

RATIONALITY OF ENRIQUES-FANO THREEFOLD OF GENUS FIVE

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We prove rationality of a non-Gorenstein Fano 3-fold of Fano index one and degree eight having terminal cyclic quotient singularities and Picard group $\mathbb{Z}$, which can be described as a quotient of a double cover of $\mathbb{P}^3$ ramified in smooth quartic surface by an involution fixing eight different points.

In this paper we will prove the following result.

THEOREM 1. *Let $\psi : V \rightarrow \mathbb{P}^3$ be a double cover ramified in smooth quartic surface, τ be an involution of 3-fold V fixing finite number of points and $X = V/\tau$. Then X is rational.*

Remark 2. In notations of Theorem 1 variety V is a smooth Fano 3-fold of index 2 and degree 2, $-K_V = \psi^*(\mathcal{O}_{\mathbb{P}^3}(2))$, $-K_V^3 = 16$, involution τ fixes 8 points, 3-fold X has eight singular points of type $\frac{1}{2}(1, 1, 1)$, $-K_X$ is not a Cartier divisor, $-K_X \sim_{\mathbb{Q}} H$, where H is an ample Cartier divisor, $H^3 = 8$ and general element in $|H|$ is a smooth Enriques surface.

Example 3. Let $V \subset \mathbb{P}(1^3, 2)$ be given by $u^2 = x^4 + y^4 + z^4 + t^4$, where x, y, z and t are coordinates of weight 1 and u is a coordinate of weight 2, τ be an involution on $\mathbb{P}(1^3, 2)$, such that $\tau(x : y : z : t : u) = (x : y : -z : -t : -u)$. Then projection of 3-fold V to $\mathbb{P}^3$ is a double cover ramified in smooth quartic surface, 3-fold V is τ -invariant and $\tau|_V$ fixes 8 points given by $u = z = t = 0$ and $u = x = y = 0$.

G.Fano tried to obtain in [1-4] the biregular classification of 3-folds, whose curve sections are canonical curves. In smooth case hyperplane sections of such 3-folds are K3 surfaces due to adjunction. Natural generalization of 3-folds studied by G.Fano are 3-folds containing an ample Cartier divisor, which is a smooth K3 surface. In smooth case such 3-folds are exactly 3-folds having ample anticanonical divisor due to [5], which is the modern definition of a Fano 3-fold given by V.Iskovskikh. Complete classification of smooth Fano 3-folds were obtained in [5-10], where 105 families of Fano 3-folds were found. Later G.Fano studied projective 3-folds in [11], whose hyperplane sections are smooth Enriques surfaces¹, where he gave a list of six such 3-folds, including exceptional ample but not very ample cases, and claimed the list is complete. However, proofs in [11] are incorrect and the classification is

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All varieties are assumed to be projective, normal and defined over field $\mathbb{C}$.

¹Similar problem can be considered for any surface of Kodaira dimension zero, but in abelian and bi-elliptic cases the corresponding problem has just a trivial answer even in ample case (see [12-17]).

not complete as was shown in [18-19]. Moreover, as was remarked in [18-19] every 3-fold, whose hyperplane sections are smooth Enriques surfaces, must be singular, which passed these 3-folds into oblivion for almost half of the century. Actually G.Fano claimed in [11] that these 3-folds have exactly eight quadruple singular points, whose tangent cone is a cone over the Veronese surface. In modern language such singular points are terminal cyclic quotient singularities of type $\frac{1}{2}(1, 1, 1)$ and locally isomorphic to a quotient of $\mathbb{C}^3$ by an involution, which fixes just the origin.

Definition 4. 3-fold X is called an Enriques-Fano 3-fold of genus $g(X) = -\frac{1}{2}K_X^3 + 1$ and degree $-K_X^3$ if X has canonical singularities, $-K_X$ is not a Cartier divisor and $-K_X \sim_{\mathbb{Q}} H$, where H is an ample Cartier divisor on X .

The following result was proved in [20].

THEOREM 5. *Let X be an Enriques-Fano 3-fold and $-K_X \sim_{\mathbb{Q}} H$, where H is a Cartier divisor on X . Then general enough surface in linear system $|H|$ is an Enriques surface with canonical singularities, which is smooth if singularities of X are isolated and $-K_X^3 \neq 2$.*

COROLLARY 6. *Degree and genus of an Enriques-Fano 3-fold are natural.*

The following result was proved in [16-17].

THEOREM 7. *Let X be a normal 3-fold and $H \subset X$ be an Enriques surface with canonical singularities, such that H is an ample Cartier divisor on X . Then $-2K_X \sim 2H$ and X is either an Enriques-Fano 3-fold or a contraction of a section of $\text{Proj}(\mathcal{O}_H \oplus \mathcal{O}_H(H|_H))$.*

Therefore except generalized cones the Fano-Enriques 3-folds are exactly 3-folds, which contain Enriques surface with Du Val singularities as an ample Cartier divisor.

Remark 8. Let X be an Enriques-Fano 3-fold, and H be a Cartier divisor on 3-fold X , such that $-K_X \sim_{\mathbb{Q}} H$. Then $2(H + K_X) \sim 0$ and $V = \text{Spec}(\mathcal{O}_X \oplus \mathcal{O}_X(H + K_X))$ is called a canonical cover of X . One can show that V is a Fano 3-fold with Gorenstein canonical singularities, there is a natural double cover $\pi : V \rightarrow X$ ramified in finite number of points, which are exactly non-Gorenstein points of X , $-K_V \sim \pi^*(H)$, double cover π induces etale double cover of every Enriques surface with canonical singularities in $|H|$, which coincides with a natural etale double cover of an Enriques surface by a K3 surface.

Enriques-Fano 3-folds are singular and non-Gorenstein by definition. Therefore Enriques-Fano 3-folds with terminal cyclic quotient singularities are non-Gorenstein analogues of smooth Fano 3-folds. However, canonical covers of the Enriques-Fano 3-folds having terminal cyclic quotient singularities are smooth. Therefore singular points of every Enriques-Fano 3-fold having terminal cyclic quotient singularities are of type $\frac{1}{2}(1, 1, 1)$, but as was observed in [21] the singular Riemann-Roch formula in [22] implies that the number of such singular points is exactly eight as claimed by G.Fano! Moreover the classification of smooth Fano 3-folds enabled to obtain in [23-24] the following result.

THEOREM 9. *Let X be an Enriques-Fano 3-fold having terminal quotient cyclic singularities and V be its canonical cover. Then V is one of the following smooth Fano 3-folds:*

- (1) *the complete intersection of a quadric and quartic in $\mathbb{P}(1^5, 2)$ and $g(X) = 2$;*

- (2) the complete intersection of three quadrics in $\mathbb{P}^6$ and $g(X) = 3$;
- (3) the blow up of the smooth hypersurface of degree 4 in $\mathbb{P}(1^4, 2)$, whose center is a smooth elliptic curve, which is cut by two hypersurfaces of degree one, and $g(X) = 3$;
- (4) $\mathbb{P}^1 \times S_2$, where S_2 is a smooth del Pezzo surface of degree 2, and $g(X) = 4$;
- (5) the double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ramified in a divisor of degree $(2, 2, 2)$ and $g(X) = 4$;
- (6) the blow up of the smooth complete intersection of two quadrics in $\mathbb{P}^5$, whose center is a smooth elliptic curve, which is cut by two hyperplane sections, and $g(X) = 5$;
- (7) the hypersurface of degree 4 in $\mathbb{P}(1^4, 2)$ and $g(X) = 5$;
- (8) the complete intersection of three divisors of degree $(1, 1)$ in $\mathbb{P}^3 \times \mathbb{P}^3$ and $g(X) = 6$;
- (9) $\mathbb{P}^1 \times S_4$, where S_4 is a smooth del Pezzo surface of degree 4 and $g(X) = 7$;
- (10) the divisor of degree $(1, 1, 1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $g(X) = 7$;
- (11) the blow up of the cone over the smooth quadric surface $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ in a disjoint union of the vertex and a smooth elliptic curve on $\mathbb{P}^1 \times \mathbb{P}^1$ and $g(X) = 8$;
- (12) the complete intersection of two quadrics in $\mathbb{P}^5$ and $g(X) = 9$;
- (13) $\mathbb{P}^1 \times S_6$, where S_6 is a smooth del Pezzo surface of degree 6, and $g(X) = 10$;
- (14) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $g(X) = 13$.

The following result was proved in [25].

THEOREM 10. *Let X be a terminal Enriques-Fano 3-fold. Then there is a small deformation $g : U \rightarrow \Delta$ over the one-parameter unit disk Δ with an origin O , such that for any point $s \in \Delta \setminus O$ fiber $g^{-1}(s)$ is an Enriques-Fano 3-fold having terminal cyclic quotient singularities and $g^{-1}(O) \cong X$.*

Therefore terminal Enriques-Fano 3-folds are classified by Theorems 9-10. However the classification of Enriques-Fano 3-folds having arbitrary canonical singularities is far from being complete. The following result is proved in [26].

THEOREM 11. *Let X be an Enriques-Fano 3-fold. Then $-K_X^3 \leq 92$ and $g(X) \leq 47$.*

The bounds in Theorem 11 do not seem to be perfect. The expected genus bound for an arbitrary Enriques-Fano 3-fold is 13. Recently, the following result was proved in [27].

THEOREM 12. *Let X be an Enriques-Fano 3-fold, such that X has isolated singularities and $-K_X \sim_{\mathbb{Q}} H$, where H is a very ample Cartier divisor on X . Then $g(X) \leq 26$.*

By definition Enriques-Fano 3-folds are Fano 3-folds with canonical singularities having integer Fano index. However, apriori the Fano index of an Enriques-Fano 3-fold does not need to be equal to one. The following result was proved in [28].

THEOREM 13. *Let $X \subset \mathbb{P}^n$ be an Enriques-Fano 3-fold, such that X has only isolated singularities and $-K_X \sim_{\mathbb{Q}} H$, where H is a hyperplane section of 3-fold X , which is a smooth Enriques surface. Then $H \subset \mathbb{P}^{n-1}$ can not be a r -th Veronese embedding of a linearly normal smooth Enriques surface $S \subset \mathbb{P}^k$ for $r \geq 2$.*

Actually the following result holds.

PROPOSITION 14. *Let X be an Enriques-Fano 3-fold and $-K_X \sim_{\mathbb{Q}} nH$ for $n \in \mathbb{N}$,*

where H is an ample Cartier divisor on 3-fold X . Then equality $n = 1$ holds.

Proof. Suppose $n \neq 1$. Let $D = K_X + nH$ be a Weil divisor on X and S be a general element in linear system $|H|$. Then S has canonical singularities due to [29] and X has canonical Gorenstein singularities near S due to [30]. Moreover S is a del Pezzo surface by adjunction formula. In particular $D|_S \sim 0$, because $D|_S$ is a Cartier divisor and $D|_S \sim_{\mathbb{Q}} 0$, but Picard group of S has no torsion. Let $k \gg 0$, surface Y be a general element in linear system $|kH|$ and $C = Y \cap S$. Then equalities

$$D|_Y \cdot H|_Y = D|_S \cdot kH|_S = D \cdot C = 0$$

hold, which implies $D|_Y \equiv 0$ and $D|_C \sim 0$ by Hodge index theorem, because $kH|_Y$ is an ample Cartier divisor on surface Y and $D|_Y \cdot D|_Y \not\leq 0$. Consider exact sequence

$$0 \rightarrow H^0(\mathcal{O}_Y(D|_Y)) \rightarrow H^0(\mathcal{O}_C) \rightarrow H^1(\mathcal{O}_Y((D - H)|_Y)),$$

where Cartier divisor $(H - D)|_Y$ is ample on Y . Now Kawamata-Vieweg vanishing theorem implies $H^1(\mathcal{O}_Y((D - H)|_Y)) = 0$. Moreover $H^0(\mathcal{O}_C) = \mathbb{C}$, because curve C is an ample divisor on Y . Thus $H^0(\mathcal{O}_Y(D|_Y)) = \mathbb{C}$, which implies $D|_Y \sim 0$. Sequence of sheaves

$$0 \rightarrow \mathcal{O}_X(D) \otimes \mathcal{O}_X(-kH) \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{O}_Y \rightarrow 0$$

is exact, because sheaf $\mathcal{O}_X(D)$ is locally free in the neighbourhood of surface Y , but outside of surface Y the above sequence is trivial. Therefore cohomology sequence of groups

$$0 = H^0(\mathcal{O}_X(D) \otimes \mathcal{O}_X(-kH)) \rightarrow H^0(\mathcal{O}_X(D)) \rightarrow H^0(\mathcal{O}_Y) \rightarrow H^1(\mathcal{O}_X(D) \otimes \mathcal{O}_X(-kH))$$

is exact. However sheaf $\mathcal{O}_X(D)$ is reflexive by [31] and there is an exact sequence

$$0 \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0,$$

where $\mathcal{E}$ is locally free and $\mathcal{F}$ is torsion free sheaves. Hence sequence

$$H^0(\mathcal{F} \otimes \mathcal{O}_X(-kH)) \rightarrow H^1(\mathcal{O}_X(D) \otimes \mathcal{O}_X(-kH)) \rightarrow H^1(\mathcal{E} \otimes \mathcal{O}_X(-kH))$$

is exact, where $H^0(\mathcal{F} \otimes \mathcal{O}_X(-kH)) = 0$, because $\mathcal{F}$ is torsion free, $H^1(\mathcal{E} \otimes \mathcal{O}_X(-kH)) = 0$, because X is normal. Thus $H^1(\mathcal{O}_X(D) \otimes \mathcal{O}_X(-kH)) = 0$, which implies $H^0(\mathcal{O}_X(D)) = \mathbb{C}$ and K_X is a Cartier divisor, which contradicts the definition of X . $\square$

Interest in Fano 3-folds was inspired by Lüroth Problem. G.Fano claimed non-rationality of a smooth quartic 3-fold in [32] and non-rationality of a smooth cubic 3-fold in [33], but unirationality of smooth cubic 3-fold was well known and unirationality of some smooth quartic 3-folds was proved in [34] later (see [35-36]), but perhaps G.Fano knew some examples of unirational smooth quartic 3-folds. However, rigorous proofs of these claims of G.Fano was obtained only in [37-38]. Now the question of rationality of smooth Fano 3-folds is almost completely settled (see [36-41]). Actually interest in Enriques-Fano 3-folds was inspired by the following result in [42-43] conjectured by F.Enriques and G.Fano.

THEOREM 15. *Let $X \subset \mathbb{P}^4$ be so-called Enriques 3-fold given by equation*

$$x_1x_2x_3x_4(x_0^2 + x_0 \sum_{i=1}^4 a_i x_i + \sum_{i,j=1}^4 b_{ij} x_i x_j) + c_1 x_2^2 x_3^2 x_4^2 + c_2 x_1^2 x_3^2 x_4^2 + c_3 x_1^2 x_2^2 x_4^2 + c_4 x_1^2 x_2^2 x_3^2 = 0$$

of degree six, where x_i is a homogeneous coordinate on $\mathbb{P}^4$, a_i , b_{ij} and c_i are general enough complex numbers. Then 3-fold V is non-rational.

G.Fano thought the Enriques 3-fold is not even unirational, but L.Roth proved in [44] that it is unirational and claimed that it is non-rational, because the fundamental group of its desingularization contains $\mathbb{Z}_2$. On the other hand J.P.Serre proved in [45] that smooth unirational variety is simply connected. The apparent gap in [44] was explained in [46] by showing that Enriques 3-fold always has non-ordinary singular points, which are in modern language images of non-Gorenstein points of its normalization.

Remark 16. Normalization of the Enriques 3-fold is an Enriques-Fano 3-fold of degree six and genus four having eight singular points of type $\frac{1}{2}(1, 1, 1)$, whose canonical cover is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ramified in a divisor of degree $(2, 2, 2)$.

So some Enriques-Fano 3-folds are non-rational. The following result was proved in [47].

THEOREM 17. *Let X be an Enriques-Fano 3-fold and $-K_X^2 \geq 10$. Then X is rational.*

In general Fano varieties tends to be *more rational* whenever their degrees increase and singularities become worse. For instance classification of smooth Fano 3-folds implies that smooth Fano 3-fold is rational once its degree is greater than 24. Therefore Theorem 17 is rather natural and Theorem 1 fills the gap between Theorems 15 and 17.

COROLLARY 18. *Let X be a non-rational Enriques-Fano 3-fold having terminal cyclic singularities and V be its canonical cover. Then V is either the complete intersection of a quadric and quartic in $\mathbb{P}(1^5, 2)$ or the complete intersection of three quadrics or the double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ramified in a divisor of degree $(2, 2, 2)$.*

CONJECTURE 19. *Let X be an Enriques-Fano 3-fold with terminal cyclic singularities, such that its canonical cover is either the complete intersection of three quadrics or the complete intersection of a quadric and quartic in $\mathbb{P}(1^5, 2)$. Then X is non-rational.*

In the rest of the paper we will prove Theorem 1. Let $\psi : V \rightarrow \mathbb{P}^3$ be a double cover ramified in smooth quartic surface $S \subset \mathbb{P}^3$, τ be an involution of 3-fold V fixing eight points and $X = V/\tau$. Then X is an Enriques-Fano 3-fold of degree 8 and genus 5.

LEMMA 20. *There are homogeneous coordinates $(x_0 : x_1 : x_2 : x_3)$ on $\mathbb{P}^3$, such that action of τ on $\mathbb{P}^3$ is $\tau(x_0 : x_1 : x_2 : x_3) = (x_0 : x_1 : -x_2 : -x_3)$ and $S \subset \mathbb{P}^3$ is given by*

$$q_4(x_0, x_1) + a_2(x_0, x_1)x_2^2 + b_2(x_0, x_1)x_2x_3 + c_2(x_0, x_1)x_3^2 + p_4(x_2, x_3) = 0,$$

where q_4 and p_4 are polynomials of degree 4, a_2 , b_2 and c_2 are polynomials of degree 2.

Proof. See 6.1.2 in [23]. $\square$

Thus V can be considered as a hypersurface of degree 4 in $\mathbb{P}(1^4, 2)$, whose equation is

$$y^2 = q_4(x_0, x_1) + a_2(x_0, x_1)x_2^2 + b_2(x_0, x_1)x_2x_3 + c_2(x_0, x_1)x_3^2 + p_4(x_2, x_3),$$

where x_i is a homogeneous coordinate of weight 1 and y is a homogeneous coordinate of weight 2, τ acts on $\mathbb{P}(1^4, 2)$ as $\tau(x_0 : x_1 : x_2 : x_3 : y) = (x_0 : x_1 : -x_2 : -x_3 : -y)$, fixed points of $\tau : V \rightarrow V$ are given by $x_0 = x_1 = y = 0$ and $x_2 = x_3 = y = 0$, where q_4 and p_4 are polynomials of degree 4, a_2, b_2 and c_2 are polynomials of degree 2.

Remark 21. Linear system $|-K_V|$ is generated by $\lambda y + \sum a_{ij}x_i x_j$, where $\lambda, a_{ij} \in \mathbb{C}$.

Let $\Lambda \subset |-K_V|$ be a linear subsystem given by forms

$$b_{00}x_0^2 + b_{01}x_0x_1 + b_{11}x_1^2 + b_{22}x_2^2 + b_{23}x_2x_3 + b_{33}x_3^2$$

and $\Sigma \subset |-K_V|$ be a linear subsystem given by linear combinations

$$\mu y + c_{02}x_0x_2 + c_{03}x_0x_3 + c_{12}x_1x_2 + c_{13}x_1x_3,$$

where $\mu, c_{ij}, b_{ij} \in \mathbb{C}$. Then linear system Λ is free, base locus of linear system Σ consists of eight fixed points of τ , surfaces in Λ and Σ are τ -invariant, general element S_Λ of linear system Λ is a smooth K3 surface, such that τ acts on S_Λ without fixed points, general element S_Σ of linear system Σ is a smooth K3 surface, such that τ fixes on S_Σ exactly fixed points of τ . Let $H_\Lambda \subset X$ be a quotient S_Λ/τ and $H_\Sigma \subset X$ be a quotient S_Σ/τ . Then H_Λ is a smooth Enriques surface, H_Σ is a K3 surface with eight simple double points, H_Λ is an ample Cartier divisor on 3-fold X , H_Σ is a Weil divisor on 3-fold X , $2H_\Sigma$ is a Cartier divisor on 3-fold X , $2H_\Lambda \sim 2H_\Sigma$, surface H_Σ is a general element of anticanonical linear system of Weil divisors $|-K_X|$ and surface H_Λ is a general element of linear system $|H_\Lambda|$.

Remark 22. Linear system $|H_\Lambda|$ is free and gives a double cover $\phi_{|H_\Lambda|} : X \rightarrow \mathbb{P}^5$, whose image is a complete intersection of two quadrics $B^2 = AC$ and $E^2 = DF$, in appropriate homogeneous coordinates $(A : B : C : D : E : F)$ on $\mathbb{P}^5$. Linear system $|H_\Sigma|$ gives rational map $\phi_{|H_\Sigma|} : X \dashrightarrow \mathbb{P}^4$, whose image is a quadric $XZ = YT$, where $(X : Y : Z : T : W)$ are appropriate homogeneous coordinates on $\mathbb{P}^4$, such that degree of $\phi_{|H_\Sigma|}$ in generic point is two and blow up of singular points of 3-fold X resolves the indeterminacy of map $\phi_{|H_\Sigma|}$.

Let Ξ be a pencil on V given by linear forms $\alpha x_0 + \beta x_1$ and Ω be a pencil on V given by linear forms $\gamma x_2 + \delta x_3$ for $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Then base locus of pencil Ξ consists of smooth elliptic curve $C_\Xi \subset V$ given by $x_0 = x_1 = 0$, base locus of pencil Ω consists of smooth elliptic curve $C_\Omega \subset V$ given by $x_2 = x_3 = 0$, surfaces in Ξ and Ω are τ -invariant, general surface $S_\Xi \in \Xi$ and general surface $S_\Omega \in \Omega$ are smooth del Pezzo surface of degree two, involution τ fixes on surface S_Ξ four base points $y = x_0 = x_1 = 0$, involution τ fixes on surface S_Ω exactly four base points $y = x_2 = x_3 = 0$. Let $H_\Xi \subset X$ be a quotient S_Ξ/τ and surface $H_\Omega \subset X$ be a quotient S_Ω/τ . Then both H_Ξ and H_Ω are del Pezzo surfaces of degree one having four simple double points, Weil divisors H_Ξ and H_Ω are not Cartier divisors on 3-fold X , but both $2H_\Xi$ and $2H_\Omega$ are Cartier divisors and $2H_\Xi \sim 2H_\Omega \sim H_\Lambda$, H_Ξ and H_Ω are general surfaces in linear systems of Weil divisors $|H_\Xi|$ and $|H_\Omega|$ correspondingly.

LEMMA 23. Let E_{Ξ} and E_{Ω} be images of smooth elliptic curves C_{Ξ} and C_{Ω} on 3-fold X correspondingly. Then E_{Ξ} and E_{Ω} are smooth rational curves, E_{Ω} intersects H_{Ξ} transversally in one smooth point O , such that point O is a unique base point of linear system $|-K_{H_{\Xi}}|$, which is cut on H_{Ξ} by linear system $|H_{\Omega}|$.

Proof. All claims are implied by the explicit construction of 3-folds V and X . $\square$

LEMMA 24. Let P_i be four singular points of 3-fold X for $i = 1, 2, 3, 4$, which are images of the four fixed points of involution τ given by equations $y = x_0 = x_2$, and let H_{Ω}^i be a unique surface in $|H_{\Omega}|$ containing P_i . Then curve $H_{\Xi} \in H_{\Omega}^i \subset |-K_{H_{\Xi}}|$ is a singular curve of arithmetic genus one having one double point P_i .

Proof. The claim is implied by Lemme 23. $\square$

Consider a composition $g : Y \rightarrow X$ of blow ups of points P_i with a blow up of proper transform of smooth rational curve E_{Ξ} . Denote F_i and E exceptional divisors of birational morphism g , such that $g(F_i) = P_i$ and $g(E) = E_{\Xi}$. Let $H \subset Y$ be a proper transform of irreducible surface H_{Ξ} , Θ be a proper transform on 3-fold Y of $|H_{\Xi}|$ and $H_i \subset Y$ be a proper transform of surface H_{Ω}^i . Then H is a smooth del Pezzo surface of degree one, linear system Θ is a free pencil, which gives morphism $\phi_{\Theta} : Y \rightarrow \mathbb{P}^1$, surface H is a general fiber of ϕ_{Θ} , curves $E \cap H$, $F_i \cap H$, $H_i \cap H$ are irreducible, smooth and rational. Moreover

$$F_i \cdot F_i \cdot H = -2, E \cdot F_i \cdot H = 1, E \cdot E \cdot H = -1, H_i \cdot H_i \cdot H = -1, E \cdot H_i \cdot H = 0, F_i \cdot H_i \cdot H = 2$$

and $F_i \cdot F_j \cdot H = 0$ in case $i \neq j$. Thus we may consider 3-fold Y as a smooth del Pezzo surface of degree one defined over a field $\mathbb{C}(x)$ of rational functions on $\mathbb{P}^1$ and consider all divisors H_i , F_i and E as curves on surface Y defined over the field $\mathbb{C}(x)$ as well, such that

$$F_i \cdot F_i = -2, E \cdot F_i = 1, E \cdot E = -1, H_i \cdot H_i = -1, E \cdot H_i = 0, F_i \cdot H_i = 2$$

and $F_i \cdot F_j = 0$ in case $i \neq j$. Moreover $H_i \cdot H_j = 1$ for $i \neq j$ and all curves H_i intersect in the base point of linear system $|-K_Y|$, equivalence $-K_Y \sim H_i + F_i$ holds and $K_Y^2 = 1$.

PROPOSITION 25. Rank of Picard group of del Pezzo surface Y is at least 5.

Proof. Equalities $-K_Y \cdot F_i = 0$, $F_i \cdot F_i = -2$ and $F_i \cdot F_j = 0$ for $i \neq j$ imply the claim. $\square$

Birational theory of geometrically rational surfaces defined over an algebraically non-closed field was studied in [48-57], where the following three results was proved.

THEOREM 26. Let W be a smooth minimal geometrically irreducible and geometrically rational surface defined over a perfect field $\mathbb{F}$. Then either $\text{Pic}(W) \cong \mathbb{Z}$ and W is a smooth del Pezzo surface or $\text{Pic}(W) \cong \mathbb{Z} \otimes \mathbb{Z}$ and there is a conic bundle $\pi : W \rightarrow Z$.

THEOREM 27. Let W be a smooth minimal geometrically irreducible and geometrically rational surface defined over a perfect field $\mathbb{F}$. Then surface W is rational over $\mathbb{F}$ if and only if surface W has a $\mathbb{F}$ -point and inequality $K_W^2 \geq 5$ holds.

THEOREM 28. Let W be a smooth geometrically irreducible and geometrically rational surface defined over a C_1 -field $\mathbb{F}$. Then surface W has a $\mathbb{F}$ -point.

Therefore smooth del Pezzo surface Y has a $\mathbb{C}(x)$ -point. Actually set of all $\mathbb{C}(x)$ -points of surface Y is huge due to [58]. Moreover surface Y is not minimal by Proposition 25 and there is a birational morphism $f : Y \rightarrow U$, where U is a smooth minimal geometrically rational surface defined over field $\mathbb{C}(x)$, which also has a $\mathbb{C}(x)$ -point.

Remark 29. $K_U^2 \geq K_Y^2 + \text{rank}(\text{Pic}(Y)) - \text{rank}(\text{Pic}(U)) \geq \text{rank}(\text{Pic}(Y)) - 1 \geq 4$.

We may assume $\text{Pic}(Y) \cong \mathbb{Z}^5$, because otherwise surface U is rational by Theorem 27, which implies rationality of 3-fold X . In particular group $\text{Pic}(Y) \otimes \mathbb{Q}$ is generated by divisors F_i and $-K_Y$. Over an algebraic closure $\overline{\mathbb{C}(x)}$ of field $\mathbb{C}(x)$ morphism $f : Y \rightarrow U$ is a composition of contractions of smooth rational curves, whose self-intersection is -1 , which implies $K_U^2 \geq 5$ in case when f contracts at least 4 curves over $\overline{\mathbb{C}(x)}$, but f contracts at least 3 curves over $\overline{\mathbb{C}(x)}$. Hence we may assume f contracts exactly 3 curves over $\overline{\mathbb{C}(x)}$, equality $K_U^2 = 4$ holds and $\text{Pic}(U) = \mathbb{Z} \otimes \mathbb{Z}$, which implies that each curve contracted by birational morphism f over $\overline{\mathbb{C}(x)}$ must be $\text{Gal}(\overline{\mathbb{C}(x)}/\mathbb{C}(x))$ -invariant.

COROLLARY 30. $f = f_1 \circ f_2 \circ f_3$, where f_i is a contraction of geometrically irreducible smooth rational curve defined over $\mathbb{C}(x)$, whose self-intersection is -1 .

Consider smooth rational curve $C \subset Y$ defined over $\mathbb{C}(x)$, such that $C^2 = -1$. Then

$$C \sim_{\mathbb{Q}} -\epsilon K_Y - \sum_{i=1}^4 k_i F_i,$$

where ϵ and k_i are rational numbers. Moreover, we have

$$C \cdot F_i \in \mathbb{Z}_{\geq 0} \Rightarrow k_i \in \frac{1}{2}\mathbb{Z}_{\geq 0}, \quad -K_Y \cdot C = 1 \Rightarrow \epsilon = 1, \quad C^2 = -1 \Rightarrow \sum_{i=1}^4 k_i^2 = 1,$$

which implies $C \sim_{\mathbb{Q}} -K_X - F_i \sim H_i$ or $C \sim_{\mathbb{Q}} -K_X - \sum_{i=1}^4 \frac{1}{2}F_i \sim_{\mathbb{Q}} E$. Thus irreducible curve C must coincide with either H_i or E . Hence all geometrically irreducible curves on surface Y having negative self-intersections are F_i , H_i and E . However intersection form of any three curves among nine curves F_i , H_i and E is not negatively defined, which contradicts Corollary 30. Therefore Theorem 1 is completely proved.

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