{A}_4, \quad \mathrm{A}_2\mathrm{A}_1, \quad \mathrm{A}_1, \quad \varnothing \tag{B.7}$$

in the case when $d = 1, 2, 3, 4, 5, 6, 8, 9$, respectively. Vice versa, if $\uprho(X) = 1$ and $\mathrm{Type}(X)$ is one of the types (B.7)*, then $\mathrm{Cl}(X) \cong \mathbb{Z}$.*

Proof. The first assertion easily follows from Proposition B.1. To prove the second one, we may assume that $d \leqslant 5$, since the remaining cases are easy. Let $\mathrm{Cl}(X, P)$ be the local Weil divisor class group of the (unique) singular point $P \in X$. Then in the cases (B.7) we have

$$\mathrm{Cl}(X, P) \cong \mathbb{Z}/d\mathbb{Z},$$

see e.g. [5, Satz 2.11]. Therefore, for any line $L \subset X$ the divisor dL is Cartier. Since $dL \sim_{\mathbb{Q}} -K_X$, we have $dL \sim -K_X$. Now, the assertion follows from Lemma 2.9(iii) and Proposition 3.4. $\square$

For every $d \in \{2, \ldots, 6\}$, let X_d be the Du Val del Pezzo surface of degree d such that $\mathrm{Cl}(X_d) \cong \mathbb{Z}$. As we already mentioned earlier, the surface X_d contains exactly one line L_d and $\mathrm{Cl}(X_d) = \mathbb{Z}[L_d]$. Take a point $P_d \in L_d \setminus \mathrm{Sing}(X_d)$. Let $\sigma_d \colon \widetilde{X}_d \to X_d$ be the blow up of the point P_d, and let $\widetilde{L}_d$ be the proper transform on $\widetilde{X}_d$ of the line L_d. Then

$$\widetilde{L}_d^2 = -1 + \frac{1}{d} < 0,$$

and $K_{\widetilde{X}_d} \cdot \widetilde{L}_d = 0$. Therefore, there exists a crepant contraction $\varphi_d \colon \widetilde{X}_d \to X'_{d-1}$ of the curve $\widetilde{L}_d$, where X'_{d-1} is a singular Du Val del Pezzo surface such that $\mathrm{Cl}(X'_{d-1}) \cong \mathbb{Z}$ and $K^2_{X'_d} = K^2_{\widetilde{X}} = d-1$. Thus, if $d \neq 2$, then $X'_{d-1} \cong X_{d-1}$ by Proposition B.1. If $d = 2$, then X'_1 is one of the two surfaces described in Remark B.5, so that we also let $X_1 = X'_1$. Hence, we obtain the following diagram:

$$\begin{array}{ccccccccccc}
& \widetilde{X}_6 & & \widetilde{X}_5 & & \widetilde{X}_4 & & \widetilde{X}_3 & & \widetilde{X}_2 & \\
{}^{\sigma_6}\swarrow & & \searrow^{\varphi_6}\ {}^{\sigma_5}\swarrow & & \searrow^{\varphi_5}\ {}^{\sigma_4}\swarrow & & \searrow^{\varphi_4}\ {}^{\sigma_3}\swarrow & & \searrow^{\varphi_3}\ {}^{\sigma_2}\swarrow & & \searrow^{\varphi_2} \\
X_6 & \dashrightarrow & X_5 & \dashrightarrow & X_4 & \dashrightarrow & X_3 & \dashrightarrow & X_2 & \dashrightarrow & X_1
\end{array}$$

It allows us to reconstruct all surfaces in Proposition B.1 starting from $X_6 = \mathbb{P}(1, 2, 3)$.

Each birational transformation $X_{d-1} \dashrightarrow X_d$ is $\mathrm{Aut}^0(X_{d-1})$-equivariant, so that it gives a natural embedding $\mathrm{Aut}^0(X_{d-1}) \hookrightarrow \mathrm{Aut}^0(X_d)$ such that $\mathrm{Aut}^0(X_{d-1})$ is just the stabilizer of the point P_d. Moreover, there following two assertions hold:

- if $d \geqslant 3$, then $\mathrm{Aut}^0(X_d)$ transitively acts on $L_d \setminus \mathrm{Sing}(X_d)$;
- if $d = 2$, then $\mathrm{Aut}^0(X_2) \cong \mathbb{G}_{\mathrm{m}}$ has two orbits in $L_2 \setminus \mathrm{Sing}(X_2)$, an open orbit and a closed orbit that consists of a single point, which explains two possibilities in Remark B.5.

The construction allow also to compute $\mathrm{Aut}^0(X)$ in the cases 1^o, 5^o, 14^o, 24^o, 36^o of Big Table.

Corollary B.8. *Let X be a Du Val del Pezzo surface such that $\mathrm{Cl}(X) \cong \mathbb{Z}$, and let $d := K_X^2$. Then the group $\mathrm{Aut}^0(X)$ is isomorphic to*

$$\mathbb{B}_3, \quad \mathbb{B}_2 \times \mathbb{G}_{\mathrm{m}}, \quad \mathbb{G}_{\mathrm{a}}^2 \rtimes \mathbb{G}_{\mathrm{m}}, \quad \mathbb{G}_{\mathrm{a}} \rtimes_{(3)} \mathbb{G}_{\mathrm{m}}, \quad \mathbb{G}_{\mathrm{m}}$$

in the case when $d = 6, 5, 4, 3, 2$, respectively. If $d = 1$, then $\mathrm{Aut}^0(X) \cong \mathbb{G}_{\mathrm{m}}$ if X is given by (B.2)*, and $\mathrm{Aut}^0(X) = \{1\}$ if X is given by* (B.3).

References

[1] J. Alper, H. Blum, D. Halpern-Leistner, C. Xu, *Reductivity of the automorphism group of K-polystable Fano varieties*, to appear in Invent. Math.

[2] I. Arzhantsev, U. Derenthal, J. Hausen, A. Laface, *Cox rings*, Cambridge University Press, 2015.

[3] D. Bindschadler, L. Brenton, *On singular 4-manifolds of the homology type of CP^2*, J. Math. Kyoto Univ. **24** (1984), 67–81.

[4] L. Brenton, *On singular complex surfaces with negative canonical bundle, with applications to singular compactifications of $\mathbf{C}^2$ and to 3-dimensional rational singularities*, Math. Ann. **248** (1980), 117–124.

[5] E. Brieskorn, *Rationale Singularitten komplexer Flchen*, Invent. Math. **4** (1968), 336–358.

[6] J. Bruce, T. Wall, *On the classification of cubic surfaces*, J. LMS **19** (1979), 245–256.

[7] D. Coray, M. Tsfasman, *Arithmetic on singular Del Pezzo surfaces*, Proc. LMS **57** (1988), 25–87.

[8] M. Demazure, *Surfaces de del Pezzo. I. II. III. IV. V.*, Lect. Notes Math. **777** (1980), 21–69.

[9] U. Derenthal, D. Loughran, *Singular del Pezzo surfaces that are equivariant compactifications*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklova **377**, (2010), 26–43.

[10] U. Derenthal, D. Loughran, *Equivariant compactifications of two-dimensional algebraic groups*, Proc. Edinb. Math. Soc. **58** (2015), 149–168.

[11] I. Dolgachev, *Classical algebraic geometry. A modern view*, Cambridge Univ. Press, Cambridge 2012.

[12] I. Dolgachev, V. Iskovskikh, *Finite subgroups of the plane Cremona group*, Progress in Mathematics **269** (2009), 443–548.

[13] P. Du Val, *On isolated singularities of surfaces which do not affect the conditions of adjunction,I-III*, Proc. Camb. Philos. Soc. **30** (1934), 453–459, 460–465, 483–491.

[14] M. Furushima, *Singular del Pezzo surfaces and analytic compactifications of 3-dimensional complex affine space $\mathbf{C}^3$*, Nagoya Math. J. **104** (1986), 1–28.

[15] J. Hausen, E. Herppich, H. Süß, *Multigraded factorial rings and Fano varieties with torus action*, Documenta Math. **16** (2011), 71–109.

[16] E. Herppich, *On Fano varieties with torus action of complexity 1*, Proc. EMS **57** (2014), 737–753.

[17] F. Hidaka, K. Watanabe, *Normal Gorenstein surfaces with ample anti-canonical divisor*, Tokyo J. Math. **4** (1981), 319–330.

[18] C. M. Hui, *Plane quartic curves*, Ph.D. Thesis, University of Liverpool, 1979.

[19] B. Kunyavskij, A. Skorobogatov, M. Tsfasman, *The combinatorics and geometry of Del Pezzo surfaces of degree four*, Russ. Math. Surv. **40** (1986), 131–132.

[20] G. Martin, C. Stadlmayr, *Weak del Pezzo surfaces with global vector fields*, preprint, arXiv:2007.03665 (2020).

[21] Y. Matsushima, *Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kaehlérienne*, Nagoya Math. J. **11** (1957), 145–150.

[22] M. Miyanishi, D.-Q. Zhang, *Gorenstein log del Pezzo surfaces of rank one*, J. Algebra **118** (1988), 63–84.

[23] D. Morrison, *The birational geometry of surfaces with rational double points*, Math. Ann. **271** (1985), 415–438.

[24] Y. Odaka, C. Spotti, S. Sun, *Compact moduli spaces of Del Pezzo surfaces and Kähler–Einstein metrics*, J. Differential Geom. **102** (2016), 127–172.

[25] B. Poonen, F. Rodriguez-Villegas, *Lattice polygons and the number* 12, American Math. Monthly **107** (2000), 238–250.

[26] Yu. Prokhorov, *Lectures on complements on log surfaces*, Mathematical Society of Japan **10** (2001), 130 pages.

[27] M. Reid, *Young person's guide to canonical singularities*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math. **46** (1987) 345–414.

[28] Y. Sakamaki, *Automorphism groups on normal singular cubic surfaces with no parameters*, Trans. Amer. Math. Soc. (2010) **362**, 2641–2666.

[29] C. T. C. Wall, *Geometry of quartic curves*, Math. Proc. of the Cambridge Phil. Society **117** (1995), 415–423.

[30] Q. Ye, *On Gorenstein log del Pezzo surfaces*, Japanese J. of Math. (2002) **28**, 87–136.

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