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The method of degeneracy and the irrationality of three-dimensional varieties with a pencil of del Pezzo surfaces

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One of the deepest and most interesting problems of algebraic geometry is that of the rationality of algebraic varieties. Global holomorphic differential forms are natural birational invariants of a smooth variety that completely solve this problem for algebraic curves and surfaces [1]. However, there are irrational varieties in the three-dimensional case that are very close in many respects to being rational yet have discrete invariants that do not suffice for the definition of rationality. For example, there are three-dimensional irrational varieties that are unirational [1], [2]. Nevertheless it is not easy to obtain similar examples. There are at present four methods for proving the irrationality of rationally connected varieties [2]. In this note we give a simple proof of the following well-known result.

Theorem 1. *Let $X \subset \mathbb{P}^1 \times \mathbb{P}^3$ be a subvariety that is a general effective divisor of bidegree $(m, 3)$, and V a two-sheeted covering of $\mathbb{P}^1 \times \mathbb{P}^2$ with branching at a general divisor of bidegree $(m, 4)$. Then both V and X are irrational for $m \geqslant 2$.*

Note that X can be represented as a fibring into cubic surfaces, and V as a fibring into del Pezzo surfaces of degree 2. It is known that X and V are unirational, and rational when $m \leqslant 1$. The irrationality of V for $m = 2$ follows from [3], and that of X for $m \geqslant 2$ is proved in [4]. The birational rigidity and irrationality of X and V for $m \geqslant 3$ are proved in [5]. If follows from [1] and [6] that X is irrational for $m \geqslant 5$ and V is irrational for $m \geqslant 2$. All the structures of Mori fibrings on X are described in [7] in the case $m = 2$, and its irrationality follows from this.

To prove Theorem 1 we shall use the following result from [8], of which a modern proof is contained in [1] and which generalizes the standard technique of degeneracy [2].

Theorem 2. *Let $\xi\colon Y \to Z$ be a proper flat morphism with fibres that are irreducible and reduced. Then there is a countable set of closed subvarities $Z_i \subset Z$ such that for any closed point $s \in Z$ the fibre $\xi^{-1}(s)$ is a ruled variety if and only if $s \in \bigcup Z_i$.*

Take a line $L \subset \mathbb{P}^3$ and a point $O \in \mathbb{P}^2$. Consider the smooth surface $S = \alpha^{-1}(L)$ and the smooth curve $C = \beta^{-1}(O)$, where $\alpha\colon \mathbb{P}^1 \times \mathbb{P}^3 \to \mathbb{P}^3$ and $\beta\colon \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ are the projections onto the second factor. Let $\widehat{X} \subset \mathbb{P}^1 \times \mathbb{P}^3$ be a general effective divisor of bidegree $(m, 3)$ that contains S, and let $\pi\colon \widehat{V} \to \mathbb{P}^1 \times \mathbb{P}^2$ be a two-sheeted covering with branching at a sufficiently general divisor D of bidegree $(m, 4)$ that has a singularity of type $\mathbb{A}_1 \times \mathbb{C}$ at a generic point of C.

Remark 3. $\widehat{X}$ and $\widehat{V}$ can be obtained as flat proper deformations of X and V respectively. Moreover, $\widehat{X}$ and $\widehat{V}$ are rationally connected [1]. On the other hand, a rationally connected three-dimensional variety is ruled if and only if it is rational. Thus it follows from Theorem 2 that Theorem 1 will be proved if we can show that $\widehat{X}$ and $\widehat{V}$ are irrational for $m \geqslant 2$.

A geometrically constructed degeneracy has the following meaning. The generic fibres of the natural projections of X and V onto $\mathbb{P}^1$ are smooth del Pezzo surfaces of degree 3 and 2 respectively defined over the field $\mathbb{C}(x)$, which have Picard group $\mathbb{Z}$ for $m \geqslant 1$. On the other hand, a generic fibre of the projection of $\widehat{X}$ onto $\mathbb{P}^1$ is a smooth cubic surface in $\mathbb{P}^3$ which contains a line. The projection of this line endows $\widehat{X}$ with the structure of a fibring into conics. A generic fibre of the projection of $\widehat{V}$ onto $\mathbb{P}^1$ is a del Pezzo surface of degree 2 which has one ordinary double point. The pencil of anticanonical curves on this surface passing through the double point endows $\widehat{V}$ with the structure of a fibring into conics.

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Let $\gamma\colon U \to \mathbb{P}^1 \times \mathbb{P}^3$ be a blow-up of the surface $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\chi\colon W \to \mathbb{P}^1 \times \mathbb{P}^2$ a blow-up of C. Put $\widetilde{X} = \gamma^{-1}(\widehat{X})$ and $\widetilde{D} = \chi^{-1}(D)$, and let $\widetilde{\pi}\colon \widetilde{V} \to W$ be a two-sheeted covering with branching at the divisor $\widetilde{D} \subset W$. Local computations show that $\widetilde{X}$ and $\widetilde{V}$ are non-singular by the genericity of the choice of the divisors $\widehat{X}$ and $\widehat{D}$. Note that, by construction, $\widetilde{X}$ and $\widetilde{V}$ are resolutions of singularities of $\widehat{X}$ and $\widetilde{V}$ respectively. Also, $\widehat{X}$ is necessarily singular for $m \geqslant 1$, but it is always nodal by the generality of the choice of $\widehat{X}$, and $\gamma|_{\widetilde{X}}$ is a small resolution.

Projection from the line L induces a morphism $\tau\colon \widetilde{X} \to \mathbb{P}^1 \times \mathbb{P}^1$, and projection from the point O induces a morphism $\eta\colon \widetilde{V} \to \mathbb{P}^1 \times \mathbb{P}^1$. By construction, each of τ, η is a fibring into conics. Also, $\mathrm{Pic}(\widetilde{X}) \cong \mathrm{Pic}(\widetilde{V}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ by Lefschetz' Theorem.

Lemma 4. *Let Λ and Ξ be divisors on $\mathbb{P}^1 \times \mathbb{P}^1$ that are divisors of the degeneracies of the fibrings τ and η respectively. Then the bidegrees of Λ and Ξ are $(5, 3m)$ and $(6, 2m)$ respectively.*

The proof is a straightforward computation.

Theorem 5. *Let $\omega\colon Y \to Z$ be a fibring into conics, where Y is a non-singular three-dimensional variety, $\mathrm{Pic}(Y/Z) = \mathbb{Z}$, and the surface $\mathbb{Z}$ is either $\mathbb{F}_r$ or $\mathbb{P}^2$. Then the rationality of Y implies that $|2K_Z + \Delta| = \varnothing$, where Δ is the divisor of the degeneracy of ω.*

Thus it follows from Lemma 4 and Theorem 5 that $\widetilde{X}$ and $\widetilde{V}$ are irrational for $m \geqslant 2$, and this proves Theorem 1 (see Remark 3). It should be mentioned that our method is easily generalized to many non-singular three-dimensional varieties fibred into del Pezzo surfaces of degrees 1, 2, and 3. For such varieties fibred into del Pezzo surfaces of degree 4, degeneracy is not required, since such varieties can always be represented as fibrings into conics. In the latter case, the question of rationality is studied in [9]. All three-dimensional varieties fibred into del Pezzo surfaces of degree at least 5 are rational [10].

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