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Extremal Metrics on del Pezzo Threefolds

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Abstract—We prove the existence of Kähler–Einstein metrics on a nonsingular section of the Grassmannian $\mathrm{Gr}(2,5)\subset\mathbb{P}^9$ by a linear subspace of codimension 3 and on the Fermat hypersurface of degree 6 in $\mathbb{P}(1,1,1,2,3)$. We also show that a global log canonical threshold of the Mukai–Umemura variety is equal to 1/2.

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1. INTRODUCTION

Let X be a variety[1] with at most log canonical singularities (see [20]), and let D be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on the variety X. Then the number

$$\mathrm{lct}(X,D)=\sup\big\{\lambda\in\mathbb{Q}\ \big|\ \text{the log pair } (X,\lambda D) \text{ is log canonical}\big\}\in\mathbb{Q}\cup\{+\infty\}$$

is called the log canonical threshold of the divisor D (see [8]).

Suppose that X is a Fano variety with at most log terminal singularities (see [19]).

Definition 1.1. The global log canonical threshold of the Fano variety X is the number

$$\mathrm{lct}(X)=\inf\big\{\mathrm{lct}(X,D)\ \big|\ D \text{ is an effective } \mathbb{Q}\text{-divisor on } X \text{ such that } D\sim_{\mathbb{Q}}-K_X\big\}\geq 0.$$

Recall that every Fano variety X is rationally connected (see [27]). Thus, the group $\mathrm{Pic}(X)$ is torsion free. Hence

$$\mathrm{lct}(X)=\sup\left\{\lambda\in\mathbb{Q}\ \middle|\ \begin{array}{l}\text{the log pair } (X,\lambda D) \text{ is log canonical}\\ \text{for every effective } \mathbb{Q}\text{-divisor } D\sim_{\mathbb{Q}}-K_X\end{array}\right\}.$$

Example 1.2. Let X be a smooth hypersurface in $\mathbb{P}^n$ of degree m, where $2\leq m\leq n$. Then

$$\mathrm{lct}(X)=\frac{1}{n+1-m}$$

if $m<n$ (see [5]). Thus, we have $\mathrm{lct}(\mathbb{P}^n)=1/(n+1)$. Suppose that $n=m$. By [5]

$$1\geq\mathrm{lct}(X)\geq\frac{n-1}{n}.$$

It follows from [4] and [12] that if X is general, then

$$\mathrm{lct}(X)\geq\begin{cases}1 & \text{if } n\geq 6,\\ 22/25 & \text{if } n=5,\\ 16/21 & \text{if } n=4,\\ 3/4 & \text{if } n=3.\end{cases}$$

One has $\mathrm{lct}(X)=1-1/n$ if X contains a cone of dimension $n-2$.

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[1] All varieties are assumed to be complex, algebraic, projective, and normal.

Example 1.3. Let X be a rational homogeneous space such that $-K_X \sim rD$ and

$$\mathrm{Pic}(X) = \mathbb{Z}[D],$$

where D is an ample divisor and $r \in \mathbb{Z}_{>0}$. Then $\mathrm{lct}(X) = 1/r$ (see [17]).

Example 1.4. Let X be a quasismooth hypersurface in $\mathbb{P}(1, a_1, \ldots, a_4)$ of degree $\sum_{i=1}^{4} a_i$ such that X has at most terminal singularities, where $a_1 \leq a_2 \leq a_3 \leq a_4$. Then

$$-K_X \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}(1,a_1,\ldots,a_4)}(1)\big|_X$$

and there are 95 possibilities for the quadruple (a_1, a_2, a_3, a_4) (see [18]). One has

$$1 \geq \mathrm{lct}(X) \geq \begin{cases} 16/21 & \text{if } \ a_1 = a_2 = a_3 = a_4 = 1, \\ 7/9 & \text{if } \ (a_1, a_2, a_3, a_4) = (1, 1, 1, 2), \\ 4/5 & \text{if } \ (a_1, a_2, a_3, a_4) = (1, 1, 2, 2), \\ 6/7 & \text{if } \ (a_1, a_2, a_3, a_4) = (1, 1, 2, 3), \\ 1 & \text{in the remaining cases} \end{cases}$$

if X is general (see [10, 12, 6]).

Example 1.5. Let X be smooth del Pezzo surface. It follows from [11] that

$$\mathrm{lct}(X) = \begin{cases} 1 & \text{if } \ K_X^2 = 1 \text{ and } |-K_X| \text{ contains no cuspidal curves}, \\ 5/6 & \text{if } \ K_X^2 = 1 \text{ and } |-K_X| \text{ contains a cuspidal curve}, \\ 5/6 & \text{if } \ K_X^2 = 2 \text{ and } |-K_X| \text{ contains no tacnodal curves}, \\ 3/4 & \text{if } \ K_X^2 = 2 \text{ and } |-K_X| \text{ contains a tacnodal curve}, \\ 3/4 & \text{if } \ X \text{ is a cubic in } \mathbb{P}^3 \text{ with no Eckardt points}, \\ 2/3 & \text{if } \ \text{either } X \text{ is a cubic in } \mathbb{P}^3 \text{ with an Eckardt point or } K_X^2 = 4, \\ 1/2 & \text{if } \ X \cong \mathbb{P}^1 \times \mathbb{P}^1 \text{ or } K_X^2 \in \{5, 6\}, \\ 1/3 & \text{in the remaining cases.} \end{cases}$$

Let $G \subset \mathrm{Aut}(X)$ be an arbitrary subgroup.

Definition 1.6. The global G-invariant log canonical threshold $\mathrm{lct}(X, G)$ of the Fano variety X is the number

$$\sup\left\{ \epsilon \in \mathbb{Q} \;\middle|\; \begin{array}{l} \text{the log pair } \left(X, \dfrac{\epsilon}{n}\mathcal{D}\right) \text{ has log canonical singularities for every} \\ G\text{-invariant linear system } \mathcal{D} \subset |-nK_X| \text{ and every } n \in \mathbb{Z}_{>0} \end{array} \right\}.$$

If the Fano variety X is smooth and G is compact, then it follows from [7, Appendix A] that

$$\mathrm{lct}(X, G) = \alpha_G(X),$$

where $\alpha_G(X)$ is the invariant introduced in [25]. We have $\mathrm{lct}(X) \leq \mathrm{lct}(X, G) \in \mathbb{R} \cup \{+\infty\}$.

Remark 1.7. Suppose that the subgroup G is finite. Then

$$\mathrm{lct}(X, G) = \sup\left\{ \lambda \in \mathbb{Q} \;\middle|\; \begin{array}{l} \text{the log pair } (X, \lambda D) \text{ is log canonical for every} \\ \text{effective } G\text{-invariant } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_X \end{array} \right\}.$$

Indeed, it is enough to show that if $\mathcal{D} \subset |-mK_X|$ is a G-invariant linear system such that the log pair $(X, c\mathcal{D})$ is not log canonical for some $c \in \mathbb{Q}_{\geq 0}$, then there is a G-invariant effective $\mathbb{Q}$-divisor $B \sim_{\mathbb{Q}} -mK_X$ such that the log pair (X, cB) is not log canonical. Put $k = |G|$. Suppose that the log pair $(X, c\mathcal{D})$ is not log canonical. Let $D \in \mathcal{D}$ be a general divisor. Then the log pair

$$\left(X, \frac{c}{k}\sum_{g\in G} g(D)\right)$$

is not log canonical either (see the proof of [20, Theorem 4.8]), which implies the required assertion.

Example 1.8. The simple group $\mathrm{PGL}(2, \mathrm{F}_7)$ is a group of automorphisms of the quartic

$$x^3y + y^3z + z^3x = 0 \subset \mathbb{P}^2 \cong \mathrm{Proj}(\mathbb{C}[x, y, z]),$$

which gives an embedding $\mathrm{PGL}(2, \mathrm{F}_7) \subset \mathrm{Aut}(\mathbb{P}^2)$. One has $\mathrm{lct}(\mathbb{P}^2, \mathrm{PGL}(2, \mathrm{F}_7)) = 4/3$ (see [24, 11]).

Example 1.9. Let X be the cubic surface in $\mathbb{P}^3$ given by the equation

$$x^3 + y^3 + z^3 + t^3 = 0 \subset \mathbb{P}^3 \cong \mathrm{Proj}(\mathbb{C}[x, y, z, t]),$$

and let $G = \mathrm{Aut}(X) \cong \mathbb{Z}_3^3 \rtimes \mathrm{S}_4$. Then $\mathrm{lct}(X, G) = 4$ by [11].

The following result is proved in [25, 23, 13] (cf. [7, Appendix A]).

Theorem 1.10. *Suppose that X has at most quotient singularities, the group G is compact, and the inequality*

$$\mathrm{lct}(X, G) > \frac{\dim(X)}{\dim(X) + 1}$$

holds. Then X admits an orbifold Kähler–Einstein metric.

Remark 1.11. Let $G \subset \mathrm{Aut}(X)$ be a reductive subgroup and $G' \subset G$ the maximal compact subgroup of G. Then a restriction to G' of any irreducible representation of G remains irreducible as a complex representation of G'. This implies that all linear systems on X that are invariant with respect to G are also invariant with respect to G' (the converse holds by obvious reasons). In particular, $\mathrm{lct}(X, G) = \mathrm{lct}(X, G')$.

Theorem 1.10 has many applications (see Examples 1.2, 1.4, and 1.9).

Example 1.12. Let X be one of the following smooth Fano varieties:

- a Fermat hypersurface in $\mathbb{P}^n$ of degree $n/2 \leq d \leq n$ (cf. Example 1.9);
- a smooth complete intersection of two quadrics in $\mathbb{P}^5$ that is given by

$$\sum_{i=0}^{5} x_i^2 = \sum_{i=0}^{5} \zeta^i x_i^2 = 0 \subseteq \mathbb{P}^5 \cong \mathrm{Proj}(\mathbb{C}[x_0, \ldots, x_5]),$$

  where ζ is a primitive sixth root of unity;
- a hypersurface in $\mathbb{P}(1^{n+1}, q)$ of degree pq that is given by the equation

$$w^p = \sum_{i=0}^{5} x_i^{pq} \subseteq \mathbb{P}(1^{n+1}, q) \cong \mathrm{Proj}(\mathbb{C}[x_0, \ldots, x_n, w])$$

  such that $pq - q \leq n$, where $\mathrm{wt}(x_0) = \ldots = \mathrm{wt}(x_n) = 1$, $\mathrm{wt}(w) = q \in \mathbb{Z}_{>0}$, and $p \in \mathbb{Z}_{>0}$.

Let $G = \mathrm{Aut}(X)$. Then G is finite and the inequality $\mathrm{lct}(X, G) \geq 1$ holds (see [25, 23]).

The numbers $\mathrm{lct}(X)$ and $\mathrm{lct}(X, G)$ also play an important role in birational geometry. For instance, the following result holds (see [11]).

Theorem 1.13. *Let X_i be a Fano variety, and let $G_i \subset \mathrm{Aut}(X_i)$ be a finite subgroup such that the variety X_i is G_i-birationally superrigid* (*see* [7]) *and the inequality $\mathrm{lct}(X_i, G_i) \geq 1$ holds, where $i = 1, \ldots, r$. Then the following assertions hold*:

- *there is no $(G_1 \times \ldots \times G_r)$-equivariant birational map $\rho\colon X_1 \times \ldots \times X_r \dashrightarrow \mathbb{P}^n$*;
- *every $(G_1 \times \ldots \times G_r)$-equivariant birational automorphism of $X_1 \times \ldots \times X_r$ is biregular*;
- *for every $(G_1 \times \ldots \times G_r)$-equivariant rational dominant map $\rho\colon X_1 \times \ldots \times X_r \dashrightarrow Y$ whose general fiber is a rationally connected variety, there a commutative diagram*

$$\begin{array}{ccc} X_1 \times \ldots \times X_r & \overset{\rho}{\dashrightarrow} & \\ {\scriptstyle\pi}\downarrow & & \searrow \\ X_{i_1} \times \ldots \times X_{i_k} & \underset{\xi}{\dashrightarrow} & Y \end{array}$$

  where ξ is a birational map, π is a natural projection, and $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, r\}$.

Varieties satisfying all hypotheses of Theorem 1.13 do exist.

Example 1.14. The simple group $\mathfrak{A}_6$ is a group of automorphisms of the sextic

$$10x^3y^3 + 9zx^5 + 9zy^5 + 27z^6 = 45x^2y^2z^2 + 135xyz^4 \subset \mathbb{P}^2 \cong \mathrm{Proj}(\mathbb{C}[x, y, z]),$$

which induces an embedding $\mathfrak{A}_6 \subset \mathrm{Aut}(\mathbb{P}^2)$. Then $\mathbb{P}^2$ is $\mathfrak{A}_6$-birationally superrigid and the equality $\mathrm{lct}(\mathbb{P}^2, \mathfrak{A}_6) = 2$ holds (see [24, 11]). Thus, there is an induced embedding $\mathfrak{A}_6 \times \mathfrak{A}_6 \cong \Omega \subset \mathrm{Bir}(\mathbb{P}^4)$ such that Ω is not conjugate to any subgroup in $\mathrm{Aut}(\mathbb{P}^4)$ by Theorem 1.13.

Let V be a smooth Fano threefold (see [19]) such that $-K_V \sim 2H$, where H is an ample Cartier divisor that is not divisible in $\mathrm{Pic}(V)$.

Remark 1.15. The variety V is called a del Pezzo variety, since a general element in the linear system $|H|$ is a smooth del Pezzo surface.

It is well-known that V is one of the following varieties:

- V_1, i.e., a hypersurface in $\mathbb{P}(1, 1, 1, 2, 3)$ of degree 6;
- V_2, i.e., a hypersurface in $\mathbb{P}(1, 1, 1, 1, 2)$ of degree 4;
- V_3, i.e., a cubic surface in $\mathbb{P}^3$;
- V_4, i.e., a complete intersection of two quadrics in $\mathbb{P}^5$;
- V_5, i.e., a section of the Grassmannian $\mathrm{Gr}(2, 5) \subset \mathbb{P}^9$ by a linear subspace of codimension 3 (all such sections are isomorphic);
- W, a divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1, 1)$;
- V_7, i.e., a blow-up of $\mathbb{P}^3$ at a point;
- the product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Remark 1.16. In [7] the values of the global log canonical thresholds of smooth del Pezzo threefolds were found:

$$\mathrm{lct}(V) = \begin{cases} 1/4 & \text{if } V \text{ is a blow-up of } \mathbb{P}^3 \text{ at a point,} \\ 1/2 & \text{in the remaining cases.} \end{cases}$$

Concerning Kähler–Einstein metrics on V, the following is known:

- V_7 does not admit a Kähler–Einstein metric (see [26]);
- V_4 admits a Kähler–Einstein metric (see [9], cf. Example 1.12);
- W and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ admit Kähler–Einstein metrics, since their automorphism groups are reductive and act on them transitively (see Theorem 1.10 and Remark 1.11);
- there are examples of varieties $V_2 \subset \mathbb{P}(1,1,1,1,2)$ and $V_3 \subset \mathbb{P}^4$ with large automorphism groups (see Example 1.12) that admit Kähler–Einstein metrics.

The question of existence of Kähler–Einstein metrics on the varieties V_1 and V_5 has not been studied in the literature yet (cf. a remark before [9, Theorem 3.2]).

The main purpose of this paper is to prove the following assertions.

Theorem 1.17. *Let G be a maximal compact subgroup in* $\mathrm{Aut}(V_5)$. *Then*

$$\mathrm{lct}(V_5, G) = \mathrm{lct}(V_5, \mathrm{Aut}(V_5)) = 5/6.$$

Theorem 1.18. *Let V_1 be a hypersurface in* $\mathbb{P}(1,1,1,2,3)$ *given by the equation*

$$w^2 = t^3 + x^6 + y^6 + z^6 \subset \mathbb{P}(1,1,1,2,3) \cong \mathrm{Proj}(\mathbb{C}[x,y,z,t,w]),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = \mathrm{wt}(z) = 1$, $\mathrm{wt}(t) = 2$, *and* $\mathrm{wt}(w) = 3$. *Then* $\mathrm{lct}(V_1, \mathrm{Aut}(V_1)) \geq 1$.

Note that the latter results combined with Theorem 1.10 imply the existence of Kähler–Einstein metrics on the variety V_5 and on the Fermat hypersurface of degree 6 in $\mathbb{P}(1,1,1,2,3)$.

Remark 1.19. Let V_1 be a smooth hypersurface in $\mathbb{P}(1,1,1,2,3)$ of degree 6. Assume that $\mathrm{lct}(V_1, G) \geq 1$, where G is a subgroup in $\mathrm{Aut}(V_1)$. Then

- the linear system $|H|$ does not contain G-invariant surfaces,
- the linear system $|H|$ does not contain G-invariant pencils (cf. the proof of [24, Theorem 1.2]),
- the variety V_1 is G-birationally superrigid (see [1, 2]).

Remark 1.20. The methods we use to prove Theorem 3.2 (see below) are similar to those of [23]. Nevertheless, some statements of [23] (say, [23, Corollary 4.2] or a standard method for excluding zero-dimensional components of a subscheme of log canonical singularities) cannot be directly applied in our case, since the group $\mathrm{Aut}(V_1)$ *never* acts on V_1 without fixed points.

The structure of the paper is as follows. Section 2 contains some auxiliary statements. In Section 3 we prove Theorem 1.18. In Section 4 we prove Theorem 1.17. The methods of Section 4 can be applied without significant changes to one more interesting Fano threefold, the so-called Mukai–Umemura variety (see [14] and Remark 5.2). To complete the picture, in Section 5 we calculate the global log canonical threshold of the Mukai–Umemura variety without any group action.

2. PRELIMINARIES

Let X be a variety with log terminal singularities. Let us consider a $\mathbb{Q}$-divisor $B_X = \sum_{i=1}^{r} a_i B_i$, where B_i is a prime Weil divisor on the variety X and a_i is an arbitrary nonnegative rational number. Suppose that B_X is a $\mathbb{Q}$-Cartier divisor such that $B_i \neq B_j$ for $i \neq j$.

Let $\pi\colon \overline{X} \to X$ be a birational morphism such that $\overline{X}$ is smooth. Put

$$B_{\overline{X}} = \sum_{i=1}^{r} a_i \overline{B}_i,$$

where $\overline{B}_i$ is a proper transform of the divisor B_i on the variety $\overline{X}$. Then

$$K_{\overline{X}} + B_{\overline{X}} \sim_{\mathbb{Q}} \pi^*(K_X + B_X) + \sum_{i=1}^{n} c_i E_i,$$

where $c_i \in \mathbb{Q}$ and E_i is an exceptional divisor of the morphism π. Suppose that

$$\left(\bigcup_{i=1}^{r} \overline{B}_i\right) \cup \left(\bigcup_{i=1}^{n} E_i\right)$$

is a divisor with simple normal crossings. Put

$$B^{\overline{X}} = B_{\overline{X}} - \sum_{i=1}^{n} c_i E_i.$$

Definition 2.1. The singularities of (X, B_X) are log canonical (respectively, log terminal) if

- the inequality $a_i \leq 1$ holds (respectively, the inequality $a_i < 1$ holds),
- the inequality $c_j \geq -1$ holds (respectively, the inequality $c_j > -1$ holds)

for every $i = 1, \ldots, r$ and $j = 1, \ldots, n$.

One can show that Definition 2.1 does not depend on the choice of the morphism π. Put

$$\mathrm{LCS}(X, B_X) = \left(\bigcup_{a_i \geq 1} B_i\right) \cup \left(\bigcup_{c_i \leq -1} \pi(E_i)\right) \subsetneq X$$

and let us call $\mathrm{LCS}(X, B_X)$ the locus of log canonical singularities of the log pair (X, B_X).

Definition 2.2. A proper irreducible subvariety $Y \subsetneq X$ is said to be a center of log canonical singularities of the log pair (X, B_X) if one of the following conditions is satisfied:

- either the inequality $a_i \geq 1$ holds and $Y = B_i$,
- or the inequality $c_i \leq -1$ holds and $Y = \pi(E_i)$

for some choice of the birational morphism $\pi\colon \overline{X} \to X$.

Let $\mathbb{LCS}(X, B_X)$ be the set of all centers of log canonical singularities of (X, B_X). Then

$$Y \in \mathbb{LCS}(X, B_X) \qquad \Rightarrow \qquad Y \subseteq \mathrm{LCS}(X, B_X)$$

and $\mathbb{LCS}(X, B_X) = \varnothing \;\Leftrightarrow\; \mathrm{LCS}(X, B_X) = \varnothing \;\Leftrightarrow\;$ the log pair (X, B_X) is log terminal.

Remark 2.3. We can use similar constructions and notation for any log pair $(X, \lambda\mathcal{D})$, where $\mathcal{D}$ is a linear system and λ is a nonnegative rational number.

The set $\mathrm{LCS}(X, B_X)$ can be naturally equipped with a scheme structure (see [23, 8]). Put

$$\mathcal{I}(X, B_X) = \pi_*\left(\sum_{i=1}^{n} \lceil c_i \rceil E_i - \sum_{i=1}^{r} \lfloor a_i \rfloor \overline{B}_i\right),$$

and let $\mathcal{L}(X, B_X)$ be a subscheme that corresponds to the ideal sheaf $\mathcal{I}(X, B_X)$.

Remark 2.4. The scheme $\mathcal{L}(X, B_X)$ is usually called the subscheme of log canonical singularities of the log pair (X, B_X), and the ideal sheaf $\mathcal{I}(X, B_X)$ is usually called the multiplier ideal sheaf of the log pair (X, B_X).

It follows from the construction of the subscheme $\mathcal{L}(X, B_X)$ that

$$\mathrm{Supp}(\mathcal{L}(X, B_X)) = \mathrm{LCS}(X, B_X) \subset X.$$

The following result is known as the Shokurov vanishing theorem (see [8]) or the Nadel vanishing theorem (see [21, Theorem 9.4.8]).

Theorem 2.5. *Let H be a nef and big $\mathbb{Q}$-divisor on X such that $K_X + B_X + H \sim_{\mathbb{Q}} D$ for some Cartier divisor D on the variety X. Then for every $i \geq 1$*

$$H^i\big(X, \mathcal{I}(X, B_X) \otimes D\big) = 0.$$

The following result is known as the Shokurov connectedness theorem.

Theorem 2.6. *Suppose that $-(K_X + B_X)$ is nef and big. Then* $\mathrm{LCS}(X, B_X)$ *is connected.*

Proof. It follows from Theorem 2.5 that the sequence

$$\mathbb{C} = H^0(\mathcal{O}_X) \to H^0\big(\mathcal{O}_{\mathcal{L}(X,B_X)}\big) \to H^1(\mathcal{I}(X, B_X)) = 0$$

is exact. Thus, the locus

$$\mathrm{LCS}(X, B_X) = \mathrm{Supp}(\mathcal{L}(X, B_X))$$

is connected. $\square$

One can generalize Theorem 2.6 in the following way (see [8, Lemma 5.7]).

Theorem 2.7. *Let $\psi\colon X \to Z$ be a morphism. Then the set*

$$\mathrm{LCS}\big(\overline{X}, B^{\overline{X}}\big)$$

is connected in a neighborhood of every fiber of the morphism $\psi \circ \pi\colon X \to Z$ in the case when

- *the morphism ψ is surjective and has connected fibers,*
- *the divisor $-(K_X + B_X)$ is nef and big with respect to ψ.*

The following result is a corollary of Theorem 2.5 (see [23, Theorem 4.1]).

Lemma 2.8. *Suppose that $-(K_X + B_X)$ is nef and big and* $\dim(\mathrm{LCS}(X, B_X)) = 1$. *Then*

- *the locus* $\mathrm{LCS}(X, B_X)$ *is a connected union of smooth rational curves,*
- *the locus* $\mathrm{LCS}(X, B_X)$ *does not contain a cycle of smooth rational curves,*
- *any intersecting irreducible components of the locus* $\mathrm{LCS}(X, B_X)$ *meet transversally.*

Let P be a point in X. Let us consider an effective divisor

$$\Delta = \sum_{i=1}^{r} \varepsilon_i B_i \sim_{\mathbb{Q}} B_X,$$

where ε_i is a nonnegative rational number. Suppose that

- the divisor Δ is a $\mathbb{Q}$-Cartier divisor,
- the equivalence $\Delta \sim_{\mathbb{Q}} B_X$ holds,
- the log pair (X, Δ) is log canonical at the point $P \in X$.

Remark 2.9. Suppose that (X, B_X) is not log canonical at the point $P \in X$. Put

$$\alpha = \min\left\{\frac{a_i}{\varepsilon_i} \;\middle|\; \varepsilon_i \neq 0\right\}.$$

Note that α is well defined, because there is $\varepsilon_i \neq 0$. Then $\alpha < 1$, the log pair

$$\left(X, \sum_{i=1}^{r} \frac{a_i - \alpha\varepsilon_i}{1 - \alpha} B_i\right)$$

is not log canonical at the point $P \in X$, the equivalence

$$\sum_{i=1}^{r} \frac{a_i - \alpha\varepsilon_i}{1-\alpha} B_i \sim_{\mathbb{Q}} B_X \sim_{\mathbb{Q}} \Delta$$

holds, and at least one irreducible component of the divisor $\mathrm{Supp}(\Delta)$ is not contained in

$$\mathrm{Supp}\left(\sum_{i=1}^{r} \frac{a_i - \alpha\varepsilon_i}{1-\alpha} B_i\right).$$

The following result is an easy corollary of Remark 2.9.

Lemma 2.10. *Let X be a smooth Fano variety such that $\mathrm{Pic}(X) = \mathbb{Z}[H]$ for some divisor $H \in \mathrm{Pic}(X)$, and let $G \subset \mathrm{Aut}(X)$ be a subgroup. Let λ be a rational number such that*

- $\mathrm{lct}(X, D) \geq \lambda/n$ *for every G-invariant divisor $D \in |nH|$,*
- $\mathrm{lct}(X, \mathcal{D}) \geq \lambda/n$ *for every G-invariant linear subsystem $\mathcal{D} \subset |nH|$ that has no fixed components.*

Then

$$\mathrm{lct}(X, G) \geq \lambda.$$

Proof. Suppose that $\mathrm{lct}(X, G) < \lambda$. Then there are a natural number n and a G-invariant linear subsystem $\mathcal{D} \subset |nH|$ such that the log pair

$$\left(X, \frac{\lambda}{n}\mathcal{D}\right)$$

is not log canonical. Put $\mathcal{D} = F + \mathcal{M}$, where F is a fixed part of the linear system $\mathcal{D}$ and $\mathcal{M}$ is a G-invariant linear system that has no fixed components.

Let $M_1, \ldots, M_r$ be general divisors in $\mathcal{M}$, where $r \gg 0$. Then

$$\left(X, \frac{\lambda}{n}\left(F + \frac{\sum_{i=1}^{r} M_i}{r}\right)\right)$$

is not log canonical by [20, Theorem 4.8].

Since $\mathrm{Pic}(X) = \mathbb{Z}[H]$, we have $F \sim n_1 H$ and $\mathcal{M} \sim n_2 H$ for some $n_1, n_2 \in \mathbb{Z}_{>0}$ such that $n_1 + n_2 = n$. By Remark 2.9, we see that the log pair

$$\left(X, \frac{\lambda}{n_2 r}\sum_{i=1}^{r} M_i\right)$$

is not log canonical, because F is G-invariant. Then the log pair

$$\left(X, \frac{\lambda}{n_2}\mathcal{M}\right)$$

is not log canonical by [20, Theorem 4.8], which is a contradiction. □

The following simple calculation will be useful in Section 4.

Lemma 2.11. *Let $\dim(X) = 3$; let $C \subset X$ be an irreducible reduced curve and $P \in C$ a point such that*

$$\mathrm{Sing}(C) \not\ni P \notin \mathrm{Sing}(X).$$

Let $L \subset X$ be a curve such that $P \notin \mathrm{Sing}(L)$ and D a $\mathbb{Q}$-divisor on X such that $C \subset \mathrm{Supp}(D) \not\supset L$. Assume that L and C are tangent at P. Then

$$\mathrm{mult}_P(D \cdot L) \geq 2\,\mathrm{mult}_C(D).$$

Proof. Let $\pi\colon \widetilde{X} \to X$ be a blow-up of the point P, and let E be an exceptional divisor of π. Denote by $\widetilde{L}$, $\widetilde{C}$, and $\widetilde{D}$ the proper transforms on $\widetilde{X}$ of the curves L and C and the divisor D, respectively. Then the intersection

$$\widetilde{L} \cap \mathrm{Supp}(\widetilde{D})$$

contains some point $\widetilde{P} \in E$, since L and C are tangent at P. Hence

$$\begin{aligned}\mathrm{mult}_P(D \cdot L) = \mathrm{mult}_P(D) + \mathrm{mult}_{\widetilde{P}}(\widetilde{D} \cdot \widetilde{L}) \geq \mathrm{mult}_C(D) + \mathrm{mult}_{\widetilde{P}}(\widetilde{D})\\ \geq \mathrm{mult}_C(D) + \mathrm{mult}_{\widetilde{C}}(D) = 2\,\mathrm{mult}_C(D). \quad \square\end{aligned}$$

3. VERONESE DOUBLE CONE

We will use the following notation: if $\mathcal{D}$ is a (nonempty) linear system on the variety X, then $\varphi_{\mathcal{D}}$ denotes the rational map defined by $\mathcal{D}$.

Let V be a smooth Fano threefold such that $(-K_V)^3 = 8$ and

$$\mathrm{Pic}(V) = \mathbb{Z}[H]$$

for some $H \in \mathrm{Pic}(V)$. Then V is a hypersurface in $\mathbb{P}(1,1,1,2,3)$ of degree 6.

The linear system $|H|$ has the only base point $O \in V$ and defines a rational map

$$\varphi_{|H|}\colon V \dashrightarrow \mathbb{P}^2$$

with irreducible fibers; a general fiber of $\varphi_{|H|}$ is an elliptic curve.

Remark 3.1. We will refer to the subvarieties of V that are swept out by the fibers of $\varphi_{|H|}$ as *vertical* subvarieties.

Let $G \subset \mathrm{Aut}(V)$ be a subgroup. Note that G is finite, its action on V extends to $\mathbb{P}(1,1,1,2,3)$, and G naturally acts on $\mathbb{P}(|H|) \cong \mathbb{P}^2$. Moreover, the following conditions are equivalent:

- G has no fixed points on $\mathbb{P}(|H|) \cong \mathbb{P}^2$;
- G has no invariant lines on $\mathbb{P}(|H|) \cong \mathbb{P}^2$;
- $|H|$ contains no G-invariant surfaces;
- $|H|$ contains no G-invariant pencils (cf. the proof of [24, Theorem 1.2]);
- V is G-birationally superrigid (see [1, 2]).

Let $\mathcal{B}$ be a linear subsystem in $|-K_X|$ generated by divisors of the form

$$\lambda_0 x^2 + \lambda_1 y^2 + \lambda_2 z^2 + \lambda_3 xy + \lambda_4 xz + \lambda_5 yz = 0,$$

where x, y, and z are coordinates of weight 1 on $\mathbb{P}(1,1,1,2,3)$. The statement of Theorem 1.18 is implied by the following result.

Theorem 3.2. *Suppose that the linear system $\mathcal{B}$ contains no G-invariant divisors. Then* $\mathrm{lct}(V,G) \geq 1$.

Proof. Assume that $\mathrm{lct}(V,G) < 1$. Then the linear system $|H|$ does not contain G-invariant divisors, but there exists an effective G-invariant $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_V$ such that

$$\mathbb{LCS}(V, \lambda D) \neq \varnothing$$

for some $1 > \lambda \in \mathbb{Q}$. The set $\mathrm{LCS}(V, \lambda D)$ is G-invariant.

Lemma 3.3. *The set $\mathbb{LCS}(V, \lambda D)$ does not contain divisors.*

Proof. Easy. $\square$

Lemma 3.4. *The set* $\mathbb{LCS}(V, \lambda D)$ *does not contain curves.*

Proof. Let $C \subset \mathrm{LCS}(V, \lambda D)$ be a G-invariant curve. Then for any point $P \in C$ one has $\mathrm{mult}_P D > 1/\lambda$.

Lemma 2.8 implies that C has a nonvertical component. Then $\deg(\phi_{|H|}(C)) \geq 3$, since the linear system $\mathcal{B}$ does not contain G-invariant surfaces.

Let S be a general surface in $|H|$. Put

$$S \cap C = \{P_1, \ldots, P_s\},$$

where $P_1, \ldots, P_s$ are distinct points. Then $s \geq 3$. Moreover, one has $s > 3$ if $O \in C$. So it is easy to see that one may assume the following:

- $O \notin \{P_1, P_2, P_3\}$,
- $\phi_{|H|}(P_1)$, $\phi_{|H|}(P_2)$, and $\phi_{|H|}(P_3)$ are distinct points.

The surface S is a del Pezzo surface. One has

$$D|_S \sim_{\mathbb{Q}} -2K_S$$

and $-K_S^2 = 1$. The log pair $(S, \lambda D|_S)$ is not log terminal at P_1, P_2, and P_3. Theorem 2.5 implies that the sequence

$$\mathbb{C}^2 = H^0(\mathcal{O}_S(-K_S)) \to H^0\big(\mathcal{O}_{\mathcal{L}(S,\lambda D|_S)}\big) \to H^1\big(\mathcal{I}(S, \lambda D|_S) \otimes \mathcal{O}_S(-K_S)\big) = 0$$

is exact, since the scheme $\mathcal{L}(S, \lambda D|_S)$ is zero-dimensional by Lemma 3.3. In particular, the support of the subscheme $\mathcal{L}(S, \lambda D|_S)$ contains at most two points, which is a contradiction. $\square$

So $\mathrm{LCS}(V, \lambda D)$ is zero-dimensional. Theorem 2.6 implies that $\mathrm{LCS}(V, \lambda D)$ consists of a single point $P \in V$.

Lemma 3.5. $P = O$.

Proof. Assume that $P \neq O$. Then the G-orbit of P is nontrivial, since so is the G-orbit of $\varphi_{|H|}(P)$, which is a contradiction. $\square$

Let $\pi\colon \overline{V} \to V$ be a blow-up of the point O with an exceptional divisor E; let $\overline{D}$ be a proper transform of D on $\overline{V}$. Then the log pair

$$\big(\overline{V}, \lambda\overline{D} + (\lambda\,\mathrm{mult}_O(D) - 2)E\big)$$

is not log canonical in the neighborhood of E. On the other hand, one has $\mathrm{mult}_O(D) \leq 2$, since otherwise $\mathrm{Supp}(\overline{D})$ would contain all fibers of the elliptic fibration $\varphi_{|\pi^*(H)-E|}$. Hence the set

$$\mathrm{LCS}\big(\overline{V}, \lambda\overline{D} + (\lambda\,\mathrm{mult}_O(D) - 2)E\big)$$

contains some G-invariant subvariety $Z \subsetneq E$ and is contained in $E \cong \mathbb{P}^2$.

Lemma 3.6. *One has* $\dim(Z) = 0$.

Proof. Suppose that $\dim(Z) = 1$. Let L be a general line in $E \cong \mathbb{P}^2$. Then

$$2 \geq \mathrm{mult}_O(D) = L \cdot \overline{D} \geq \deg(Z)\,\mathrm{mult}_Z(\overline{D}) > \frac{\deg(Z)}{\lambda} > \deg(Z).$$

Hence Z contains a G-invariant line. But $|H|$ does not contain G-invariant surfaces, which gives a contradiction. $\square$

So we see that the G-invariant set $\mathrm{LCS}(\overline{V}, \lambda\overline{D})$ consists of a finite number of points. By Theorem 2.7 the set $\mathrm{LCS}(Y, \lambda\overline{D})$ consists of a single point, since the divisor $-(K_Y + \lambda\overline{D})$ is π-ample. But G acts on E without fixed points, since $|H|$ contains no G-invariant pencils. The contradiction concludes the proof of Theorem 3.2. $\square$

4. QUINTIC DEL PEZZO THREEFOLD

Let V_5 be a smooth Fano variety such that

$$\mathrm{Pic}(V_5) = \mathbb{Z}[H]$$

and $H^3 = 5$. One has $-K_{V_5} \sim 2H$ (see, for example, [19]). Let $\mathcal{W} \cong \mathbb{C}^3$ be a vector space endowed with a nondegenerate quadratic form q. Then the variety V_5 is isomorphic to the variety of triples of pairwise orthogonal (with respect to q) lines in $\mathcal{W}$ (see [19]). In particular, there is a natural action of the group $\mathrm{SO}_3(\mathbb{C})$ (or $\mathrm{SL}_2(\mathbb{C})$) on the variety V_5.

Remark 4.1. One can show that $\mathrm{Aut}(V_5) = \mathrm{PSL}_2(\mathbb{C})$. By Remark 1.11, to prove Theorem 1.17 it suffices to check that $\mathrm{lct}(V_5, \mathrm{PSL}_2(\mathbb{C})) = 5/6$.

The variety V_5 has a natural $\mathrm{PSL}_2(\mathbb{C})$-equivariant stratification:

$$V_5 = U \cup \Delta \cup C,$$

where U is an open orbit that consists of triples of pairwise distinct lines, Δ is a two-dimensional orbit that consists of the triples (l_1, l_1, l_2), where $q(l_1, l_1) = 0$ and $q(l_1, l_2) = 0$, and C is a one-dimensional orbit that consists of the triples (l, l, l), where $q(l, l) = 0$.

The linear system $|H|$ defines an embedding $V_5 \subset \mathbb{P}^6$. Under this embedding the curve C is a rational normal curve of degree 6, and Δ is swept out by the lines that are tangent to C.

Lemma 4.2. *One has* $\mathrm{lct}(V_5, \Delta) = 5/6$.

Proof. The surface Δ is smooth outside C and has a singularity along C that is locally isomorphic to $T \times \mathbb{A}^1$, where T is a germ of a cuspidal curve. □

In particular, $\mathrm{lct}(V_5, \mathrm{PSL}_2(\mathbb{C})) \leq 5/6$.

Lemma 4.3. *Let* $\mathcal{D} \subset |nH|$ *be a* $\mathrm{PSL}_2(\mathbb{C})$*-invariant linear system on* V_5 *such that* $\Delta \not\subset \mathrm{Bs}(\mathcal{D})$. *Then* $\mathrm{lct}(X, \mathcal{D}) \geq 1/n$.

Proof. Suppose that $\mathrm{lct}(X, \mathcal{D}) < 1/n$. Then there exists a $\mathrm{PSL}_2(\mathbb{C})$-invariant subvariety $Z \subsetneq X$ such that

$$\mathrm{mult}_Z(D) > n,$$

where D is a general divisor in $\mathcal{D}$. Since $\Delta \not\subset \mathrm{Bs}(\mathcal{D})$, the subvariety Z is the curve C. Let P be a general point of C and L be the tangent line to C at P. Then $L \not\subset \mathrm{Supp}(D)$. By Lemma 2.11 one has

$$2n = D \cdot L \geq \mathrm{mult}_P(D \cdot L) > 2n,$$

which is a contradiction. □

Lemmas 2.10, 4.2, and 4.3 imply that $\mathrm{lct}(V_5, \mathrm{PSL}_2(\mathbb{C})) \geq 5/6$, and hence $\mathrm{lct}(V_5, \mathrm{PSL}_2(\mathbb{C})) = 5/6$.

5. THE MUKAI–UMEMURA THREEFOLD

Let X be a smooth Fano threefold such that

$$\mathrm{Pic}(X) = \mathbb{Z}[-K_X],$$

the equality $-K_X^3 = 22$ holds, and $\mathrm{Aut}(X) \cong \mathrm{PSL}(2, \mathbb{C})$. It is well known that the variety having such properties is unique (see [22, 3]).

Proposition 5.1. *The equality* $\mathrm{lct}(X) = 1/2$ *holds.*

Proof. Let $U \subset \mathbb{C}[x, y]$ be a subspace of forms of degree 12. Consider $U \cong \mathbb{C}^{13}$ as the affine part of $\mathbb{P}(U \oplus \mathbb{C}) \cong \mathbb{P}^{13}$, and let us identify $\mathbb{P}(U)$ with the hyperplane at infinity.

The natural action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}[x,y]$ induces an action on $\mathbb{P}(U\oplus\mathbb{C})$. Put

$$\varphi = xy(x^{10}-11x^5y^5-y^{10})\in U$$

and consider the closure $\overline{\mathrm{SL}(2,\mathbb{C})\cdot[\varphi+1]}\subset\mathbb{P}(U\oplus\mathbb{C})$. It follows from [22] that

$$X\cong\overline{\mathrm{SL}(2,\mathbb{C})\cdot[\varphi+1]},$$

and the natural embedding $X\subset\mathbb{P}(U\oplus\mathbb{C})\cong\mathbb{P}^{13}$ is induced by $|-K_X|$.

It is well known (see [19, Theorem 5.2.13]) that the action of $\mathrm{SL}(2,\mathbb{C})$ on X has the following orbits:

- the three-dimensional orbit $\Sigma_3=\mathrm{SL}(2,\mathbb{C})\cdot[\varphi+1]$;
- the two-dimensional orbit $\Sigma_2=\mathrm{SL}(2,\mathbb{C})\cdot[xy^{11}]$;
- the one-dimensional orbit $\Sigma_1=\mathrm{SL}(2,\mathbb{C})\cdot[y^{12}]$.

The orbit Σ_3 is open. The orbit $\Sigma_1\cong\mathbb{P}^1$ is closed. One has $\overline{\Sigma}_2=\Sigma_1\cup\Sigma_2$, and

$$X\cap\mathbb{P}(U)=\Sigma_1\cup\Sigma_2.$$

Put $R=X\cap\mathbb{P}(U)$. It follows from [22] that

- the surface R is swept out by lines on $X\subset\mathbb{P}^{13}$,
- the surface R contains all lines on $X\subset\mathbb{P}^{13}$,
- for any lines $L_1\subset R\supset L_2$ such that $L_1\neq L_2$, one has $L_1\cap L_2=\varnothing$,
- the surface R is singular along the orbit $\Sigma_1\cong\mathbb{P}^1$,
- the normalization of the surface R is isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$,
- for every point $P\in\Sigma_1$, the surface R is locally isomorphic to

  $$x^2=y^3\subset\mathbb{C}^3\cong\mathrm{Spec}(\mathbb{C}[x,y,z]),$$

  which implies that $\mathrm{lct}(X,R)=5/6$.

The structure of the surface R can be described as follows. We see that

$$\Sigma_2=\big\{[(ax+by)(cx+dy)^{11}]\;\big|\;ad-bc=1\big\}\subset\mathbb{P}(U),$$

which implies that there is a birational morphism $\nu\colon\mathbb{P}^1\times\mathbb{P}^1\to R$ that is defined by

$$\nu\colon [a:b]\times[c:d]\mapsto[(ax+by)(cx+dy)^{11}]\in R,$$

which is a normalization of the surface R.

Let V_5 be a smooth Fano threefold such that

$$-K_{V_5}\sim 2H$$

and $H^3=5$, where H is a Cartier divisor on V_5. Then $|H|$ induces an embedding $V_5\subset\mathbb{P}^6$ (see Section 4).

Let $L\cong\mathbb{P}^1$ be a line on X. Then

$$\mathcal{N}_{L/X}\cong\mathcal{O}_{\mathbb{P}^1}(-2)\oplus\mathcal{O}_{\mathbb{P}^1}(1).$$

Let $\alpha_L \colon U_L \to X$ be a blow-up of the line L, and let E_L be the exceptional divisor of α_L. Then it follows from Theorem 4.3.3 in [19] that there is a commutative diagram

$$\begin{array}{ccc} U_L & \overset{\rho_L}{\dashrightarrow} & W_L \\ {\scriptstyle \alpha_L}\big\downarrow & & \big\downarrow{\scriptstyle \beta_L} \\ X & \underset{\psi_L}{\dashrightarrow} & V_5 \end{array}$$

where ρ_L is a flop in the exceptional section of $E \cong \mathbb{F}_3$, the morphism β_L contracts a surface $D_L \subset W_L$ to a smooth rational curve of degree 5, and ψ_L is a double projection from the line L.

Let $\overline{D}_L \subset X$ be the proper transform of the surface D_L. Then $\mathrm{mult}_L(\overline{D}_L) = 3$ and $\overline{D}_L \sim -K_X$. It follows from [15] that $X \setminus \overline{D}_L \cong \mathbb{C}^3$.

It follows from [16] that there is an open subset $\breve{D}_L \subset \overline{D}_L$ that is given by

$$\mu_0 x^4 + (\mu_1 yz + \mu_2 z^3)x^3 + (\mu_3 y^3 + \mu_4 y^2 z^2 + \mu_5 yz^4)x^2 + (\mu_6 y^4 z + \mu_7 y^3 z^3)x + \mu_8 y^6 + \mu_9 y^5 z^2 = 0$$

in $\mathbb{C}^3 \cong \mathrm{Spec}(\mathbb{C}[x,y,z])$, where the point $L \cap \Sigma_1 \in \breve{D}_L$ is given by the equations $x = y = z = 0$ and

$$\begin{array}{llll} \mu_0 = -2^8 \times 5^2, & \mu_1 = 2^9 \times 3^3 \times 5, & \mu_2 = -2^6 \times 3^4 \times 5, & \mu_3 = -2^8 \times 3^3 \times 7, \\ \mu_4 = -2^4 \times 3^4 \times 127, & \mu_5 = 2^9 \times 3^5, & \mu_6 = 2^2 \times 3^6 \times 89, & \mu_7 = -2^8 \times 3^6, \\ & \mu_8 = -3^6 \times 5^3, & \mu_9 = 2^5 \times 3^7. & \end{array}$$

Put $O_L = \Sigma_1 \cap L$. Then $\mathrm{mult}_{O_L}(\overline{D}_L) = 4$, and it follows from [20, Proposition 8.14] that

$$\mathrm{LCS}\Big(X, \frac{1}{2}\overline{D}_L\Big) = O_L$$

and $\mathrm{lct}(X, \overline{D}_L) = 1/2$. Thus, we see that $\mathrm{lct}(X) \leq 1/2$.

Suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor

$$D \sim_{\mathbb{Q}} -K_X$$

such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$. By Remark 2.9, we may assume that $R \not\subset \mathrm{Supp}(D)$, because $\mathrm{lct}(X, R) = 5/6$.

Let C be a line in X such that $C \not\subset \mathrm{Supp}(D)$. Then

$$1 = D \cdot C \geq \mathrm{mult}_{O_C}(D)\,\mathrm{mult}_{O_C}(C) = \mathrm{mult}_{O_C}(D),$$

which implies that $O_C \notin \mathrm{LCS}(X, \lambda D)$. In particular, we see that $\Sigma_1 \notin \mathrm{LCS}(X, \lambda D)$.

Let Γ be an irreducible curve in $\mathrm{Supp}(D)$ such that $O_C \in \Gamma$. Then

$$\mathrm{mult}_\Gamma\Big(\frac{1}{2}\overline{D}_C + \lambda D\Big) = \frac{\mathrm{mult}_\Gamma(\overline{D}_C)}{2} + \lambda\,\mathrm{mult}_\Gamma(D) \leq \frac{\mathrm{mult}_\Gamma(\overline{D}_C)}{2} + \lambda\,\mathrm{mult}_{O_C}(D) < 1,$$

because $\lambda < 1/2$ and $\mathrm{Sing}(\overline{D}_C) = C$, because $\overline{D}_C \neq R$. Thus, we see that

$$\Gamma \not\subseteq \mathrm{LCS}\Big(X, \frac{1}{2}\overline{D}_C + \lambda D\Big) \supseteq \mathrm{LCS}(X, \lambda D) \cup O_C,$$

which is impossible by Theorem 2.6, because $O_C \notin \mathrm{LCS}(X, \lambda D)$ and $\lambda < 1/2$. $\square$

Remark 5.2. It follows from [14] that

$$\mathrm{lct}(X, \mathrm{SO}_3(\mathbb{R})) = \frac{5}{6},$$

which implies, in particular, that X has a Kähler–Einstein metric. This equality can be obtained by arguing as in the proof of Theorem 1.17 (the only difference is that we do not need to use Lemma 2.11 here).

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