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COMPLEMENTS ON SURFACES

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The main result is a boundedness theorem for n-complements on algebraic surfaces. In addition, this theorem is used in a classification of log Del Pezzo surfaces and birational contractions for threefolds.

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1. Introduction

An introduction to complements can be found in Sec. 5 in [23] (also see the end of this section).

1.1. Example. Let $S = \mathbb{P}_E(F_2)$ be a ruled surface over a nonsingular curve E of genus 1, which corresponds to a nonsplitting vector bundle F_2/E of rank 2 (Theorem 5(i) in [3]). The ruling is denoted by $f\colon S \to E$. It has a single section $\mathbb{P}_E(F_1)$ in its linear and even numerical equivalence class (by 2.9.1 in [22], cf. arguments below). We identify the section with E.

We note that $E|_E \sim \det F_2 \sim 0$, where $\sim$ denotes *linear* equivalence (cf. Proposition 2.9 in [10]). On the other hand, $(K_S + E)|_E \sim K_E \sim 0$ according to the adjunction formula. Therefore, $(K_S + 2E)|_E \sim 0$ (Lemma 2.10 in [10]). But $K_S + 2E \equiv 0/E$, where $\equiv$ denotes *numerical* equivalence. Therefore, $K_S + 2E = f^*((K_S + 2E)|_E) \sim 0$ (2.9.1 in [22]). Equivalently, we can choose $-K_S = 2E$ as an anticanonical divisor. In particular, $-K_S$ is nef because $E^2 = 0$.

The latter also implies that the cone $\overline{\mathrm{NE}}(S)$ is two-dimensional and has two extremal rays:

- the first ray R_1 is generated by a fiber F of the ruling f and
- the second ray R_2 is generated by the section E.

We contend that E is the only curve in R_2.

Indeed, if there is a curve $C \neq E$ in R_2, then $C \equiv mE$, where $m = (C.F)$ is the degree of C as a multisection of f. We can assume that m is minimal. Then, as above, $C \sim mE$, which induces another fibering with the projection $g\colon S \to \mathbb{P}^1$. All fibers of g are nonsingular curves of genus 1. However, some of them may be multiple. According to our assumptions, any multiple fiber G_i has multiplicity $m_i = m \geq 2$, and we have at least two of them. Therefore, if $G_1 = E$, then another multiple fiber $G_2 \equiv E$ gives another section in R_2. This is impossible.

Therefore, in general, a nef anticanonical divisor $-K$ may not be semiample, and we may have no complements when $-K$ is just nef. The same can occur in the log case.

1.2. Example. Now let E be a fiber of an elliptic fibering $f : S \to Z$ such that

- E is a nonsingular elliptic curve and
- m is the multiplicity of f in E, i.e., m is the minimal positive integer with $mE \sim 0$ locally$/Z$.

Then, for a canonical divisor K, $K \sim aE$ locally$/Z$ with a unique $0 \leq a \leq m-1$. Moreover, in the characteristic 0, $a = m-1$, or $K + E \sim 0$, which means that $K + bE$ with some real $0 \leq b \leq 1$ has a 1-complement locally$/Z$. However, if the characteristic p is positive, we can construct such a fibering with $m = p^k$ and any $0 \leq a \leq m-1$ (e.g., see Example 8.2 in [11] for $m = p^2$ and $0 \leq a \leq p-1$). Therefore, $K + bE$ with b close to 1 has no n-complements with bounded n because such a complement would have the form $K + E \sim (a+1)E$ and $n(K+E) \sim n(a+1)E \not\sim 0$ when $n, a \ll p$ and $m \geq p$. Therefore, $K + E$ can be an n-complement having a rather high index $n|m$. However, it is high in the Archimedean sense, but n is small p-adically when $m = p^k$.

We therefore assume everywhere below that the characteristic is 0.

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1.3. Conjecture on complements. We consider log pairs (X, B) with boundaries B such that $K + B$ is log canonical. Then complements in any given dimension d are *bounded.* This means that there exists a finite set N_d of natural numbers such that any contraction $f : X \to Z$ satisfying certain conditions that we discuss below and having $\dim X \leq d$ has an n-complement locally/Z for some *index* $n \in N_d$.

Of course, we must assume that

(EC) there exists at least some n-complement.

In particular, if $B = 0$, then the linear system $|-nK| \neq \emptyset$ (cf. Corollaries 1.16–1.17 and (NV) in Remark 2.6) and its generic element have good singularities in terms of complements (see explanations before Corollary 1.16). Perhaps, the existence of complements (EC) is sufficient. The following conditions are more realistic and important for applications:

(SM) the multiplicities b_i of B are *standard,* i.e., $b_i = (m-1)/m$ for a natural number m or $b_i = 1$ (as for $m = \infty$);

(WLF) $(X/Z, B)$ is *weak log Fano,* which means that $-(K + B)$ is nef and big/Z.

We note that these conditions imply (EC) according to the proof of Proposition 5.5 in [23]. By Example 1.1, the condition that $-(K + B)$ is just nef is not sufficient even for (EC).

1.4. Main Theorem. *The complements in dimension two are bounded under the condition (WLF) and*

(M) *the multiplicities b_i of B are* standard, *i.e., $b_i = (m-1)/m$ for a natural number m or $b_i \geq 6/7$.*

More precisely, for almost all contractions and all contractions of relative dimension zero and one, we can take a complementary index in $RN_2 = \{1, 2, 3, 4, 6\}$.

Here, "almost all" means up to a bounded family of contractions in terms of moduli. This really concerns spaces of moduli in the global case or that of log Del Pezzos. The moduli themselves may not be the usual ones according to Remark 1.13. By the *global* case, we mean that $Z = \text{pt}$. The other cases are *local*/Z.

1.5. Definition (cf. Theorem 5.6 in [23]). We say that a complement $K + B^+$ is *nonexceptional* if it is not Kawamata log terminal when $Z = \text{pt}$. and is not purely log terminal on a log terminal resolution when $Z \neq \text{pt}$. The former is called *global,* and the latter *local.* The corresponding log pair $(X/Z, B)$ is called *nonexceptional* if it has a nonexceptional complement. For instance, surface Du Val singularities of types $\mathbb{A}_*$ and $\mathbb{D}_*$ are nonexceptional even though they have trivial complements that are canonical (cf. Example 1.7). They respectively have other 1- and 2-complements that are nonexceptional.

On the other hand, the pair $(X/Z, B)$ is *exceptional* if each of its complements $K+B^+$ is *exceptional.* The latter means that $K + B^+$ is Kawamata log terminal when $Z = \text{pt}$. and is *exceptionally* log terminal when $Z \neq \text{pt}$. The exceptional log-terminal property means the purely log-terminal property on a log terminal resolution. The Du Val singularities of exceptional types $\mathbb{E}_6, \mathbb{E}_7$, and $\mathbb{E}_8$ are really exceptional from this standpoint as well (cf. Example 5.2.3 in [23]).

If the pair $(X/Z, B)$ is exceptional and $Z \neq \text{pt}$., then for any complement $K + B^+$, any possible divisor (at most one, as can be proved) with log discrepancy 0 for $K + B^+$ has the center over the given point in Z. Otherwise, we can find another complement $B^{+'} > B^+$ that is nonexceptional (see the Proof of Theorem 3.1: General Case in Sec. 3).

For instance, in the main theorem, the n-complements with n not in RN_2 are over $Z = \text{pt}$. and exceptional, as we see later in Theorems 3.1 and 4.1. The theorem states that the global exceptions are bounded. Some of them are discussed in Sec. 5. In higher dimensions, it is conjectured that such complements and the corresponding pairs $(X/\text{pt}., B)$ under conditions (WLF) and (SM) are bounded. Of course, this is not true if we drop (WLF) (see Example 1.7). However, it may hold formally and even in the local case under certain conditions, as suggested in Remark 1.14.

We note that N_2 in the conjecture on complements differs from RN_2 in the main theorem in exceptional cases, which are still not completely classified. Nonetheless, we define corresponding *exceptional* indexes as $EN_2 = N_2 \setminus RN_2$. We say that RN_2 is the *regular* part of N_2 in the conjecture for dimension two. We note that $RN_2 = N_1 = RN_1 \cup EN_1$, where $RN_1 = \{1, 2\}$ and $ER_1 = \{3, 4, 6\}$.

1.6. Conjecture. In general, we can define $EN_d = N_2 \setminus N_{d-1}$ and conjecture that $RN_d = N_{d-1}$ or, equivalently, that EN_d really corresponds to exceptions. Evidence supporting this is related to our method in the proof of the main theorem and Borisov–Alekseev's conjecture [2].

Some extra conditions on multiplicities are needed: (SM) or an appropriate version of (M) (cf. (M)′ in Remark 4.7.2, (M)″ in 7.1.1, and 2.3.1).

Basic examples of complements can be found in 5.2 in [23].

1.7. Example-Problem. *Trivial complements.* Let X/Z be a contraction with only log canonical singularities on X and with $K \equiv 0/Z$, e.g., Abelian variety, K3 surface, Calabi–Yau threefold, or a fibering of them. It is known that K is then semiample locally/Z: $K \sim_{\mathbb{Q}} 0$ or $nK \sim 0$ for some natural number n (at least in the log-terminal case, cf. Remark 6-1-15(2) in [14]). Such minimal n is known as the *index* of X/Z, to be more precise, over a point $P \in Z$. Indices are *global* when $Z = P =$ pt., and *local* otherwise. Therefore, if we consider X/Z as a log pair with $B = 0$, then (EC) is fulfilled for the above n. In the global case, the conjecture on complements suggests that we can find such n in N_d for a given dimension $d = \dim X$. Moreover, in that case, we can replace $B = 0$ by B under (SM) according to Monotonicity 2.7 below and assuming that X can be seminormal. In the local case, we need an additional assumption on the presence of a log canonical singularity/P, i.e., there exists an exceptional or nonexceptional divisor E with log discrepancy 0 for $K + B$ and with $\mathrm{center}_X(E)/P$. Otherwise, a complement can be *nontrivial*: $B^+ > B$.

This really holds in dimensions one and two by Corollary 1.9 below. In dimension three, it is known that any global index n divides the Beauville number

$$b_3 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$$

when $X/$ pt. has at most terminal singularities and $B = 0$. We note that a similar two-dimensional number is $b_2 = 12$. Its proper divisors give $N_1 = RN_2$. We therefore conjecture that N_2 consists of the (proper) divisors of the Beauville number. Perhaps we have something like this in higher dimensions.

Of course, if indices are bounded in a given dimension d, we have a universal index I_d as their least common multiple. The case of the conjecture on complements under consideration in dimension d is equivalent to the existence of such I_d. As usual in mathematics, it is known only that $I_1 = 12$ and that I_2 exists (see Corollary 1.9). We suggest that $I_2 \approx b_3$. In particular, this means that b_3 corresponds to nonexceptional cases. We also note that in the global case, according to our definition, contractions and their complements are nonexceptional when they are Kawamata log terminal, for instance, Abelian varieties. These varieties have unbounded moduli in any dimension $d \geq 2$, but their index is 1. Therefore, from the formal standpoint of Remark 1.14 below, we treat them as a regular case.

We note that we are still discussing the case that is opposite to the assumption (WLF) in the main theorem. Indeed, if we consider a birational contraction X/Z, then the indices are not bounded, for instance, when $X = Z$ is a neighborhood of a quotient singularity. Nonetheless, the above should hold when X is really log canonical in P as we suppose. In the surface case, such singularities with $B = 0$ are known as *elliptic*, and they have a Kodaira dimension zero in a certain sense. Their universal index is 12. The conjecture on complements implies the existence of such a universal number in any dimension.

An inductive explanation can be presented in the log canonical case (cf. Remark 1.14). Using the log minimal model program (LMMP), it is possible to reduce to the case where B has a reduced component E. Then, by the adjunction formula, the index of $K + B$ near E coincides with the same for the restriction $K_E + B_E = (K + B)|_E$. It is therefore not surprising that indices for dimension $d - 1 = \dim E$ should divide indices in dimension d. If the above restriction is epimorphic for Cartier multiples of $K_E + B_E$, then both indices coincide over a neighborhood of $P \in Z$. This holds, for example, when X/Z is birational. We see in Sec. 3 that this works in fiber cases as well.

Finally, we note that in the fiber case, the complements can be nontrivial when the above condition on log canonicity is not fulfilled. Let X be smooth and $B = 0$. Then the index of K in a neighborhood of a fiber $F \subset X$ over P can be arbitrarily high. The point is that the contraction X/Z itself is not smooth in those cases and the fiber F is multiple. For instance, K and F have multiplicity m near a fiber F of type mI_0 in an elliptic fibering. However, there exists the 1-complement $K + F$ in a neighborhood of F. This, in particular, explains why complements are fruitful even in such a well-known case as elliptic fiberings. More details can be found in Sec. 3. According to Example 1.2, this does not work well in the positive characteristic p. It might work modulo p factors.

Each *trivial* complement, i.e., a complement with $B^+ = B$, defines a cover $\widetilde{X} \to X$ of degree d, which is unramified in divisors with respect to B: a covering is *B-unramified* if it preserves the boundary B, i.e., the inequality $\mathrm{mult}_{D_i} B = b_i \geq (m_j - 1)/m_j$, where m_j is any ramification multiplicity$/D_i$ (2.1.1 in [23]), holds for each prime D_i in X. The conjecture in this case states boundedness of such degrees. In terms of a local or global algebraic fundamental group $\pi(X/Z)$, they correspond to normal subgroups of finite index and with a cyclic quotient. A general conjecture here states that $\pi(X/Z)$ is quasi-Abelian and even finite in exceptional cases (see Remark 1.14). The structure of the fundamental group is interesting even for nonsingular X.

We know that the condition (EC) is satisfied in the previous example. Therefore, according to the general philosophy and to the conjectures there, the following results are not surprising.

1.8. Corollary. *We can replace the condition (WLF) in the main theorem with (EC).*

This is proved in Secs. 3 and 4. More advanced results can be found in Remark 4.7. We consider only a *trivial* case here.

Proof: Numerically trivial case under (SM). We suppose that $K + B \equiv 0/Z$ and it has a trivial complement $B^+ = B$ (cf. Monotonicity 2.7). Moreover, we consider the global case where $Z = \mathrm{pt.}$ (for the local case, see Sec. 3). We must bound the index of $K + B$.

It is well known that if $B = 0$ and X has only canonical singularities, then the index of K divides $I_1 = 12$. On the other hand, the pairs (X, B) with $B \neq 0$ and surfaces X with noncanonical singularities are bounded when they are ε-log canonical for any fixed $\varepsilon > 0$, for instance, for $\varepsilon \geq 1/7$ (Theorem 7.7 in [2]). Therefore, their indices are bounded as well. We note that by (SM), the possible multiplicities b_i satisfy the descending chain condition (d.c.c.).

It follows that we must bound the indices in the non-ε-log canonical case for an appropriate ε. We take $\varepsilon = 1/7$. We can suppose that after a log terminal resolution, B has the multiplicity $b = b_i > 6/7$ in a prime divisor $D = D_i$. The resolution has the same indices (cf. Lemma 5.4 in [23]). Moreover, for the regular complements or for almost all (S, B), all $b_i = 1$, when $b_i \geq 6/7$, and the index is in RN_2. Therefore, there exists a real $c > 0$ such that all $b_i = 1$ when $B_i \geq c$, and we then have a regular complement. To reduce this to the main theorem, we apply the LMMP. (Cf. Tsunoda's Theorem 2.1 in [29], which states that if B is reduced and $K + B \equiv 0$, then $K + B$ has an index ≤ 66.)

We have the extremal contraction $f : X \to Z$ because $K + B - bD$ is negative on a covering family of curves. If f is birational, we replace X with Z. Then D is not contracted, because $(K + B - bD.D) = -bD^2 > 0$.

If Z is a curve, then f is a ruling. Further, according to a one-dimensional result of our corollary applied to the generic fiber, $b_i = 1$, and $K + B$ has a regular index near the generic fiber of f. We note that D is a multisection of f with multiplicity at most 2. If D is a 1-section, we reduce the problem to a one-dimensional case after an adjunction, $K_D + B_D = (K + B)|_D$. In that case, we use the same arguments as in Example 1.1 (cf. Lemma 2.20). We can take a complement with an even index when $K + B$ has the index 2 near the generic fiber.

Otherwise, D is an irreducible double section of f. In that case, the index is in RN_2 near each fiber. The same holds for the entire X, except for the case where $B = D$, $K_D + B_D = (K + B)_D \sim 0$, and $2(K + B) \sim 0$ (see Lemma 2.21).

Finally, if $Z = \mathrm{pt.}$, we can apply the main theorem for $B - \delta D$ for some $\delta > 0$. The main difficulties are here.

1.9. Corollary. *Under (SM), I_2 exists. Moreover, for the trivial complements, each $b_i = 1$ when $b_i \geq I_2/(I_2 + 1)$, and for almost all trivial complements, each $b_i = 1$ when $b_i \geq 6/7$.*

In the global case where (X, B) is not Kawamata log terminal and in the local case, we can replace the inexplicit I_2 with $I_1 = 12$ and (SM) with (M). Then each $b_i = 1$ when $b_i \geq 6/7$.

If we have an infinite number of exceptional divisors with log discrepancies 0 over P, then we can even replace $I_1 = 12$ with 2.

Proof: Global case. We suppose that $Z = \text{pt.}$ (we consider other cases in Sec. 3). Then the results follow from the above arguments and from Theorem 7.7 in [23]. The non-Kawamata log terminal case corresponds to regular complements by the main theorem (cf. the inductive theorem and Theorem 4.1 below).

For global and local cases with an infinite number of exceptional divisors with log discrepancies 0, see 2.3.2.

1.10. Corollary. *Under (SM), let $P \in (X, B)$ be a threefold log canonical singularity having a trivial complement as in Example 1.7. Then its index divides I_2. Moreover, each $b_i = 1$ when $b_i \geq I_2/(I_2+1)$.*

If (X, B) has at least two exceptional divisors$/P$ with log discrepancy 0, then we can replace the inexplicit I_2 with $I_1 = 12$. The same holds for one exceptional or nonexceptional divisor with center passing through P but not equal to P.

If (X, B) on some resolution has a triple point in an exceptional locus of divisors$/P$ with log discrepancy 0, then we can even replace $I_1 = 12$ with 2.

Proof. According to the very definition of a trivial complement, we have an exceptional divisor E/P with log discrepancy 0. By the LMMP (cf. Theorem 3.1 in [24]), we can make a crepant resolution $g : Y \to X$ of just this E, and E is in the reduced part of the boundary B^Y. By the adjunction, $K_E + B_E = (K_Y + B^Y)|_E \equiv 0$, and B_E has only standard coefficients. Therefore, it has a trivial complement. Then, by Corollary 1.9 and by the local inverse adjunction, $K_Y + B^Y$ has the same Cartier index, and it divides I_2 when the restriction $K_E + B_E = (K_Y + B^Y)|_E$ is Kawamata log terminal. We can use a covering trick here as well. Then, by the Kawamata–Viehweg vanishing, the restrictions $m(K_E + B_E) = m(K_Y + B^Y)|_E$ are epimorphic when m divides the index of $K_E + B_E$. Therefore, $K + B$ has the same index, and it divides I_2.

Other special cases follow from special cases in Corollary 1.9 (cf. Corollary 5.10 in [23]). Then the index divides $I_1 = 12$. A difficulty here is related to an application of the above surjectivity to a seminormal surface E. However, its combinatoric is quite simple.

We can simulate examples on well-known varieties as in the next example.

1.11. Example. Let $X = \mathbb{P}^d$ and $B = \sum b_i L_i$, where the prime divisors L_i are hyperplanes in a generic position and there are at most l of them. (The latter holds when $\min\{b_i \neq 0\} > 0$.) Then, for any boundary coefficients $0 \leq b_i \leq 1$, the log pair (X, B) is log terminal. If we have a reduced component L_i in B, we can restrict our problem on L_i. Therefore, really new complements are related to the case where all $b_i < 1$, which we assume below. We note that if all b_i are rational (or $\sum b_i < d + 1$), the condition (EC) follows from the inequality $\sum b_i \leq d + 1$. The existence of an n-complement is equivalent to another inequality,

$$\deg\lfloor (n+1)B \rfloor = \sum \lfloor (n+1)b_i \rfloor / n \leq d + 1.$$

The conjecture on complements states that we can choose such n in a finite set N_d. The space

$$T_d^l = \{(b_1, \ldots, b_l) \mid b_i \in [0,1] \text{ and } \sum b_i \leq d+1\}$$

is compact, and the vectors $v = (b_1, \ldots, b_l)$ without n-complements form a union of convex rational polyhedra: $b_i \geq c_i$. In the intersection, they give a vector v without any n-complements; we can also assume that it is maximal, $\sum b_i = d{+}1$, and $b_i > 0$. But this is impossible, because there exists an infinite set of approximations with natural numbers n such that $(n+1)b_i = N_i + \delta_i, i = 1, 2$, where $|\delta_i - b_i| < \varepsilon \ll 1$, and N_i is an integer. Indeed, then each $\lfloor (n+1)b_i \rfloor = N_i$, and

$$\sum \lfloor (n+1)b_i \rfloor / n = \sum N_i / n = \sum b_i + \sum (b_i - \delta_i)/n \leq d + 1 + l\varepsilon/n.$$

Hence, $\sum \lfloor (n+1)b_i \rfloor / n \leq d+1$ when $l\varepsilon < 1$, because d is an integer. We can find the required approximation from another one, $nb_i = N_i + \varepsilon_i$, where $\varepsilon_i = \delta_i - b_i$, $|\varepsilon_i| < \varepsilon \ll 1$, and N_i is an integer. It has the required solutions n according to the Kronecker theorem (Theorem IV in Sec. 5 of Chap. 3 in [7]) with $L_i = b_i x$ and $\alpha_i = 0$. Indeed, for every integer number u_i, $\sum u_i \alpha_i = 0$ is an integer.

Explicitly, this is known up to $d = 2$: for $d = 1$, see Example 5.2.1 in [23]; for $d = 2$ under (SM) with $n \leq 66$, see Prokhorov, Example 6.1 in [20] (cf. Tsunoda's Theorem 2.1 in [29]).

We can generalize this and suppose that d is any positive rational (and, maybe even, real) number, the hyperplanes are in a nongeneric position, and with hypersurfaces rather than hyperplanes of fixed degrees. Moreover, we can consider complements with n divided by a given natural number m (cf. Lemma 2.29). (We can consider this even for $l = \infty$, as for $d = 1$. However, the boundedness is unknown for $l = \infty$ and $d > 1$.)

There exists a corresponding local case; we consider a contraction X/Z with a fiber $F = \mathbb{P}^d$ and $B = \sum b_i L_i$, where prime divisors L_i intersect F in hyperplanes in a generic position.

Of course, more interesting cases are related to a nongeneric position and have hypersurfaces rather than hyperplanes. They also give nontrivial examples of trivial complements. For instance, if $C \subset \mathbb{P}^2$ is a plane curve of degree 6 with one simple triple point singularity, then $K + C/2$ is log terminal and has a trivial 2-complement. The corresponding double cover $\widetilde{X} \to \mathbb{P}^2$ produces a K3 surface $\widetilde{X}$ with a single Du Val singularity of type $\mathbb{D}_4$ over the triple singularity of C. For a generic C, after a resolution of $\widetilde{X}$, we obtain a nonsingular K3 surface Y with a Picard lattice of rank 5 and an involution on Y that is identical on the lattice. We note that a generic curve C is a generic trigonal curve of genus 7. But this is a different tale.

1.12. Example-Problem. Other interesting complements correspond to *Galois quotients*. Let $G : X$ be an (effective) action of a finite group G on a log pair $(X/Z, B)$ with a boundary B under (SM), and let $-(K+B)$ be nef. For instance, $X = \mathbb{P}^n$ or a Fano variety, Abelian variety, Calabi–Yau threefold, or an identical contraction of a nonsingular point. Then on the quotient $f : X \to Y = X/G$, we have a unique boundary B_Y such that $K+B = f^*(K_Y+B_Y)$. Hence, $(Y/Z, B_Y)$ is a log pair of the same type as $(X/Z, B)$. The (minimal) complement index n in this case is an invariant of the action of the group. In the global one-dimensional case $X = \mathbb{P}^1$, there exists an action of a finite group $G \subset PGL(1)$ with $n \in N_1$. All exceptional cases have the quotient description. But in dimension $d \geq 1$, it may be that some complement indices do not correspond to quotients of $PGL(d)$ and not all exceptional cases have a quotient description for $X = \mathbb{P}^d$ or some other nonsingular Del Pezzos. Even in the one-dimensional case, $Y = \mathbb{P}^1$ with $B = (n-1)P/n + (m-1)Q/m$ and $n > m \geq 2$ does not correspond to a Galois quotient. In higher dimensions, we expect more asymmetry as in a modern cosmology. But among the symmetrical minority, we may encounter real treasures.

Quotients X/G for finite groups of automorphisms of K3 surfaces X reflect the geometry of the pairs (X, G). According to Nikulin, such pairs (X, G) are bounded when G is nontrivial (cf. Theorem 18.5 in [5]), in particular, with a nonsymplectic action. On the other hand, each Abelian surface X has an involution. Therefore, pairs $(X, \mathbb{Z}_2)$ are not bounded in that case. However, we expect that the pairs (X, G) with $K \equiv 0$ and nonsingular X are bounded when (X, G) is quite nontrivial (nonregular). For instance, X has the irregularity 0 and $G \not\cong \mathbb{Z}_m$ with proper divisors $m|12$. We can say that such pairs (X, G) are *exceptional* for dimension two. Indeed, if we have such a group G, then it has a fixed point, according to the classification of algebraic surfaces. If the fixed point is not isolated, it produces a nontrivial boundary on X/G. Otherwise, it gives a non–Du Val singularity on X/G when the action is nonsymplectic. Then X/G with a boundary belongs to a bounded family by Corollary 1.8 and Theorem 7.7 by Alekseev [2]. Perhaps this sometimes holds for symplectic actions as well as for $K3$ surfaces.

1.13. Remark. The moduli spaces mentioned after the main theorem can have real parameters corresponding to boundary coefficients. If we want the usual algebraic moduli, we can forget boundaries or impose a condition such as (SM). A bit more generally, we can suppose that 1 is the only accumulation point for possible boundary multiplicities.

1.14. Remark. In addition to Conjecture 1.6, we suggest that regular $RN_d = N_{d-1}$ is sufficient for global nonexceptional and any local complements in dimension d. We verify this in this paper for $RN_2 = N_1$. An explanation is related to Lemma 5.3 in [23]. If we really have log canonical singularities, we can induce the problem from a lower dimension (cf. Proof of Theorem 5.6 in [23] and Induction Theorem 2.3 below). This means that in this case, we only need indices from N_{d-1} for dimension d. The same holds if we can increase B to B' preserving all requirements on $K + B$ for $K + B'$ and $K + B'$ has a log canonical singularity. A good choice is to simulate construction of complements. Therefore, we take $B' = B^+ = B + H/n$, where $H \in |-nK - \lfloor (n+1)B \rfloor|$ for some n. If $K + B^+$ is log canonical, we have an n-complement. This complement is bounded when $n \in N_d$ or $n \in N_{d-1}$. Otherwise, we have a log singular case where H is called a *singular* element and $K + B^+$ is a *singular* complement. Keel and McKernan referred to such an H as a tiger, but it looks more like a must. In this case, we can consider a weighted combination $B' = aB + (1-a)B^+$, and for an appropriate $0 < a < 1$, the log divisor $K + B'$ is log canonical but not Kawamata log terminal. Therefore, we have no reduction to dimension $d-1$ when we have no singular elements or such complements. These cases correspond to exceptions. In particular, $|-nK - \lfloor (n+1)B \rfloor| = \emptyset$ for each $n \in RN_d$ in the exceptional case. However, $|-nK - \lfloor (n+1)B \rfloor| \neq \emptyset$ for some $n \in RN_d$ in the regular cases. (Cf. Corollaries 1.16–18 for $d = 2$ below.)

In the local case, singular elements are easy to construct by adding a pullback of a hyperplane section of the base Z. In the global case, singular elements define an ideal sheaf of the type of Nadel-multipliers ideal sheaf. No singular elements and $B = 0$ implies the existence of a Kähler–Einstein metric with a good convergence in singularities (cf. [18]). It is known that X is then stable in the sense of Bogomolov (cf. Proposition 1.6 in [28]) and exceptions with $B = 0$ should be bounded. An algebraic counterpart of this idea is the Borisov–Alekseev conjecture. Therefore, we can expect the same for exceptions with $B \neq 0$, at least under (SM). We prove this for dimension two in Sec. 4. We discuss some exceptions in Sec. 5.

We think the same or something close holds in any dimension (which will be discussed elsewhere), as well as for exceptional and nonexceptional cases in the *formal sense*. This means that they have an appropriate index: some $n \in RN_d$ in the regular case and only some $n \in ER_d$ in the exceptional case. In the local cases, we replace d by $d-1$. For instance, any Abelian variety is exceptional according to Definition 1.5, but they are nonexceptional from the formal standpoint. Their moduli are unbounded, as expected, in the nonexceptional case. On the other hand, an elliptic fiber of type III or any other exceptional type is formally exceptional. In a certain sense, their moduli are bounded. However, the log terminal singularities of this type, i.e., with the same graph of a minimal resolution, are unbounded, but again bounded if we suppose the ε-log terminal property. For instance, the Du Val or surface canonical singularities of the exceptional types are bounded up to a certain degree (cf. Corollaries 7.3–4). This is a local version of the Borisov–Alekseev conjecture.

1.15. Remark. Complements and their indices allow us to classify contractions. In particular, we divide them into exceptional and regular. This implies the same in some special situations.

For instance, any finite (and even reductive) group representation or, more generally, action on a Fano or on an algebraic variety with a numerically seminegative canonical divisor can be treated as exceptional or regular, and they have such an invariant as the index in accordance with their quotients in Example 1.12. In particular, this works for subgroups of PGLs. In the case of $G \subset PGL(1)$, all the exceptional subgroups correspond to exceptional boundary structures on $\mathbb{P}^1 \cong \mathbb{P}^1/G$. Crystallographic groups give other possible examples.

We can apply the same ideas to a classification of surface quotients, log terminal or canonical, or elliptic singularities, or their higher-dimensional analogues. The same holds for elliptic fiberings or other fiberings with complements.

Quite possibly, the most important applications of complements are still to come; perhaps they are related to classifications of contractions of threefolds X seminegative with respect to K. They may help to choose an appropriate model for Del Pezzo, elliptic fiberings, and conic bundles. A strength of these methods is that they apply in the most general situation when we have log canonical singularities and contractions are extremal in an algebraic sense, or even just contractions. This also shows a weakness because the distance to very special applications, when we have such restrictions as terminal singularities and/or extremal-property contractions, can be quite long and difficult. In addition, exceptional cases are still not classified completely and explicitly.

We give a primitive sample in 5.1.3.

Another application to a log uniruledness is given in [15] (cf. Remark 6.9.)

We now explain the statement of the main theorem and outline a plan of its proof. We also derive some corollaries. Let S be a normal algebraic surface and $C + B$ be a boundary (or subboundary) in it. We assume that $C = \lfloor C + B \rfloor$ is the reduced part of the boundary and $B = \{C + B\}$ is its fractional part. Let $f : S \to Z$ be a contraction. We fix a point $P \in Z$. We must find $n \in RN_2$ such that $K + C + B$ has an n-complement locally over a neighborhood of $P \in Z$ or prove that other possibilities are bounded. More precisely, the latter ones are only global: $Z = P$ and $(S, C + B)$ are bounded. In particular, the underlying

S are bounded. On the other hand, for $(S/Z, C+B)$ having an n-complement, there exists an element $D \in |-nK-nC-\lfloor(n+1)B\rfloor|$ such that for $B^+ = \lfloor(n+1)B\rfloor/n + D/n$, $K+C+B^+$ has only log canonical singularities (cf. Definition 5.1 in [23]). In the local case, a linear system means a local one$/P$.

As in Corollary 1.8 we can prove more.

1.16. Corollary. *Let $(S/Z, C+B)$ be a quasi log Del Pezzo,* i.e., a pair under (EC) with nef$/Z$ $-(K+C+B)$, *and let (M) hold for B. Then for almost all of them, there exists an index $n \in RN_2$ such that $|-nK-nC-\lfloor(n+1)B\rfloor| \neq \emptyset$ or, equivalently, we have a nonvanishing $R^0f_*\mathcal{O}(-nK-nC-\lfloor(n+1)B\rfloor) \neq 0$ in P. More precisely, the same holds for $n \in RN_1 = \{1,2\}$ when $K+C+\lfloor(n+1)B\rfloor/n$ is not exceptionally log terminal,* i.e., there exists an infinite set of exceptional divisors$/P$ with the log discrepancy 0.

Moreover, we need the condition (M) in only the global case with only Kawamata log terminal singularities, in particular, with $C=0$, and the exceptions for RN_2 belong to it. In this case, we can choose a required n in N_2.

The last two statements are proved in Sec. 4, and a more precise picture is given in Example 7.10. We also note that n in the local case in the corollary depends on P. We prove that N_2 is finite, but we still have no explicit description of it. By Monotonicity 2.7, $\lfloor(n+1)B\rfloor \geq nB$ under the condition (SM), or even (M) if $n \in RN_2$.

1.17. Corollary. *Again, let conditions (EC) and (M) hold. Then for almost all log pairs $(S/Z, B+C)$, there exists an index $n \in RN_2$ such that $|-n(K+C+B)| \neq \emptyset$ or, equivalently, we have a nonvanishing $R^0f_*\mathcal{O}(-n(K+C+B)) \neq 0$ in P. More precisely, the same holds for $n \in RN_1 = \{1,2\}$ when $K+C+B$ is not exceptionally log terminal.*

The exceptions for RN_2 belong to only the global case with Kawamata log terminal singularities, in particular, with $C=0$. In this case, we can choose n in N_2, but we should require that $b_i \geq m/(m+1)$ with the maximal $m \in N_2$ instead of $b_i \geq 6/7$ in (M).

In particular, if the boundary $C+B$ is reduced, i.e., $B=0$, then $|-n(K+C)| \neq \emptyset$ or $R^0f_*\mathcal{O}(-n(K+C)) \neq 0$ in P.

The last nonvanishing is nontrivial even when S is a Del Pezzo surface with quotient singularities because it states that $|-nK| \neq \emptyset$ or $h^0(S, -nK) \neq 0$ for bounded n. Such Del Pezzo surfaces form an unbounded family; moreover, the indices of K for them are unbounded, as are their ranks of the Picard group.

1.18. Corollary. *Again, let (EC) and (M) hold. A log pair $(S/Z, B+C)$ is exceptional when $Z =$ pt., $(S, B+C)$ is Kawamata log terminal, in particular, $C=0$, and for each $n \in RN_2$, $|-nK-\lfloor(n+1)B\rfloor| = \emptyset$ or $|-n(K+B)| = \emptyset$; equivalently, we have a vanishing $h^0(S, -nK-\lfloor(n+1)B\rfloor) = 0$ or $h^0(S, -n(K+B)) = 0$.*

In particular, if the boundary B is reduced, i.e., $B=0$, then $|-nK| = \emptyset$ or $h^0(S, -nK) = 0$.

Quite soon, in the proof of the global case in the inductive theorem, we see that an inverse holds, at least in the weak Del Pezzo case and in the formal sense. This leads to the following question.

1.19. Question. Under (WLF) and (SM), does any $(X/Z, B)$ with a formal regular complement have a real nonexceptional regular complement in the sense of Definition 1.5? In general, this is not completely true ($\mathrm{reg}(X, B^+) = 1$), for instance, locally for non–log terminal singularities with 2-complements of type $\mathbb{E}2_1^0$ (see Sec. 6); the singularity is exceptionally log terminal. It holds for log terminal singularities of types $\mathbb{A}_m$ and $\mathbb{D}_m$. But it is unknown, even for surfaces $X = S/$pt.

The inverse does not hold if we drop (WLF), for instance, for Abelian varieties, as mentioned in Example 1.7. However, it looks possible for $\mathbb{P}^1 \times E$ where E is an Abelian, in particular, an elliptic curve. In this case, $-K$ has a positive numerical dimension. Moreover, each 1-complement in such a case has log singularities. Alas, these complements are not quite nonexceptional as well. We explain this in Sec. 7 in terms of $\mathrm{reg}(S, B^+)$, which specifies the question for 1- and 2-complements with $\mathrm{reg}(S, B^+) = 1$.

The above corollaries show that it is easier to construct a required $D \in |-nK-nC-\lfloor(n+1)B\rfloor|$ with small n when $K+C+B$ has more log singularities. An expansion of this fact is related to the inductive theorem in Sec. 2, an analogue of arguments in Theorem 5.6 in [23]. As in the last theorem, under conditions of the inductive theorem, we extend or lift D and a complement from its one-dimensional restriction or projection. Therefore, we say that we have an *inductive* case or complement, and the latter has regular indices and is nonexceptional in the global case.

We can then try to change $(X/Z, B)$ such that a type of complement is preserved and the new $(X/Z, B)$ satisfies the inductive theorem. For instance, we can increase B as is done for local complements in Sec. 3. Here, in the global case, a problem arises with standard multiplicities. Fortunately, all the other cases are global and Kawamata log terminal, where we use a reduction to the inductive theorem or to a Picard number-1 case which, in addition, is 1/7-log terminal in points. In the latter case, we apply Alekseev's results [2]. We discuss this in Sec. 4.

2. Inductive Complements

2.1. Example. Let $S = \mathbb{P}_E(F_3)$ be a ruled surface over a nonsingular curve E of genus 1, which corresponds to a nonsplitting vector bundle F_3/E of rank 2 (see p. 141 in [5]) with an odd determinant. The ruling is denoted by $f\colon S \to E$. It has a single section $\mathbb{P}_E(F_1)$ in its linear class by a Riemann–Roch and a vanishing below. We identify the section with E.

We note that $E|_E \sim \det F_2 \sim O$ for a single point $O \in E$ (cf. Proposition 2.9 in Chap. 5 in [10]). Therefore, we have a natural structure of an elliptic curve on E with O as a zero. On the other hand, $(K_S + E)|_E \sim K_E \sim 0$ according to the adjunction formula. Therefore, $(K_S + 2E - f^*O)|_E \sim 0$ (see Lemma 2.10 in Chap. 5 in [10]). But $K_S + 2E \equiv 0/E$. Therefore, as in Example 1.1, $K + 2E - f^*O \sim 0$, and we can choose $-K_S = 2E - f^*O$ as an anticanonical divisor. In particular, $K_S^2 = 0$, and $-K_S$ is nef because F_3 is not splitting. The latter also implies that the cone $\overline{\mathrm{NE}}(S)$ is two-dimensional and has two extremal rays:

- the first ray R_1 is generated by a fiber F of the ruling f and
- the second ray R_2 is generated by $-K_S$.

Because $-K_S = 2E - f^*O$, the ray R_2 has no sections. We contend that *R_2 has three unramified double sections C_i, $1 \le i \le 3$. Each of them is $C_i \sim 2E - f^*(O + \theta_i) = -K_S - f^*\theta_i$, where θ_i is a nontrivial element of the second order in* $\mathrm{Pic}(E)$. *Hence, K_S has 2-complements C_i: $2(K_S + C_i) \sim 0$ but $K_S + C_i \not\sim 0$ but no 1-complements. The sections are nonsingular curves of genus 1. Moreover, R_2 is contractible with* $\mathrm{cont}_{R_2} : S \to \mathbb{P}^1$ *having 4-sections also nonsingular and of genus 1 as generic fibers. The curves C_i are the only multiple (really, double) fibers of* cont_{R_2}.

Because $-K_S + mE$ is ample for integer $m > 0$, we have vanishings $h^i(S, mE) = h^i(K_S - K_S + mE) = 0$ for $i > 0$ according to Kodaira. Hence, by the Riemann–Roch,

$$h^0(S, 2E) = 2E(2E - K_S)/2 = 3.$$

Similarly, $h^0(S, E) = 1$. On the other hand, a restriction of $|2E|$ on E is epimorphic and free because $h^1(S, E) = 0$ and the restriction has degree 2 on E. Therefore, $|2E|$ is free and defines a finite morphism $g : S \to \mathbb{P}^2$ of degree 4. According to the restrictions, E goes to a line $L = g(E)$, and a generic fiber $F = f^*P$ of f goes to a conic $Q = g(F)$, which is tangent to L in $g(P)$. Otherwise, $|2E - F| \neq \emptyset$, and we have a single double section $C \equiv -K_S$ passing through $Q \sim 2O - P$. This defines a contraction onto E, which is impossible by Kodaira's formula (Theorem 12.1 in Chap. 5 in [5]), because $-K_S$ is nef. Therefore, $Q = g(F)$ is a conic, and g embeds F onto $Q \subset \mathbb{P}^2$. Then, by the projection formula, $g^*(L|_Q) = (g^*L)|_F = 2E|_F = 2P$ locally over $g(P)$. Therefore, Q is tangent to L in $g(P)$. We note that the same image gives a fiber $F' = f^*P'$ for $P' \sim 2O - P$ because F_3 is invariant for an automorphism (even involution) $i : S \to S$ induced by the involution $P \mapsto P'$ on E. The latter holds by the uniqueness of F_3 with the determinant O.

We therefore have a one-dimensional linear system $|4E - f^*O|$, whose generic element is a nonsingular 4-section of genus 1. This gives $g = \mathrm{cont}_{R_2}$. Because all nonmultiple fibers of g are isomorphic, $g_*K_{S/E} \equiv \chi(\mathcal{O}_S) = 0$. By Kodaira's formula, we hence have degenerations. (Equivalently, the second symmetrical product of F_3 is not spanned on E; cf. p. 3 in [17].) Double fibers are the only possible degenerations because their components are in R_2 but are not sections of f. They give double sections C_i of f in R_2. On the other hand, $2K_S \sim -4E + 2f^*O \sim -g^*P$ for a generic $P \in \mathbb{P}^1$. Therefore, $K_S \sim_{\mathbb{Q}} -g^*P/2$. In particular, K_S has the 2-complement $g^*P/2$. Again, by Kodaira, we have three double fibers: C_i, $1 \le i \le 3$. We note that $2C_i \sim g^*P \sim -2K_S \sim 4E - 2f^*O$. We thus obtain all properties of C_i, except for $\theta_i \neq 0$. This follows from a monodromy argument because we have exactly three such $\theta_i \in \mathrm{Pic}(E)$ with $2\theta_i = 0$ and three double sections C_i of f in R_2. (The corresponding double cover C_i/E is given by θ_i.)

2.2. Corollary. *Let S/E be an extremal ruling over a nonsingular curve of genus* 1. *It can be given as a projectivization $S = \mathbb{P}_E(F)$ for a vector bundle F/E of rank* 2. *Then S/E or F/E has*

- *a splitting type if and only if K_S has a* 1*-complement,*
- *an exception in Example* 2.1 *if and only if K_S has a* 2*-complement, and*
- *an exception in Example* 1.1 *if and only if K_S does not have complements at all.*

In addition, the cone $\mathrm{NE}(S)$ *is closed and generated by two curves or extremal rays:*

- *the first ray R_1 is generated by a fiber F of the ruling S/E and*
- *the second ray R_2 is given by one of curves G with $G^2 \leq 0$ given by a splitting in a splitting case, $K_S + G + G' \sim 0$; other cases are discussed in Examples* 1.1 *and* 2.1.

Proof. We only need to consider the splitting case where $F = V \oplus V'$ is a direct sum of two line bundles. We then have two disjoint sections $G = \mathbb{P}_E(V)$ and $G' = \mathbb{P}_E(V')$. We can suppose that $G^2 \leq 0$. By the adjunction formula, $K_S + G + G' \equiv 0$ and, moreover, ~ 0 by the arguments in Example 1.1. Therefore, K has a 1-complement, and G generates R_2. The latter holds for some C with $C^2 \leq 0$. We assume that the curve C is an irreducible multisection of f. If $C^2 < 0$, then $C = G$. Otherwise, C is disjoint from G and G', and $(K_S.C) = (K_S + G + G'.C) = 0 = (K_S + G + G'.G)$. Therefore, $C = G$. If $C^2 = G^2 = 0$, then R_2 is generated by G as well.

2.3. Inductive Theorem. *Let $(S/Z, C + B)$ be a surface log contraction such that*

(NK) *$K + C + B$ is not Kawamata log terminal, for instance, $C \neq 0$, and*

(NEF) *$-(K + C + B)$ is nef.*

Then it has a regular complement locally/Z, i.e., $K + C + B$ has a 1-, 2-, 3-, 4- or 6-complement, *under (WLF) of Conjecture 1.3 or with (M) under any one of the following conditions:*

(RPC) *$\overline{\mathrm{NE}}(S/Z)$ is rationally polyhedral with contractible faces/Z or*

(EEC) *there exists an* effective *complement,* i.e., a boundary $B' \geq B$ such that $K + C + B'$ is log canonical and $\equiv 0/Z$ *or*

(EC)+(SM) *or*

(ASA) anti *log canonical divisor $-(K + C + B)$* semiample/Z *or*

(NTC) *there exists a* numerically trivial contraction $\nu : X \to Y/Z$, i.e., ν contracts the curves $F \subset S/Z$ with $(K + C + B.F) = 0$.

2.3.1. *We can drop (M) in the theorem, but then it states just a boundedness of n-complements. More precisely, $n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 35, 36, 40, 41, 42, 43, 56, 57\}$.*

2.3.2. *If there exist an infinite number of exceptional divisors with log discrepancy* 0*, then we have a* 1*- or* 2*-complement. If we drop (M), we have* 6*-complements in addition.*

There exist (formally) nonregular complements in the inductive theorem when (M) is not assumed. Similarly, we have examples with only 6-complements in 2.3.2.

2.4. Example. Let $f : S \to \mathbb{P}^1$ be an extremal ruling $\mathbb{F}_n$ with a negative section C. We take a divisor $B = V + H$ with a vertical part $f^*(E)$, where $E \geq 0$, and a horizontal part $H = \sum d_i D_i$ with $d_i \in \mathbb{Z}/m \cap (0, 1)$ for some natural number m. The latter can be given, for instance, as generic sections of f.

We suppose that the different $B_C = (B)_C$ has only the standard multiplicities, $V = f^* B_C$, $\deg(K_C + B_C) = 0$, and H is disjoint from C. Then $K + C + B$ as $K_C + B_C$ has n-complements only for n such that n is divided by the index of $K_C + B_C$ (cf. Monotonicity Lemmas 2.7 and 2.17 below). Then we do not have n complements for $m = (n + 1)$ when $K + C + B \equiv 0$ (cf. Monotonicity 2.16.1 below).

We therefore have a nonregular $n(n+1)$-complement when K_C+B_C has only one regular n-complement, and $n(n+1)\geq 7$. Moreover, we can have no regular complements in this case. This holds, for instance, when $n=6$.

In general, nonregular complements in the inductive theorem are similar, as we see in its proof, with the following modifications. The ruling cannot be extremal, S can be singular, and B_C can have nonstandard multiplicities. To find complements in such cases, we use Lemmas 2.20 and 2.28–29.

If we take V with the horizontal multiplicities 1/3 and H such that (C,B_C) has just 2-complements, then $K+C+B$ has 6-complements as the minimal.

The following result clarifies the relations of conditions in the inductive theorem.

2.5. Proposition. *If $K+C+B$ is log canonical and nef/Z, then*

$$\text{(WLF)} \Longrightarrow \text{(RPC)} \Longrightarrow \text{(NTC)} \Longleftrightarrow \text{(ASA)} \Longleftrightarrow \text{(EEC)} \Longleftarrow \text{(EC)} + \text{(SM)}$$

with the following exceptions: for (WLF) $\Longrightarrow$ (RPC),

(EX1) *the contraction $f: S \to Z$ is birational, and up to a log terminal resolution, $C=C+B$ is a curve with nodal singularities of arithmetic genus* 1, */P, $K+C\equiv 0/Z$, and S has only canonical singularities outside C;*

for (EEC) $\Longrightarrow$ (NTC),

(EX2) $Z=$ pt., *$K+C+B$ has the numerical dimension one, B' and E are unique, $(K+C+B')$ has a log* nontorsion *singularity of* genus 1 *and of* numerical dimension *one,* i.e., on a crepant log resolution, $C+B'=C$ forms a curve with only nodal singularities, with the connected components of genus 1, and $E|_C\equiv 0$, but not $\sim_{\mathbb{Q}}$.

Nonetheless, (WLF) $\Longrightarrow$ (NTC) *always, and* (WLF) $\Longrightarrow$ (RPC) *always in the analytic category or in the category of algebraic spaces. In (EX2), there exists a* 1*-complement.*

2.6. Remark-Corollary. In particular, for surfaces, (WLF) always implies (ASA). In other words, if $-(K+C+B)$ is log canonical, nef, and big/Z, then it is semiample. This is well known when $K+B+C$ is Kawamata log terminal (see Remark 3-1-2 in [14]; cf. arguments in the proof below). For log canonical singularities, it is an open question in dimensions three and higher. In dimension two, we can replace the last two conditions with a nonvanishing:

(NV) $-(K+C+B)\sim_{\mathbb{R}} E/Z$ where E is effective,

and $E^2>0$. (See Definition 2.5 in [24] for a definition of $\sim_A$.) We also note that (EEC) implies (NV) but not vice versa (cf. Example 1.1). Therefore, in general, nef $-(K+C+B)/Z$ is not semiample if it is not big and not $\equiv 0/Z$. In the latter case, $E=0$, and according to the semiampleness conjecture for $K+C+D/Z$, we expect semiampleness (see Conjecture 2.6 in [24]; cf. Remark 6-1-15(2) in [14] and Remark 1.7). This is the main difficulty in constructing complements: (EC). In dimension two, at least (NV) *holds when $-(K+C+B)/Z$ is nef*. However, as in Example 1.1, $K+C+B+E$ may not be log canonical. Does such a nonvanishing hold in higher dimensions? In any case, it implies a log generalization of a Campana–Peternell problem in dimension two (see 11.4 in [6] and [17]; cf. [9]): $-(K+C+B)$ is ample/Z if and only if $-(K+C+B)$ is positive on all curves $F\subset Y/Z$. Indeed, then $E^2>0$, and $-(K+C+B)$ is ample by the Nakai–Moishezon Corollary 5.4 in [5], or we can use the implication (WLF) $\Longrightarrow$ (ASA). Therefore, we have a complement in a weak form (EEC). Again by, Example 1.1, we cannot replace the above positivity with a weaker version: nef and $(K+C+B.F)=0$ only for a finite set of curves. As we can see in a proof of (NV), the nef property of $-(K+C+B)$ implies (EEC) in most cases and in a linear form. For instance, the only exception is S of Example 1.1 when S is not rational. Do exceptions exist when S is rational? It appears that $-(K+C+B)$ satisfies (ASA) in most of these cases as well. The exceptions again are unknown to the author. For threefolds, similar questions are more difficult, for example, the Campana–Peternell problem. In dimension two, the most difficult cases are related to nonrational or nonrationally connected surfaces or, more precisely, to extremal fiberings over curves E of genus 1. They are projectivizations of rank 2 vector bundles/E. Therefore, similar cases are of prime interest for threefolds: projectivizations X of rank-3 vector bundles/E, of rank 2 over Abelian or K3 surfaces. Is their cone $\mathrm{NE}(X)$ always closed rationally polyhedral and generated by curves as for Fanos? What are the complements for K? (Cf. Corollary 2.2.)

According to the same example, we really need at least a contractibility of extremal faces in (RPC) for (EEC) or (EC) and, moreover, for regular complements in the theorem. As in (EX1), we sometimes have just the rational polyhedral property but not contractibility of any face. If $-(K+C+B)$ is only nef, the cone may not be locally polyhedral near $(K+C+B)^{\perp}$ (Example 4.6.4 in [8]). The same example gives an exception of (EX2).

In dimension $d \geq 3$, (ASA) is better than (NTC): (ASA) implies (NTC) easily, but the converse is harder, as we see below. In addition, in the proof of the inductive theorem below, we can see that a semiampleness of $-(K+B)$ on S is sufficient. Fortunately, this is good for induction in higher dimensions. This will be developed elsewhere. It appears that the same should hold for (NTC); then it would be best in this circumstance. The condition (NTC) also implies the Campana–Peternell problem (but not a solution of the latter).

Proof. We first verify that (WLF) implies (NTC) and, except for (EX1), (RPC). For the former, it is sufficient to prove (ASA). For the latter, it is sufficient to prove that $\overline{\mathrm{NE}}(S/Z)$ is rationally polyhedral and, except for (EX1), with the contractible faces near a contracted face $\overline{\mathrm{NE}}(S/Z)\cap(K+C+B)^{\perp}$. Indeed, the cone satisfies both properties locally outside the face by the LMMP. A contraction in any face of $\overline{\mathrm{NE}}(S/Z)\cap(K+C+B)^{\perp}$ preserves this outside property.

To prove (ASA) in the $\mathbb{Q}$-factorial case, we can use a modern technique: restrict a Cartier multiple of $-(K+C+B)$ on the log singularities of $K+C+B$. Then a multiple of $-(K+C+B)$ is free in such singularities because any nef Cartier divisor is free on a point, on a rational curve, and on a curve of arithmetic genus 1 with at most nodal singularities, assuming in the last case that the divisor is canonical when it is numerically trivial. We meet only this case after the restriction. We can then apply traditional arguments to eliminate the base points outside the log singularities.

For surfaces, however, we can use more direct arguments, which work in the positive characteristics. Because $-(K+C+B)$ is big, we have a finite set of curves F/P with $(K+C+B.F)=0$. We verify that each of them generates an extremal ray and (RPC) or (EX1) hold near them. After a log crepant blowup (Theorem 3.1 in [24]), we assume that S has only rational singularities whereas $K+C+B$ is log terminal. In particular, C has only nodal singularities.

In most cases, we can easily verify (RPC) and derive (ASA) from this as shown below. For instance, if $-(K+C+B)$ is ample, it follows from the LMMP. Otherwise, we change $C+B$ such that it holds except for three cases: (EX1);

i. C is a chain of rational curves/P, $(K+C+B.C+B')=0$ in $C+B'$ is contractible to a rational elliptic singularity, where $C+B'/P$ is the connected component of C in B; or

ii. $Z=$ pt., and there exists a ruling $g: S \to C$ with a section C, with $C^2<0$, which is a nonsingular curve of genus 1, and S has only canonical singularities, and outside C and B, B has no components in fibers and does not intersect C.

In (EX1), we obviously have the rational polyhedral property, but contractions of some faces may not exist in general. In the last two cases, we verify (RPC) directly near $(K+C+B)^{\perp}$.

According to Lemma 6.17 in [24], we suppose that $-(K+C+B)=D=\sum d_iD_i$ is an effective divisor with irreducible curves D_i. Moreover, we suppose that $\operatorname{Supp} D$ contains an ample divisor H. Therefore, if $K+C+B+\delta D$ is log canonical for some $\delta>0$, then $K+C+B+\delta D-\gamma H$ gives a required boundary $C+B{:}=C+B+\delta D-\gamma H$. For instance, this works when $K+C+B$ is Kawamata log terminal. Therefore, $C\neq 0$ in the exceptional cases considered below. The curve C is connected by Lemma 5.7 and the proof of Theorem 6.9 in [23] (see also Theorem 17.4 in [16]).

More precisely, $K+C+B+\delta D$ is not log canonical when $\operatorname{Supp} D$ has a component C_i that is in the reduced part C as well (cf. 1.3.3 in [23]). Moreover, any component C_i of C is $/P$, in D, $(K+C+B.C_i)=0$, and $C_i^2<0$. If C_i is not/P or C_i/P with $(K+C+B.C_i)<0$, we can decrease the boundary multiplicity in C_i. By an induction and the connectedness of C, this gives a reduction to the Kawamata log terminal case. If $C_i^2\geq 0$, we can do the same. Because H and D intersect C_i, $C_i \subseteq \operatorname{Supp} D$.

In most cases, C is a chain of nonsingular rational curves because C is connected with nodal singularities and

$$\deg K_C \leq (C.K + C) \leq (C.K + C + B) = -(C.D) \leq 0.$$

The only exception arises when $\deg K_C = 0$, C has the arithmetic genus 1, $B = 0$, and S is nonsingular in a neighborhood of C.

To verify (RPC) locally, we must verify (in the exceptional cases) that $D = -(K + C + B)$, that each nef $\mathbb{R}$-divisor D near $-(K + C + B)$ is semiample, and that only curves F/Z with $(K + C + B.F) = 0$ are contracted by D. Indeed, D is big as $-(K + C + B)$. Therefore, we can assume that D is effective as above and $D \geq H$. Therefore, D is numerically trivial only on a finite set of curves F and $F^2 < 0$. We can also assume that D is quite close to $-(K + C + B)$, namely, $(K + C + B.F) = 0$. In other words, we must verify that any set of curves F with $(K + C + B.F) = 0$ is contractible in an algebraic category. If the singularities after the contraction are rational, we can pull down D to a $\mathbb{R}$-Cartier divisor. Therefore, the pull-down of D is ample according to Nakai–Moishezon (Corollary 5.4 in [5]). Hence, D is semiample. Of course, this applies directly when D is a $\mathbb{Q}$-divisor. Otherwise, we can represent it as weighted combination of such divisors (cf. Step 2 in the Proof of Theorem 2.7 in [24]) because a small perturbation of coefficients of D preserves its positivity on all curves. (Essential ones are components of $\operatorname{Supp} D$.) The cone is rational finite polyhedral near $K + C + B$ because the set of curves F is finite.

If the curves F with exceptional curves on a minimal resolution form a tree of rational curves, we could then contract any set of curves F by Grauert and Artin's criteria (Theorems 2.1 and 3.2 in [5]). In addition, the singularities are rational log canonical after contraction. This works in case *i* when C is a chain of nonsingular rational curves C_i/P. Indeed, we can contract inductively because $F^2 < 0$. We do this first for F not in C. Because the boundary multiplicity in F is <1, F is a nonsingular rational curve, and by a classification of the log terminal singularities, F with the curves of a resolution forms a tree of rational curves. Therefore, F is contractible (by the LMMP as well). Moreover, this preserves the log terminality and rationality of singularities. By Lemma 5.7 in [23], C forms a chain of rational curves again. Finally, C is the whole curve where $K + C + B \equiv 0/Z$. Because all singularities are log terminal, we have the required resolution of C. We can then contract C inductively, too. This effectively gives case *i* because $B = 0$ near C. Otherwise, we can decrease B and replace $C + B$ by a Kawamata log terminal boundary.

We now suppose that C is a curve$/P$ with only nodal singularities of arithmetic genus 1. If $K+C+B \equiv 0/P$, then we have the exceptional case (EX1) because $-(K + C + B)$ is big$/P$. Because S is nonsingular in a neighborhood of C, it is easy to verify that other curves$/P$ on a minimal resolution form a chain of rational curves intersecting simply C. Moreover, they are (-1)- or (-2)-curves. Therefore, we can contract any set of such curves with any proper subset of C according to Artin. The only problem here is to contract the whole C. Alas, this is not always true in the algebraic category, e.g., after a monoidal transform in the generic point of C. The cone in this case is rational polyhedral, but contractions may not be defined for some faces, including the components of C. Nonetheless, $-(K + C + B)$ is semiample because $K + C + B$ is semiample in this case.

We now suppose that $K+C+B \not\equiv 0/Z$. Then C is nonsingular, and we have case *ii*. Indeed, we have an extremal contraction $S \to Y/Z$ negative with respect to $K + B$. It is not to a point or onto a curve because $(K + C + B.C) = (K + C.C) = 0$, and, accordingly, C cannot be a section if C is singular. Therefore, the contraction is birational. By a classification of such contractions, it contracts a curve that does not intersect C. Then, we again have $K + C + B \not\equiv 0/Z$. Induction on the Picard number$/Z$ gives a contradiction. If C is nonsingular, the only possible case is where, after a finite number of birational contractions, we have an extremal ruling that induces a ruling g as in case *ii* with section C. Then, in particular, $Z = \mathrm{pt}$.

The object S has only canonical singularities outside C. There are also no components $\operatorname{Supp} B$ in fibers of g and intersecting C. In other words, $K+C+B$ is canonical in points, and there are no components of B in fibers of g. Indeed, as we know, there are no components of B in fibers of g that intersect C. After extremal contractions$/C$, this holds for any component in fibers of g. Therefore, after a blowup in a noncanonical point, we obtain a contradiction. This implies that it is really case *ii*.

We now verify (RPC) near $(K+C+B)^{\perp}$. As above in case *i*, we must verify that any set of curves F with $(K+C+B.F) = 0$ is contractible. Again, as in case *i*, any such F in addition to C is rational. Hence, such an

F belongs to a fiber of g. Therefore, we have a finite number of them. As above, this implies the polyhedral property for $\overline{\mathrm{NE}}(S)$. In addition, a required contraction corresponds to a face in $\overline{\mathrm{NE}}(S) \cap (K+C+B)^{\perp}$, and the latter is generated by curves in fibers of g and, perhaps, C.

We only want to verify the existence of the contraction in this face. If C does not belong to such a face, it follows from the relative statement$/C$. Otherwise, after contracting the curves of the face in fibers, we suppose that F is generated by C. Any birational contractions with disjoint contracted loci commute. Therefore, it is sufficient to establish a contraction of C, after contractions of the curves in fibers of g, which does not intersect C. Equivalently, we can assume that the fibers of g are irreducible. Then S is nonsingular, g is extremal, and $\rho(S) = 2$, where $\rho(S)$ denotes the Picard number. Because $C^2 < 0$, we contract C by $h\colon S \to S'$ to a point at least in the category of normal algebraic spaces. However, we can pull down $K+C+B$ to a $\mathbb{R}$-Cartier divisor $-D' = K_{S'} + h(C+B)$ because $(K+C+B)|_C$ is semiample over a neighborhood of $h(C)$. This follows from the existence of a 1-complement locally (cf. Corollary 5.10 in [23]) and even globally, as we see later in the big case of the inductive theorem.

Because D' is nef and big and $\rho(S') = 1$, D' is ample and $-(K+C+B) = h^*D'$ is semiample. This completes the proof of (WLF) $\Longrightarrow$ (RPC).

By definition, (RPC) implies (NTC).

Because any semiample divisor D defines a contraction that contracts the curves F with $(D.F) = 0$, (ASA) implies (NTC). The converse and other arguments in this proof are related to the semiampleness of log canonical divisors, Theorem 11.1.3 in [16] (it assumes $\mathbb{Q}$-boundaries that can be improved up to $\mathbb{R}$-boundaries as in Theorem 2.7 in [24]). Therefore, let $\nu : X \to Y/Z$ be a numerical contraction. Then, by the semiampleness$/Z$, $K+C+B = \nu^*D$ for a $\mathbb{R}$-divisor on Y that is numerically positive on each curve of Y/Z. It is then easy to see that D is ample$/Z$ and hence semiample$/Z$, except for the case where ν is birational and $Z = \mathrm{pt}$. However, this case is also known because for complete surfaces, $-(K+C+B)$ is ample when $(K+C+B.F) < 0$ on each curve $F \subset S$, which follows from (NV) as explained in Remark-Corollary 2.6.

In the corollary, we only need to verify (NV) when $-(K+C+B)$ is nef$/Z$. We do this now. After a crepant resolution, we assume that $K+C+B$ is log terminal. When $-(K+C+B)$ is numerically big, then (NV) follows from Lemma 6.17 in [23]. On the other hand, if the numerical dimension of $-(K+C+B)$ is $0/Z$, then $-(K+C+B) \sim_{\mathbb{R}} 0/Z$ by Theorem 2.7 in [24]. In other cases, $Z = \mathrm{pt.}$, and $D = -(K+C+B)$ has numerical dimension one, i.e., D is nef, $D \not\equiv 0$, and $D^2 = 0$. We assume that (NTC) and (ASA) do not hold. Otherwise, $-(K+C+B)$ is semiample, as we already know, and (NV) holds. We now reduce (NV) to the case where D is a $\mathbb{Q}$-divisor or, equivalently, B is a $\mathbb{Q}$-divisor (cf. proof in the big case of the Inductive Theorem). If $C+B$ has a big prime component F, then $-(K+C+B-\varepsilon F)$ satisfies (WLF), and $-(K+C+B)$ is not assumed (NTC). On the other hand, each component F of B with $F^2 < 0$ can be contracted by the LMMP or by Artin. If this contraction is crepant, we preserve the numerical dimension, the log terminality, and (NV). Otherwise, $(K+C+B.F) < 0$, and we can decrease the multiplicity of B in F, which, as in the above big case, gives (NV) by (WLF). Therefore, we assume that each F in B with $F^2 < 0$ is contracted, i.e., for each prime component F in B, $F^2 = 0$. For the same reason, such components F are disjoint. Hence, if we slightly decrease each irrational b_i to a rational value, we obtain a $\mathbb{Q}$-boundary $B' \le B$ and a divisor $-(K+C+B')$ that is nef and has the same numerical dimension. Assuming (NV) but not (NTC), we have a unique effective $D' \sim_{\mathbb{Q}} -(K+C+B')$. By the uniqueness, D has positive multiplicities in components F for any such change. Moreover, they are greater than or equal to the change. Therefore, $D \sim_{\mathbb{Q}} -(K+C+B)$, and (NV) holds.

We now assume that D is a $\mathbb{Q}$-divisor. After a minimal crepant resolution, we also assume that S is nonsingular. Then for positive Cartier multiples mD, we have the vanishing $h^2(S,mD) = h^0(S,K-mD) = 0$, at least for $m \gg 0$, because D has a positive numerical dimension. Therefore, by the Riemann–Roch,

$$\begin{aligned} h^0(S,mD) &\ge h^1(S,mD) + mD(mD-K)/2 + \chi(\mathcal{O}_S) \\ &= -mDK/2 + \chi(\mathcal{O}_S) = mD(C+B) + \chi(\mathcal{O}_S) \ge \chi(\mathcal{O}_S). \end{aligned}$$

In particular, $h^0(S,mD) \ge 1$ when $\chi(\mathcal{O}_S) \ge 1$, for instance, when S is rational. Assuming now that S is nonrational, we verify that after crepant blow-downs, S is an extremal or minimal ruling over a nonsingular curve of genus 1. Indeed, we have a ruling $g : S \to E$ over a nonsingular curve of genus 1 or higher because

$D = K + C + B \neq 0$ and $D \neq 0/E$. If g is not extremal, we have a divisorial contraction in a fiber of g, which is positive with respect to D. Hence, after a contraction, we have (WLF). Because S is not rationally connected, this is only possible when $C \neq 0$ by Theorem 9 in [27]. If $C \neq 0$, a classification of log canonical singularities and Corollary 7 in [27] imply that we have a connectedness by rational and elliptic curves in C. Hence, E has genus 1, and C is a section of g. If S is not extremal, then we have a crepant blow-down of a (-1)-curve in a fiber of g.

In addition, $\chi(\mathcal{O}_S) = 0$, and for each F in $C+B$, $(D.F) = 0$ by the above Riemann–Roch. In particular, $C + B$ does not have components in fibers. The cone $\mathrm{NE}(S) = \overline{\mathrm{NE}}(S)$ is a closed angle with two sides:

- R_1 is generated by a fiber of the ruling g and
- R_2 is generated by a multisection F with $F^2 \leq 0$.

We note that $(D.R_1) > 0$ and $D = -(K + C + B)$ is nef. Hence, if $F^2 < 0$, we can take $F = C$ as a section. In this case, we find an effective divisor $D = B' - B$ by increasing a component of $B \neq 0$ or by taking another disjoint section F otherwise. Then $K + C + B + D = K + C + B' \sim_{\mathbb{R}} 0$ by Theorem 2.7 on semiampleness in [24], because $K + C + B'$ is log canonical (Theorem 6.9 in [23]). Finally, if $F^2 = 0$, then $D = -(K + C + B)$, and any component of $C + B$ generates R_2 as well. Moreover, each curve in R_2 is nonsingular and of genus 1. They are disjoint in R_2. There exist at most two of them when (NTC) is not assumed. As above, $D = B' - B$ for some $B' > B$ such that $K + C + B' \equiv 0$. Using [3], it is possible to verify that $K + C + B' \sim 0$ for an appropriate choice or $K + C + B'$ is log canonical, except for the case in Example 1.1. In the latter case, we have a single curve E in R_2, $C + B = aE$ with $a \in [0, 1]$, and $K \sim -2E$. Hence, $D = -(K + C + B) \sim (2 - a)E > 0$. This completes the proof of (NV).

We now suppose (EEC): there exists an effective divisor $E = B' - B = (K + C + B') - (K + C + B) \equiv -(K + C + B)/Z$ which is nef. If E is numerically big$/Z$, then E is semiample$/Z$. This implies (NTC). In other cases, the numerical dimension of E is 1 or $0/Z$. In the latter case, $K + C + B \equiv -E = 0/Z$ is semiample. This implies (ASA) and (NTC). Hence, we assume that E has the numerical dimension $1/Z$ and $Z = \mathrm{pt}$. We can then reduce this situation to the case where E is an isolated reduced component in $C + B'$. Otherwise, E is contractible by the LMMP. Indeed, if $K + C + B'$ is purely log terminal near a connected component F of E, we can increase $C+B'$ in F: for small $\varepsilon > 0$, a log canonical divisor $K+C+B'+\varepsilon F \equiv \varepsilon F$ is log canonical and semiample. This implies (NTC). After a crepant blowup, we can suppose that $K+C+B'$ is log terminal, and a log singularity is on each connected component F of E. Then, essentially by Artin, we can contract the log terminal components in F. Therefore, E is reduced in $C + B'$, and each connected component F of E has the numerical dimension one. We verify that F is semiample when F is not isolated in $C + B'$. This implies (NTC). Let D be another component of B' intersecting F and having a positive multiplicity b in B. We can then subtract a nef and big positive linear combination $\delta D + \varepsilon F$ from $K+C+B'$, which gives the nef and big anti-log divisor $-(K + B'') = -(K + C + B') + \delta D + \varepsilon F$. That is (WLF), which implies (RPC) and (NTC).

Hence, we must now consider the case where $C+B'$ has $F = \mathrm{Supp}\, E$ as a reduced and isolated component. We also assume that $K + C + B'$ is log terminal. Then F has only nodal singularities.

Because E is nef and of numerical dimension one, $E|_F \equiv 0$.

If $E|_F \not\sim_{\mathbb{R}} 0$, we have (EX2) but not (NTC). Nonetheless, in these cases, $K+C+B' \sim 0$ or, equivalently, $K + C + B$ has a 1-complement. In particular, $C + B'$ is reduced. Indeed, if $E|_F \not\sim_{\mathbb{R}} 0$, then F is a curve of arithmetic genus 1. To find the index of $K + C + B'$, we can replace S with its terminal resolution when S is nonsingular. If S is rational, we reduce the problem to a minimal case where $F = C = C + B' \sim -K$. Hence, $K + C + B'$ has index 1. Otherwise, S is not rationally connected. Hence, F is a nonsingular curve with at most two components of genus 1, and S has a ruling $g : S \to G$ over a nonsingular curve G of genus 1. If g has a section G in F, then C has another section G', whereas $C = C+B' = G+G'$ and $K+C+B' = K+C \sim 0$. Indeed, after contractions of (-1)-curves, which do not intersect a component of F, where $E|_F \equiv 0$ but $\not\sim_{\mathbb{R}} 0$, we preserve the last property on an extremal g. It has a section G in F such that $E|_G \not\sim_{\mathbb{R}} 0$ or, equivalently, $G|_G \not\sim_{\mathbb{Q}} 0$. According to [3], this is a splitting case, i.e., G generates the extremal ray R_2, and it has only two curves G and another section G'. The latter holds because R_2 is not contractible. Therefore, we can assume that $F = C + B'$ is a double section of g. This is possible only when the ruling g is minimal. Otherwise, after a contraction of a (-1)-curve in a fiber of g intersecting F, we obtain big $F \equiv -K$. By (ASA), this is impossible for $-K$ because K has no log singularities and S is hence rationally connected. Therefore, S/G is extremal with another extremal ray R_2 generated by F, and $F^2 = 0$. According to a classification of minimal rulings over G, this is a nonsplitting case because (NTC) is not assumed. By Examples 1.1 and 2.1, this is impossible for the other rulings: either we have no complements or we have (NTC) by Corollary 2.2.

We note that (NTC) does not hold in (EX2), for example, because of the uniqueness of B' and E. In these cases, $K + C + B' \sim 0$ and not only $\equiv 0$ by Corollary 2.2. In addition, B' and E are unique in (EX2) because otherwise a weighted linear combination of different complements B' gives a complement $C+B'$ that has no log singularity in a component of F.

In other cases, $E|_F \sim_{\mathbb{R}} 0$. Let G be a connected component of E in a reduced part of $C + B'$. We prove that a multiple of G is then movable at least algebraically. This implies (NTC) because $K + B + C' + \varepsilon G'$ is semiample for small $\varepsilon > 0$ and a divisor G' disjoint from $\operatorname{Supp}(C + B')$, including E, which is algebraically equivalent to a multiple of G. This implies $\equiv$ as well. We note that after log terminal contractions for $K+C+B'$, the LMMP implies that $C+B'$ is reduced or (NTC) holds. Indeed, if B' has a prime component D with multiplicity $0 < b < 1$, then D is in B and disjoint from F. After log terminal contractions of such divisors with $D^2 < 0$, we assume that $D^2 \geq 0$ for others. Because $D^2 > 0$ implies (WLF) and (RPC) for $K + C + B' - \varepsilon D$, $D^2 > 0$ implies (NTC) for E. The same holds for $D^2 = 0$ by the semiampleness of $K+C+B'+\varepsilon D$. Hence, $C+B'$ is reduced after contractions. We can now use a covering trick (Example 2.4.1 in [23]), because $K + C + B' \sim_{\mathbb{Q}} 0$. If S is rational, the latter holds for any numerically trivial divisor. If S is nonrational, then, as in (EX2) above, we verify that $K + C + B' \sim 0$ or $\sim_{\mathbb{Q}} 0$ of index 2 by Corollary 2.2. Therefore, after an algebraic covering, which is ramified only in $C+B'$, we suppose that $K+C+B'$ has index 1. A new reduced boundary is an inverse image of $C+B'$. The same holds for G. After a crepant log resolution, we assume that S is nonsingular and G is a Cartier divisor. The curve G has only nodal singularities and genus 1. Contracting the (-1)-curves in G, we assume that each nonsingular rational component of G is a $(-m)$-curve with $m \geq 2$. Then all such curves are (-2)-curves, and we can take a reduced G. The latter is obvious in other cases, too. Indeed, after a normalization, we suppose that D is a $\mathbb{Q}$-divisor with multiplicities ≤ 1 and with one component D with multiplicity 1. Then $K + C + B' - G$ is log terminal near G, $\equiv 0$ on G, and has multiplicity 0 in D. Because each such D is not a (-1)-curve, the log divisor $K + C + B' - G$ is trivial near G, and G is a reduced Cartier divisor. According to our assumptions and reductions, $G|G \sim_{\mathbb{Q}} 0$. We prove that a multiple mG is linearly movable. First, we suppose that S is rational. Let $mG|G \sim 0$ for an integer $m > 0$. Then by a restriction sequence on G, we have a nonvanishing $h^1(S, (m-1)G) \neq 0$ when G is linearly fixed and the restriction is not epimorphic. Hence, by the Riemann–Roch,

$$\begin{aligned} h^0(S,(m-1)G) &\geq h^1(S,(m-1)G) + (m-1)G((m-1)G - K)/2 + \chi(\mathcal{O}_S) \\ &\geq 1 + 0 + 1 = 2, \end{aligned}$$

and $(m-1)G$ is linearly movable. In other cases, S is not rationally connected. Moreover, F is a nonsingular curve of genus 1 or a pair of them, and there exists a ruling $g : S \to G'$ to a nonsingular curve G' of genus 1. If G is a section of g, then we can suppose that S is extremal/G' after contractions of (-1)-curves, disjoint from G, in fibers $/G'$. By [3], (EEC), and Example 1.1, this is a splitting case, i.e., g has a section G' in $C + B'$, and G' is disjoint from G. Then, as in Example 1.1, we verify that $G - G' \sim_{\mathbb{Q}} 0$ because g is extremal and $(G-G')|_G = G|_G \sim_{\mathbb{Q}} 0$. This means that a multiple of G is linearly movable. In other cases, $C+B' = F = G$ is a double section of g. As above, S/G' is minimal because S is not rationally connected. Then G is again linearly movable when S has a splitting type. If G is a double section, then $C + B' = G$, and we have (NTC) by Corollary 2.2. Indeed, we have no such case after the covering trick.

As in Proposition 5.5 in [23], (ASA) implies (EEC).

Finally, (EC)+(SM) implies (EEC) by the monotonicity result below.

2.7. Monotonicity Lemma. *Let $r = (n-1)/n$ for a natural number $n \neq 0$. Then for any natural number $m \neq 0$,*

$$\lfloor (m+1)r \rfloor / m \geq r.$$

Moreover, for $n \geq m + 1$,

$$\lfloor (m+1)r \rfloor / m = 1.$$

Proof. Indeed, because mr has the denominator n and $r = 1 - 1/n$,

$$\lfloor (m+1)r \rfloor = \lfloor mr + r \rfloor \geq mr.$$

This implies the inequality.

Because $\lfloor (m+1)r \rfloor / m$ has the denominator m, and it is always ≤ 1, we obtain the equation.

Proof of the Inductive Theorem: Big case. We suppose that $-(K + C + B)$ is big (as in Theorem 5.6 in [23]). (Cf. Theorem 19.6 in [16].)

The proof is based on the Kawamata–Viehweg vanishing and the connectedness of the log singularities (Lemma 5.7 and Theorem 6.9 in [23]). Kawamata states that the vanishing works even for $\mathbb{R}$-divisors. In our situation, it is easy to replace B with a new $\mathbb{Q}$-divisor $\leq B$ with the same regular complements. We can make a small decrease of each irrational b_i in prime D_i with $(K + B + C.D_i) < 0$. Finally, B is rational after this procedure. Equivalently, each b_i in D_i with $(K + B + C.D_i) = 0$ is rational. Because $-(K + C + B)$ is big, each such D_i is contractible, and the corresponding b_i is rational.

A higher-dimensional version is similar (cf. Proof of Theorem 7.1). Other cases in the inductive theorem are more subtle and need more preparation. (A higher-dimensional version will be presented elsewhere.) But we first derive some corollaries.

2.8. Corollary. *If (S, B) is a weak log Del Pezzo with $(K + B)^2 \geq 4$, then it has a regular complement.*

Proof. We must find $B' > 0$ such that $(S, B + B')$ is a weak log Del Pezzo but $K + B + B'$ is not Kawamata log terminal. By the Riemann–Roch formula and arguments in the proof of Lemma 1.3 in [22], it follows from the inequality

$$\begin{aligned}(N(-K - B).N(-K - B) - K)/2 &= N(N+1)(K+B)^2/2 + (-K - B.B)/2 \\ &\geq 2N(N+1) > 2N(2N+1)/2.\end{aligned}$$

This implies nonvanishings as in Corollaries 1.16–18, in particular, the following nonvanishing.

2.9. Corollary. *If $(S, 0)$ is a weak Del Pezzo with log canonical singularities and $K^2 \geq 4$, then $|-12K| \neq \emptyset$ or $h^0(X, -12K) \neq 0$.*

2.10. Corollary. *A weak log Del Pezzo (S, B) is exceptional only when $(K + B)^2 < 4$.*

2.11. Definition. Let D be a divisor of a complete algebraic variety X. We say that D has a *type of numerical dimension* m if m is the maximum of the numerical dimensions for effective $\mathbb{R}$-divisors D' with $\operatorname{Supp} D' \subseteq \operatorname{Supp} D$. Similarly, we define a *type of linear dimension* where we replace the numerical dimension of D' with the Iitaka dimension of D'. We note that both $0 \leq m \leq \dim X$. A *big* type is a type with $m = \dim X$.

Here are the basic properties.

- The linear type m is the Iitaka dimension of D when D is an effective $\mathbb{R}$-divisor.
- Such D is movable if and only if D has the linear type $m \geq 1$.
- The linear type is a birational invariant for log *isomorphisms*, i.e., we add exceptional divisors for extractions and contract only divisors in D. In particular, this holds for the extractions and flips.
- For an arbitrary extraction, the numerical type does not decrease.
- For an arbitrary log transform, the linear type does not decrease.

We can prove more for surfaces.

2.12. Proposition-Definition. Let $X = S$ be a complete surface and D be a divisor. Then the numerical type of D is not higher than that of the linear type.

Moreover, if $(S, C + B)$ is a log surface such that

- $K + C + B$ log terminal,
- $K + C + B \equiv 0$, and

• $\operatorname{Supp} D$ is *divisorially* disjoint from $LCS(S, C+B)$, i.e, they have no divisorial components in common, then the numerical type of D is the same as the linear one. In addition to the big type, we have a *fiber* type for a type of dimension one and an *exceptional* type for a type of dimension zero.

More precisely, D is a fiber *geometrically*, i.e., there exists a fiber contraction $g : S \to Y$ with an algebraic fiber $D' = g^*P$ for P in a nonsingular curve Y, having $\operatorname{Supp} D' \subseteq \operatorname{Supp} D$, if and only if D has a fiber type. In addition, D is supported in fibers of g, and g is defined uniquely by D.

Respectively, D is exceptional *geometrically*, i.e., there exists a birational contraction of D to a zero-dimensional locus if and only if D has an exceptional type.

The condition $K + C + B \equiv 0$ can be replaced by (ASA).

Proof. We use the semiampleness (Theorem 2.7 in [24]) for $K + C + B + \varepsilon D' \sim_{\mathbb{R}} D'$ with small $\varepsilon > 0$. We note that $K + C + B \sim_{\mathbb{R}} 0$.

If D has a fiber type, we have a fibering $g : S \to Y$ with a required fiber $D' = g^*P$. If D has a horizontal irreducible component D'', then $D' + \varepsilon D''$ has a big type for small $\varepsilon > 0$. *Horizontal* means not in fibers; *vertical*, in fibers.

If D is of an exceptional type, we can contract D by Artin or by the LMMP (cf. with the proof of Proposition 2.5).

Finally, if $K+C+B$ satisfies (ASA), we have a numerical complement $K+C+B' \equiv 0$ with the required properties.

Proof of the Inductive Theorem: Strategy. In this section, we assume that $Z = \text{pt}$. We discuss the local cases where $\dim Z \geq 1$ in Sec. 3, where we obtain better results.

Using a log terminal blowup (Lemma 5.4 in [23] and Example 1.6), we reduce the problem to the case where $K + C + B$ is log terminal. By our assumption, it has a nontrivial reduced component $C \neq 0$. We want to induce complements from lower dimensions (in most cases, from C).

First, we prove the theorem in

Case I: C is not a chain of rational curves.

We then assume that C is a chain of rational curves.

By Proposition 2.5, we can assume (NTC) or, equivalently, (ASA). Therefore, we have a numerical contraction $\nu : S \to Y$ for $-(K + C + B)$, where Y is a nonsingular curve or a point, the latter for a *while*. We have regular complements in the exceptional cases. Let κ^* be the numerical dimension of $-(K + C + B)$ or, equivalently, $\dim Y$. We construct complements in different cases according to the configuration of $C + B$ with respect to ν. For $\kappa^* = 1$, we distinguish two cases:

Case II: ν has a multisection in C;

Case III: C is in a fiber of ν.

For $\kappa^* = 0$, most of the cases are inserted in the above cases. A pair $(S, K + C + B)$ with $\kappa^* = 0$ is considered as Case II, where $D = \operatorname{Supp} B$ has a big type. As we see in the proof below, ν naturally arises when we need it, and such a ν is a contraction to a curve. Such a pair is considered as Case II, where D has a fiber type and $C + D$ is of a big type. Equivalently, C has a multisection for $\nu = g$ with g of Proposition 2.12. Moreover, C then has a double section of g. In this case, we have a complement by Lemma 2.21. Such a pair is considered as Case III, where D has a fiber type but now $C + B$ and D sit only in fibers of $\nu = g$. One component of D gives a (geometric) fiber of ν. The contraction $\nu = g : S \to Y$ here plays the same role as ν in Case III with $\kappa^* = 1$. Finally,

Case IV: D has an exceptional type.

We then contract the boundary B to points.

Case I could be distributed in the other cases, but we prefer to simplify the geometry of C, which slightly simplifies the proofs of Cases II–IV.

We try to reduce each case to the big case for $K + C + B$ or for $K + C + B'$ with $B' = \lfloor (n+1)B \rfloor / n$ when a complement index n is suggested. There are two obstacles here. First, we cannot change B or B' in such a way, for instance, in Case III with curves of genus 1 in fibers. This leads to a separation into cases. Second, we cannot preserve complements of some indices, for instance, when decreasing B, as Lemma 2.16

shows. Here is a situation as in Example 2.4, and it is a main difficulty in Case II, when we try to induce a (regular) complement from C. We use Lemmas 2.20 and 2.21 below to resolve this difficulty. In the cases where we cannot induce regular complements, we find others by Lemmas 2.27–29. This occurs only when (M) is not assumed.

In Case III, the most difficult situation, where ν has curves of genus 1 in fibers, is quite concrete. We then use the Kodaira classification of degenerate fibers. We also *indirectly* use a Kodaira formula (p. 161 in [5]) for the canonical divisor K.

A possible alternative approach to Cases II–III is to directly use an analogue of Kodaira's formula,

$$K + C + B \sim_{\mathbb{R}} \nu^*(K_Y + \nu_*((K + C + B)_{S/Y}) + B_Y), \tag{1}$$

for a certain boundary B_Y, which can be found locally/Y. We assume that $K + C + B \equiv 0/Y$. Therefore, this is a fibering of genus-1 log curves. In an arbitrary dimension $n = \dim S$, formula (1) with a boundary D instead of $\nu_*((K + C + B)_{S/Y}) + B_Y$ was proposed, but not proved, in the first draft of the paper. Formula (1) also plays an important role in a proof of an adjunction in codimension n. In this context, a similar formula was proved in our surface case by Kawamata [13], where B_Y corresponds to a divisorial part and $f_*((K + C + B)_{S/Y})$ to a moduli part. However, we have three difficulties in its application. First, a relative log canonical divisor $(K+C+B)_{S/Y}$ is nowhere defined, and its properties are nowhere to be found. Second, the divisorial part is given, but not very explicitly. Third, we have $\sim_{\mathbb{R}}$ or $\sim_{\mathbb{Q}}$ for $\mathbb{Q}$-boundaries. Therefore, we must control indices for a complement on (Y, B_Y) and for $\sim_{\mathbb{Q}}$ in (1) for an induced complement.

Finally, in Case IV, we decrease C (cf. proof of Corollary 1.8) and use a covering trick. In this case, we have trivial regular complements.

A modification of a log model may change *types*, i.e., possible indices, of complements. Nonetheless, for blow-down, we can always induce a complement of the same index from above by Lemma 5.4 in [23]. The converse does not hold in general. For instance, there are no complements after many blowups in generic points. Lemmas 2.13 and 4.4 give certain sufficient conditions for when we can induce complements from below.

2.13. Lemma (Cf. Lemma 5.4 in [23]). *In the notation of Definition* 5.1 *in [23], let $f\colon X \dashrightarrow Y$ be a birational contraction such that $K_X + S + \lfloor (n+1)D \rfloor / n$ is numerically nonnegative on a* sufficiently general *curve/Y in each exceptional divisor of f. Then*

$$K_Y + f(S + D) \text{ } n\text{-complementary} \implies K_X + S + D \text{ } n\text{-complementary}.$$

By a *sufficiently general* curve in a variety Z, we mean a curve that belongs to a covering family of curves in Z (cf. Conjecture in [25]). We note that for such a curve C and any effective $\mathbb{R}$-Cartier divisor D, $(D.C) \geq 0$.

2.14. Example-Corollary. If $S + D = S$ is in a neighborhood of exceptional locus for f and $K_X + S + D$ is nef/Y, then we can pull back the complements, i.e., for any integer $n > 0$,

$$K_Y + f(S + D) \text{ } n\text{-complementary} \implies K_X + S + D \text{ } n\text{-complementary}.$$

2.15. Negativity of a proper modification (Cf. Negativity 1.1 in [23]). *Let $f : X \to S$ be a proper modification morphism and D be a $\mathbb{R}$-Cartier divisor. If*

i. *f contracts all components E_i of D with negative multiplicities,* i.e., such components are exceptional *for f, and*

ii. *D is numerically nonpositive on a* sufficiently general *curve/S in each exceptional divisor E_i in* i,

then D is effective.

Proof. First, it is a local problem/S.

Second, according to Hironaka, we can assume that f is projective/S and X is nonsingular, in particular, $\mathbb{Q}$-factorial. The pullback of D satisfies the assumptions. The proper inverse image of a sufficiently general curve is again sufficiently general.

Third, there exists an effective Cartier divisor H in S such that *iii.* the support of f^*H contains the components of D with negative multiplicities and

iv. $f^{-1}H$ is positive on the sufficiently general curves in *ii* (this is meaningful because $f^{-1}H$ is Cartier).

For example, we can take H as a general hyperplane through the direct image of an effective and relatively very ample divisor$/S$ and through the images of the components of D with negative multiplicities.

According to *iii–iv*, Cartier divisor $E = f^*H - f^{-1}H$ is effective with positive multiplicities for the components of D having negative multiplicities and

v. negative on the sufficiently general curves in *ii*.

Therefore, there exists a positive real r such that $D + rE \geq 0$ and has an exceptional component E_i with 0 multiplicity unless $D \geq 0$. However, then $(D + rE.C) \geq 0$ for a general curve C in E_i, which contradicts *ii* and *v*.

Proof of Lemma 2.13. We take a crepant pullback:

$$K_X + D^{+X} = f^*(K_Y + D^+).$$

It satisfies 5.1.2–3 in [23] as $K_Y + D^+$, and we must verify 5.1.1 in [23] only for the exceptional divisors. For them, it follows from our assumption and Negativity 2.15.

2.16. Monotonicity Lemma. *Let r be real and n be a natural number. Then*

$$\lfloor (n+1)(r-\varepsilon) \rfloor / n = \lfloor (n+1)r \rfloor / n$$

for any small $\varepsilon > 0$ if and only if $r \notin \mathbb{Z}/(n+1)$.

2.16.1. *We note that for $r = k/(n+1) > 0$, we have $\lfloor (n+1)r \rfloor / n = k/n > r$. Therefore, $r > 0$ is not in $Z/(n+1)$ if $\lfloor (n+1)r \rfloor / n = k/n \leq r$.*

Proof of the Inductive Theorem: Case I. By (ASA) and Proposition 5.5 in [23], we can assume that $K+C+B \equiv 0$. Then by Theorem 6.9 in [23] or by the LMMP and Lemma 5.7 in [23], C has a single connected component, except for the case where C consists of two nonsingular disjoint irreducible components C_1 and C_2 such that S has a ruling $g : S \to C_1 \cong C_2/Y$ with sections C_1 and C_2 (cf. Theorem 6.1 below). By Lemma 5.7 in [23], the existence of the ruling follows from the LMMP applied to $K + C_2 + B$. A terminal model cannot be a log Del Pezzo surface, because C_i is always disjoint and nonexceptional during the LMMP (cf. Theorem 6.7).

In this case, we can construct complements to $K + C + B$ with a given $C = C_1 + C_2$. (Cf. Lemma 2.20 and 2.21 below.) Making a blowup of a generic point of C_1 and then contracting the complement component of the fiber, we reduce to the case where C_2 is big. Then for small rational $\varepsilon > 0$, $-(K + C + B - \varepsilon C_2) = -(K+C+B)+\varepsilon C_2$ is nef and big, which gives the required complement by the big case above. More precisely, we have the same complement as $K_{C_1} + (B)_{C_1}$ on C_1, where $(B)_{C_1}$ denotes a different (Adjunction 3.1 in [23]). The blowups preserve complements by Lemma 5.4 in [23]. For contractions, we can use Example 2.14. In this case, it is easily verified directly as well.

We therefore assume that C is connected. We also assume that each component of C is a nonsingular rational curve.

Otherwise, C is a (nonsingular, if we want) irreducible curve of genus 1 with $B = 0$ and nonsingular S in a neighborhood of C (Properties 3.2 and Proposition 3.9 in [23]). In this case, we suppose that S is nonsingular everywhere after a minimal resolution. We chose a B that is crepant for the resolution. Then the LMMP and a classification of contractions in a two-dimensional minimal model program gives a ruling $f: S \to C'$ with a surjection $C \to C'$ or $S = \mathbb{P}^2$ with a cubic $C + B = C$. Because a blowup in any point of $\mathbb{P}^2$ gives a ruling, $\mathbb{P}^2$ is the only possible case, where $S = \mathbb{P}^2$ and $C = C + B \sim -K_{\mathbb{P}^2}$. Hence, $K + C + B = K + C \sim 0$ on $S = \mathbb{P}^2$, and we have a 1-complement. Similarly, a 1- or 2-complement holds in the case with the ruling f and double covering $C \to C'$ (cf. Lemma 2.21). We want to verify only that $B = 0$ and $2(K + C + B) = 2(K + C) \sim 0$ but not only $\equiv 0$. After contractions of exceptional curves of the first kind intersecting C in fibers of f, we can suppose that f is extremal, i.e., with irreducible fibers. This preserves all types of complements by Example 2.14. We also note that the same reduction holds for the ruling $f: S \to C' = C$ having a section C.

Because $K+C+B \equiv 0$ and the covering $C \to C'$ is double, we have no boundary components in fibers of f and $B = 0$. Therefore, if C is rational, then f is a rational ruling $\mathbb{F}_n$, and $K+C+B = K+C \sim 0$. Otherwise, C' has genus 1, and $(K+C)$ is 1- or 2-complementary by Corollary 2.2. If S is a surface of Example 2.1, then $C = C_i$ and we have a 2-complement. The surface S of Example 1.1 is impossible by (ASA). If f is a splitting case, $-K_S \sim G + G'$. Therefore, $C \equiv G + G'$, as well as $C \sim 2G$ and $2G'$ because $C \cap G = C \cap G' = G \cap G' = \emptyset$. Therefore, $2C \sim 2(G+G') \sim -2K_S$, which gives a 2-complement (but not a 1-complement).

Next, we consider the case where $C = C'$ is a nonsingular curve of genus 1 and a section of f. The boundary $B = \sum b_i D_i \neq 0$ has only horizontal components D_i. The curves D_i are nonrational. Hence, $D_i^2 \geq 0$ by the LMMP. On the other hand, we have a trivial n-complement for $(K+C+B)|_C = K_{C'} \sim 0$ for any natural number n. Therefore, we have an n-complement when we have $D_i^2 > 0$ with multiplicity $b_i \notin \mathbb{Z}/(n+1)$. Indeed, we can then construct an n-complement, as in the big case, for $(K+C+B-\varepsilon D_i)$ with small $\varepsilon > 0$. For $(K+B+C)$, we have the same complement by Monotonicity Lemma 2.16. Because $\mathbb{Z}/2 \cap \mathbb{Z}/3 = \emptyset$ in the unit interval $(0,1)$, $K+C+B$ is 1- or 2-complementary when some $D_i^2 > 0$. Otherwise, all $D_i^2 = 0$. By Corollary 2.2, the curves D_i and C' are in R_2, and are all disjoint nonsingular curves (of genus 1). Hence, it is sufficient to find an n-complement in the generic fiber$/Z$, which we have for $n = 1$ or 2 by Example 5.2.1 in [23].

Finally, we suppose that C is a (connected) wheel of a rational curve. Then, by the arguments in the case where C is a nonsingular curve of genus 1, we see that S is a rational ruling $S \to C'$ with a double covering $C \to C'$ or $S = \mathbb{P}^2$ with a cubic $C + B = C$. In both cases, we have a 1-complement.

2.17. Monotonicity Lemma. *Let $r < 1$ be a rational number with a positive integer denominator n and m be a positive integer such that $n|m$. Then*

$$\lfloor (m+1)r \rfloor / m \leq r.$$

Moreover, the equality holds if and only if $r \geq 0$.

Proof. Let $r = k/n$. Then

$$\lfloor (m+1)r \rfloor / m = \lfloor (m+1)k/n \rfloor / m = (km/n + \lfloor k/n \rfloor)/m = r + \lfloor r \rfloor / m \leq r,$$

and the equality holds if and only if $r \geq 0$.

2.18. Corollary. *Let m be a natural number and D be a subboundary of index m in codimension one and without a reduced part, i.e., mD is integral with multiplicities $< m$. Then*

$$mD \geq \lfloor (m+1)D \rfloor,$$

and the equality holds if and only if D is a boundary.

2.19. Lemma. *Let $C = \lfloor C+B \rfloor$ be the reduced component in a boundary $C+B$ on a surface S and $C' \subseteq C$ be a complete curve such that*

i. $K+C+B$ is (formally) log terminal in a neighborhood of C',
ii. $(K+C+B)|_{C'}$ has an n-complement, and
iii. $-(K+C+B)$ is nef on C'.

Then $(C_i.K+C+D) \leq 0$ on each component $C_i \subseteq C'$ with $D = \lfloor (n+1)B \rfloor / n$, and $K+C+D$ is log canonical in a neighborhood of C'.

In *i*, *formally* means locally in an analytic or etale topology. This can be defined formally as well.

Proof. First, we can suppose that C' is connected.

Second, by the proof of Theorem 5.6 in [23] (cf. Proof of the big case in the Inductive Theorem and that of the local case in Sec. 3), the lemma holds when C' is contractible because then we have an n-complement in a neighborhood of C'.

Third, it is sufficient for an analytic contraction because under our assumptions in most cases, it is algebraic by Artin or by the LMMP. It works, and we obtain a rational singularity after the contraction when C' is not isolated in $C+B$. Otherwise, $B=0$ and $C+B=C+D=C'$. Then the lemma follows from *i* and *iii*.

Finally, the contraction exists when a certain numerical condition on the intersection form on C' is satisfied. This is negative after sufficiently many monoidal transforms in generic points of C'. Such a crepant pullback of $K+C+D$ preserves the assumptions *i–iii* and the statements.

2.20. Lemma. *Let $(S, C+B)$ be a complete log surface with a ruling $f\colon S \to Z$ such that*

i. *there exists a section $C_1 \hookrightarrow S$ of f which is in the reduced part C,*
ii. *$(K+C+B)|_C$ has an n-complement for some natural $n>0$,*
iii. *$C+D=C+\lfloor (n+1)B)\rfloor/n$ gives an n-complement near the generic fiber of f,* i.e., *$K+C+D$* is numerically trivial on it,
iv. *$-(K+C+B)$ is nef, and*
v. *$K+C+B$ is (formally) log terminal in a neighborhood of C if we do not assume that $K+C+B$ is log canonical everywhere but just $C+B\geq 0$ outside C.*

Then $K+C+B$ has an n-complement.

Proof. The above numerical property *iv* and Theorem 6.9 in [23] imply that $K+C+B$ is log canonical everywhere (cf. the proof of Theorem 5.6 in [23]).

Making a crepant log blowup, we can assume that $K+C+B$ is log terminal everywhere, essentially by Lemma 5.4 in [23]. Because f is a ruling, $\overline{\mathrm{NE}}(S/Z)$ is rational polyhedral and generated by curves in fibers of f (cf. (EX1) in 2.5). We note that any contraction of a curve $E \not\subseteq C$ in fibers of f preserves *i–v*: *ii* by Lemma 5.3 in [23] because the boundary coefficients of $(K+C+B)|_C$ are not increasing. This implies v as well.

We simultaneously consider the boundary $C+D$ as in *iii*. After contractions of curve $E \not\subseteq C$ in fibers of f with $(E.K+C+D)\geq 0$, we can suppose that $-(K+C+D)$ is nef. Indeed, this is true for the fibers and on section C_1. Because $K+C+D\equiv 0/Z$, applying the LMMP to f, we can suppose that f is extremal. Then $\overline{\mathrm{NE}}(S)$ is generated by a fiber and a section. We note that $(C_i.K+C+D)\leq 0$ for the curves $C_i \subseteq C$ by Lemma 2.19.

It is sufficient to construct an n-complement after such contractions by Lemma 2.13.

The boundary coefficients of D belong to $\mathbb{Z}/n$. In addition, by *iii*

vi. $K+C+D$ is numerically trivial$/Z$.

Therefore, again by Lemma 2.13, we can assume that the fibers of f are irreducible or in C. Because C_1 is a section, we increase $C+D$ in fibers to B^+ such that $(K+C+B^+)|_{C_1}$ is given by an n-complement in *ii*. We contend that $K+B^+$ gives an n-complement of $K+C+B$, too.

First, we note that 5.1.1 in [23] holds by construction.

Second, as above, $K+B^+\equiv 0$ because this is true for the fibers and section C_1.

Third, $K+B^+$ is log canonical in a neighborhood of C_1 by inverse adjunction (Corollary 9.5 in [23]). Therefore, as above, $K+B^+$ is log canonical everywhere, i.e., 5.1.2 in [23].

Finally, we must verify that $n(K+B^+)\sim 0$. In particular, this means that nB^+ is integral. Because the log terminal singularities are rational as well as any contractions of curves in fibers of f, we can replace $(S/Z, C+B^+)$ by any other crepant birational model. For instance, we can suppose that S is nonsingular and all fibers of f are irreducible. Then S is a nonsingular minimal ruling$/Z$ with the section C_1. In that case, $n(K+B^+)$ is integral and ~ 0 by the contraction theorem because these hold for $n(K+B^+)|_{C_1}$. The latter is preserved after any crepant modification above.

2.21. Lemma. *Lemma* 2.20 *holds even if we drop* iii *and simultaneously change* i *in it to*

• *there exists a curve C_1 in C with a covering $C_1 \to Z$ of degree $d \geq 2$,*

except for the case $C + B = C_i$ in Example 2.1, when n is odd. Moreover, then $d = 2$ always. In the exceptional case, we have a $2n$-complement for any natural number n.

2.22. Lemma. *Let $f\colon X \to Y$ be a conic bundle contraction with a double section C. If a divisor $D \equiv 0$ over its generic points of codimension two in Y and for any component C_i of C, $C_i \not\subseteq D$, then the different D_{C^ν} is invariant under the involution I given by the double covering $C^\nu \to Y$ on the normalization C^ν.*

2.22.1. *The same holds for C, assuming that $K+C$ is log canonical in codimension two* (cf. Theorem 12.3.4 in [16]).

Proof-Commentary. First, taking hyperplane sections, we reduce the lemma to the case of a surface ruling $X \to Y$ with a double curve C^ν over Y.

Second, we can drop D, because it is pulled back from Y.

Finally, according to the numerical definition of the different [23] and because $K + C \equiv 0/Y$, we can replace X by any crepant model (X, D). In particular, we can suppose that X is nonsingular with an extremal ruling f. According to M. Noether, we can assume that C is nonsingular as well. Then $D \equiv 0/Y$ because it is supported in fibers of f. Therefore, we can drop D again.

In this case, the different is 0 and invariant.

The same works for 2.22.1. We need the log canonical condition on $K+C+D$ only to define $(K+C+D)|_C$.

2.23. Lemma. *Let C_1 be a component of a seminormal curve C with a finite Galois covering $f\colon C_1 \to C'$ of* a main type $\mathbb{A}$, *and let B be a Weil $\mathbb{R}$-boundary supporting in the normal part of C and Galois invariant on C_1. Then $K + B$ has an n-complement that is Galois invariant on C_1 if and only if it has an n-complement.*

Proof-Commentary. The type $\mathbb{A}$ means that we have branchings at most over two points Q_1 and $Q_2 \in C'$ in each irreducible component of C'.

According to Example 5.2.2 in [23], $K + B$ has an n-complement if and only if

$$K + \lfloor B \rfloor + \lfloor (n+1)\{B\} \rfloor / n$$

is nonpositive on all components of C. This is a numerical condition that can be preserved if we first replace C with C_1 and even with any irreducible component with the Galois covering $f : C = C_1 \to C'$ given by the stabilizer of this component. We include intersections with other components into the boundary with multiplicity 1. We also assume that f is not an isomorphism.

If C is singular, $B^+ = 0$ is invariant. If C is nonsingular, by Monotonicity Lemma 2.17, we suppose that nB is integral. If $K + B \equiv 0$, then we have the required complement by the above criterion, and it is invariant by our conditions.

Otherwise, $\deg(K + B) < 0$, and C is rational. Then under our conditions on the branchings, we have them only over two points Q_1 and $Q_2 \in C'$. In addition, we have unique points P_1/Q_1 and P_2/Q_2 with maximal ramification indices $\deg f - 1$. Indeed,

$$-2 = \deg K_C = (\deg f)(K_{C'} + (\frac{r_1 - 1}{r_1})Q_1 + (\frac{r_2 - 1}{r_2})Q_2) = -(\deg f)(\frac{1}{r_1} + \frac{1}{r_2}),$$

where $r_i | \deg f$ is the ramification multiplicity in P_i/Q_i. Therefore, we can maximally extend B in P_1 and P_2 preserving the following properties:

- B is Galois invariant,
- nB is integral, and
- $\deg(K + B) \leq 0$.

Then $\deg(K + B) = 0$, because $\deg K = -2$. By the numerical condition, we are done.

2.24. Example. For other types of Galois action, we can lose Lemma 2.23.

We consider, for example, a type $\mathbb{D}$. In this case, we have a Galois covering $f\colon C \to C'$ such that

- the curves C and C' are isomorphic to $\mathbb{P}^1$,
- f is branching over three points Q_1, Q_2, and Q_3,
- f has two branching points $P_{1,1}$ and $P_{1,2}/Q_1$ with multiplicities d, where $\deg f = 2d \geq 4$, and
- f has d simple branching points $P_{i,1}, ..., P_{i,d}/Q_i$ with $i = 2$ and 3.

Therefore, if $d = 2m + 1$ is odd and $n = d$, then for

$$B = \frac{m}{d}(P_{1,1} + P_{1,2}) + \frac{1}{d}\left(\sum P_{2,i}\right),$$

$\deg(K+B) = -2+2m/d+d/d = -1/d$, whereas B is invariant. However, any d-complement is $B+(1/d)P$, which is not invariant for any choice of the point P.

Proof of Lemma 2.21. By *iv* in 2.20, we have $d = 2$. Therefore, $D = 0$ near a generic fiber of f, and *iii* in 2.20 is satisfied. Hence, by Lemma 2.20, we can assume that C_1 is irreducible.

By *v* in 2.20 and Lemma 3.6 in [23], $C_1 = C_1^\nu$ is *nonsingular*, except for the case where C_1 is a Cartesian leaf, i.e., an irreducible curve of arithmetic genus 1 with a single nodal singularity. In such a case, $C + B = C + D = C_1$, and $K + C + B \equiv 0$ by the LMMP. This gives an n-complement for any n because S is rational.

The double covering $C_1 \to Z$ is given by an involution $I\colon C_1 \to C_1$. By *iii* in 2.20, $K + C + B \equiv 0/Z$. Therefore, any contraction in fibers/Z is crepant. They preserve the different $(C - C_1 + B)_{C_1}$. Therefore, we can suppose that S/Z is extremal. Then $D = C - C_1 + B = (K+C+B) - (K+C_1) \equiv -(K+C_1) \equiv 0/Z$ and according to Lemma 2.22, $(C - C_1 + B)_{C_1}$ is invariant under I. Therefore, it has an invariant n-complement by Lemma 2.23 when C_1 is rational.

Otherwise, $C_1 = C + B$ is a nonsingular curve of genus 1, and we again have an invariant n-complement for any n: 0.

We can then construct B^+ as in the proof of Lemma 2.20 and, after a reduction to extremal f, verify that B^+ gives an n-complement, except for the case where S is nonrational and, by Corollary 2.2, n is odd. In the latter case, by the same corollary, we are in the situation of Example 2.1. Indeed, in a splitting case, $n(K + C + D) = n(K + C + B) = n(K + C_1) = n(K_S + G + G') \sim 0$ for any n. On the other hand, in the exceptional case, $2n(K + C + D) = 2n(K + C + B) = 2n(K + C_1) = 2n(K_S + C_i) \sim 0$ for any n.

2.25. Lemma. *If, in a surface neighborhood S of a point P, we have a boundary $B = C + \sum b_i D_i$ with distinct prime divisors D_i such that*

i. *C is a reduced and irreducible curve through P,*
ii. *each $b_i = (m-1)/m$ for some integer $m > 0$, and*
iii. *$K + B$ is log terminal,*

then

$$(K + B)|_C = \frac{n-1}{n}P$$

for some integer $n > 0$, and $m \mid n$.

2.25.1. *Formally, at most one component D_i with $b_i > 0$ passes through P. If we replace* iii *with*

iv. *$K + B$ is log canonical but not formally log terminal in P,*

then

$$(K + B)|_C = P,$$

and, formally, at most two components D_i with $b_i > 0$ pass through P. Moreover, both have multiplicities $b_i = 1/2$ when we have two of them.

Proof. By Theorem 5.6 in [23] and *i–iii*, $K+B$ has a 1-complement $K+C+\sum D_i$ in a neighborhood of P. Therefore, $K + C + \sum D_i$ is log canonical there.

Therefore, by Corollary 3.10 in [23] in a neighborhood of P, a single divisor D_i passes through P, and $D_i|_C = 1/l$, where l is the index of P. Therefore,

$$(K + B)|_C = \left(\frac{l-1}{l} + \frac{m-1}{ml}\right)P = \frac{n-1}{n}P$$

with $n = lm$ (cf. the proof of Lemma 4.2 in [23]).

Statement 2.25.1 follows from the above calculations and Corollary 9.5 in [23].

2.26. Corollary. *If a pair (X, B) is log canonical and B satisfies (M) or (SM), then for any reduced divisor C in the reduced part $\lfloor B \rfloor$, the different $(B - C)_C$ respectively satisfies (M) or (SM).*

Proof. Hyperplane sections reduce the proof to the surface case $X = S$. It is sufficient to consider the log terminal case locally. For (M), we note that $(l7 - 1)/l7 \geq 6/7$ for any natural number $l = n/7$.

2.27. Lemma. *Let $b_i \in (0, 1)$ and n be a natural number. Then the pairs* i, ii *and* iii, iv *of the conditions below are equivalent:*

i. $\sum b_i \leq 1$,
ii. $\sum \lfloor (n+1) b_i \rfloor / n > 1$,
iii. $b_i \in \mathbb{Z}/(n+1)$, *and*
iv. $\sum b_i = 1$.

Proof. The inequality *ii* holds if and only if for natural numbers k_i, $b_i \geq k_i/(n+1)$ and $\sum k_i \geq n+1$. Hence, by *i* we have *iii* and *iv*: $1 \geq \sum b_i \geq \sum k_i/(n+1) \geq 1$. The converse follows from the same computation.

2.28. Corollary. *In the notation of Example* 5.2.2 *in [23], let $X = C$ be a chain of rational curves with the boundary B. Then (X, B) is* 1*-complementary but not* 2*-complementary only when*

- *all* $b_i' \in \mathbb{Z}/3$ *and* $\deg B' = 1$, *or*
- *all* $b_i'' \in \mathbb{Z}/3$ *and* $\deg B'' = 1$.

In either of these cases, we have 4*- and* 6*-complements.*

Proof. The proof is as in Example 5.2.2 in [23].

2.29. Lemma. *In the notation of Example 5.2.1 in [23], let $X = C$ be an irreducible rational curve with the boundary B and $n \in N_1$ be the minimal complementary index for (X, B). Then (X, B) is $(n+1)m$-complementary for a bounded m. More precisely, if $\deg B < 2$, then*

for $n = 1$, $m \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11\}$;
for $n = 2$, $m \in \{1, 2, 3, 4, 5, 6, 7, 8, 10\}$;
for $n = 3$, $m \in \{1, 3, 4, 5, 6\}$;
for $n = 4$, $m \in \{2, 3, 4, 5, 6, 8\}$; *and*
for $n = 6$, $m \in \{3, 4, 5, 6, 8\}$.

Proof-Remark. The proof is as in Example 5.2.2 in [23]. But it is better to use a computer. There exists a program by Anton Shokurov for $\deg B < 2$, which can be easily modified for $\deg B = 2$.

On the other hand, we can decrease B. Then by Lemma 2.16, we have the same complementary indices $(n+1)m$ as in the case with $\deg B < 2$, except for the case where $\deg B = 2$ and B has the index $(n+1)m+1$. Then we have an $((n+1)m + 1)$-complement.

From a theoretical standpoint, under the assumption that the number of elements in $\operatorname{Supp} B$ is bounded, we can prove this result for any n, if we verify that for any B, there exists an $(n+1)m$-complement. Of course, here is the difficult case for $\deg B = 2$ and B having irrational multiplicities, which has been done in Example 1.11.

Finally, we expect that the lemma holds for arbitrary n and without the assumption that n is the minimal complementary index. For the latter, if $\deg B < 2$, a computer check shows that

for $n = 2$, $m \in \{1, 2, 3, \ldots, 15, 16, 18\}$ and
for $n = 3$, $m \in \{1, 2, 3, \ldots, 24, 25, 27\}$.

Addition in a proof of 2.3.1. To obtain the indices in 2.3.1, we unify $(n+1)m$ and $(n+1)m+1$ for n and m in Lemma 2.29.

Addition in a proof of 2.3.2. To obtain the indices in 2.3.2, we add a 6-complement of Lemma 2.28, which follows from the proof below.

Proof of the Inductive Theorem: Case II. Here, we assume that C has a multisection C_1 of ν. If C_1 is not a section, we have a regular complement from C_1 by Lemma 2.21 and Examples 5.1.1–2 in [23]. In the exceptional cases, we take $n=1$ or 2, which gives regular complements again.

Hence, we can assume that C has a single section C_1. We note that C is then connected by Theorem 6.9 in [23], and by our assumptions, C is a chain of rational curves. We also assume that B has a big type when $\kappa^*=0$.

We also note that $B\neq 0$: B has horizontal components for any contraction on a curve when $\kappa^*=0$. Otherwise, C is a double section.

Let $B_C=(B)_C$ be the different for the adjunction $(K+C+B)|_C=K_C+B_C$. According to our construction, the numerical dimension of $-(K+C+B)$ is then equal to that of $-(K_C+B_C)$, i.e., equal to κ^*, at least on C_1.

We suppose that K_C+B_C is n-complementary and $\kappa^*=1$. Let b_i be a multiplicity of B in a horizontal component D_i such that $b_i\notin\mathbb{Z}/(n+1)$. Then $-(K+C+B)+\varepsilon D_i=-(K+C+B-\varepsilon D_i)$ is big and has the same n-complements by Monotonicity Lemma 2.16. Therefore, $K+C+B$ is n-complementary unless all horizontal $b_i\in\mathbb{Z}/(n+1)\cap(0,1)$. In this case, we find a complement for another index $m\in(n+1)\mathbb{N}$ by Lemmas 2.17 and 2.20. We have an m-complement for such m by Corollary 2.28 and Lemma 2.29. If C is not irreducible, we have just regular complements. Example 2.4 shows that we need nonregular complements as well.

In addition, for multiplicities under (M) or (SM), we have only regular complements. Indeed, then $b_i=1/2$ because $K+C+B\equiv 0/Z$. By Example 5.2.2 in [23], we have a (formally) regular complement for $K+C+B$, except for the case with $n=1$, when K_C+B_C is 1-complementary but not n-complementary for all other regular indices n. We note that B_C also satisfies (M) and (SM) by Corollary 2.26. Therefore, it is possible only when C is irreducible (otherwise we have a 2-complement) and B_C is reduced. Additionally, by Example 5.2.1 in [23] and in its notation, $b_1\geq b_2\geq 1/2$ and all other $b_i=0$. Hence, we have a 2-complement, which concludes the case under (M) and (SM).

The same holds for an appropriate ν when $\kappa^*=0$. Indeed, we have (RPC) by Proposition 2.5 because $(K+B+C)-\varepsilon D'$ satisfies (WLF) for some D' with $\operatorname{Supp} D'=\operatorname{Supp} B$ and small $\varepsilon>0$. We then use arguments in the proof of Lemma 2.20. We mean that we contract exceptional curves E with $(K+C+B'.E)>0$, where $B'=\lfloor(n+1)B\rfloor/n$. This preserves the situation and n-complements for the same reasons. Also, (RPC) is preserved. For a terminal model, we have either a fiber contraction such that $(K+C+B'.E)>0$ for the generic fiber or $-(K+C+B')$ is nef on the model. In the former case, the contraction induces a contraction ν as for $\kappa^*=1$. Indeed, C is a section of ν. Otherwise, C is in a fiber of ν. After the above contractions, $(K+C+B'.C)>0$, which contradicts Lemma 2.19.

In the other cases, we assume that $-(K+C+B')$ is nef after such contractions. However, $K+C+B'$ may not be log terminal but just log canonical and only near C. Therefore, if we want to use Lemma 2.20 and the big case, we need the following preparation. We contract all connected components of the exceptional type in B'. By Proposition 2.12, we then have a semiample divisor D' with $\operatorname{Supp} D'=\operatorname{Supp} B'$. We contend that Lemmas 2.20–21 and the big case can be applied in that situation with C instead of $\lfloor C+B'\rfloor$. In the lemmas, we suppose that C has a multisection for a given contraction $f: S\to Z$ and $K+C+B'\equiv 0/Z$. To verify the lemmas and the big case, we replace $K+C+B'$ with $K+C+B'-\varepsilon D'$. The condition *ii* in 2.20 follows from that for $K+C+B$ and $K+C+B'$ by Lemma 5.3 in [23]. Because $K+C+B'\equiv 0/Z$, we have *iii* in 2.20 by Lemma 2.17. Conditions *iv* and *v* follow from the construction. For instance, *v* holds because we otherwise have a log canonical, but not log terminal, point $P\notin\operatorname{Supp} B'$. Then P is not log terminal for $K+C$ and $K+C+B$. But that is impossible by the LMMP for $K+C+B$ versus $K+C+B'$. Finally, $K+C+B'$ and $K+C+B'-\varepsilon D'$ have the same n-complements for small $\varepsilon>0$ by Monotonicity Lemma 2.16.

We continue the case with nef $-(K+C+B')$. If $-(K+C+B')$ is big, we have an n-complement as in the big case as well for $K+C+B'$ and $K+C+B$ by our construction. If $-(K+C+B')$ has the numerical dimension one, we have a numerical contraction $f=\nu : S \dashrightarrow Z$ and $K+C+B'$ is n-complementary, as in the above case with $\kappa^*=1$, when f has a multisection in C. The same holds for the same reasons (cf. the proof in Case III below) if B' has a horizontal component. If B' is in fibers of f, then B (an image of B) is also in fibers of f, because $K+C+B \equiv K+C+B' \equiv K+C \equiv 0/Z$. But B has a big type. Therefore, B' has a horizontal component.

Finally, $K+C+B' \equiv 0$. By Monotonicity Lemma 2.16, we have an n-complement when B' is of a big type. As above, if B' has a fiber type and is in fibers, this is only possible when C is a multisection for g given by B'. Then $K+C+B'$ and $K+C+B$ are n-complementary by Lemmas 2.20 and 2.21. Because B has a big type, the case where $B'=0$ and $K+C=K+C+B' \equiv 0$ is impossible.

Proof of the Inductive Theorem: Case III. If $\kappa^*=1$ and we have a horizontal element D_i in B, we can reduce this case to the big case for $K+C+B-\varepsilon D_i$ when D_i is chosen properly. As in the proof of Lemma 2.19, we can find an n-complement, with $n \in RN_2$, near C (cf. Proof of the local case in the Inductive Theorem in Sec. 3). In particular, by 2.16.1, we have a horizontal D_i with the multiplicity b_i of B such that $b_i \notin \mathbb{Z}/(n+1)$ (cf. Lemma 2.27). We can then use the above reduction by Lemma 2.16.

If $\kappa^*=0$, we have ν given by B, and B has only vertical components.

Therefore, we assume below that B has only vertical components with respect to $\nu : S \dashrightarrow Y$ and ν is a fibering with curve fibers of genus 1. Also, C is vertical, and according to Kodaira, a modification of its fiber has the type I_b^*, II, II*, III, III*, or IV, IV*. Near such fibers, we have $n=$ 2-, 6-, 4-, or 3-complements respectively (cf. Classification 3.2). In most cases, this can be extended to an n-complement on S. In other cases, we have $n(n+1)$-complements. Under (M) or (SM), in the latter cases, $n=1$, and we have regular $n(n+1)=2$.

We can prove it as in the proofs of Lemma 2.20 and Case II. We consider contractions of curves $E \not\subseteq C$ with $(K+C+B'.E) \geq 0$ for $B'=\lfloor (n+1)B \rfloor /n$. In particular, B' has multiplicities in $\mathbb{Z}/n$. Because $K+C+B=K+C+B'=K \equiv 0/Y$, we can make ν extremal outside a fiber C, i.e., all other fibers are irreducible. Then $K+C+B'$ is positive on fibers of some fibering $f : S \dashrightarrow Z$ or $-(K+C+B')$ is nef.

Indeed, we can contract components of C preserving the numerical properties of $K+C+B'$ because $K+C+B' \equiv 0/Y$ on each component of C. A terminal model is extremal, and its cone has two extremal rays:

- the first ray, R_1, is generated by a fiber F of ν and
- the second ray, R_2, is generated by a multisection section E.

If $(K+C+B'.E)>0$, then E induces the required fibering f. We must verify that if E is contracted to a point, then we have the required fiber contraction on S. The former contraction induces a birational contraction $f : S \dashrightarrow Z$, for a birational inverse image of E. After that, $K+C+B'$ is nef and big. Moreover, it is also positive on each curve $E \not\subseteq C$. Again, we can find extremal contractions of such E in S by subtracting C. Finally, they give the required fibering because a terminal model cannot have the Picard number 1 by Lemma 2.19.

In the case of such a fiber contraction (ruling), we return to the original S. By Lemma 2.19, C has a section of induced $f : S \dashrightarrow Z$. Therefore, the horizontal multiplicities of B are in $\mathbb{Z}/(n+1)$ by Lemma 2.27. We then use Lemma 2.20. After contractions in a fiber of C for ν, we have a fiber C with the same B_C because $K+C+B \equiv 0/Y$. On the other hand, by Corollary 2.26, B_C satisfies (M) and (SM) whereas $\deg(K_C+B_C)=0$. Therefore, by Monotonicity Lemmas 2.7 and 2.17, K_C+B_C is n-complementary if and only if B_C has the index n. Therefore, K_C+B_C is mn-complementary for any natural number m. For $n:=(n+1)n$, we then have *ii–iii* in Lemma 2.20 on S. Therefore, $K+C+B$ is $n(n+1)$-complementary. Under (M) or (SM), $n=1$.

In other cases, $-(K+C+B')$ is nef after the above contractions. We also assume that all fibers of ν, except for the fiber C, are irreducible. By Theorem 6.9 in [23] after a complement (cf. with the proof of Case I), $K+C+B'$ is log terminal, except for the case where C and $C'=\lfloor B' \rfloor$ are irreducible curves, and

$K+C+B' \equiv 0$. Then $K+C+B'$ and $K+C+B$ are n-complementary as in the proof of Case I. Therefore, we suppose that $K+C+B'$ is log terminal.

If $K+C+B' \equiv 0$, then B' has a fiber type, and we have an n-complement as in Case II. Indeed, we can use Lemma 2.20 when there exists a ruling $f: S \dashrightarrow Y$. Otherwise, by the LMMP with $K+C+B-\varepsilon D$ for an algebraic fiber $D=\nu^*(\nu(C))$, we have a birational contraction of a curve $E \not\subseteq C$ with $(K+C+B'.E)=0$. Because E is a multisection of ν and B' has a fiber type before the contraction, it has a big type afterwards. Therefore, decreasing B' after contraction, we have an n-complement as in the big case by Lemmas 2.16 and 2.13.

Finally, if $(K+C+B'.E)<0$, we increase B' by adding vertical components but not in C such that the new B' again has multiplicities in $\mathbb{Z}/n$ and $(K+C+B'.E)=0$. Then $K+C+B' \equiv 0$, which is the above case. To verify $K+C+B' \equiv 0$, we note that ν is also a numerical contraction for $K+C+B'$ with old B', and the new $B' := B+\nu^*D$ for an effective divisor D on Y.

Before increasing B', we classify and choose an appropriate model for the fiber C. We suppose that C is minimal: all (-1)-curves of C are contracted when S is nonsingular near C. This is possible only when C is reducible and the (-1)-curve is not an edge in the chain of C.

Then fiber C has one of the following types (see [5] and Classification 3.2):

I_b^*. A minimal resolution of fiber C has a graph $\widetilde{D}_{4+b}$, where $b+1 \geq 1$ is the number of irreducible components of C. All curves of the resolved fiber are (-2)-curves. $B_C=(1/2)(P_1+P_2+P_3+P_4)$.
II. $B_C=(1/2)P_1+(2/3)P_2+(5/6)P_3$.
III. $B_C=(1/2)P_1+(3/4)(P_2+P_3)$.
IV. $B_C=(2/3)P_1+(2/3)P_2+(2/3)P_3$.

Curve C is irreducible exactly in the cases I_0^* and II–IV. Moreover, in cases II–IV, C is a (-1)- or (-2)-curve, which splits our classification into Kodaira's cases II–IV and II*–IV* respectively (cf. Classification 3.2). Respectively, for I_b^* and II–IV, $n=2$, 6, 4, and 3. Type I_b disappeared by our conditions on C.

When C is a (-1)-curve, we transform the fiber C of type II–IV into a standard one F_0 of type II–IV in Kodaira's classification (p. 158 in [5]). Then S is nonsingular near F_0, F_0 has multiplicity 1 for ν, and near F_0, modified $C=(5/6)F_0$, $(3/4)F_0$, and $(2/3)F_0$ respectively. The log singularity C is now hidden in a point on F_0, and the new C is not reduced. By Lemma 2.13, we preserve the n-complements. During the crepant modification, all boundary multiplicities have the denominator n.

If C is a (-2)-curve, then for types II–IV, all curves of the resolved fiber are (-2)-curves, too. The fiber $F_0=nC$ has the respective multiplicities 6, 4, and 3.

In the case I_b^*, $F_0=2C$. This is an algebraic fiber.

We now choose E. Because $K+C+B'$ is not nef, there exists an extremal contraction $f: S \to Z$ negative with respect to $K+C+B'$. It is not to a point.

First, we suppose that f gives a ruling with a generic fiber E, i.e., E is a 0-curve or $(K.E)=-2$. Then $(C+B'.E) \leq 2-1/n$ because E only crosses $C+B'$ in nonsingular points, and $(K+C+B'.E) \leq -(1/n)$ when <0. If E is a section of ν, we can add a few copies of the generic fiber F of ν with multiplicity $1/n$ to $K+C+B'$.

Otherwise, E is a double section of ν, except for the cases II*–IV*, which we consider later. This follows from the inequality $(C+B'.E) \leq 2-1/n$ because ν has a multiplicity ≤ 2 in F_0 and even 1 in cases II–IV. On the other hand, $\mathrm{mult}_{F_0} C \geq (n-1)/n \geq 2/3$ in cases II–IV. Therefore, E is a double section, and if $B' \neq 0$, then $(B'.E) \leq 1/n$: in case I_b^*, $(B'.E)=(C+B'.E)-(C.E) \leq 1-1/n=1/2$, and in cases II–IV, $(B'.E)=(C+B'.E)-(((n-1)/n)F_0.E) \leq (2-1/n)-2(n-1)/n=1/n$. Therefore, such $B'=(1/n)F_1$, and $(F_1.E)=1$ where $F_1 \neq F_0$ is an irreducible fiber of ν. We then increase B' to $(2/n)F_1$. If $B'=0$, we increase B' by $(1/n)F$ in a generic fiber F/Y.

If f is birational, f contracts a (-1)-curve E ((-1)-curve on a minimal resolution). Again, in cases I_b^* and II–IV, $0>(C+K+B'.E) \geq -(1/n)$. Moreover, E is a section of ν, and we can increase B' by adding $(1/n)F$ when $K+C+B'$ has the index n near E. In case I_b^*, E passes through P_i.

In the other cases, $K+C+B'$ does not have the index n somewhere near section E. It gives a singularity of S on E outside C. On the above log model, E has another singularity P_3 of S. (In cases II–IV, section E crosses F_0 only in a (single) nonsingular point of F_0.) Hence, by a classification of surface log contractions, there exists just a single singularity outside C, and we increase B' only in a corresponding fiber F_1. We then only need to verify that $K + C + B'$ has index n near F_1. After a crepant modification, we suppose that S is minimal/Y and nonsingular near F_1. On the other hand, $(K + C + B'.E) = 0$, and $K + C + B'$ has the index n outside F_1. Because E is a section, $K + C + B$ has the index n near E or in a point $F_1 \cap E$. Using a classification of degenerations due to Kodaira, we find that $K + C + B'$ has the index n everywhere in fiber F_1 and in S.

Finally, we consider the cases with II*–IV*. When f is a ruling, there exists a (-1)-curve E in a singular fiber of f. In this case, we choose E through P_3. If f is birational, we again have such a curve E. In both cases, C passes through a singular point $P_j \in C$ of S. This time, $0 > (C + K + B'.E) \geq -(n-1)/n$. More precisely, if E passes through the point P_j of type $\mathbb{A}_i$, then $(C+K+B'.E) \geq -i/(i+1)$. Because E is not of a big type, on a minimal resolution, it can intersect only $(-m)$-curves of the resolution with $m \geq i+1$, and the equality holds only when f is a ruling. But then $(C+K+B'.E) \geq -i/(i+1)+(i-1)/(i+1) = -1/(i+1)$, and $\geq -\, i/(i+1)(i+2)$ when E is contractible. In particular, if E is a section, then $i+1 = n$ and $-(n-1)/n(n+1) > -1/n$. Therefore, E has at most two singularities of S with the m.l.d. < 1 (the m.l.d. is the minimal log discrepancy) and no intersections with B' in other points in such a case. Therefore, we can complete B' as above when E is a section.

We also note that in the ruling case, E is a section, except for the case where E has only one singularity P_3 of type $\mathbb{A}_3$ and E intersects the middle curve on a minimal resolution of this point. Then $n = 4$, E is double section, B' intersects E simply in a single branching point $Q \in E/Y$, and $(K + C + B'.E) = 1/2$ in P_3 and $1/4$ in Q. Such an intersection means the intersection of E on a minimal crepant resolution with the boundary. Therefore, we complete B' in the corresponding fiber. Then $(K + C + B'.E) = 1/2$ in Q also for the new B'. We must verify that for such B', $K + B'$ has the index n. This can be done on a Kodaira model C'. If E passes through a singular point on a Kodaira model C' after contractions to the central curves as in our models I_b^* and II–IV, then for the fiber C' of type I_b^*, E passes through a simple Du Val singularity, $B' = (1/2)C'$, and $K + (1/2)C'$ has the index 4. Another possible type, assuming that C' is not reduced in B', is only of type IV* with $B' = (3/4)C'$ and E. Again, $K + (1/2)C'$ has the index 4. If E passes transversally through a nonsingular point of C', then the multiplicity of C' is 2, $B' = (1/2)C'$, and $K + (1/2)C'$ has the index 4. If E does not transversally pass through C', the multiplicity of C' is 1, $(C'.E) = 2$ near C', $B' = (1/4)C'$, and $K + (1/4)C'$ has the index 4.

In the other cases, E is contractible, and E is an l-section with $2 \leq l \leq 4$. According to the classification of such contractions, E has a simple single intersection with the exceptional curve over P_j, and it is an edge curve. Moreover, E has at most two singularities. In the latter case, $B' \neq 0$ only in the two corresponding fibers C and C' because $-i/(i+1)(i+2) \geq 1/6$ and $(K+C+B'.E) < 0$. In addition, $(K+C+B'.E) = 1/(i+1)$ near P_j, and $l = n/(i+1)$. We increase B' to the numerically trivial case. Near C', $(K+C+B'.E) = i/(i+1)$ for the new B'. For such B', we verify that $K + B'$ has the index n near C'. If $i = 1$, $n = 4$, and $l = 2$, it was proved above in the ruling case. If $i = 1$, $n = 6$, and $l = 3$, we can proceed similarly. Because E is a 3-section, the type I_b^* is only possible when $B' = (1/3)C'$ with E passing a singular point. Then $K + B'$ has the index 6. For type I_b, $B' = (1/6)C'$ when C' has the multiplicity 1, and $B' = (1/2)$ when C' has the multiplicity 3. In both cases, $K + B'$ has the index 6. The same holds for types II–IV whereas the multiplicity of C' is 1. In the case III*, $B = (2/3)C'$, and $K + (2/3)C'$ has the index 6. In the case IV*, $B = (1/2)C'$, and $K + (1/3)C'$ has the index 6.

In the other case with two singularities, $i = 2$, $n = 6$, and $l = 2$. For type I_b, $B' = (2/3)C'$ when the multiplicity of C' is 2, and $B' = (1/3)C'$ when the multiplicity is 1. The same holds for types II–IV, whereas the multiplicity of C' is 1. For type I_b^*, $B' = (1/3)C'$, whereas E does not pass through singularities. Types II*–IV* are impossible in this situation. In all these cases, $K + C + B'$ has the index 6.

Finally, P_j is the only singularity of S on E. Then it is sufficient to construct B' near E. Therefore, it is possible to do this when E has the only (maximally branching) point with simple intersection with a fiber somewhere over Y. On the other hand, if $B' \neq 0$ is in a fiber C' with a nonbranching point, then near C', $B' \geq (1/n)C'$ and $(B'.E) \geq l/n \geq 2/n$. Therefore, $n \geq 4$, and for $n = 4$, $(K + C + B'.E) = 1/4$ in P_j, and $l = 2$ is impossible. Therefore, $n = 6$, $(K + C + B'.E) = 1/3$ in P_j, and we can increase C' up to

$(1/3)C'$ when B' does not have other components. Otherwise, we have a third fiber C'' with a simple only intersection and a 6-complement.

In all other cases, we have two branchings of E/Y in one fiber C', $l = 4$, and these branchings are in a fiber C' with $B' \geq (1/n)C'$. (Cf. Lemma 2.23.) Then $n = 6$, $P_j = P_2$, $(K + C + B'.E) = 2/3$ in P_2, and this case is impossible.

Proof of the Inductive Theorem: Case IV. Here, we suppose that $K + C + B \equiv 0$ and B has an exceptional type. Then B_C satisfies (SM) by Lemma 2.26 after a contraction of B. On the other hand, $\deg(K_C + B_C) = (K + C + B.C) = 0$. Therefore, we have an n-complement on C and near C (exactly) for n such that B_C has the index n. We take such an n.

Therefore, $K + C + B = K + C + B' \equiv 0$ and has the index n near C where $B' = \lfloor (n+1)B \rfloor /n$. This follows from Proposition 3.9 in [23] when B is contracted. To establish that $K+C+B$ gives an n-complement, we must verify that $K + C + B$ has the index n everywhere in S.

After a contraction of B, we assume that $C + B = C$. Using the LMMP for K, we reduce the situation to the case where there exists an extremal fiber contraction $f : S \to Z$.

If F is to a curve Z, we use Lemma 2.21. Perhaps we change n to $2n$ when $n = 1$ or 3.

Otherwise, $Z =$ pt., S has the Picard number 1, and C is ample. We then use a covering trick with a cyclic n-covering $g : T \to S$. On T, $K_T + D = g^*(K + C)$ has the index 1 near ample $D = g^{-1}C$. By Proposition 3.9 in [23], D is a nonsingular curve (of genus 1), and T is nonsingular near D. Hence, T is rational, and $K_T + D \sim 0$ (cf. proof of Case I). Therefore, $K + C + B$ has the index n.

Proof of 2.3.2: Global case. The proof is from the proof of the inductive theorem.

3. Local Complements

In the *local case*, when $Z \geq 1$, we can drop most of the assumptions in the main and inductive theorems and in the other results.

3.1. Theorem. *Let $(S/Z, C + B)$ be a surface log contraction such that*

- $\dim Z \geq 1$ *and*
- $-(K + C + B)$ *is nef.*

Then it has a regular complement locally/Z, i.e., $K + C + B$ has a 1-, 2-, 3-, 4- or 6-complement.

Proof of Theorem 3.1: **Special case.** Here, (NK) of the inductive theorem is assumed in the following strict form:

(MLC) (*maximal log canonical*) $K + C + B$ is not Kawamata log terminal *in* a fiber/P near which we would like to find a complement, i.e., C has a component in the fiber or an exceptional divisor with log discrepancy 0 for $K + C + B$ having the center in the fiber.

But as in Theorem 3.1, we drop the other assumptions in the inductive theorem except for (NEF).

After a log terminal blowup, we assume that $K + C + B$ is log terminal, and by our assumption, $C \neq 0$. Moreover, C has a component in a fiber of f.

According to the big case, we suppose that Z is a nonsingular curve, and $K + C + B \equiv 0/Z$. Hence, f is a fibering of log curves of genus 1. We note that C is connected near the fiber by the LMMP and Lemma 5.7 in [23].

Let $B_C = (B)_C$ be the different for the adjunction $(K + C + B)|_C = K_C + B_C$. Then $K_C + B_C \equiv 0/P$. We take a regular n such that $K_C + B_C$ is n-complementary near the fiber. We contend that $K + C + B$ is n-complementary near the fiber, too.

The divisor $-(K + C + B')$ is nef on a generic fiber of f where $B' = \lfloor (n+1)B \rfloor$. Indeed, by the LMMP and Proposition 2.19, the same holds for the fiber after a contraction of nonreduced components of B in the fiber/P.

We decrease $C + B$ in a horizontal component when one exists. Then, by Lemma 2.16 and 2.16.1, for an appropriate choice of the horizontal component, we have the same n-complements and use the big case.

Therefore, we can assume that $C+B$ has no horizontal components. Therefore, f is a fibering of curve of genus 1.

Then, as in Case IV in the proof of the inductive theorem, B has an exceptional type. As there, we can verify that B_C satisfies (SM), $K_C+B_C \equiv 0$, and K_C+B_C has index n. Moreover, $K+C+B$ has the (local) index n near the fiber$/P$, and $B' = B$.

Therefore, to verify that $K+C+B$ has a (trivial) n-complement near the fiber, we only need to verify that $K+C+B$ has a *global* index n, i.e.,

$$n(K+C+B) \sim 0/P. \tag{2}$$

The log terminal contractions, i.e., contractions of the components in B, preserve the (formal) log terminal property of $K+C+B$ and (2) according to 2.9.1 in [22] and 3-2-5 in [14].

Therefore, we assume after contractions that $C+B=C$ is the fiber$/P$ and $K+C$ is (formally) log terminal. Such a model of the fiber is its *weak log canonical model*. It is not unique. For instance, we can blow up a nodal point of C. However, it is unique if we impose the following *minimal* property:

- all (-1)-curves of C are contracted when S is nonsingular near C.

Such a model is called *log minimal*. Its uniqueness follows from the MMP and a classification below (see Classification 3.2). We also note that it can be non–log terminal, but it is always formally log terminal.

We check (2) for the minimal n such that K_C+B_C has the index n, i.e., for the index of K_C+B_C.

This essentially follows from Kodaira's classification of elliptic fibers (Sec. 7 of Chap. 5 in [5]) and his formula for a canonical divisor of an elliptic fibering (Theorem 12.1 in Chap. 5 in [5] and [21]; see also Classification 3.2 below). The latter gives a (nonstandard) classification of the degenerations for a fibering with the generic curve of genus 1.

First, we add types ${}_m\mathrm{I}_b$, $b \geq 0$, to the types of log models in the proof of Case III in the inductive theorem. In the following cases, S is nonsingular near C.

${}_m\mathrm{I}_0$. C is a nonsingular curve of genus 1, and $f^*P = mC$.

${}_m\mathrm{I}_1$. C is an irreducible rational curve of genus 1 with one node, and $f^*P = mC$.

${}_m\mathrm{I}_b$. C is a wheel of $b \geq 2$ irreducible nonsingular rational curves C_i, and $f^*P = mC$. Each C_i is a (-2)-curve.

3.2. Classification of degenerations in genus 1 (Kodaira). *Any degeneration of nonsingular curves of genus* 1 *has a log minimal model of one of the following types up to a birational transform:* ${}_m\mathrm{I}_b$, ${}_m\mathrm{I}_b^*$, II, II*, III, III*, *or* IV, IV*. *Each of these models has a unique birational transform into a Kodaira model with the same label.*

In addition, for the log model of type ${}_m\mathrm{I}_b$, ${}_m\mathrm{I}_b^*$, II, II*, III, III*, *or* IV, IV*, $K+C$ *has the respective index* 1, 2, 6, 4, *or* 3.

We note that $K+C$ is log terminal, except for the type ${}_m\mathrm{I}_1$ when C is a Cartesian leaf.

Proof. Adding a multiplicity of the fiber, we can suppose (MLC) (cf. below Proof of Theorem 3.1: General case).

Then, according to the proof of the special case in Theorem 3.1, we have a log minimal model C/P.

According to a classification of formally log terminal singularities, C is a connected curve with only nodal singularities. On the other hand, (C, B_C) has a log genus 1. Therefore, C has an arithmetic genus ≤ 1.

If the genus is 0, then C is a chain of rational curves, and the possible types were given in the proof of Case III in the inductive theorem.

If C is not irreducible, then by (SM) and because $K_C+B_C \equiv 0$, B_C is the same as in the type I_b^*. Therefore, each P_i is a simple double singularity of S, and a minimal resolution gives a graph $\widetilde{D}_{4+b}$. This fiber has the type I_b^* per Kodaira. In that case, $n=2$, $f^*P = 2C$, and $K \sim 0/P$. Therefore, $2(K+C) \sim 0/P$.

The same holds when C is irreducible, and B_C is the same as in type I_b^*.

In the other cases with genus 0, C is an irreducible nonsingular rational curve. Then $\deg B_C = 2$, and under (SM), we have only B_C as in types I_b^*, II, III, or IV as above. We only need to consider the types II–IV. In all of them, we have three singularities P_i. If C is a (-1)-curve, each of them is simple, i.e., has a resolution with one irreducible curve. Otherwise, C is a (-2)-curve, and the singularities are Du Val. This gives the respective Kodaira types II–IV and II*–IV*. For instance, the curves in a minimal resolution of points P_i in type IV are (-3)-curves. We can now easily transform fibers of the types II, III, and IV into the same per Kodaira because C is a (-1)-curve. In type III, C is transformed into three (-2)-curves with a simple intersection in a single point. The latter is a blow-down of the old C. (Cf. p. 158 in [5].) After the transform, $C = ((n-1)/n)F_0$, where F_0 is the modified fiber (cf. Proof of the Inductive Theorem: Case III).

For types I_b^* and II*-IV*, the transform is a minimal resolution.

Then, for Kodaira types II–IV and II*–IV*, $K \sim 0/P$ and $F_0 \sim 0/P$. (More generally, $D \sim 0/P$ for any integral D such that $D \equiv 0/P$ and for the types I_b^*, II–IV, and II*–IV*. This follows because F_0 does not have nontrivial unramified coverings.) Hence, on Kodaira's model, $n(K+C) = n(K + ((n-1)/n)F_0) = nK + (n-1)F_0 \sim 0/P$. An alternative approach is discussed at the end of the proof.

In the other cases, $B_C = 0$, and the genus is 1. Because $K + C$ is formally log terminal, S is nonsingular near C (3.9.2 in [23]), and C is a curve with only nodal singularities and of arithmetic genus 1. In particular, $n = 1$.

If C is irreducible, then for some natural number m, $f^*P = mC$, and $K \sim (m-1)C/P$ by Kodaira's formula (p. 158 in [5]; cf. formula (1) with $B_Y = P$ in Sec. 2). Therefore, $K + C \sim 0/P$.

Similarly, we can handle the next case where C is reducible. Because $K_C + B_C \equiv 0$, the irreducible components C_i of C form a wheel as in ${}_m\mathrm{I}_b$. By the minimal property, K/P is nef. On the other hand, $K \equiv 0$ in the generic fiber. Hence, $K \equiv 0/P$, each C_i is a (-2)-curve, and $f^*P = mC$. Therefore, in this case, a log minimal model C/P coincides with a Kodaira model of the type ${}_m\mathrm{I}_b$, $K \sim (m-1)C/P$, and $K + C \sim 0/P$.

Finally, we can also use a covering trick (Sec. 2 in [23]) to reduce the proof of (2) to the case with $n = 1$ or to the type ${}_m\mathrm{I}_b$. The latter is a *crucial* fact: $K + C \sim 0/P$ for the type ${}_m\mathrm{I}_b$. It can be induced from dimension one.

Proof of Corollary 1.9: **Local case.** This is a special case because any local trivial complement satisfies (MLC). On the other hand, any regular n divides $I_1 = 12$.

Proof of 2.3.2: **Local case.** The proof is from Example 5.2.2 in [23] because the complements are induced from the one-dimensional case.

Proof of Theorem 3.1: **General case.** According to the big case in the inductive theorem, we suppose that Z is a nonsingular curve and $K + C + B \equiv 0/Z$.

By Lemma 5.3 in [23], we can increase B. We do it such that $K + C + B + pf^*P$ is *maximally log canonical* for some real $p \geq 0$:

- $K + C + B + pf^*P$ is log canonical but
- $K + C + B + p'f^*P$ is not so for any $p' > p$.

Such a p exists, and (MLC) is equivalent to these conditions.

Proof of Corollary 1.8, **Main and Inductive Theorems: Local case.** The proof follows from Theorem 3.1.

4. Global Complements

4.1. Theorem. *Let $(S, C + B)$ be a complete algebraic log surface such that*

(*M*) *of the main theorem holds and*

(NEF) $-(K + C + B)$ *is nef.*

Then its complements are bounded under any one of the following conditions:

(WLF) *of Conjecture* 1.3,

(RPC) *of the inductive theorem,*

(EEC) *of the inductive theorem,*

(EC)+(SM) *of Conjecture* 1.3,

(ASA) *of the inductive theorem, or*

(NTC) *of the inductive theorem.*

More precisely, for almost all such $(S, C+B)$, *we can take a regular index in* RN_2. *The nonregular complements are exceptional in the sense of Definition* 1.5.

4.2. Lemma. *There exists* $c > 0$ *such that all* $b_i \leq 1-c$ *for any log surface* (S, B) *under the assumptions of Theorem 4.1 and such that* $\rho(S) = 1$, S *is* $1/7$*-log terminal, and* (S, B) *does not have regular complements.*

Proof. If $B = 0$, any $c > 0$ fits. Otherwise, S is log Del Pezzo. Such an S is bounded according to Theorem 6.9 in [2]. By (M), the same holds for $(S, \operatorname{Supp} B)$.

Therefore, we can assume that S is fixed, as are the irreducible components of $\operatorname{Supp} B = \cup D_i$. We consider a domain

$$\mathcal{D} = \left\{D = \sum d_i D_i \mid K + D \text{ is log canonical and } -(K+B) \text{ is nef}\right\}.$$

It is a *closed* polyhedron by Property 1.3.2 in [23] and by a polyhedral property of $\overline{\mathrm{NE}}(S)$. We take $c = 1-d$, where $d = \max\{d_i\}$ for $D \in \mathcal{D} \cap \{D \geq B\}$.

Proof: Strategy. We seek the exceptions. We therefore assume that (S, B) does not have 1-, 2-, 3-, 4-, and 6-complements. We prove that $(S, C+B)$ belongs to a bounded family. Equivalently, $(S, \operatorname{Supp}(C+B))$ is bounded. Moreover, we verify that complements are bounded and exceptional as well.

We suppose (ASA) or (NTC) by Proposition 2.5. For the exceptions in the proposition, we have regular complements.

According to Theorem 2.3 in [24] and Lemma 5.4 in [23], we can suppose that $(S, C+B)$ is log terminal. In particular, S has only rational singularities. Therefore, S is projective. Moreover, $(S, C+B)$ is then Kawamata log terminal by the inductive theorem. In particular, $C = 0$. The change preserves (ASA).

In addition, we suppose that $K+B$ is $1/7$-log terminal in the closed points of S. Otherwise, we make a crepant blowup of the exceptional curves E with a log discrepancy $\leq 1/7$. This preserves all our assumptions. We have a finite set of such E by Corollary 1.7 in [24]. In other words, $K+B$ is now $1/7$-log terminal in the closed points.

If $K+B$ is $1/7$-log terminal everywhere or, equivalently, if B does not have an irreducible component D_i with $\operatorname{mult}_{D_i} B \geq 6/7$ and satisfies (SM), then $S = (S, B)$ is bounded according to (M) and Theorem 6.9 in [2], except for the case where $B = 0$ and S has only canonical singularities. In the former case, we have a bounded complement. If (S, B) satisfies (WLF), we construct a complement as in Proposition 5.5 in [23]. Similarly, we proceed in the other cases by (ASA). Because (S, B) is bounded in a strict sense, i.e., in an algebraic moduli sense, the freeness of $-(K+B)$ is bounded. In the case where $B = 0$ and S has only a canonical singularity, $(S, 0)$ has a regular complement according to (ASA) and the classification of surfaces. In such a case, we can even suppose that S is nonsingular.

We can now assume that B has an irreducible component D_i with $\operatorname{mult}_{D_i} B \geq 6/7$. We then reduce all required boundednesses to the case with the minimal Picard number $\rho = \rho(S) = 1$. We find a birational contraction $g: S \to S_{\min}$ such that $S_{\min}$ has all the above properties and $\rho(S_{\min}) = 1$. Moreover, g does not contract the irreducible components D_i with $\operatorname{mult}_{D_i} B \geq 6/7$, and

(BPR) there exists a boundary $B' \geq B$ with $\operatorname{Supp} B'$ in divisors D_i having $\operatorname{mult}_{D_i} B \geq 6/7$ such that g contracts only curves E with log discrepancies ≤ 1 for $K_{\min} + B'_{\min}$ and $-(K_{\min} + B'_{\min})$ is nef, where $K_{\min} = K_{S_{\min}}$ and $B'_{\min} = g(B')$.

In particular, $B_{\min} = g(B) \neq 0$. This reduction is called a *minimization*. It uses the inductive theorem and the main lemma below.

By the LMMP, $-(K_{\min} + B_{\min})$ is nef. Hence, $B_{\min}$ and $-K_{\min}$ are ample because $\rho(S_{\min}) = 1$ and $B_{\min} \neq 0$. Therefore, $S_{\min}$ is a log Del Pezzo surface. Because $K_{\min} + B_{\min}$ is $1/7$-log terminal in the closed points, $S_{\min}$ does the same by Monotonicity 1.3.3 in [23]. Therefore, according to Alekseev, we have a bounded family of such Del Pezzo surfaces (Theorem 6.9 in [2]). For $\operatorname{Supp} B_{\min}$, we have only a bounded family of possibilities because all $b_i \geq 1/2$ and $\rho(S_{\min}) = 1$.

The condition (BPR) above guarantees a *boundedness for the partial resolution* g. First, by the inductive theorem, $K_{\min}+B'_{\min}$ is Kawamata log terminal, and $B'_{\min}$ is reduced because $K_{\min}+B'_{\min}$ does not have regular complements as $K_{\min}+B_{\min}$. Hence, the multiplicities of $B'_{\min}$ and B are *universally* bounded according to Lemma 4.2 and (M): all $b_i \le 1-c$ for some $c>0$. Therefore, $(S, \operatorname{Supp} B)$ is bounded because it resolves only exceptional (for $S_{\min}$) divisors E with log discrepancies ≤ 1 for $K_{\min}+B'_{\min}$ by (BPR) (cf. Second Main Theorem and Corollary 6.22 in [24]).

This is done more explicitly in Theorem 5.1 below (see the following remark for another approach).

4.3. Remark. In the strategy above, $(S, \operatorname{Supp} B)$ is bounded according to Theorem 6.9 in [2] and Lemma 4.2. Indeed, $K+B$ is ε-log terminal for any $c > \varepsilon > 0$. However, we prefer the more effective and explicit property (BPR) (cf. Proof of Theorem 5.1 in Sec. 5).

In the same style as Lemma 2.13, we can prove its improvement.

4.4. Main Lemma. *In the notation of Definition* 5.1 *in [23], let* $f\colon X \to Y$ *be a birational contraction such that*

i. K_X+S+D *is numerically nonnegative on a* sufficiently general *curve/*Y *in each exceptional divisor of* f *and*

ii. *for each multiplicity* $d_i = \operatorname{mult}_{D_i} D$ *of a prime divisor* D_i *in* D, $\lfloor (n+1)d_i \rfloor /n \ge d_i$ *when a divisor* D_i *nonexceptional on* Y *intersects an exceptional divisor of* f.

Then

$$K_Y + f(S+D) \text{ } n\text{-complementary} \implies K_X+S+D \text{ } n\text{-complementary}.$$

In addition, we can assume that D *is just a subboundary.*

4.5. Example-Corollary. By Monotonicity Lemma 2.7, *ii* in 4.4 holds when all coefficients are standard, i.e., they satisfy (SM).

Respectively, *i* in 4.4 holds when $K+S+D$ is nef/Y.

By the main lemma, we can then pull back the complements, i.e., for any integer $n>0$,

$$K_Y + f(S+D) \ n\text{-complementary} \implies K_X+S+D \ n\text{-complementary}.$$

Proof of the Main Lemma. We take a crepant pullback:

$$K_X + D^{+X} = f^*(K_Y + D^+).$$

It satisfies 5.1.2-3 in [23] as K_Y+D^+, and we must verify 5.1.1 in [23] only for the exceptional divisors. For them, it follows from our assumption and Negativity 2.15. Indeed, on the exceptional prime divisors D_i, $D^{+X} \ge D$ and has multiplicities in $\mathbb{Z}/n$. Hence, $D^{+X} \ge S + \lfloor (n+1)D \rfloor /n$ according to Monotonicity Lemma 2.17 above and Lemma 5.3 in [23]. Indeed, for any multiplicity $d_i^+ < 1$ in D^{+X}, we have $d_i^+ \ge \lfloor (n+1)d_i^+ \rfloor /n \ge \lfloor (n+1)d_i \rfloor /n$.

Proof of Theorem 4.1: **Minimization.** Let D denote a boundary with the coefficients

$$d_i = \begin{cases} 1 & \text{if } b_i \ge 6/7, \\ b_i & \text{otherwise.} \end{cases}$$

Hence, by Monotonicity Lemma 2.7, for any $n \in RN_2$, we have

- $\lfloor (n+1)B \rfloor /n = \lfloor D \rfloor + \lfloor (n+1)\{D\} \rfloor /n \ge D \ge B$.

Therefore, $K+D$ is log canonical. Indeed, an n-complement (S, B^+) with $n \in RN_2$ exists locally by Corollary 5.9 in [23]. In addition, $B^+ \ge \lfloor (n+1)B \rfloor /n \ge D$. Hence, by Monotonicity 1.3.3 in [23], $K+D$ is log canonical.

Because B has a multiplicity $b_i \ge 6/7$, D has a nontrivial reduced part, and $K+D$ is not Kawamata log terminal.

By the inductive theorem, $-(K+D)$ does not satisfy (ASA), because $K+D$, like $K+B$, does not have regular complements.

Moreover, we contend that if $\rho = \rho(S) > 1$, then for any $\mathbb{R}$-divisor F such that $D \geq F \geq B$ and $-(K+F)$ is semiample, there exist an exceptional curve E and a divisor B' such that

- $(K+D.E) > 0$ and $\operatorname{mult}_E B \leq 5/6$,
- $D \geq B' \geq F$, $(K+B'.E) = 0$, and $-(K+B')$ is semiample.

We then contract E to a point, $h : S \to Z$, and replace (S, B) by $(Z, h(B))$. On the first S, we take $F = B$. We then take $F = h(B')$. The contraction preserves the properties. In particular, $(Z, h(B))$ does not have regular complements by the main lemma. We contract only curves with $(K+B.E) \leq 0$ and, by the local case and (M), with $b_i \leq 5/6$. Indeed, near E, we have a regular complement (S, B^+), $B^+ \geq D$, $\operatorname{mult}_E B^+ = \operatorname{mult}_E D = 1$, and $(K+D.E) \leq (K+B^+.E) = 0$. Hence, $K+B$ is always 1/7-log terminal, and we do not contract the curves with $b_i \geq 6/7$. Contracted E, or any other exceptional divisor of S with a log discrepancy $\leq$1 for $K+B'$, has the same log discrepancy for $K_Z + h(B')$. By 1.3.3 in [23], these discrepancies do not increase for $K+F'$ with any $F' \geq h(B')$. Therefore, all contracted E have log discrepancies $\leq$1 for $K+B'$. Finally, an induction on ρ gives the required $S_{\min} = S$ with $\rho = 1$.

We find E case by case with respect to the numerical dimension κ^* of $-(K+B)$.

First, (WLF) holds when $\kappa^* = 2$. We then have (RPC). In particular, $-(K+B)$ is not nef, because it is not semiample. Then there exists an exceptional curve E with $(E.K+D) > 0$. Because $\rho > 1$, we otherwise have a fiber extremal contraction $S \to Z$ that is positive with respect to $K+D$. The latter is impossible by (M) because $-(K+B)$ is nef. Therefore, we must find B' and E with above properties. We take a closed polyhedron

$$\mathcal{D} = \{B' \mid D \geq B' \geq F \text{ and } -(K+B') \text{ is nef}\}.$$

It is polyhedral by (RPC). We take a maximal B' in $\mathcal{D}$. Then $-(K+B')$ satisfies (ASA). It is Kawamata log terminal by the inductive theorem because $B' \geq B$. Therefore, we cannot increase B' only because $(K+B'.E) = 0$ for some extremal curve and is positive when we increase B'. By (M), this is possible only for birational contractions. The properties (WLF) and (RPC) are preserved.

Second, $\kappa^* = 1$, and we have a numerical contraction $\nu : S \to Y$ for $K+B$. By (M), the horizontal multiplicities satisfy (SM), and $D = B$ in the horizontal components. Therefore, $-(K+D) \equiv 0/Y$. Otherwise, we have a vertical exceptional curve E with $(K+D.E) > 0$. As above, we contract E. This time, we can take $B' = B$ because $K+B \equiv 0/Y$. After such contractions, $-(K+D) \equiv 0/Y$ and is not nef, because it is not semiample. We note that Y is rational because $K+B$ is negative on the horizontal curves. As above, we have no extremal fiber contractions positive with respect to $K+D$. Therefore, we have an exceptional (horizontal) curve E with $(E.K+D) > 0$. After that contraction, we have (WLF) and do as above.

Third, $\kappa^* = 0$ or $K+B \equiv 0$. In this case, we take $B' = B$ and only need to contract some E with $(K+D.E) > 0$. If B has a big type, we again have (WLF) and (RPC). If B' has a fiber type, then by Proposition 2.12, we have a fibration $S \to Y$ of genus 1 curves whereas B' and D have only vertical components. As above, after contractions, we suppose that $K+D \equiv 0/Y$. Because $B \neq 0$ and forms a fiber, we have a horizontal extremal curve E with the required properties. After its contraction, we have (WLF). Finally, B has an exceptional type. Decreasing B in the nonstandard multiplicities, we can find E, which is outside $\lfloor D \rfloor$ but intersects $\lfloor D \rfloor$. Therefore, $(K+D.E) > 0$. If we change the type of B after a contraction of such E, we return to the corresponding type, big or fiber.

Proof of Theorem 4.1: Bounded complements. We show that complements are bounded. Because $(S, \operatorname{Supp} B)$ is bounded, it is sufficient to establish that complements are bounded for all

$$B' \in \mathcal{D} = \{\operatorname{Supp} B' = \operatorname{Supp} B, -(K+B') \text{ is nef and log canonical}\}.$$

We note that each $K+B'$ is semiample by Proposition 2.12 because $K+B$ is semiample and Kawamata log terminal and (NV) of Remark 2.6 holds for $K+B'$. Therefore, for each $\mathbb{Q}$-boundary B', we have an n-complement such that nB' is integral. We therefore have n-complements near B' by Monotonicity Lemma 2.16. Hence, we have bounded complements according to Example 1.11. Indeed, we can restrict our problem on any ample nonsingular curve; as is seen later in Sec. 5, the cases with nonstandard coefficients are reduced to a case with $\rho(S) = 1$.

A more explicit approach is given in Theorem 5.1.

Proof of Theorem 4.1: Exceptional complements. As in the strategy, we assume (ASA), a log terminal property for $K+B$, and an absence of regular complements. We then have a (nonregular) complement (S, B^+). Here, we verify that $K+B^+$ is Kawamata log terminal for any (such) complement.

After a crepant blowup, we suppose that $K+B^+$ has a reduced component and derive a contradiction. Let D denote a boundary with the coefficients

$$d_i = \begin{cases} 1 & \text{if } b_i^+ = 1, \\ 6/7 & \text{if } 1 > b_i^+ \geq 6/7, \\ b_i & \text{otherwise.} \end{cases}$$

Then $B^+ \geq D$ and D satisfies (SM). By Monotonicity Lemma 2.7, $\lfloor (n+1)B \rfloor / n \leq \lfloor (n+1)D \rfloor / n$ in the nonreduced components of D and B^+ for any $n \in RN_2$. Hence, by Lemmas 5.3-4 in [23], (S, D) does not have regular complements. Hence, $-(K+D)$ does not satisfy (ASA) by the inductive theorem.

We then contend that $\rho > 1$ and we have an exceptional curve E with $(K+D.E) > 0$ and automatically $\operatorname{mult}_E D < 1$. Indeed, if $\rho = 1$, then $K+B^+ \equiv 0$ and is log canonical, and $B^+ \neq 0$ and K are ample. Hence, $-(K+D)$ is nef because $B^+ \geq D$. This is impossible by the inductive theorem.

Therefore, $\rho > 1$. If we have an exceptional curve E with $(K+D.E) > 0$, we contract this curve. Again, we have no regular complements by Example 4.5. Such a contraction is to a rational singularity because E is not in $\lfloor D \rfloor$.

We prove, case by case, that such an E exists, except for the case where $K+B \equiv 0$.

Indeed, if we have (WLF) or $\kappa^* = 2$, then we have (RPC), and the latter can be preserved after a crepant log resolution above. We take a weighted linear combination of B and B^+. In addition, $-(K+C)$ is not nef when (ASA) fails. Therefore, we have an extremal contraction $S \to Z$ that is positive with respect to $K+D$, and $\dim Z \geq 1$. If Z is a curve, $K+B^+ \equiv 0/Z$, but $K+D$ is numerically positive$/Z$, which is impossible for $B^+ \geq D$ as in the above case. Hence, we have E. An induction on ρ and contractions of such E give a contradiction in this case.

We now suppose that $\kappa^* = 1$ and we have a numerical contraction $\nu : S \to Y$ for $K+B$. By (M), the horizontal multiplicities satisfy (SM), and $B^+ = D = B$ in the horizontal components. Therefore, $-(K+D)$ is nef on the horizontal curves. Moreover, it is nef. Otherwise, we have a vertical exceptional curve E with $(K+D.E) > 0$. As above, we contract E. Finally, $-(K+D)$ is nef, and $-(K+D) \equiv 0$ by the Inductive Theorem. This is possible only when $B^+ = D$. But then we have (ASA), which does not hold in our case also. Therefore, we obtain $\kappa^* = 0$ or $K+B \equiv 0$.

Now, let D denote a support of the nonstandard multiplicities in B. If D has a big type, then we obtain (WLF) for $K+B-\varepsilon D'$ for some effective $\mathbb{R}$-divisor D' with $\operatorname{Supp} D' \leq D$ by Proposition 2.12. Again, we do not have regular complements by Monotonicity Lemma 2.16: the nonstandard multiplicities $>6/7$ under (M). We do the same when D has a fiber type. Finally, D has an exceptional type, and we contract D to points. (Cf. Proof of Corollary 1.8: Numerically trivial case in Sec. 1.)

Therefore, $K+B \equiv 0$, satisfies (SM), but does not have regular complements by Lemma 4.4 and Monotonicity Lemma 2.7. Then we have only the trivial complements. In that case $B^+ = B$, which contradicts the Kawamata log terminal property of $K+B$. This gives a contradiction to our assumption on the existence of a nonregular nonexceptional complement (S, B^+).

Proof of the Main Theorem: Global case. The proof follows from Theorem 4.1.

Now, we slightly improve Proposition 2.5.

4.6. Proposition. *If $K+C+B$ is log canonical and nef$/Z$, then*

$$\text{(EC)} \Longrightarrow \text{(NTC)} \Longleftrightarrow \text{(ASA)}$$

with the exception (EX2) of Proposition 2.5. *Nonetheless, there exists a* 1*-complement in (EX2).*

Moreover, we can replace (EC) by its weaker form, i.e., (EC) for S:

(EC)′ *there exists a boundary B' such that $K+B'$ is log canonical and $\equiv 0/Z$,*

Proof. Let (S, B') be as in (EC)'. If we replace $C + B$ by a weighted linear combination of $C + B$ and B', we can suppose that $K + C + B$ and $K + B'$ have the same log singularities:

- the exceptional and nonexceptional divisors with log discrepancy 0 and
- the exceptional and nonexceptional divisors with log discrepancies <1.

We note that (EX2) means that B' is unique and $B' \geq B + C$, i.e., (EEC) holds.

After a log terminal resolution, we suppose that $K + B'$ is log terminal. By the above properties, a support D of curves, where $C + B > B'$, is divisorially disjoint from $\mathrm{LCS}(S, B')$. If D has an exceptional type, we can contract it when $K + C + B \equiv 0$ on D. Then $B' \geq B$, and we have (EEC), which implies the proposition by Proposition 2.5. If $K + C + B$ is negative somewhere on D, then (WLF) and (RPC) hold for $K + C + B - \varepsilon D'$ with some $\varepsilon > 0$, and D' having $\operatorname{Supp} D' \leq D$.

On the other hand, if D has a big type, (WLF) and (RPC) hold for $K+C+B-\varepsilon D'$ for some $\varepsilon > 0$, nef, and big D' having $\operatorname{Supp} D' \leq D$. Here, we can have one exception (EX1) when $K + C + B$ satisfies (NTC).

In addition, the proposition holds when $K + C + B \equiv 0$ by its semiampleness.

Finally, D has a fiber type, $-(K + C + B)$ has a numerical dimension one, and $Z = \mathrm{pt.}$ (the global case). If a fibering given by D does not agree with $K + C + B$, i.e., $(K + C + B.F) < 0$ on the generic fiber, then $K + C + B - \varepsilon F'$ satisfies (WLF) for a divisor F' with $\operatorname{Supp} F' \leq D$, which defines the fibering. Otherwise, the fibering gives a numerical contraction for $K + C + B$.

Proof of Corollary 1.8: **Global case.** The proof follows from Theorem 4.1 and Proposition 4.6.

Proof of Corollary 1.16. Again in the global case, the proof follows from Theorem 4.1 and Proposition 4.6. In the local case, we use Theorem 3.1.

4.7. Remark-Corollary. We can improve most of the above results as well.

4.7.1. In the main and inductive theorems, we can replace (WLF) with (EC)' of Proposition 4.6.

We expect that the main theorem and Corollary 1.8 hold without (M), as does the inductive theorem. Of course, exceptional complements may then be unbounded (cf. Example 2.4).

4.7.2. By Monotonicity Lemma 2.7, we can replace (SM) in Corollaries 1.9–10 with

(M)' the multiplicities b_i of B are standard, i.e., $b_i = (m-1)/m$ for a natural number m or $b_i \geq I/(I+1)$, where I is the maximal among the indexes under (SM): $I|I_2$.

(Cf. Classification 7.1.1 below.)

5. Exceptional Complements

In this section, we begin a classification of the exceptional complements. By the main and inductive theorems, they arise only in the global case (S, B) when $K + B$ is Kawamata log terminal. By Remark 4.7.1, we can assume just (EC)' and (M) as additional conditions. In a classification, we describe such (S, B), which are also called *exceptional*, and their *minimal* complements. Here, we do this completely in a few cases. The importance of a complete classification of the exceptions is illustrated in Sec. 7. We will continue the classification elsewhere.

Because the exceptional complements are bounded, the invariant

$$\delta(S, B) = \#\{E \mid E \text{ is an exceptional or nonexceptional divisor with log discrepancy } a(E) \leq 1/7 \text{ for } K + B\}$$

is also bounded. It is independent of crepant modifications.

5.1. Theorem. *The invariant* $\delta \leq 2$.

5.1.1. *If* $\delta = 0$, *then* (K, B) *is* $1/7$*-log terminal, and* B *has only multiplicities in* $\{0, 1/2, 2/3, 3/4, 4/5, 5/6\}$. *However, the m.l.d. of* $K + B$ *is only* $>1/7$.

A minimum of such m.l.d.'s exists, but it is not yet known explicitly.

In the other cases, where $\delta \geq 1$, the m.l.d. of $K+B$ is $\geq 1/7$, and after a crepant resolution, we assume that $K+B$ is $1/7$-log terminal in the closed points. To classify the original (S,B), we must find crepant birational contractions of the $1/7$-log terminal pairs (S,B). To classify the latter pairs, we consider their minimizations $g: S \to S_{\min}$ as in the strategy of the proof of Theorem 4.1. In this section, some results on $(S_{\min}, B_{\min})$ and their classification are given. According to the strategy, it is sufficient to prove Theorem 5.1 for $(S_{\min}, B_{\min})$. Therefore, we assume in this section that

- $\rho(S) = 1$,
- $K+B$ is $1/7$-log terminal in the closed points,
- B has a multiplicity $b_i \geq 6/7$,
- $-(K+B)$ is nef, but
- $K+D$ is ample for $D = \lfloor (n+1)B \rfloor / n$ for any $n \in RN_2$.

To find all such $1/7$-log terminal pairs (S,B) with $\rho(S) > 1$, we must find $K+B' \equiv 0$ with $B' \geq B$ and $\rho(S) = 1$. The former pairs are crepant partial resolutions of (S,B') (see the strategy and (BPR) in Sec. 4).

Let $C = \lfloor D \rfloor$ denote a support of the curves C_i with $\operatorname{mult}_{C_i} B \geq 6/7$ and D be the same as in the Minimization of Sec. 4. Let F be the rest of B or, equivalently, be the fractional part of D: $F = \{D\} = \sum b_i D_i$ for D_i with $b_i = \operatorname{mult}_{D_i} B \leq 5/6$. By the inductive theorem, $K+D$ is ample for such D.

5.1.2. *For $\delta = 1$, a curve C is irreducible and has only nodal singularities and at most one node. The arithmetic genus of C is ≤ 1. The divisor F does not pass the node.*

Abe found a classification in the *elliptic* case where C has the arithmetic genus 1 [1].

5.1.3. *For $\delta = 2$, $C = C_1 + C_2$, where C_1 and C_2 are irreducible curves with only normal crossings in nonsingular points of S. The divisor F does not pass $C_1 \cap C_2$, and $b_1 + b_2 < 13/7$, where $b_i = \operatorname{mult}_{C_i} B$. The constant c below is as in Lemma 4.2.*

For C, we have only the configurations

(I_2) $C = C_1 + C_2$ and the curves C_i form a wheel and

(A_2) $C = C_1 + C_2$ and the curves C_i form a chain.

Moreover, the curves C_i are nonsingular rational $m_1 \geq m_2 \geq 0$-curves, except for the case (A_2^6) below. In the case (A_2), the only possible cases are the following:

(A_2^1) *$S = \mathbb{P}^2$, and C_1 and C_2 are straight lines. $F = \sum d_i D_i$, and $1 < \sum d_i \deg D_i \leq 3 - b_1 - b_2$, assuming that $K+B$ is log terminal.*

(A_2^2) *S is a quadratic cone, C_1 is its section, and C_2 is its generator, $2b_1 + b_2 \leq 8/3$. $F = (2/3)D_1$, where D_1 is another section not passing the vertex. The constant $c = 1/21$.*

(A_2^3) *S is a normal rational cubic cone, C_1 is its section, and C_2 is its generator, $3b_1 + b_2 \leq 7/2$. $F = (1/2)D_1$, where D_1 is also a section. Both the sections C_1 and D_1 do not pass the vertex, and $\#C_1 \cap D_1 \geq 2$. The constant $c = 1/14$.*

(A_2^4) *S has $B = (6/7)(C_1 + C_2) + (1/2)D_1$, $m_1 = 1$, and $m_2 = 0$. S has only two singularities, $P_1 \in C_1$ and $P_2 \in C_2$, and D_1 is a nonsingular rational 1-curve with a single simple intersection with C_2, a single simple intersection with C_1, and another single intersection with C_1 in P_1. The singularity P_i is Du Val of type $\mathbb{A}_i$.*

(A_2^5) *S has $B = (6/7)(C_1 + C_2) + (1/2)D_1$, $m_1 = 1$, and $m_2 = 0$. S has only two singularities, $P_1 \in C_1$ and $P_2 \in C_2$, and D_1 is a nonsingular rational 1-curve with a single simple intersection with C_2, a single simple intersection with C_1, and another single intersection with C_1 in P_1. The singularity P_1 is simple with a (-3)-curve in a minimal resolution, and the singularity P_2 is Du Val of type $\mathbb{A}_3$. The constant $c = 1/7$, (S,B) is 14-complementary, and the complement is trivial.*

(A_2^6) *S has $B = (6/7)(C_1 + C_2)$ and only two singularities, P_1 and $P_2 \in C_2$. The curve C_1 has the arithmetic genus 1 and has only nodal singularities, at most one. The curve C_2 is a rational nonsingular (-1)-curve. The singularities P_i are Du Val of type $\mathbb{A}_i$. The constant $c = 1/7$, (S,B) is 7-complementary, and the complement is trivial.*

In the case (I_2), the only possible cases are the following:

(I_2^1) *S is a quadratic cone, and C_1 and C_2 are its two distinct sections, $b_1+b_2\le 7/4$. $F=(1/2)L$, where L is a generator of cone S. The constant $c=3/28$.*

(I_2^2) *S has $B=(6/7)(C_1+C_2)$, $m_1=1$, $m_2=2$, and S has only two singularities $P_i\in C_i$. The singularities P_i are Du Val of type $\mathbb{A}_i$. The constant $c=1/7$, (S,B) is 7-complementary, and the complement is trivial.*

5.2. Proposition. *Under the assumptions in this section, $K+D$ is formally log terminal, except for the case where P is nonsingular and near P, and $D=C+(1/2)C'$ with nonsingular irreducible curves C and C' having a simple tangency;* $\mathrm{mult}_C B<13/14$.

We note that the latter log singularity appears only on cones: the cases (A_2^{1-3}) when D_1 is tangent to C_1.

Proof. By the proof of the minimization in Sec. 4, $K+D$ is log canonical and (1/7-)log terminal outside C. Therefore, we must verify a log terminal property formally (locally in an analytic topology) in the points $P\in C$.

First, we suppose that $C\ne D$ in a neighborhood (even Zariski) of P. Then $K+C$ is purely log terminal, and C is nonsingular by Lemma 3.6 in [23]. But S can have a singularity of index m in P (Proposition 3.9 in [23]). If we formally have two distinct prime divisors (two branches) D_1 and D_2 through P in $\mathrm{Supp}(D-C)$, then by 2.25.1, $K+D$ is log canonical in C only when $b_1=b_2=1/2$, and in a neighborhood of P, $K+D=K+C+(1/2)(D_1+D_2)$. (We recall that all nonreduced $b_i=(n-1)/n$ with $n=1,2,3,4,5$, or 6.) Therefore, if P is singular, then by a classification of surface log canonical singularities, the curves E of a minimal resolution form a chain, whereas a birational transform of C intersects one end of the chain simply and that of D_i intersects another end simply. The intersection points are outside the intersections of the curves E. Therefore, the log discrepancy $a=(E,K+B)$ in any E for $K+B$ is the same as $a(E,K+bC+D_1)$ for $K+bC+D_1$, where $b=\mathrm{mult}_C B\ge 6/7$ near P. Therefore, by Corollary 2.26, $a=a(E,K+B)=1-\mathrm{mult}_P(bC)_{D_1}\le 1-\mathrm{mult}_P((6/7)C)_{D_1}\le 1-6/7=1/7$ for an exceptional divisor E. This is impossible by our assumptions. Hence, P is nonsingular. Then the monoidal transform in P gives E with the same property.

Hence, we can formally have at most one irreducible component (or a single branch) C' of $\mathrm{Supp}(D-C)$ through P. By the form of it and a classification of log canonical singularities [12], $K+D$ is log terminal in P, except for the case where C' has multiplicity 1/2 in D, and B intersects only an end curve in a minimal resolution of P or $D=C$ near P. Again, as in the above case, we have a contradiction with the assumptions, except for the case where P is nonsingular and C' has a simple tangency with C in P. Such a singularity is 1/7-log terminal for $K+B=bC+(1/2)C'$ only when $b<13/14$.

Finally, we suppose that $C=D$ near P. By Theorem 9.6 (6) in [12], P has the type $\mathbb{D}_m$ with $m\ge 3$ when C is formally irreducible in P. Then it has an exceptional divisor E on a minimal resolution with $a(E,K+B)\le 1-b\le 1/7$, which is again impossible. Indeed, $a(E,K+C)=0$, $a(E,K)\le 1$, and $a(E,K+B)=a(E,K+bC)=a(E,K+C)+(1-b)\,\mathrm{mult}_E C=(1-b)a(E,K)\le 1/7$.

Otherwise, by Theorem 9.6 (7) in [12], C has two branches in P, and by the 1/7 log terminal property in P, P is again nonsingular. Hence, $K+B=K+C$ is formally log terminal here.

5.3. Corollary. *The curve C has only nodal singularities and only in nonsingular points of S. Also, C is connected. Moreover, each irreducible component of C intersects all other such components.*

Proof. The first statement follows from Proposition 5.2. Because $\rho=1$, each curve on S is ample, which proves the rest.

Let g be the arithmetic genus of C.

5.4. Proposition. *The arithmetic genus $g\le 1$.*

In the proof of this proposition and in the proof of Theorem 5.1 below, we use the following construction.

We reconstruct S as a nonsingular minimal rational model S'. We make a minimal resolution $S^{\min} \to S$ of S and then contract (-1)-curves: $S^{\min} \to S'$, where S' is minimal. Then $S' = \mathbb{P}^2$ or $S' = \mathbb{F}_m$ because S and S' are rational.

By Corollary 5.3, the resolution $S^{\min} \to S$ preserves C up to an isomorphism. A birational transform of a curve C_i or another one on S' is denoted again by C_i or as the other one respectively. According to the LMMP, (S', B') is log canonical, and $-(K_{S'} + B')$ is nef because the same holds for $K + B$ and its crepant blowup $K^{\min} + B^{\min}$, where $K^{\min} = K^{S^{\min}}$ and B' is the image of $B^{\min}$. An image of $B^{\min}$ is not less than a birational image of B. Therefore, $-(K_{S'} + B)$ is nef for $S' = \mathbb{P}^2$.

5.4.1. *Moreover, on a minimal rational model S', $g(C) \leq 1$, and $g(C') = 0$ for each (proper) $C' \subset C$, except for the case* (A_2^6) *in* 5.1.3.

Proof. We suppose that $g \geq 2$.

Because it has to be a tree of nonsingular rational curves, $S^{\min} \to S'$ cannot contract all C. However, we can increase g after contraction of some components of C and other curves on $S^{\min}$.

If $S' = \mathbb{P}^2$, then $-(K_{S'} + B)$ and $-(K_{S'} + (6/7)C)$ are nef, and $\deg C \leq 3$, which is only possible for $g \leq 1$.

Therefore $S' = \mathbb{F}_m$ with $m \geq 2$. Indeed, the original $S' \not\cong \mathbb{F}_0$, because $\rho = 1$. Therefore, if the final $S' = \mathbb{F}_0$, we had a contraction of a (-1)-curve before. We can then reconstruct S as $S' = \mathbb{F}_1$ and therefore as $\mathbb{P}^2$.

By Corollary 5.3, we have at most one fiber F of $\mathbb{F}_m$ in C. If a unique *negative* section Σ is not in C, then $\sigma = \operatorname{mult}_\Sigma B' \leq 2 - 2 \times (6/7) = 2/7 < 1/3$ because C is not a section of $\mathbb{F}_m$ over the generic point of Σ. (Otherwise, C is rational with only double singularities, and $g = 0$.) By the nef property of $-(K + B)$, we have the inequality $0 \geq (K_{S'} + B'.\Sigma) \geq (K_{S'} + (2/7)\Sigma.\Sigma)$. This implies that Σ is a (-2)-curve and $m = 2$. If we had a contraction of a (-1)-curve before, we can reconstruct S' as the above $S = \mathbb{P}^2$. Hence, $(S' = S^{\min}, B' = B^{\min})$ is the minimal resolution of S, and $C \cap \Sigma = \emptyset$ (cf. Lemma 5.6 below). Because $\rho(S) = 1$, S is a quadratic cone (or a quadric of rank 3 in $\mathbb{P}^3$) with a double section $C = D$ ($\sim -K$) not passing through its vertex. Therefore, by the adjunction, $g = 1$.

Finally, Σ is a component of C; we then have another component Σ' in C, which is also a section/Σ. If $C = \Sigma + \Sigma'$ has $g \geq 2$, then $(\Sigma.\Sigma') \geq 3$. This is impossible because $0 \geq (K_{S'} + B'.\Sigma) \geq (K_{S'} + \Sigma + (6/7)\Sigma'.\Sigma) = \deg(K_\Sigma + (6/7)(\Sigma'|_\Sigma)) \geq -2 + 3(6/7) = 4/7$. One last case is $C = \Sigma + \Sigma' + F$; C has $g \geq 2$ only when $(\Sigma' + F.\Sigma) \geq 3$ because $\Sigma \cap \Sigma' \cap F = \emptyset$ by the log canonical property: $3 \times (6/7) > 2$. We can then act as above replacing $\Sigma' + F$ by Σ'.

We now prove 5.4.1. If C has a component of the arithmetic genus $g \geq 1$, for example, C_1, then according to the above, $g = 1$, $C = C_1 + C_2$, where C_2 is nonsingular rational (m_2)-curve, and C_1 intersects C_2 in one point. Moreover, if $m_2 \geq 0$, then C_2 is not exceptional on S', and C in S' has genus ≥ 2, which is impossible, as we know. On the other hand, $(K_X.C_2) < 0$ because $\rho(S) = 1$. Therefore, $m_2 = -1$. However, this is only possible for the case (A_2^6) by the following lemma.

5.5. Lemma. *Each nonsingular irreducible rational (proper) component $C_i \subset C$ is movable on a minimal resolution of S, i.e., C_i is an m-curve with $m \geq 0$, except for the case* (A_2^6) *in* 5.1.3.

5.6. Lemma. *Let P be a log singularity (S, B) such that*

i. *$B \geq bC > 0$, where C is an irreducible curve through P,*
ii. *B is a boundary, and*
iii. *P is a singularity of S.*

Let E be a curve on a minimal resolution of P intersecting the proper inverse image of C, and let $d = 1 - a(E, K + B)$ be the multiplicity of the boundary on the resolution for the crepant pullback.

Then $d \geq ((m-1)/m)b$ where $m = -E^2$. Moreover, $d \geq (1/2)b$ always, and the equality holds only when P is simple Du Val, $B = bC$ near P, and $K + C$ is log terminal in P. Otherwise, $d \geq (2/3)b$. In addition, $d \geq ((m-1)/m)b$ when P is log terminal for $K + C$ of index m. In this case, components of B with standard multiplicities can also be included in C (cf. Lemma 2.25).

Proof. Let C be the inverse image of C in $S^{\min}$. Because the resolution is minimal, all multiplicities of $B^{\min}$ are nonnegative. Therefore, we can consider a contraction of only E and suppose that $B = bC$. We can then find d from the equation $(K + bC + dE.E) = 0$. If E is singular or nonrational, $d \geq 1 \geq (1/2)b$ and even $d \geq (2/3)b$ because $b \leq 1$ by *ii*.

Otherwise, E is a $(-m)$-curve with $m \geq 2$. Hence, $d = (m-2)/m + (C.E)(1/m)b \geq ((m-2)/m + (1/m))b = ((m-1)/m)b \geq (1/2)b$ because $0 \leq b \leq 1$. Moreover, the equality holds only for $m = 2$, $B = bC$ near P, and $(C.E) = 1$ in P when $b > 0$. The next calculation shows that P is a simple Du Val singularity when $b > 0$ and $d = (1/2)b$.

If $m \geq 3$, then $d \geq (2/3)b$. The same holds if we replace E with a pair of intersected (-2)-curves.

Finally, $d(b) = 1 - a(E, K+B)$ is a linear function of b. Therefore, it is sufficient to verify the last inequality for $b = 0$ and 1. For $b = 0$, $a \leq 1$ and $d \geq 0$ by *iii*. This gives the required inequality. For $b = 1$, $a(E, K+B) \leq a(E, K+C) = 1/m$ by 3.9.1 in [23]. Hence, $d \geq (m-1)/m \geq ((m-1)/m)b$.

Components with standard coefficients can also be included in C by Lemma 2.25.

Proof of Lemma 5.5. Because $-K$ is ample on S, we should only eliminate the case where C_i is a (-1)-curve.

By the inequality $C_i^2 > 0$ on S and by Proposition 5.2, C_i has at least two singularities P_1 and P_2. They are distinct from the intersection points $P = (C \setminus C_i) \cap C_i$. Such an intersection point P exists because of $C_i \neq C$ and by Corollary 5.3.

Therefore, we can calculate $(C_i.K + B)$ on a minimal resolution $S^{\min}/S$. Again, let C' and C_i denote the respective proper inverse images of $C' = C \setminus C_i$ and C_i. Over P_1 and P_2, we have the respective single (nonsingular rational) curves E_1 and E_2 intersecting C_i in $S^{\min}$. Let b, d, and $b_1 \leq b_2$ be the multiplicities of $B^{\min}$ in C' (in any component through P), C_i, E_1, and E_2 respectively. Then by our assumptions, $b, d \geq 6/7$. On the other hand, $b_1 \geq (1/2)d$ because P_1 is singular, and $b_1 = d/2$ holds only for a simple Du Val singularity in P_1. Because $C_i^2 > 0$, P_2 is not such a singularity, and $b_2 \geq (2/3)d$ by Lemma 5.6. Otherwise, we have a third singularity P_3 of S in C_i, and $b_3 = \operatorname{mult}_{E_3} B^{\min} \geq (1/2)d$, where E_3 intersects C_i in $S^{\min}$.

For three or more points, $(K+B.C_i) \geq -1+b-d+b_1+b_2+b_3 \geq -1+b+d/2 \geq -1+6/7+3/7 = 2/7 > 0$. Therefore, we have only two singularities and $0 \geq (K + B.C_i) = (K^{\min} + B^{\min}.C_i) \geq (K^{\min} + bC' + dC_i + b_1E_1 + b_2E_2.C_i) \geq -1+b-d+b_1+b_2 \geq -1+b-d+(1/2)d+(2/3)d = -1+b+(1/6)d \geq -1+6/7+1/7 = 0$. Hence, we have the equality; $C_i = C_2$, $C' = C_1$, $B = (6/7)(C_1+C_2)$, P_1 is a simple Du Val singularity, and P_2 is a Du Val singularity of type $\mathbb{A}_2$. Otherwise, it is a simple singularity with (-3)-exceptional curve because $b_2 = (2/3)d$ and $C_2^2 > 0$. However, this is impossible for $m_2 = -1$.

Therefore, we can resolve singularities P_1 and P_2 by the (-2)-curves E_1 and E_2, E_3 respectively, where E_2 intersects C_2 on the resolution. If we successively contract C_2, E_2, and E_3, we transform E_1 into a (1)-curve that is tangent to the transform of C_1 of order 3. Hence, $S' = \mathbb{P}^2$, and C_1 is a cubic in it. Finally, $K + B \equiv 0$, and we have a trivial 7-complement according to the computation.

Proof of Theorem 5.1. C has at most $\delta = 3$ irreducible components C_i by Proposition 5.4. Otherwise, $g \geq 2$, and the equality holds when C has four nonsingular rational components with one intersection point for each pair of components by Corollary 5.3.

Therefore, we prove 5.1.1–2 and can assume that C has at least $\delta \geq 2$ components $C_i, 1 \leq i \leq \delta$.

The 1/7-log terminal property of $K + B$ implies that F does not pass $C_1 \cap C_2$ and $b_1 + b_2 < 13/7$.

By Proposition 5.2, F does not pass the nodes of C: $2 \times (6/7) + 1/2 > 2$. In particular, F does not pass $C_i \cap C_j$ for $i \neq j$.

Except for the case (A_2^6), each component C_i is nonsingular and rational. By 5.4.1, this holds on S'. For S, it is then implied by Lemma 5.5.

Therefore, excluding the case (A_2^6) in what follows, we have only the following three configurations of nonsingular rational curves C_i:

(I_3) $C = C_1 + C_2 + C_3$ and the curves C_i form a wheel,
(I_2) $C = C_1 + C_2$ and the curves C_i form a wheel, and
(A_2) $C = C_1 + C_2$, and the curves C_i form a chain.

This follows from Corollary 5.3 and Proposition 5.4.

To eliminate some of these cases, we prove that each C_i with two nodes P_1 and P_2 in C is an m_i-curve with $m_i \geq 1$. We know (or can suppose) that C_i is nonsingular and rational or an m_i-curve. Moreover, by Lemma 5.5, $m_i \geq 0$. We suppose that $m_i = 0$ and derive a contradiction.

Because $C_i^2 > 0$ on S, C_i has at least one singularity P of S, $P \neq P_1$ and P_2.

As above, we can calculate $(K + B.C_i)$ on a minimal resolution $S^{\min}/S$. Let C' and C'' denote the respective branches of $C \setminus C_i$ in P_1 and P_2. We identify them with their proper inverse images on $S^{\min}$. Over P in $S^{\min}$, we have one (nonsingular rational) curve E intersecting C_i. Let b', b'', d, and b be the multiplicities of $B^{\min}$ in C', C'', C_i, and E. Then by our assumptions, $b', b'', d \geq 6/7$. On the other hand, $b \geq (1/2)d$ by Lemma 5.6.

We therefore obtain a contradiction: $0 \geq (K+B.C_i) = (K^{\min} + B^{\min}.C_i) \geq (K^{\min} + b'C' + b''C'' + dC_i + bE.C_i) \geq -2 + b' + b'' + b \geq -2 + b' + b'' + (1/2)d \geq -2 + 6/7 + 6/7 + 3/7 = 1/7$.

We are now ready to verify that $g(C) = 0$ and we have the case (A_2), except for the two cases (I_2^1) and (I_2^2) in 5.1.3 with the configuration (I_2). We want to eliminate the case (I_3) and the other cases in (I_2). According to what was proved above and the construction, S' again has the same curves C_i as components of C: m_i'-curves with $m_i' \geq m_i \geq 1$.

First, we consider $S' = \mathbb{P}^2$. Because $4 \times (6/7) > 3$, they are all 1-curves in the case (I_3). Hence, there are no contractions of (-1)-curves onto $C \subset S'$ for $S^{\min} \to S'$. In particular, we preserve curve E over any singularity $P \in C$ of S or a curve E with a standard multiplicity $0 < \operatorname{mult}_E B < 1$. Either has a multiplicity $b \geq 3/7$. Therefore, we obtain $0 \geq \deg(K_{S'} + b_1C_1 + b_2C_2 + b_3C_3 + bE) \geq -3 + b_1 + b_2 + b_3 + b \geq -3 + 3 \times (6/7) + 3/7 = 0$. Hence, $b = 3/7$ and at most one of curves C_i, for example, C_1, has a singularity. This is impossible when $S \neq S'$ because S is then a rational cone by the condition $\rho(S) = 1$ and because $S^{\min} = \mathbb{F}_1$ when $S' = \mathbb{P}^2$. Therefore, $S = S' = \mathbb{P}^2$ and $F = 0$. But (S, B) is then 1-complementary to $B^+ = C$, which contradicts our assumptions.

We can do the same in the case (I_2), when the (1)-curve C_i does not have singularities of S, because then $S = S' = \mathbb{P}^2$ with $F = 0$ and 1-complement $B^+ = C$. The case where C_1 and C_2 are both (1)-curves on S' is only possible when $g(C) = 0$. Therefore, we have a (1)-curve, for example, C_1, on S and on S' with a single simple Du Val singularity P_1 of S. This is only possible in the case (I_2^2). More precisely, $m_1 = m_1' = 1$, $m_2 = 2$, $m_2' = 4$, $B' = (6/7)(C_1 + C_2) + (3/7)E_1$, and the line E_1 is tangent to the conic C_2 at a point $P \notin C_1 \cap C_2$ on S'. The inverse transform $S' - \to S$ can be done as follows. The surface $S^{\min}$ is obtained by successive monoidal transforms: first, in P which gives the (-1)-curve E_2, then in $E_1 \cap E_2$, which gives the (-1)-curve E_3, and then in $E_1 \cap E_3$, which gives the (-1)-curve E_4. The curves E_1, E_2, and E_3 are (-2)-curves on $S^{\min}$, and $B^{\min} = (6/7)(C_1 + C_2) + (3/7)E_1 + (2/7)E_2 + (4/7)E_3$. To obtain S, we contract E_1 to P_1 and E_2 and E_3 to P_2.

We now suppose that $S' = \mathbb{F}_m$ with $m \geq 2$ but never $= \mathbb{P}^2$. Because C_i are m_i-curves with $m_i \geq 1$, they are sections of $\mathbb{F}_m/\Sigma$, $C_i \neq \Sigma$, and only in the case (I_2^1).

Indeed, as in the proof of Proposition 5.4, $\sigma \leq 2/7$, and Σ is an exceptional (-2)-curve in S. Hence, $S^{\min} = S' = \mathbb{F}_2$, and S is a quadric of rank 3 having just one singularity. Moreover, because $\sigma \leq 2/7$, this is only possible in the case (I_2^1) with two conic sections (not through the singularity) C_1 and C_2. However, $\deg B \leq 4$ with respect to $C_1 \sim C_2$. Therefore, $F = (1/2)L$, where L is a generator of the quadric, $c = 3/28$, and $b_1 + b_2 \leq 7/4$. Because $\mathbb{F}_0 \neq S^{\min}$, we reduce $S' = \mathbb{F}_0$ to one of the above cases.

In particular, we have proved that $\delta \leq 2$. In our assumptions, $\delta = 2$. We also know that C_1 and C_2 are m_1 and m_2-curves and, for example, $m_1 \geq m_2 \geq 0$. Excluding the case (I_2) in what follows, we suppose (A_2): $\#C_1 \cap C_2 = 1$.

First, we consider cases with $S' = \mathbb{F}_m$ but never $= \mathbb{P}^2$, in particular, $m \geq 2$.

As above, $S' = \mathbb{F}_m$ is possible only when C_1 is a section and C_2 is a fiber of $\mathbb{F}_m/\Sigma$. If both C_1 and C_2 are sections of $S' = \mathbb{F}_m$ and $m \geq 2$, then $m = 2, S^{\min} = S' = \mathbb{F}_2$, and $\Sigma \neq C_1$ and C_2. Therefore, $(C_1.C_2) \geq 2$, which is impossible under (A_2). By the same reason, C_1 is a section but not a multisection.

Because $C_2 \subset S'$ is a 0-curve, $m_2 = m_2' = 0$, and there are no contractions on C_2. Therefore, by Proposition 5.2, $K + D$ is log terminal near C_2 on S. Otherwise, $F = \{D\} \geq (1/2)D_1$, where the curve D_1 is tangent to C_2. Because $(K + D.C_2) > 0$, we have a singular point of S or one more (nontangency) intersection point with F on C_2. This gives one more curve D_2 on S' with $d_2 = \mathrm{mult}_{D_2} B' \geq (1/2)b_2 \geq 3/7$ by Lemma 5.6. But this is impossible: $0 \geq (K+B.C_2) \geq -2+b_1+2(1/2)+d_2 \geq -2+6/7+1+3/7 = 2/7$.

Because $K+D$ is log terminal near C_2 on S and $(K+D.C_2) > 0$, in total, we have at least two singularities of S on C_2 or intersection points P_1 and P_2 with F. Moreover, one of them is neither

($\widetilde{\mathrm{A}}_1$) a simple Du Val singularity of S near the singularity $F = 0$ nor

($\widetilde{\mathrm{A}}_1^*$) a simple (in a nonsingular point of S) intersection point with a component D_i of F with $\mathrm{mult}_{D_i} F = 1/2$ near the point $F = (1/2)D_i$.

Otherwise, in total, we have three singularities or intersection points with F, which is impossible because then $0 \geq (K_{S'}+B'.C_2) \geq -2+6/7+3\times(3/7) = 1/7 > 0$ by Lemma 5.6.

We assume that P_2 is neither ($\widetilde{\mathrm{A}}_1$) nor ($\widetilde{\mathrm{A}}_1^*$). Therefore, if P_2 is a nonsingular point of S, then F has a component D_2 passing through P_2 with $\mathrm{mult}_{D_2} F = (i_2-1)/i_2$ and $i_2 \geq 3$. Moreover, near P_2, D_2 has a simple intersection with C_2 and $F = ((i_2-1)/i_2)D_2$. In addition, P_1 has type ($\widetilde{\mathrm{A}}_1$) or ($\widetilde{\mathrm{A}}_1^*$). Otherwise, some $S' = \mathbb{P}^2$. Indeed, $0 \geq (K+B.C_2) \geq -2+6/7+2\times(4/7) = 0$. Then $K+B$, $K^{\min}+B^{\min}$, and $K_{S'}+B' \equiv 0$, $b_1 = b_2 = 6/7$, and the modifications are crepant. By Lemma 5.6, $K+D$ has the index 3 in each P_i. Moreover, $B' = (6/7)(C_1+C_2)+(4/7)(E_1+E_2)$ for the divisors E_1 and E_2 on $S^{\min}$ and S' over P_1 and P_2. Hence, each P_i is a singularity, and $\rho(S^{\min}) \geq 3$. Therefore, we can suppose that $S' = \mathbb{F}_m$ and $m \geq 3$. Then $\Sigma = E_i$ for some E_i. This gives a contradiction: $4/7 = \sigma \geq 1/3+(1/3)b_2 = 1/3+2/7$.

Assuming that P_1 has type ($\widetilde{\mathrm{A}}_1$) and this is the only singularity of S on C_2, we verify then that (S,B) has type (A_2^2). Indeed, a fractional component F of D with multiplicity $(l-1)/l$, $l \geq 3$, intersects C_2. Because $\rho(S) = 1$, S is a quadric cone, $S^{\min} = S' = \mathbb{F}_2$, and $0 \geq (K+B.C_2) \geq -2+b_1+(1/2)b_2+(l-1)/l \geq -2+6/7+3/7+(l-1)/l$. Hence, $(l-1)/l \leq 5/7$, and $l = 3$, which gives the case (A_2^2). In particular, $F = (2/3)D_1$ for a section D_1 not passing P_1.

The next case, where P_1 has type ($\widetilde{\mathrm{A}}_1$) and P_2 is a singularity of S, is reduced to $\mathbb{P}^2$. Indeed, $\rho(S^{\min}) \geq 3$, and we can suppose that $m \geq 3$. Then $\Sigma \neq E_1$, where E_1 is the exceptional curve/P_1 on $S^{\min}$ or S', because E_1 is a section of $S' = \mathbb{F}_m$ with $E_1^2 \geq -2$ and even $\geq m$ on S'. On the other hand, $0 \geq (K+B.C_2) \geq -2+b_1+\sigma+(1/2)b_2 \geq -2+6/7+\sigma+3/7$, and $\sigma \leq 5/7$. But because $(K+B'.\Sigma) \leq 0$, $5/7 \geq \sigma \geq (m-2)/m+b_2/m \geq (m-2)/m+(1/m)(6/7)$, which gives $m \leq 4$. As above, after a modification, we assume that $m \leq 3$. Therefore, $m = 3$. The curves C_1 and E_1 do not intersect Σ simultaneously. Otherwise, we have a contradiction: $5/7 \geq \sigma \geq 1/3+(b_1+b_2+d_1)/3 \geq 1/3+5/7$, where $d_1 = \mathrm{mult}_{E_1} B' = (1/2)b_2$. Therefore, $(E_1.C_1) \geq 3$, and the intersection points $E_1 \cap C_1$ are outside of Σ. We need to make at least three blowups in E_1 to disjoint E_1 and C_1 on $S^{\min}$. Hence, we can obtain $S' = \mathbb{P}^2$.

In the next case, P_1 has type ($\widetilde{\mathrm{A}}_1^*$). Then P_2 is a singularity of S because $C_2^2 > 0$ and $m_2 = m_2' = 0$. We suppose that P_2 has an index $l \geq 3$ for $K+D$. Then we obtain type (A_2^3). We have no other singularities on C_2 or, equivalently, $B = b_2C_2$ and S is nonsingular near each other point. Therefore, Σ is a curve in a resolution of P_2 intersecting C_2. It is a $(-m)$-curve on $S' = \mathbb{F}_m$ with $m \geq 2$. The singularity P_1 gives a fractional component: $F \geq (1/2)D_1$, where D_1 is a section of $S' = \mathbb{F}_m$. As above, $0 \geq (K+B.C_2) \geq -2+b_1+\sigma+1/2 \geq -2+6/7+\sigma+1/2$, and $\sigma \leq 9/14$. By Lemma 5.6, $9/14 \geq \sigma \geq ((l-1)/l)b_2 \geq ((l-1)/l)6/7$. Therefore, $l \leq 4$. Moreover, for $l = 4$, we have the equations $b_2 = 6/7$ and $\sigma = 9/14$. As in the last part of the proof of Lemma 5.6, this is possible only when $F = 0$ near P_2 and P_2 is a Du Val singularity of type $\mathbb{A}_3$. But then $m = 2$, and we can reconstruct S' as $\mathbb{P}^2$.

Therefore $l = 3$. Moreover, for the same reasons, this is not a Du Val singularity of type $\mathbb{A}_2$. Therefore, P_2 is a simple singularity that can be resolved by a (-3)-curve. Therefore, $m = l = 3$, and S is a cubic cone. This is the case (A_2^3). Moreover, $F = (1/2)D_1$, and both sections C_1 and D_1 do not pass the vertex. Because $K+B$ is log terminal, $\#C_1 \cap D_1 \geq 2$. Because $K+B$ has a nonpositive degree, $3b_1+b_2 \leq 7/2$ and $c = 1/14$.

Finally, we consider the case where some $S' = \mathbb{P}^2$. We suppose that C_1 and C_2 are respectively m_1- and m_2-curves on S with $m_1 \geq m_2 \geq 0$, and m_1'- and m_2'-curves on $S' = \mathbb{P}^2$ with $m_1', m_2' \geq 1$ and $m_1' \geq m_2'$ when $m_1 = m_2$.

We note that $K+D$ is log terminal in this case. By Proposition 5.2, if $K+D$ is not log terminal, then $F \geq (1/2)D_1$ where the curve D_1 is tangent to C, as it is for C and D_1 in $S' = \mathbb{P}^2$. If D_1 is a line in $\mathbb{P}^2$, then $3\times(6/7)+1/2 > 3$, and $K+B'$ is ample. Therefore, $m_1' = m_2' = 1$, and D_1 is a conic in $\mathbb{P}^2$. It was verified above that D_1 on S is not tangent to the 0-curves C_i. In other words, D_1 is tangent to C_i with $m_i = m'_i = 1$. In addition, we have a singular point of S or one more (nontangency) intersection point with F on C_i. This gives one more curve D_2 with $d_2 = \text{mult}_{D_2} B' \geq (1/2)b_i \geq 3/7$ by Lemma 5.6. But this is impossible: $0 \geq \deg(K_{S'} + B') \geq \deg(K_{S'} + b_1C_1 + b_2C_2 + (1/2)D_1 + d_2D_2) \geq -3 + b_1 + b_2 + 2(1/2) + d_2 \geq -3 + 2 \times (6/7) + 1 + 3/7 = 1/7$.

If $m_1 = m_2 = 0$, we have the contraction $S^{\min} \to S' = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ given by the linear system $|C_1 + C_2|$ on $S^{\min}$. We consider such cases later. In other cases, $m_1 \geq 1$. We verify that the latter is possible only for types (A_2^1), (A_2^4), or (A_2^5). Therefore, first we verify that $m_1 = 1$. Otherwise, $m'_1 = 4 \geq m_1 \geq 2$, and C_1 is a conic on $S' = \mathbb{P}^2$; $m'_2 = 1$, and C_2 is a line on $S' = \mathbb{P}^2$ because $4 \times (6/7) > 3$. Because $K + B'$ is log terminal, C_1 and C_2 have two intersection points R_1 and R_2. Moreover, $m_2 = 0$ because $C_1 + C_2$ has the configuration (A_2) on S by our assumptions.

As we know, in total, C_2 has two singularities of S or intersections with F. On the other hand, we have contractions of curves onto C_2 for $S^{\min} \to S'$ only over one of the points R_i. Therefore, there exists point $P_1 \neq R_1$ and R_2 on C_2 that is singular on S or belongs to F. Moreover, P_1 on S has type $(\widetilde{\mathrm{A}}_1)$ or $(\widetilde{\mathrm{A}}_1^*)$. Otherwise, by Lemma 5.6, P_1 gives a curve $D_1 \neq C_1$ and C_2 with $d_1 = \text{mult}_{E_1} B' \geq (2/3)b_2 \geq 4/7$ on $S' = \mathbb{P}^2$. Then $K + B'$ is ample because $3 \times (6/7) + 4/7 > 3$. This is impossible.

For the same reasons, $d_1 = 3/7$, P_1 has type $(\widetilde{\mathrm{A}}_1)$, D_1 is a (-2)-curve on $S^{\min}$, $b_1 = b_2 = 6/7$, $F = 0$, $K + B$, $K^{\min} + B^{\min}$, and $K_{S'} + B' \equiv 0$, and the modifications are crepant. If $\#D_1 \cap C_1 = 2$, then we should make at least two blowups in this intersection to disjoint D_1 and C_1 on $S^{\min}$. According to (A_2) for $C_1 + C_2$ on S and $S^{\min}$, we must make at least one blowup in R_1 or R_2. Therefore, $m_1 \leq 1$. This contradicts our assumptions. Therefore, D_1 is tangent to C_1. But to disjoint D_1 and C_1, we again need two blowups. That leads to the same contradiction. As is seen later, the latter case is possible for $m_1 = 1$ in type (I_2^2) or for $\rho = 2$ and $m_1 = m_2 = 0$, as is discussed below.

Therefore, $m_1 = 1$, and we have the contraction $S^{\min} \to S' = \mathbb{P}^2$ given by the linear system $|C_1|$ on $S^{\min}$; $m'_1 = m'_2 = 1$. Therefore, C_1 and C_2 are lines on minimal $S' = \mathbb{P}^2$.

If $m_2 = 1$, we have no contractions onto C for $S^{\min} \to S' = \mathbb{P}^2$ and no singularities of S on C. Then $S = \mathbb{P}^2$, and this is the first exception (A_2^1). Indeed, if, for example, C_1 has a singularity, it gives rise to a curve $E_1 \subset S^{\min}$ and $S' = \mathbb{P}^2$, which intersects C_2 on both S' and $S^{\min}$. This is impossible.

Therefore, $m_2 = 0$. Moreover, by the same arguments, C_2 has at most one singularity P_2 of S, and C_1 has a singularity P_1 of S (except for (A_2^1)). Really, S has a singularity $P_2 \in C_2$ because $m_2 = 0$; P_2 is a single singularity of S on C_2. It corresponds to the point $P_2 \in C_2 \subset S' = \mathbb{P}^2$ into which we contract a (-1)-curve only once.

Therefore, $F \geq (1/2)D_1$ for some curve $D_1 \neq C_1$ and C_2 on S. Moreover, D_1 is a curve on $S' = \mathbb{P}^2$. On the other hand, P_1 gives another curve E_1 on $S' = \mathbb{P}^2$ with $d_1 = \text{mult}_{E_1} B' \geq ((m-1)/m)b_1$, where $m \geq 2$ is the index of $K + D$ in P_1. Moreover, by construction, E_1 is a line on $S' = \mathbb{P}^2$. The curve D_1 is also a line on $S' = \mathbb{P}^2$. Otherwise, $0 \geq \deg(K_{S'} + B') \geq \deg(K_{S'} + b_1C_1 + b_2C_2 + (1/2)D_1 + d_1E_1) \geq -3 + b_1 + b_2 + 2(1/2) + d_1 \geq -3 + 2 \times (6/7) + 1 + 3/7 = 1/7$. For the same reasons, other components of F are contracted on $S' = \mathbb{P}^2$. In other words, $\operatorname{Supp} B' = C_1 + C_2 + D_1 + E_1$. Equivalently, D_1 is the only component of F that passes a nonsingular point of S on C_1. Moreover, the lines C_1, C_2, D_1, and E_1 are in a general position in $S' = \mathbb{P}^2$ because the intersection point P_3 of E_1 and D_1 does not belong to C.

The curve D_1 crosses C_1 in a single nonsingular point of S. We contend that D_1 passes P_1 on S also. Indeed, we can increase B to B'' in D_1 such that $K + B'' \equiv 0$. Then $K^{\min} + (B'')^{\min}$ and $K_{S'} + B''' \equiv 0$, and the modifications are crepant, where $B''' = b_1C_1 + b_2C_2 + d''D_1 + d'_1E_1$ with $d'_1 = \text{mult}_{E_1}(B'')^{\min}$ and with $d'' = \text{mult}_{D_1} B''$, and $(B'')^{\min}$ corresponds to the crepant resolution $S^{\min} \to S$. By the above arguments or the inductive theorem, d'_1 and $d'' < 1$. However, $\delta = d'_1 + d'' - 1 \geq 0$ if we assume that D_1 and E_1 are disjoint on $S^{\min}$. Moreover, we have contractions of curves for $S^{\min} \to S'$ onto D_1 only over the intersection point P_3 because their multiplicities in $(B'')^{\min}$ are nonnegative. The curves on $S^{\min}/P_3$ form a chain with D_1 and E_1. Therefore, D_1 is a rational n-curve with $n \leq 0$ on S with a single singularity P_3 of type $\mathbb{A}_l$. Moreover, $n = 0$ because $D_1^2 > 0$ on S. Also, $l \geq 2$ because S is not a cone: it has too many singularities. In addition, by Lemma 5.6 and our description of the modification near D_1, $\delta \geq d_2 = \text{mult}_{E_2}(B'')^{\min} \geq (2/3)d'' \geq (2/3)(1/2) \geq 1/3$, where E_3/P_3 in $S^{\min}$ intersects D_1. This gives a contradiction: $0 = \deg(K_{S'} + B''') =$

$\deg(K_{S'}+b_1C_1+b_2C_2+d''D_1+d_1'E_1) \geq -3+b_1+b_2+d''+d_1' \geq -3+2\times(6/7)+1+1/3 = 1/21$. We note another contradiction: the (-1)-curve$/P_3$, which is nonexceptional on S, does not intersect C_2 on S.

Therefore, D_1 passes P_1, and because D_1 intersects E_1 on $S^{\min}$, P_1 is a simple singularity with a single $(-l)$-curve E_1 on $S^{\min}/P_1$ by the log terminal property of $K+D$. In addition, $\operatorname{Supp} F = D_1$. Arguing as above, we can verify that P_2 is a Du Val singularity of type $\mathbb{A}_l$ or $S^{\min} \to S'$ contracts only (-2)-curves and one (-1)-curve, successive blowups of $P_2 \in S' = \mathbb{P}^2$. In addition, $F = 0$ near P_2. Hence, P_2 has the index $l+1$ for $K+D$, and $l \geq 2$ because S is not a cone (except for (A_2^1)). This gives $d_2 = \operatorname{mult}_{E_2} B' = (l/(l+1))b_2 \geq (l/(l+1))(6/7)$ by Lemma 5.6 in divisor E_2/P_2 on $S^{\min}$ intersecting C_2. Therefore, in this case, $0 \geq (K+B.C_2) \geq -2+b_1+\operatorname{mult}_{D_1} F+d_2 \geq -2+6/7+1/2+(l/(l+1))(6/7)$. Hence, $l/(l+1) \leq 3/4$ and $l \leq 3$. This gives types (A_2^4) and (A_2^5) for $l=2$ and $l=3$ respectively. The same inequality with $l=2$ gives $\operatorname{mult}_{D_1} F \leq 4/7$, and therefore $F = (1/2)D_1$. If $l=3$, $K+(6/7)(C_1+C_2)+(1/2)D_1 \equiv 0$.

Finally, we prove that the cases with $m_1 = m_2 = 0$ are impossible. First, we verify that each C_i has at least two singularities of S. Otherwise, $F \geq (1/2)D_1$ where the curve $D_1 \neq C_1$ and C_2 intersects one of these curves, for example, C_1, in a nonsingular point and only in this point. The curve C_1 has a singular point Q_1 of S, and C_2 does so for P_1 because $C_i^2 > 0$. Let F_1/Q_1 and E_1/P_1 be the respective curves on $S^{\min}$ that intersect C_1 and C_2. They give different generators F_1 and E_1 of $S' = \mathbb{P}^1 \times \mathbb{P}^1$ with multiplicities $f_1 = \operatorname{mult}_{F_1} B' \geq (1/2)b_1$ and $e_1 = \operatorname{mult}_{E_1} B' \geq (1/2)b_1$. Moreover, if $\operatorname{Supp} F$ passes Q_1 or, equivalently, a component D_i of F passes Q_1, then $f_1 \geq (1/2)b_1 + (1/2)\operatorname{mult}_{D_i} B \geq 3/7+1/4$, but $0 \geq (K+B.C_1) \geq -2+b_2+f_1+\operatorname{mult}_{D_i} B \geq -2+6/7+3/7+1/4+1/2 = 1/28$. For the same reasons, D_1 intersects C_1 in only one point. Therefore, $\operatorname{Supp} F = D_1$ does not pass Q_1. If D_1 does not intersect C_2 in a nonsingular point of S, then D_1 is also a generator $D_1 \sim F_1$. However, D_1 intersects another generator E_1 on $S' = \mathbb{P}^1 \times \mathbb{P}^1$ transversally in one point P, and $\operatorname{Supp} B' = D_1 + E_1$ near P. On $S^{\min}$, D_1 and E_1 cannot be disjoint by a chain of rational curves$/P$, because we can assume, as above, that $K+B \equiv 0$ and the modifications are crepant. Then a (-1)-curve$/P$ is a curve on S not intersecting C_1. This is impossible. Hence, either D_1 passes each singular point of S on C_2, and we have two of them, or D_1 intersects C_2 in a nonsingular point of S. The former case is impossible: $0 \geq (K+B.C_2) \geq -2+b_2+2\times((1/4)+(1/2)b_2) \geq -2+6/7+1/2+6/7 = 3/4$ as above with f_1. Therefore, D_1 intersects C on S in two nonsingular points: one on each C_i. Then $D_1 \sim F_1+E_1$ or has bi-degree $(1,1)$ on $S' = \mathbb{P}^1 \times \mathbb{P}^1$. Moreover, $B' = e_1E_1 + f_1F_1 + (\operatorname{mult}_{D_1} F)D_1$, and D_1 passes the intersection point P of E_1 and F_1 because D_1 does not pass singular points of C. Assuming, as above, that $K+B \equiv 0$ and the modifications are crepant, we can verify that $S^{\min} \to S'$ contracts only curves over P. Because $\rho(S) = 1$, all but one of these curves are contracted on S to singularities P_1, Q_1, and maybe one on D_1. However, this is only possible when we contract E_1 and F_1 after the first monoidal transform in P that does not produce singularities of S at all. Otherwise, as above, a (-1)-curve$/P$ is a curve on S not intersecting some C_i.

Therefore, F intersects C only in singular points of S. Hence, we have at least four singularities of S: $Q_1, Q_2 \in C_1$ and $P_1, P_2 \in C_2$. They give the respective different generators $F_1 \sim F_2$ and $E_1 \sim E_2$ with multiplicities $f_i = \operatorname{mult}_{F_i} B' \geq (1/2)b_1$ and $e_i = \operatorname{mult}_{E_i} B' \geq (1/2)b_2$. If for two of these multiplicities, for example, e_1 and f_1, $e_1+f_1 < 1$, then we cannot disjoint E_1 and F_1 on $S^{\min}$ under the assumption $K+B \equiv 0$. Therefore, we can suppose that e_1 and $e_2 \geq 1/2$. Then by Lemma 5.6, e_1 and $e_2 \geq (2/3)b_1 \geq 4/7$. This gives an equation in the inequality $0 \geq (K+B.C_2) \geq -2+b_1+e_1+e_2 \geq -1+6/7+2\times(4/7) = 0$. Moreover, $e_1 = e_2 = 4/7$, $B = (6/7)(C_1+C_2)$, $F = 0$, $K+B \equiv 0$, and the modifications are crepant. Indeed, because $1/4+(1/2)(6/7) = 19/28 > 4/7$, C_2 has exactly two singularities P_1 and P_2 of index 3 for $K+D$ or $K+C_2$. Therefore, they are Du Val of type $\mathbb{A}_2$.

On the other hand, S has two singularities Q_1 and $Q_2 \in C_1$. We can assume that Q_2 is not simple Du Val. If Q_1 is also not simple Du Val, then both are again Du Val of type $\mathbb{A}_2$. Therefore, $B' = (6/7)(C_1+C_2)+(4/7)(E_1+E_2+F_1+F_2)$, where F_i are curves with $d = 4/7$ over Q_i. All curves $C_1 \sim E_1 \sim E_2$ and $C_2 \sim F_1 \sim F_2$ are generators of corresponding rulings of $S' = \mathbb{P}^1 \times \mathbb{P}^1$. On $S^{\min}$, there exists a curve E' with $\operatorname{mult}_{E'} B^{\min} = 2/7$, for instance, the second curve in a minimal resolution of P_1. Such curves could only be over the intersection of $P = E_i \cap F_i$, which is impossible because they have only one curve E''/P with $a(E'', K_{S'}+B') \leq 1$. It is a blowup of P with $a(E'', K_{S'}+B') = 6/7$.

Therefore, Q_1 is simple Du Val, $f_1 = (1/2)b_1 = 3/7$, $f_2 = 2 - b_2 - f_1 = 2 - 6/7 - 3/7 = 5/7$, and $B' = (6/7)(C_1 + C_2) + (4/7)(E_1 + E_2) + (3/7)F_1 + (5/7)F_2$. However, this is possible only for a surface S with $\rho(S) = 2$. Indeed, to disjoint F_2 and E_1, we should make two successive monoidal transformations: first in $F_2 \cap E_1$, which gives E_3, and then in $F_2 \cap E_3$, which gives E_4. Both the curves E_1 and E_3 are (-2)-curves on $S^{\min}$: over points $F_1 \cap E_i$, and the contraction $S^{\min} \to S'$ is just one monoidal transform. On the other hand, F_2 is a (-4)-curve, and E_4 is a contractible (-1)-curve passing Q_2 and P_1. Hence, $\rho(S) \geq 2$. It can be verified that $\rho(S) = 2$, and after a contraction of E_4 on S, we obtain the case (I_2^2) in 5.1.3.

We can now find minimal complements in 5.1.3 as in Example 5.2.1 in [23]; however, we did it in only a few trivial cases. Indeed, by Proposition 5.2, we should care about only the numerical property, not singularities.

6. Classification of Surface Complements

We take $r \in N_2$. A classification of r-complements or of complements of index r means a classification of surface log canonical pairs $(S/Z, B)$ with $r(K + B) \sim 0/Z$. We assume that S/Z is a contraction, in the local case, according to the main theorem, such that there exists a log canonical center$/P$ (cf. Example 1.7 and Sec. 3). This classification implies a classification of log surfaces $(S/Z, B')$ with such complements. For instance, the minimal complementary index r is an important invariant of $(S/Z, B')$.

Moreover, for exceptional $r \in EN_2 = \{7, \dots\}$, $K + B$ is assumed to be Kawamata log terminal, and S is complete with $Z = \mathrm{pt}$. Such cases are bounded. They have been partially described in Sec. 5. Here, we focus on basic invariants for regular complements with $r \in RN_2 = \{1, 2, 3, 4, 6\}$.

For indices 1, 2, 3, 4, and 6, types of complements are respectively denoted by $\mathbb{A}_m^n$, $\mathbb{D}_m^n$, $\mathbb{E}3_m^n$, $\mathbb{E}4_m^n$, and $\mathbb{E}6_m^n$, where

- n is the number of reduced (and formally) irreducible components in B (over a neighborhood of a given $P \in Z$ in the local case) and
- m is the number of reduced exceptional divisors of $B^{\min}$ on a crepant minimal formally log terminal resolution (in most cases, this is a minimal resolution; cf. Example 6.2 below) $(S^{\min}, B^{\min}) \to (S, B)$ (over a neighborhood of $P \in Z$) or, equivalently, the number of exceptional divisors on the minimal resolution with log discrepancy 0.

We also assume that for the types $\mathbb{A}_m^n$ and $\mathbb{D}_m^n$, $\operatorname{Supp} B^{\min}$ is a connected singular curve; otherwise they are respectively denoted by $\mathbb{E}1_m^n$ and $\mathbb{E}2_m^n$. Equivalently, exactly in types $\mathbb{E}1 - 2_m^n$ among types $\mathbb{A}_m^n$ and $\mathbb{D}_m^n$, the number of exceptional divisors with log discrepancy 0 is finite. The same numerical invariants can be defined in any dimension up to the LMMP. But, as is seen in the next section, a simplicial space associated with reduced components is a more important invariant.

6.1. Theorem. *If $m = n = 0$, then $Z = \mathrm{pt.}$, and $K + B$ is Kawamata log terminal of a regular index $r \in RN_2$. This is possible only for types $\mathbb{E}r_0^0$. The complements of this type are bounded when $B \neq 0$ or S has a non-log terminal point* [2]. *Otherwise, $B = 0$, and S is a complete surface with canonical singularities and $rK \sim 0$* (their classification is well known up to a minimal resolution and can be found in any textbook on algebraic surfaces, e.g., [5] and [21]).

In the other types, we suppose that $m + n \geq 1$. Then they have a nonempty locus of the log canonical singularities $\mathrm{LCS}(S, B)$. It is the image of $\mathrm{LCS}(S^{\min}, B^{\min}) = \operatorname{Supp}\lfloor B^{\min} \rfloor$ and has at most two components: two only for exceptional types $\mathbb{E}r_n^m$ with $n + m = 2$ and in the global case.

If $\mathrm{LCS}(S, B)$ is connected, it is a point if and only if $n = 0$. Otherwise, it is a connected curve $C = \operatorname{Supp}\lfloor B \rfloor$ with at most nodal singularities and of arithmetic genus $g \leq 1$ for any subcurve $C' \subseteq C/P$. Moreover, if $g = 1$, then $C' = C$; this case is only possible for types $\mathbb{E}1_1^0$ and $\mathbb{A}_m^n$ with $n \geq 1$; $C = C'$ is respectively a nonsingular curve of genus 1 or a Cartesian leaf when $n = 1$, and C is a wheel of n rational curves when $n \geq 2$. In all other cases, C is a chain of n rational curves.

The singularities of (S, B) outside $\mathrm{LCS}(S, B)$ are log terminal of index r. In particular, they are only canonical when $r = 1$; moreover, they are only of type $\mathbb{A}_i$ when $m + n \geq 1$.

If $B = 0$, then $n = 0$, and $\mathrm{LCS}(S, B)$ is the set of elliptic singularities of S.

For types $\mathbb{A}_m^n$ and $\mathbb{D}_m^n$, any natural numbers m and n are possible. If $n = 0$, then we have a global case with $B = 0$, $K \sim 0$, and S has a single elliptic singularity.

In the exceptional types $\mathbb{E}\mathrm{r}_m^n$, $n + m \leq 2$, and any m and n are possible under this condition; $n + m = 2$ is only possible in the global case. The number of connected components of $\mathrm{LCS}(S, B)$ *is $m+n$. Equivalently, each such component is irreducible. Moreover, when C/P, it is a point or a nonsingular curve respectively of genus 1 for types $\mathbb{E}1_m^n$ and of genus 0 for the other types.*

Proof. The most difficult part is related to connectedness (Theorem 6.9 in [23]). The other statements follow an adjunction, except for the statement on types of canonical singularities for 1-complements $(S/Z, B)$. Essentially, this was proved in Sec. 2.

We therefore let $(S/Z, B)$ be a 1-complement. After a formal log resolution, we can then assume that $\mathrm{LCS}(S, B) = B = C$ is a reduced curve and S is nonsingular near C. We can also assume that C is minimal, i.e., does not contain (-1)-curves. We verify that the singularities of S have type $\mathbb{A}_i$ using an induction on extremal contractions. By the LMMP, we have an extremal contraction $g : S \to T/Z$ with respect to K if there exists a curve$/P$ not in C. If this is a contraction of a curve C' to a point, then it is to a nonsingular point, and S has only singularities of type $\mathbb{A}_i$ near C' because S has only canonical singularities. If this is a contraction of a fiber type, then it is a ruling that can have singularities only when C has an irreducible component C' as a double section. Then the only possible singularities are simple double. If $T =$ pt., then components of C are ample, and $S = \mathbb{P}^2$. We note that if we have no contractions, we have no singularities because the latter ones are outside C.

Of course, our notation is similar to the classical one (however, with some twists).

6.2. Example. For instance, a singularity $P \in (S/S, H)$ of type $\mathbb{A}_m$ with a generic hyperplane through P has type $\mathbb{A}_m^2$ in our notation. But type $\mathbb{D}_m$ corresponds to $\mathbb{D}_{m-2}^1$ with some reduced and irreducible H.

We have more differences for elliptic fiberings $(S/S, E)$. For instance, the Kodaira type ${}_m\mathrm{I}_\mathrm{b}$ is our $\mathbb{A}_0^b$.

Let $(S/S, L_1 + (1/2)(L_2 + L_3))$ be a singularity as in the plane $S = \mathbb{P}^2$ in the intersection of three lines L_i. Then it has type $\mathbb{D}_1^1$ because its minimal log resolution is a monoidal transform in this point.

6.3. Example. Each toric variety X has a natural 1-complement structure (X, D), where $D = \sum D_i$ with the orbit closures D_i. Therefore, the number of elements in this sum n is the number of edges in the fan.

A toric surface S with n edges has type $\mathbb{A}_m^n$. In addition, $n = \rho(S) + 2$. This characterizes toric surfaces.

6.4. Theorem. *Let $(S/Z, B)$ have a log canonical $K + B$ and nef$/Z$ divisor $-(K + B)$. Then $\rho(S/Z) \geq \sum b_i - 2$, where $\rho(S/Z)$ is the rank of the Weil group modulo the algebraic equivalence$/Z$ or just the Picard number when the singularities of S are rational. Moreover, the equality holds if and only if $K + B \equiv 0$ and S/Z is formally toric with $C = \lfloor B \rfloor \subseteq D$.*

In addition, in the case with the equality and reduced $B = C$, $(S/Z, C)$ is formally toric with $C = D$ (see Example 5.3).

Formally toric$/Z$ means formally equivalent to a toric contraction, or locally$/Z$ in an analytic topology, when the base field is $\mathbb{C}$.

Proof. First, we can assume that $\mathrm{LCS}(S, C) \neq \emptyset$. In the local case, we can do this by adding pullback divisors as in the proof of the general case in Theorem 3.1. In the global case, after contractions, we can assume that $\rho(S) = 1$. If $\mathrm{LCS}(S, C) = \emptyset$, the inequality is improved after contractions. If B has at most one component C_i with $b_i = \mathrm{mult}_{B_i} B > 0$, then $\rho(S) = 1 > b_i - 2$. Otherwise, we have at least two curves C_i and C_j with b_i and $b_j > 0$. We can also assume that $K + B \equiv 0$. If $\mathrm{LCS}(S, B) = \emptyset$, we can change b_i and b_j such that $K + B \equiv 0$ and $b_i + b_j$ is nondecreasing. Indeed, $K + B \equiv 0$ gives a linear equation on b_i and b_j. We then obtain $\mathrm{LCS}(S, B) \neq \emptyset$ or b_i or $b_j = 0$. An induction on the number of curves in $\mathrm{Supp}\, B$ gives the log singularity or the inequality.

Second, we can replace (S, C) by its log terminal minimal resolution $(S^{\mathrm{lt}}, C^{\mathrm{lt}})$ over $C = \mathrm{LCS}(S, B) \neq 0$. We preserve all the statements. The contraction is toric because it contracts curves of D.

If every curve C'/P is in C, we have the local case, and by the adjunction and Corollary 3.10 in [23], we reduce our inequality to a one-dimensional case on C''/P. In addition, for the equality, S/P is nonsingular and toric, which is possible to verify case by case. Here, we use the monotonicity $(m-1)/m+\sum k_i d_i/m \geq d_i$ when $k_i \geq 1$ and even $> d_i$ when $m \geq 2$, $k_i \geq 1$, and $1 > d_i$ (cf. Corollary 3.10 in [23]).

Third, we could assume that $K + B \equiv 0$ on C/P after birational contractions. This improves the inequality. If we return to a Kawamata log terminal case, we can find a complement $K + B' \equiv 0/Z$ with $B' \geq B$. This again improves the inequality.

By Theorem 6.9 in [23], we assume that C/P is connected. Otherwise, $Z =$ pt., $C = C_1 + C_2$, and after contractions, we can assume that each fractional component, i.e., each component of $B - C$, intersects some C_i (cf. the arguments for the connected case). We then reduce the problem to a one-dimensional case on C_i.

Therefore, we assume that there exists a curve C'/P not in C. We then have an extremal contraction $g : S \to T$ that is numerically nonnegative for a divisor H with $\operatorname{Supp} H = C$, and g is numerically negative for $K + B - \varepsilon H$ with some $\varepsilon > 0$. If C has an exceptional type, we take an H that is negative on C. Otherwise, we assume that H is nef on C and even ample in the big case. Therefore, such a g preserves C birationally when birational, or H is ample, and we consider this case as a contraction to $Z =$ pt. below. After birational contractions, f has a fiber type. If it is to a point, then $Z =$ pt., and C has at most two components that are intersected by other components of B. We can choose them and reduce the problem to a one-dimensional case as above. If g is a ruling, we can do similarly when C has an ample component. Otherwise, C is in a fiber of g. As in the first step, we can assume that we have at most one other fiber component. This implies the inequality. Otherwise, we obtain a case where C is not connected. We note that we preserve inequality only when we contract a curve C_i/P with $(K+B.C) = 0$ and $b_i = 1$. Such a transform preserves the formally toric property. Therefore, D always contains C.

We hope that in general $\rho(X/Z) \geq -\dim X + \sum b_i$, where ρ is the Weil–Picard number, i.e., the rank of the Weil divisors modulo the algebraic equivalence. Moreover, the equality holds exactly for formally toric varieties and $\lfloor B \rfloor \subseteq D$. For instance, this implies that locally $\sum b_i \leq \dim X$ when the singularity is $\mathbb{Q}$-factorial and $\rho = 0$. If the singularity is not $\mathbb{Q}$-factorial, we have the stronger inequality $\sum b_i \leq \dim X - 1$ for $B = \sum b_i D_i$ with $\mathbb{Q}$-Cartier D_i (cf. Theorem 18.22 in [16]).

6.5. Corollary. *Let $(S/Z, C)$ be as in Theorem 6.4. The following statements are equivalent:*

- *$(S/Z, C)$ is a surface 1-complement of type $\mathbb{A}_m^n$ with $n = \rho(S/Z) + 2$.*
- *$\rho(S/Z) = n - 2$, where n is the number of (formally) irreducible components in C.*
- *$(S/Z, C)$ is formally toric.*

For instance, if $\rho(S) = 1$ and $Z =$ pt., then a 1-complement (S, C) of type $\mathbb{A}_m^n$ is toric with $D = C$ if and only if $n = 3$. In the other cases, $n \leq 2$.

Most of the above results work over non–algebraically closed fields of characteristic 0.

6.6. Example. If C is a nonsingular curve of genus 0, it always has a 1-complement. But it has type $\mathbb{A}_0^2$ only when it has a k-point. Otherwise, it has type $\mathbb{A}_0^1$, and its Fano index is 1.

A complement with a connected $\operatorname{LCS}(S, B)$ can be called a *monopoly*. Other complements are *dipoles*.

6.7. Theorem. *Any exceptional complement $(S/\text{pt.}, B)$ of type $\mathbb{E}r_0^2$ has a ruling $g : S \to Z$ with a normal curve Z and with two sections in $\operatorname{LCS}(S, B)$; the genus of Z is 1 for $(r = 1)$-complements and 0 in the other cases.*

Proof. We obtain the ruling after birational contractions with respect to a curve C_i in $\operatorname{LCS}(S, B)$, $C_i^2 \leq 0$ (cf. Proof of the Inductive Theorem: Case I).

In addition, if $g(Z) \geq 1$, then $B = 0$, and we have a 1-complement (S, B).

6.8. Corollary. *Let $(S, B) \to (S', B')$ be a normalization of a connected seminormal log pair (S', B') with B' under (M). Then (S', B') has an r-complement if (S, B) has a complement of type $\mathbb{E}r_m^n$.*

Proof. The divisors B^+ in the normal part of S belong to fibers of the contraction g on components of the normalization of S after a log terminal resolution. The latter has the same r for each component of S because they are induced from curves of nonnormal singularities on S'.

For the other dipoles, LCS(S, B) is a pair of points or a point and a curve.

6.9. Remark. The ruling induces a pencil $\{C_t\}$ of rational curves through the points. Similarly, in the other cases, we can find

(PEN) *a log proper pencil* $\{C_t\}$ *of log genus*-1 *curves*, i.e., C_t does not intersect LCS(X, B), when its normalization has genus 1, and for the corresponding map $g_t : (\mathbb{P}^1, 0 + \infty) \to (X, B)$ onto C_t, $g_t(Q) \notin$ LCS(X, B) when C_t has genus 0, t is generic, and $Q \neq \infty$ and $Q \neq 0$.

This implies easy cases in the Keel–McKernan theorem on the log rational covering family (Theorem 1.1 in [15]). The difficult cases are exceptional and bounded. Perhaps it can be generalized in a weighted form for fractional boundaries or m.l.d.'s.

7. Classification of Threefold Log Canonical Singularities

7.1. Theorem. *Let* $(X/Z, B)$ *be a birational contraction* $f : X \to Z$ *of a log threefold* X

- *with boundary* B *under* (SM) *and*
- *nef* $-(K + B)$.

Then it has an n*-complement* $(X/Z, B^+)$ *over a neighborhood of any point* $P \in Z$ *such that*

- $n \in N_2$ *and*
- $K + B^+$ *is not Kawamata log terminal over* P.

7.1.1. *We can replace* (SM) *by*

(M)″ *the multiplicities* b_i *of* B *are* standard, *i.e.,* $b_i = (m - 1)/m$ *for a natural number* m *or* $b_i \geq l/(l+1)$ *where* $l = \max\{r \in N_2\}$.

7.2. Lemma. *Let* $(S/Z, B)$ *be a surface log pair and* $(S^{\min}, B^{\min})$ *be its crepant minimal resolution. If* B *satisfies* (SM), *then*

$$K + B \ n\text{-complementary} \implies K^{\min} + B^{\min} \ n\text{-complementary}.$$

Moreover, we could replace $S^{\min}$ *by any resolution* $S' \to S$ *with the subboundary* $B' = B^{S'}$.

For any $n \in N_2$, *we can replace* (SM) *by* (M)″.

Proof. The proof follows from the Main Lemma 4.4 or it can be done in the same style. The last statement follows from Monotonicity Lemma 2.7.

Sketch of proof of Theorem 7.1. First, we can assume that $K+B$ is strictly log terminal$/Z$ and B has the reduced part $S = \lfloor B \rfloor \neq \emptyset/P$. For this, we add a multiple of an effective divisor $D = f^*(H)$ for a hyperplane section H of Z through $P \in Z$. However, $B + dD$ may contradict (SM) in D. We drop D after a log terminal resolution of $(S, B + dD)$. In turn, this can spoil the nef condition for $-(K + B)$. This is preserved when D or even a divisor $D' \geq D$ below is nef$/P$. If not, we can do this after modifications in extremal rays on which $D' = B^+ - B \geq D$ is negative, where B^+ is a complement for $B + dD$, i.e., $K + B^+ \equiv 0/P$. We then drop D'. To do the log flops with respect to D', we use the LSEPD trick (10.5 in [23]). Finally, if $K + B$ is not strictly log terminal, it holds after a log terminal resolution of (S, B).

Second, $(S/Z, B_S)$ is a seminormal connected surface over a neighborhood of P where B_S is nonsingular in codimension one and again under (SM) or under (M)″ for 7.1.1. This follows from the LMMP or Theorem 17.4 in [16] and Corollary 2.26. Adjunctions $(K + B)|_{S_i}$ on each component are log terminal (3.2.3 in [23]).

We have a complement $(S/Z, B_S^+)$ on S/Z and hence on each S_i. This gives (EC)′.

Third, we can glue complements from the irreducible components of S. If one of them has no regular complements, then S is normal, and there is nothing to verify. In the other cases, we have r-complements with $r \in RN_2$. Moreover, if we have a complement of type $\mathbb{E}r_m^n$ on some component S_i, then $m = 0$ by the log terminality of the adjunction $(K+B)|_{S_i}$, and S is a wheel or a chain of its irreducible components. We can then glue complements by Corollary 6.8. Finally, we have the complements of type $\mathbb{D}_m^n$ with $r=2$. They are induced from a one-dimensional nonirreducible case where we always have a 2-complement (cf. Example 5.2.2 in [23]). (However, if we have nonstandard coefficients $b_i \leq 2/3$, we must use higher complements; cf. Lemma 2.29.)

Finally, we can act as in the proof of Theorem 5.6 in [23] (cf. Theorem 19.6 in [16]). Therefore, an r-complement of $(X/Z, B)$ is induced from the r-complement of $(S/Z, B_S)$. We can lift the r-complement on any log resolution by Lemma 7.2.

In particular, we divide threefold birational contractions into two types: exceptional and regular. By Mori's results, all small contractions in the terminal case are formally regular.

This time the exceptions are not bounded, for instance, if (X, S) is a simple compound Du Val singularity:

$$x^2+y^2+z^2+w^d=0$$

with a quadratic cone surface S given by $x^2+y^2+z^2=0$. It is (not formally) an exceptional complement. Such singularities are not isomorphic for different d, even formally. However, they have many common finite invariants: the m.l.d., the index of complement, the index of K, etc. They are bounded up to an isomorphism of a certain degree or order.

7.3. Corollary. *Under the assumptions of Theorem* 7.1, *for any* $\varepsilon > 0$, *the exceptional contractions* $(X/Z, B)$ *and their complements* $(X/Z, B^+)$ *are bounded with respect to the m.l.d. and discrepancies when* $K+B$ *is* ε*-log terminal: over a neighborhood of* $P \in Z$, *the set of m.l.d.'s* $a(\eta, B, X)$ *for points* η *and the discrepancies* $a(E, B^+, X)$, $a(E, B, X) \leq \delta$ *for any* $\delta > 0$, *is finite.*

This also holds under (M)″ *if the contraction is not divisorial. Otherwise, we should assume that the set of* $b_i \leq 1-\varepsilon$ *is finite.*

Proof. According to our assumptions, b_i belongs to a finite set, indeed, if $b_i \leq (1-\varepsilon)$. Therefore, it is sufficient to verify the finiteness for discrepancies in exceptional divisors E of X. Indeed, the latter implies that we can consider the m.l.d.'s > 3. Such do not exist.

We note that $a = a(E, B^+, X) = a(E, B^+, Y)$ form a finite set because $K+B^+$ has a finite set of indices $n \in N_2$.

The index N of S is bounded$/P$ as well because it is bounded locally on Y. Therefore, we assume that N is the universal index. To verify the local case, after a $\mathbb{Q}$-factorialization, we can suppose that X is $\mathbb{Q}$-factorial. We assume this below always. Then the boundedness follows from the boundedness of quotient singularities on S by the exceptional property. In particular, for any point $Q \in S$, the local fundamental groups of $S \setminus Q$ are bounded. Along curves in S, the index is bounded (≤ 6) according to Proposition 3.9 in [23]. After a covering branching over such curves, S does not pass through codimension-two singularities of Y. We can then argue as in the proof of Corollary 3.7 in [23] (cf. Lemma 1.1.5 in [19]). The index in such singularities is bounded by orders of cyclic quotients of the fundamental groups.

This also implies that any divisor near S has a bounded index.

We now consider discrepancies $a(E, B, X)$, especially, for $E = S/P$. If it is not exceptional, the finiteness follows from (SM) and the ε-log terminal property. The same holds for the other nonexceptional E on X. For exceptional $E = S$ and for any other exceptional E, to compute discrepancies, we choose an appropriate strictly log terminal model $g: Y \to X$ on which S is the only exceptional divisor$/X$. Then $B^Y = g^{-1}B + (1 - a(S, B, X))S = B_Y^+ - a(S, B, B)S - D$, where $D = g^{-1}(B^+ - B)$ is effective. Moreover, the multiplicities of D and $B^+ - B$ form a finite set.

We take a rather generic curve $C \subset S/X/P$. Then

$$\begin{aligned} 0 = (K_Y + B^Y.C) &= (K_Y + B_Y^+.C) - a(S, B, X)(S.C) - (D.C) \\ &= -a(S, B, X)(S.C) - (D.C) \end{aligned}$$

implies

$$a = a(S, B, X) = -(D.C)/(S.C).$$

Because $a > \varepsilon > 0$ and $(D.C) = (D|_S.C)$ belongs to a finite set, $(S.C)$ is bounded from below. On the other hand, $(S.C) < 0$. Therefore, $(S.C)$ and $a(S, B, X)$ belong to a finite set. This implies the statement for the discrepancies $a(E, B, X) = a(E, B^Y, Y) = a(E, B^+, Y) + a(S, B, X) \operatorname{mult}_E S + \operatorname{mult}_E D$ with centers near S or intersecting S.

For the other centers, it is sufficient to verify that the index of D is also bounded there. By Theorem 3.2 in [24], we must verify that the set of exceptional divisors with discrepancies $a(E, B_Y, Y) < 1 + 1/l$ is bounded (cf. the proof of Proposition 4.4 in [24]). Because D is effective and $K_Y + B^+$ has an index $\leq l$, $a(E, B_Y^+, Y) \leq 1$. We must verify that such E with centers not intersecting S are bounded. We can see that this bound has the form $\leq A(1/\varepsilon)$.

We take a terminal resolution W/Y of the above exceptional divisors for $K_Y + B_Y^+$. It does not change the intersection $(S.C)$ for any curve $C \subset S/Z/P$. (This time C may be not$/X$.) As above, we have the inequality

$$0 \geq (K + B.g(C)) = (K_Y + B^Y.C) = -a(S, B, X)(S.C) - (D.C)$$

or $(S.C) \geq -(D.C)/a(S, B, X) > -(D.C)/\varepsilon$. Therefore, (S, C) belongs to a finite set even if we assume that C is ample on C and on its other such models of S.

We then apply the LMMP to $K_W + B_W^+ - S$. More precisely, we make flops for $K_W + B_W^+$ with respect to $-S$ or in extremal rays R with $(S.R) > 0$. We thus decrease $(K_W + B_W^+ - S.C)$ and increase $(S.C)$, strictly when the support $|R|$ has a divisorial intersection with S. Therefore, the number of exceptional divisors on W/X is bounded because we contract all of them during such LMMP.

We have proved more.

7.4. Corollary. *For the exceptional and ε-log terminal $(X/Z, B)$, the fibers $f^{-1}P$ are bounded. In particular, if X/Z is small, the number of curves$/P$ is bounded.*

7.5. Corollary. *For each $\varepsilon > 0$, there exists a finite set $M(\varepsilon)$ in $(+\infty, \varepsilon]$ such that $(X/Z, B)$ is not exceptional when the m.l.d. of $(X, B)/P \geq \varepsilon$ and is not in $M(\varepsilon)$.*

Proof. The set $M(\varepsilon)$ is the set of m.l.d.'s for the exceptional complements that are $>\varepsilon$.

7.6. Corollary. *There exists a natural number n such that any small contraction X/Z of a threefold X with terminal singularities has a regular complement when it has a singularity Q/P in which K has an index $\geq n$.*

A similar bound exists for the number of curves$/P$ (cf. Corollary 7.4).

Proof. We take $n = 1/A$, where $A = \min M(1)$. We note that any terminal *singularity* of index n has the m.l.d. $= 1/n$.

This result is much weaker than Mori's on the good element in $|-K|$ when X is formally $\mathbb{Q}$-factorial$/Z$ and has at most one curve$/P$. Indeed, as Prokhorov remarked to the author, there then exists a good element $D \in |-K|$ according to Mori and Kollar. (It is unknown whether this holds when the number of curves$/P$ is >1 or when X/Z is not $\mathbb{Q}$-factorial.) Therefore, $(X/Z, D)$ and $(X/Z, 0)$ have a regular complement by Theorem 5.12 in [23]. In general, the corollary shows that exceptional cases are the most difficult in combinatorics. On the other hand, we expect few exceptions (maybe none) among them in the terminal case.

7.7. Example. Let $(X/Z, B)$ be a divisorial contraction with a surface $B = E/P$, and let $(K + B)_E = K_E + B_E$ be of type (A_2^6) or (I_2^2) in 5.1.3. Then $(X/Z, E)$ has a trivial 7-complement. In this case, the singularity P in Z is (maximal) log canonical of index 7, but it is not log terminal.

7.8. Example. Let $(X/Z, B)$ be a divisorial contraction with a surface $B = E = \mathbb{P}^2/P$, and let $(K+B)_E = K_E + B_E$ be of type (A_2^1) in 5.1.3. Moreover, let $B_E = b_1 L_1 + b_2 L_2 + (2/3)L_3 + (1/2)L_4$, where L_i are straight lines in a general position and $b_1, b_2 = 6/7$. We note that in such a situation, the coefficients b_i are always standard (Proposition 3.9 in [23]), and we have a finite choice of them.

Then $(X/Z, E)$ is not regular; it has a 42-complement. Moreover, in this case, P is a purely log terminal singularity, but *not terminal* or *canonical*, except for the case where X has only Du Val singularities along curves L_i and $K \equiv 0/Z$. In particular, P has a crepant desingularization. We do not discuss the existence of such singularities here.

Indeed, E has index 42. Therefore, for a straight line L in E, $(E.L) < 0$, and $42(E.L)$ is an integer. By the adjunction, $(K + E.L) = (K_E + B_E.L) = -5/42$ and $(E.L) \geq -5/42$ when P is canonical. Moreover, $(E.L) > -5/42$ in such cases, except for the above exception with $K \equiv 0/Z$. For $(E.L) = -5/42$, X/Z is a crepant blowup with log canonical singularities along L_i.

Hence, in the other cases, $(E.C) = -m/42$ with some integer $m = 1, 2, 3$ or 4. Then $(K.L) = -(5-m)/42$, and the discrepancy $d = d(E, 0, Z) = (5 - m)/m$. If P is terminal of index $N \geq 2$, it should have at least all $N-1$ discrepancies $i/N, 1 \leq i < N$ (Theorem 3.2 in [24]). For instance, if $m = 1$ and X has only Du Val singularities, then $d = 4$ and $N \geq 21$. On the other hand, making blowups over lines L_i, we can construct a (minimal log) resolution Y/X with at most $1 + 1 + 2 + 2 \times 6 = 16$ exceptional divisors. Then all other divisors have discrepancies >1. (Moreover, the divisors in the resolution give only a discrepancy $1/7 < 1$.) Therefore, it is not a terminal singularity. Similarly, we can exclude other cases. Therefore, P is canonical or worse.

Of course, this approach uses a classification of terminal singularities. But it can be replaced by the following arguments.

Let $m = 4$; then $d = 1/4$, and for any exceptional divisor E'/P, the discrepancy $d(E', 0, Z) = d(E', K - (1/4)E, X) = d(E', K, X) + (1/4)\operatorname{mult}_{E'} E$. For instance, L_4 is a simple Du Val singularity, and for E'/L_4 on its minimal resolution, $d(E', 0, Z) = d(E', K, X) + (1/4)\operatorname{mult}_{E'} E = 0 + (1/4)(1/2) = 1/8$. But if L_3 is a simple singularity, i.e., a divisor E'/L_3 is unique on a minimal resolution, then $d(E', 0, Z) = d(E', K, X) + (1/4)\operatorname{mult}_{E'} E = -1/3 + (1/4)(1/3) = -1/4$. In this case, P is not canonical. Otherwise, L_3 is a Du Val singularity of type $\mathbb{A}_2$. The same works for other singularities L_i when $d < 1$; they are Du Val also. Otherwise, they have a discrepancy <0. This can be verified by induction on the number of divisors on a minimal resolution. But then on a minimal resolution $g : Y \to X$,

$$\begin{aligned} -4/42 = (E.L) = (g^*E.g^{-1}L) &= (g^{-1}E.g^{-1}L) + (\operatorname{mult}_{L_i} E.L) \\ &= I + (1/2) + (2/3) + 2 \times (6/7) = I + 3 - 5/42, \end{aligned}$$

where $I = (g^{-1}E.g^{-1}L)$ is an integer. This is impossible. We can treat the other cases similarly.

7.9. Proposition-Definition. Let (X, B) be a log terminal pair. Then we can define a simplicial space $\mathrm{R}(X, B)$:

- its l-simplex is an irreducible component Δ_l in an intersection of $l + 1$ distinct irreducible components $D_{i_0}, \ldots D_{i_l}$,

$$\Delta_l \subsetneq D_{i_0} \cap \cdots \cap D_{i_l};$$

- $\Delta_{l'}$ is a face of Δ_l if $\Delta_{l'} \supsetneq \Delta_l$; and
- the intersection of two simplices Δ_l and $\Delta_{l'}$ consists of a finite set of simplices $\Delta_{l''}$ such that

$$\Delta_{l''} \supsetneq \Delta_l \cup \Delta_{l'}.$$

The simplices Δ_l give a triangulation of $\mathrm{R}(X, B)$ or a simplicial complex if and only if we have real global normal crossings in the generic points: all the intersections $D_{i_0} \cap \cdots \cap D_{i_l}$ are irreducible. The latter can be obtained for an appropriate log terminal resolution $(S'/S, B')$.

If (X, B) has a log terminal resolution $(Y/X, B_Y)$, then the *topological type of* $\mathrm{R}(Y, B_Y)$ *is independent of such a resolution*. Therefore, it is denoted by $\mathrm{R}(X, B)$. The topology of $\mathrm{R}(X, B)$ reflects the complexity of log singularities for (X, B) and, in particular, of $\mathrm{LCS}(X, B) = \lfloor B \rfloor$ when (X, B) is log terminal.

We set $\operatorname{reg}(X, B) = \dim \mathrm{R}(X, B)$.

When X/Z is a contraction, we *assume* that the components Δ_l are irreducible formally or in the analytic topology on Z, i.e., we consider irreducible branches over a neighborhood of $P \in Z$.

Proof. According to Hironaka, it is sufficient to verify that a monoidal transform in $\Delta_l \subset X$ gives a barycentric triangulation of Δ_l in $\mathrm{R}(X, B)$.

7.10. Example. If (S, B) is a surface singularity, then $\mathrm{R}(S, B)$ is a graph of $\mathrm{LCS}(S', B')$ for a log terminal resolution $(S', B') \to (S, B)$.

Moreover, $\mathrm{R}(S, B)$ is homeomorphic to a circle S^1, to a segment $[0, 1]$, to a point, or to an empty set when (S, B) is log canonical. Additionally, the case S^1 is only possible when $B = 0$ near the singularity and it is elliptic with a wheel of rational curves for a minimal resolution.

Now let $(S/Z, B)$ be an r-complement. Then

- $\mathrm{reg}(S, B) = 1$ if $(S/Z, B)$ has type $\mathbb{A}_m^n$ or $\mathbb{D}_m^n$,
- $\mathrm{reg}(S, B) = 0$ when $(S/Z, B)$ has type $\mathbb{E}r_m^n$ with $(2 \geq)m + n \geq 1$, and
- $\mathrm{reg}(S, B) = -\infty$ when $(S/Z, B)$ has type $\mathbb{E}r_m^n$ with $m + n = 0$, i.e., when $K + B$ is Kawamata log terminal.

If $(S, B) = (\mathbb{P}^2, L)$, where $L = \sum L_i$ with n lines L_i in a generic position, then $\mathrm{R}(\mathbb{P}^2, L)$ is a complete graph with n points. Therefore, it is a manifold with a boundary only when $n \leq 3$ or $-(K + L)$ is nef.

We have something similar in dimension three.

7.11. Proposition-Definition. Let $(X/Z, B)$ be an n-fold contraction. Then the space $\mathrm{R}(X, B)$ has the property

(DIM) $\mathrm{R}(X, B)$ is a compact topological space of real dimension $\mathrm{reg}(X, B) \leq n - 1 = \dim X - 1$.

We now suppose that

- $-(K + B)$ is nef$/Z$ and
- (X, B) is log canonical.

Then locally$/Z$

(CN) $\mathrm{R}(X, B)$ is connected when $-(K + B)$ is big$/Z$ or consists of two points for $n \leq 3$; moreover, the latter is possible only if $\dim Z \leq \dim X - 1 = 2$ and there exists a (birationally unique) conic bundle structure on X/Z with two reduced disjoint sections D_1 and D_2 in B' for a log terminal resolution $(X', B') \to (X, B)$; $\mathrm{R}(X, B) = \{D_1, D_2\}$;

(MB) $\mathrm{R}(X, B)$ is a manifold with a boundary; in addition, it is a manifold when (X, B) is a 1-complement and B is over a given point $P \in Z$.

In particular, with each birational contraction $(X/Z, B)$, or a singularity when $X \to Z$ is an isomorphism, we associate a connected manifold $\mathrm{R}(X, B)$ locally over a point $P \in Z$, which is called a *type* of $(X/Z, B)$. The *regularity* $\mathrm{reg}(X, B)$ characterizes its topological difficulty.

If $(X/Z, B)$ is an n-complement that is not Kawamata log terminal over P as in Theorem 7.1, then $\mathrm{R}(X, B) \neq \emptyset$ and $\mathrm{reg}(X, B) \geq 0$. If $(X/Z, B)$ is an arbitrary log canonical singularity, we associate a topological manifold of the maximal dimension (and maximal for inclusions) for some complements $\mathrm{R}(X, B^+)$ with it, and its *complete* regularity is $\mathrm{reg}(X, B^+)$.

Proof. By the very definition, (DIM) holds in any dimension.

For $n =$ threefolds, (CN) follows from the LMMP and proofs of Connectedness Lemma 5.7 and Theorem 6.9 in [23]. The connectedness when $-(K + B)$ is big$/Z$ was proved in Theorem 17.4 in [16].

Near each point $\Delta_0 = D_{i_0}$, (MB) is a local question on $\mathrm{R}(X, B)$. That $\mathrm{R}(Y, B_Y)$ satisfies (MB), however, is a global question on $Y = D_{i_0}$. More precisely, a neighborhood of Δ_0 is a cone over $\mathrm{R}(Y, B_Y)$. But we can assume (MB) for $\mathrm{R}(Y, B_Y)$ by the adjunction and an induction on $\dim X$.

7.12. Corollary. *Under the assumptions of Theorem* 7.1, *let* $(X/Z, B^+)$ *be an* n*-complement with the minimal index* n. *Then*

- $\mathrm{reg}(X, B^+) = 2$, *and* $\mathrm{R}(X, B^+)$ *is a real compact surface with a boundary only when* $n = 1$ *or* 2; $\mathrm{R}(X, B^+)$ *is closed only for* $n = 1$;
- $\mathrm{reg}(X, B^+) = 1$, *and* $\mathrm{R}(X, B^+)$ *is a real curve with a boundary only when* $n = 1, 2, 3, 4$ *or* 6; *and*
- $\mathrm{reg}(X, B^+) = 0$, *and* $\mathrm{R}(X, B^+)$ *is a point only when* $n \in N_2$; *such complements and contractions are exceptional.*

Proof. The proof follows from the proof of Theorem 7.1.

A topology of log singularities can be quite difficult when $\mathrm{reg}(X, B^+) = 2$. Below, by a real surface, we mean a connected compact manifold with a boundary of dimension two. It is *closed* when the boundary is empty.

7.13. Example. For any closed real surface S, there exists a threefold 1-complement (X, B) such that $\mathrm{R}(X, B)$ is homeomorphic to S.

First, we take a triangulation $\{\Delta_i\}$ of S.

Second, we immerse its dual into $\mathbb{P}^3$ such that each point Δ_0 is represented by a plane L_i in a generic position.

Third, we make a blowup in such intersections $L_i \cap L_j$ that does not correspond to a segment $\Delta_1 = L_i \cap L_j$. We then obtain an algebraic surface $B = \sum L_i$ such that $K + B$ is log terminal and $\mathrm{R}(X, B)$ is the triangulation. Therefore, $\mathrm{R}(X, B)$ is homeomorphic to S. Moreover, $K + B$ is numerically trivial on the one-dimensional skeleton or on each curve $C_{i,j} = \Delta_1 = L_i \cap L_j$ because we have exactly two triple-points on each Δ_1 or each Δ_1 belongs to exactly two simplices of the triangulation.

Fourth, we contract all $C_{i,j}$ and something else from B and X to a point that gives the required singularity. Indeed, after the LMMP for $K + B$, we can use the semiampleness of $K + B$ when it has a general type. We note that the birational contractions or flips do not touch $C_{i,j}$. This can be verified on each L_i by the adjunction. For the same reason, we have no surface contraction on L_i or flips intersecting B but not in B. However, terminal singularities and flips are possible outside B or inside B, which preserves our assumptions on B. To secure the big property for $K + B$, we can similarly add B' on which $K + B$ is big. Then $K + B$ has the log Kodaira dimension three when B' has more than two connected components; otherwise $(K + B + B')|_{B'}$ is not big on some of the components.

The same holds for 2-complements with an arbitrary real nonclosed surface S with a boundary. We do not need to contract birationally some L_i and replace them with $(1/2)D$ for generic $D \in |2L_i|$. This can be done by the above combinatorics when $(K + B + B')_{L_i}$ is big.

7.14. Corollary. *Let $P \in (X, B)$ be a log canonical singularity such that*

- *X is $\mathbb{Q}$-factorial, and even formally or locally in the analytic topology of X, when there exists a non-normal curve in P as a center of log canonical discrepancy 0 for $K + B$;*
- *$\{B\} \neq 0$ in P, i.e., B has a fractional component through P; and*
- *$K + \lfloor B \rfloor$ is purely log terminal near P.*

Then $\mathrm{R}(X, B)$ has type $\mathbb{B}^r$, where $\mathbb{B}^r$ is a ball of dimension $r = \mathrm{reg}(X, B)$.

Moreover, we can drop the conditions when $r \leq 0$, or we can drop the first condition when

- *B has a fractional $\mathbb{R}$-Cartier component F, i.e., $0 < F \leq \{B\}$.*

Proof. According to our assumptions, $K + \lfloor B \rfloor$ is purely log terminal in P. Then X is the only log minimal model of $(X, \lfloor B \rfloor)$ over X (1.5.7 in [23]).

We now take a log terminal resolution $(Y/X, B_Y)$ and formally consider the LMMP$/X$ for $\lfloor B_Y \rfloor$. According to the above, the final model is $(X, \lfloor B \rfloor)$ with $\mathrm{reg}(X, \lfloor B \rfloor) \leq 1$ and has a trivial homotopy type. On the other hand, $\mathrm{R}(X, B) = \mathrm{R}(Y, B_Y)$. Therefore, it is sufficient to verify that contractions and flips preserve the homotopy type. If the centers of the flip or contraction are not in $\mathrm{LCS}(X, B) = \lfloor B_Y \rfloor$, then we even have a homeomorphism. If we have a divisorial contraction of a divisor D_i in $\lfloor B_Y \rfloor$, it induces a fiber contraction $D_i \to Z/X$. If Z is a curve$/P$, then, according to our conditions, it is a curve on another component D_j in $\lfloor B_Y \rfloor$ because any contraction of Y/X is divisorial. We note that each exceptional divisor of Y/X belongs to $\lfloor B_Y \rfloor$ and $\mathrm{R}(Y, B_Y)$ is a gluing of a cone with the vertex $\Delta_0 = D_i$ over $\mathrm{R}(D_i, B_{D_i})$ and in the latter. It is homotopy to $\mathbb{B}^{r-1}$ because $-(K_Y + \lfloor B_Y \rfloor)$ is nef$/X$ on all double curves $\Delta_1 = D_i \cap D_j$. The surgery drops this cone. We have a similar picture when $Z = \mathrm{pt.}$ or Z is not over P. In the latter case, Z has at most one curve$/P$ formally. (If nonformally, algebraically, then it has at most two double curves$/P$, which, in addition, are connected, by our assumptions.) Finally, let $Y \dashrightarrow Y^+/X$ be a flip in a curve $C_{i,j} = \Delta_1 = D_i \cap D_j/P$. Then, according to our assumptions, we have a flip on a third surface D_k in B_Y with $(D_k.C_{i,j}) = 1$ on Y.

The surgery deletes the segment $\Delta_1 = C_{i,j}$ and the interior of the triangle $\Delta_2 = D_i \cap D_j \cap D_k$. Formally, it looks like a blowup in Δ_1 that is the barycentric triangulation in Δ_1, and we then "contract" the resolution divisor on a curve on D_k.

If $r = 1$, we can also assume that at the start, $\{B\} \equiv -(K_Y + \lfloor B_Y \rfloor)/Z$ is nef Z (Theorem 5.2 in [24]). Indeed, we can consider the LMMP for $B_Y + \varepsilon\{B\}$. Because $-(K_Y + \lfloor B_Y \rfloor)$ is big$/Z$, we have a birational contraction $Y \dashrightarrow Z/X$; it contracts all surfaces to a curve C by Theorem 6.7. Then any fractional component of B is positive on C, which contradicts our assumption, because Z/X is small.

If B has the fractional component F, we can apply an induction on the number of irreducible curves C_i/P on a formal $\mathbb{Q}$-factorialization X'/X. Indeed, if X'/X blows up one such curve C_i, it is irreducible rational and has at most two points in which $K + B$ is not log terminal. Therefore, we glue at most two balls $\mathbb{B}^r$ in $\mathbb{B}^{r-1}$. We note that F passes through C_i on X'. Finally, we have no such curves when X is formally $\mathbb{Q}$-factorial in P. This drops the first condition.

In general, we can verify that for $r = 2$, the formal Weil–Picard number in P is not less than $q = h^1(\mathrm{R}(X, B), \mathbb{R}) = 2 - \chi(\mathrm{R}(X, B))$ (the topological genus), where χ is the topological Euler characteristic. We can consider q the *genus* of the singularity P.

7.15. Corollary. *Under the assumptions of Theorem* 7.1, *let* $(X/Z, B^+)$ *be an* n*-complement with the minimal index* n *and* P *be* $\mathbb{Q}$*-factorial, formally, when there exists a nonnormal curve in* P *as a center of log canonical discrepancy* 0 *for* $K + \lfloor B^+ \rfloor$ *and purely log terminal* $K + B$. *Then* $\mathrm{R}(X, B)$ *has type* $\mathbb{B}^r$ *with* $r = \mathrm{reg}(X, B)$ *or* S^2. *The latter is only possible for* $n = 1$.

If P is an isolated singularity, we can drop the formal condition for an appropriate complement with $n \geq 2$. For $n = 1$, we can take a 2-complement as in the proof below. For $n \geq 3$, we can even drop the $\mathbb{Q}$-factorial property.

Proof. We only need to consider the case with $n = 1$. Then we have a reduced component D through P in B^+. If we replace D by a generic $(1/2)D'$, where $D' \in |2D|$ is rather generic, we obtain a 2-complement B' with $\mathrm{R}(X, B')$ homeomorphic to $\mathbb{B}^2$. Then $\mathrm{R}(X, B^+)$ can be obtained from this by gluing a cone over $\mathrm{R}(D, B_D)$. The latter is $[0, 1]$ or S^1. That gives respectively $\mathbb{B}^2$ or S^2.

In the investigation of the m.l.d.'s for threefolds, we can assume that the point P in (X, B) is

(T2) $\mathbb{Q}$-factorial (even formally) and terminal in codimensions one and two.

Otherwise, after a crepant resolution, we can reduce the problem to (T2) or to dimension two or one.

7.16. Corollary. *The conjecture on discrepancies* (Conjecture 4.2 in [24]) *holds for threefold log singularities* (X, B) *with* $b_i \in \Gamma$ *under* (M)″ *and* (T2), *when* $\mathrm{reg}(X, B^+) \leq 1$. *Moreover, in such a case, the only clusters of* $A(\Gamma, 3)$ *are*

(0) 0, *when* $\mathrm{reg}(X, B^+) = 0$, i.e., in the exceptional cases;

(1) 0 *and*

$$A(\Gamma', 2)$$

when $\mathrm{reg}(X, B^+) = 1$, *where* $\Gamma' = \{0, 1/2, 2/3, 3/4, 4/5, 5/6\}$ *and where we consider only two-dimensional singularities* (S, B^+) *with the boundary multiplicities in* Γ' *and* $\mathrm{reg}(S, B^+) = 0$, i.e., the exceptional case.

The cluster points are rational, our $A(\Gamma, 3)$ *is closed when* $1 \in \Gamma$, *and the only cluster of the clusters is* 0.

Sketch of Proof. We note that Γ satisfies (M)″ but may not be the d.c.c. In particular, Γ may not be standard. If $\mathrm{reg}(X, B) = 0$, then by Corollary 7.3, we have a finite subset

$$\{a \in A(\Gamma, 3) \mid a \geq \varepsilon\}$$

of the corresponding $A(\Gamma, 3)$ for any $\varepsilon > 0$. Of course, the corollary was proved for the standard Γ, but in our case, all $b_i \leq (l-1)/l$, where $l = \max\{r \in N_2\}$, according to Corollary 2.26 and because of $\mathrm{reg}(X, B^+) = 0$. Therefore, such b_i is standard because of (M)″.

If $\mathrm{reg}(X, B) = 1$, we can use the arguments in [26] and Theorem 6.7. The latter almost reduces our case to the two-dimensional situation. "Almost" means except for the edge components in the chain of the reduced part S of B^+ as in the proof of Theorem 7.1 (cf. the proof of Corollary 7.14). The clusters can be realized as m.l.d.'s for (X, B) with the reduced part $S = \lfloor B \rfloor$ and the fractional multiplicities in Γ'.

Therefore, to complete the conjecture on discrepancies for threefolds, the singularities with 2-complements of type $\mathbb{B}^2$ by Corollary 7.15 must be considered. This case is related to 1- or 2-complements. The former are closed to toric complements, where the conjecture was verified by Borisov [4]. We also see that $\text{reg}(X, B^+)$ conjecturally may have interpretations in terms of clusters: the first cluster of $A(\Gamma, n)$ with $r = \text{reg}(X, B^+)$ is $A(\Gamma, n-1)$ with $\text{reg}(X, B^+) = r - 1$; in particular, it is rational when Γ is standard.

7.17. Remark. We expect that most of the results in this section hold in any dimension and for any regularity $r = \text{reg}(X, B)$ or $\text{reg}(X, B^+)$.

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