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THE NOETHER-ENRIQUES THEOREM ON CANONICAL CURVES

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Abstract. The principal result of the present work consists in the proof that an intersection of quadrics passing through a canonical curve is a reduced variety. The possible cases when the intersection of quadrics does not coincide with the curve itself are also examined in this article.

Figures: 1. Bibliography: 8 references.

Max Noether considered in [7] the space $\Phi^{(i)}\cdot\Phi^{(1)}$ of functions of the form ω/ω_0, where the ω are regular differentials of some curve X and ω_0 is a fixed regular differential; $\Phi^{(i)}$ is the space of i-forms of functions from the space $\Phi^{(1)}$ with coefficients in a ground field k. Noether showed that the dimension of the space of relations of degree i for functions in $\Phi^{(1)}$ is equal to

$$\binom{g+i-1}{i}-(2i-1)(g-1)$$

in the nonhyperelliptic case and equal to

$$\binom{g+i-1}{i}-i(g-1)-1$$

in the hyperelliptic case, where $g=\dim\Phi^{(1)}$ is the genus of the curve X.

Enriques looked at Noether's result from a geometrical point of view (see [6] for Enriques' results). We shall consider a curve C, the image of X under a canonical transformation. It is well known that C is isomorphic to X in the nonhyperelliptic case. We will assume in what follows that X is not hyperelliptic. A relation of degree i between regular differentials corresponds to a form of degree i passing through C. Enriques proved that the number of linearly independent quadrics passing through C is equal to $(g-2)(g-3)/2$. This corresponds to Noether's result on the number of independent relations of the second degree for regular differentials. Enriques then looked at the intersection of the quadrics through C, and showed that in it will be found only the points of C, or a surface of degree $g-2$.

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In the present article, the results of Noether and Enriques will be examined in connection with the theory of schemes. The principal portion of the article is devoted to proving that the intersection of the quadrics through C is a reduced scheme. It coincides with C, or is an irreducible surface.

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§0. Formulation of the problem and some basic results

Let k be some field; all our varieties and schemes will be defined over k. Let us denote by X a complete nonsingular curve of genus $g \geq 3$. The curve will be assumed nonhyperelliptic. Then it is well known that we have a canonical immersion

$$\varkappa : X \to \operatorname{Proj}(S(H^0(X, \Omega_X))),$$

where Ω_X denotes the sheaf of regular differentials of X over k. Let $C = \kappa(X)$. To avoid the inhibiting effect of too complicated a notation, we will put $\mathbf{P}^{g-1} = \operatorname{Proj}(S(H^0(X, \Omega_X)))$. The basic properties of the canonical immersion will be recalled at the start of §2.

Let us denote by Q the closed subscheme of the space $\mathbf{P}^{g-1}$ whose ideal is generated by the forms of degree 2 in the ideal of the curve C. It is the purpose of this article to study the basic properties of the scheme Q. The principal results are contained in the following theorems.

Theorem 1. a. *Q is a projective variety; that is, Q is a reduced irreducible closed subscheme of $\mathbf{P}^{g-1}$.*

b. *The dimension of Q is either 1 or 2.*

c. *If $\dim Q = 1$, then $Q = C$.*

d. *If $\dim Q = 2$ and $g \neq 4$, then Q is a smooth surface of degree $g - 2$. In this situation, only the following possibilities can arise.*

1. *$Q \simeq P^2$, in which case we have the following exact description of the immersion of P^2 in $\mathbf{P}^{g-1}$ and of the location of C on P^2. Let us denote by $O_Q(1)$ the restriction of the sheaf $O_{P^{g-1}}(1)$ to Q. Then $\mathbf{P}^{g-1} = Q$ and $O_Q(C) = O_{P^2}(4)$ for $g = 3$; $O_Q(1) = O_{P^2}(2)$ and $O_Q(C) = O_{P^2}(5)$ for $g = 6$; in other words, in this case the curve C lying on $P^2 \simeq Q$ will be a curve of degree 5, and Q is a Veronese image of this plan. If $g \neq 3$, or 6, then $Q \not\simeq P^2$.*

2. *$Q \simeq F_n$,(1) and the following relation holds:*

$$0 \leqslant n \leqslant \min\left\{\frac{g+2}{3}, g-4\right\}; \qquad n \equiv g \pmod 2).$$

As in the previous case, the immersion of F_n in $\mathbf{P}^{g-1}$ and the locus of C in the divisor class group of the surface F_n admits an exact description:

(1)The definition and elementary properties of F_n can be found in [1] and [2] (see also §6 of this article).

$$O_Q(1) = O_{F_n}\left(b_n + \frac{g+n-2}{2} \cdot s_n\right);$$

$$O_Q(C) = O_Q(C) \otimes \Omega_Q^{-1} = O_{F_n}\left(3b_n + \frac{g+3\cdot n+2}{2} \cdot s_n\right).$$

e. *If* $g = 4$, *then either* Q *is the surface* F_0, *immersed in the same way as* F_n *in case* d2, *or* Q *is a cone with a nonsingular base curve of degree* 2.

Definition. Curves X for which $\dim Q = 2$ will be called *special.*

Theorem 2. *In order that the curve* X *be special, it is necessary and sufficient that*

a) *for* $g \geq 7$ *or* $g = 5$, *there exists an effective divisor* D *of degree* 3 *such that* $\dim H^0(X, O_X(D)) = 2$;

b) *for* $g = 6$, *either there exists an effective divisor* D *of degree* 3 *for which* $\dim H^0(X, O_X(D)) = 2$, *or there exists an effective divisor* D *of degree* 5 *for which* $\dim H^0(X, O_X(D)) = 3$;

c) *for* $g = 3, 4$, *every curve* X *is special.*

Theorem 3. a. *There exists a special curve of genus* 6 *for which* Q *is a Veronese image of the plane* P^2.

b. *If* k *is infinite and* n *satisfies the relation* (1), *then there exists a special curve* X *of genus* g *such that* $Q \simeq F_n$.

We will assume throughout the sequel that $g \geq 4$. The proof of the theorems in case $g = 3$ presents no difficulty. We will further assume that k is algebraically closed. It is clear that the validity of Theorems 1 and 2, and of part a of Theorem 3, is independent of this assumption. The proof of part b of Theorem 3 is given in §9. Essentially it relies only on Bertini's theorem for hyperplanar sections, which holds when k is infinite.

The following propositions lie at the foundation of the proof of Theorem 1.

Proposition 1. $\dim H^0(\mathbf{P}^{g-1}, I_C(2)) = (g-2)(g-3)/2$.

Proposition 2. *Let* M *be a scheme given as the intersection of* $(g-2)(g-3)/2$ *independent quadrics in the space* P^{g-2}, *which contains* $2g-2$ *isolated points lying in general position. Then* $M_{\mathrm{red}} = M$, *and if* $\#(M) > 2g-2$,(2) *then* M *is a reduced irreducible nonsingular curve of genus* 0 *and degree* $g-2$, *generating* P^{g-2}.

The proof of Proposition 1 is dealt with in §2, where the dimensions of the spaces $H^i(P^{g-1}, I_C(n))$ are actually calculated for all i and n.

§§3 and 5 are devoted to the proof of Proposition 2. This proof makes use of a detailed analysis of spaces of sections in suitable bases. The situation described in Proposition 2 arises if we consider the scheme $Q \cap H$ for the generic hyperplane

(2) $\# M$ denotes the number of points in the set M.

H. The condition on the number of quadrics defining the scheme $Q \cap H$ is realized in accordance with Proposition 1.

§§4, 6 and 7 are devoted to a detailed derivation of Theorem 1, at the root of which lie the above propositions.

§1. The necessary general information about the techniques of hyperplanar sections

1. Let I be some sheaf of ideals on the projective space P^n, and H some hyperplane having equation h. Then we have the following exact sequence:

$$0 \to I(m) \overset{\otimes h}{\to} I(m+1) \to (I \otimes O_H)(m+1) \to 0. \tag{1.1}$$

Multiplication by the local equation of H is injective on I, since it is a subsheaf of O_{P^n}. Let $Z \subset P^n$ be the subscheme determined by the ideal I.

Lemma 1.1. *If* $\mathrm{Ass}(O_Z) \cap H = \emptyset$, *then* $\mathrm{Tor}_1(O_H, O_Z) = 0$.

Proof. Let $X \in P^n$ and let f be the local equation for H at the point x. Then, tensoring the exact sequence

$$0 \to I_x \to O_{x,P^n} \to O_{x,Z} \to 0$$

with $O_{x,P^n}/f \cdot O_{x,P^n} = O_{x,H}$, we obtain an exact sequence

$$\mathrm{Tor}_1(O_{x,P^n}/f \cdot O_{x,P^n}, O_{x,Z}) \to I_x \cdot O_{x,H} \to O_{x,H}.$$

The arrow on the right is injective since f is not a divisor of 0 in $O_{x,Z}$ (f is invertible on all prime ideals associated with $O_{x,Z}$), and we therefore obtain that $\mathrm{Tor}_1(O_{x,H}, O_{x,Z}) = 0$. The lemma is proved.

Tensoring the monomorphism $I \hookrightarrow O_{P^n}$ of sheaves with O_H we obtain a homomorphism ϕ: $I \otimes O_H \to O_H$. In the general case this homomorphism will not be an immersion, but if it is, then, identifying the sheaf $I \otimes O_H$ with the image $I_H \subset O_H$, we obtain a quasi-coherent subsheaf I_H, namely, the sheaf of ideals in O_H. The following lemma indicates a sufficient condition that the above homomorphism be a monomorphism.

Lemma 1.2. *If* $\mathrm{Ass}(O_Z) \cap H = \emptyset$, *then the homomorphism* ϕ *will be an immersion and the subsheaf* $I_H = \mathrm{Im}(I \otimes O_H)$ *will be the sheaf of ideals on* H *defining the closed subscheme* $Z \cap H \subset H$.

Proof. Consider the exact sequence

$$0 \to I \to O_{P^n} \to O_Z \to 0$$

and tensor it with O_H; then with the aid of Lemma 1.1 we obtain that

$$\begin{array}{ccccccccc} \mathrm{Tor}_1(O_H, O_Z) & \to & I \otimes O_H & \to & O_{P^n} \otimes O_H & \to & O_Z \otimes O_H & \to & 0 \\ \| & & \wr\| & & \wr\| & & \wr\| & & \\ 0 & \longrightarrow & I_H & \longrightarrow & O_H & \to & O_{Z \cap H, H} & \to & 0. \end{array} \tag{1.2}$$

The lemma is proved.

In what follows, it will be necessary for us repeatedly to carry out restrictions of a certain scheme $Z \subset P^n$ to some hyperplane H. It is naturally desirable for us to consider restrictions to hyperplanes H for which the sequence (1.2) is exact. According to Lemma 1.2, it is sufficient for this that $\mathrm{Ass}(O_Z) \cap H = \emptyset$.

Lemma 1.3. *Let Z be a subvariety (that is, a reduced irreducible k-algebraic scheme) of P^n; then the following assertions are equivalent:*

a) $\mathrm{Ass}(O_Z) \cap H \neq \emptyset$,

b) $Z \subset H$.

The proof is obvious, since $\mathrm{Ass}(O_Z)$ consists of one point, the generic one.

Corollary. *If Z is a variety not satisfying one of the conditions of Lemma 1.3, then the sequence*

$$0 \to I(m) \xrightarrow{\otimes h} I(m+1) \to I_H(m+1) \to 0 \tag{1.3}$$

is exact for any integer m, where I_H is the sheaf of ideals of the variety $Z \cap H$ in O_H (the conditions of Lemma 1.3 are usually verified on the isolated points of Z).

In what follows we will use part of the

Proposition on the restriction. I. *Let Z be a variety in P^n and H a hyperplane such that $Z \not\subset H$. Then we have an exact sequence*

$$0 \to H^0(P^n, I(m)) \to H^0(P^n, I(m+1)) \to H^0(H, I_H(m+1)) \to H^1(P^n, I(m)). \tag{1.4}$$

We will make more frequent use of a weaker version of this, namely,

II. *If we supplement the above conditions by assuming that $Z \not\subset H$ for any hyperplane, then we have the following immersion for any hyperplane H:*

$$0 \to H^0(P^n, I(2)) \to H^0(H, I_H(2)). \tag{1.5}$$

This will be an isomorphism if and only if $H^0(H, I_H(2))$ is mapped to $0 \in H^1(P^n, I(1))$, which will be the case if $H^1(P^n, I(1)) = 0$, for example.

Proof. The exact sequence (1.4) is obtained by considering the cohomology sequence for the exact triple (1.3). The immersion (1.5) is obtained from the exact sequence (1.4) for $m = 1$, and from the fact that if Z does not lie in any hyperplane, then $H^0(P^n, I(1)) = 0$.

2. Let Y be a projective variety lying in P^N and having the following properties:

1) Y does not lie in any hyperplane, and $\dim Y \geq 1$.

2) The ideal of this variety is generated by forms of degree not less than n.

3) The intersection of the forms of degree n, as a topological space, coincides with Y.

We will denote by S the intersection of the forms of degree n of the scheme Y.

Condition 3) means that $S_{\mathrm{red}} = Y$. The purpose of the present section is to prove the following lemma, which gives a sufficient condition that $S = Y$.

Lemma on reducibility. *$S = Y$ if there exists a hyperplane H such that $S \cap H$ is a reduced scheme, or, in other words, S is reduced if $S \cap H$ is reduced for some hyperplane H.*

A refinement of this is Lemma 1.4, which is actually a criterion for the reducibility of the scheme S.

Lemma 1.4. *$Y = S$ if and only if $Y \cap H = S \cap H$ for some hyperplane H.*

Proof. The assertion in one direction is obvious, so let us assume that there exists a hyperplane H such that $Y \cap H = S \cap H$. Let F be some form on P^N such that $F(Y) = 0$; then it is evident that $\deg F \geq n$. We will prove by induction on $\deg F$ that F is generated by forms of degree n. In the case when $\deg F = n$, this is obvious by hypothesis. Let $\deg F > n$, and let us consider the restriction of F to H. This is a form f which, since $Y \cap H = S \cap H$, is expressed by means of forms of smaller degree restricted to H; that is, we can assume that $F = 0$ on H. Hence $F = F' \cdot h$, where h is the equation of the hyperplane H, a form of degree 1. It follows from property 1) of the scheme Y that $F'(Y) = 0$. Furthermore, $\deg F' < \deg F$, whence F' is generated by forms of degree n, by the inductive assumption. Hence the same is also true for F. This shows that the ideals of the schemes S and Y coincide; that is, $S = Y$. The lemma is proved.

3. We will elucidate in this section those problems involved with the choice of of a hyperplane, in some sense "good" as regards restriction, and also prove that for a certain class of closed subschemes of p^n, general hyperplane will be "good".

Definition 1.1. N isolated points of p^n are situated in *general position* in p^n if and only if any $k+1$ of these N points generate a subspace $P^k \subset P^n$, where $k \leq n$.

Remark. It is easy to show that, for $N \geq n+1$, N isolated points of P^n are in general position in P^n if and only if any $n+1$ of these N points generate P^n.

Lemma on the choice of a "good" hyperplane. *Let C be an irreducible and reduced curve generating P^n $(n \geq 2)$. Then for a generic hyperplane H we have $H \cdot C = \Sigma_{i=1}^{\deg C} x_i$, where the x_i are $\deg C$ distinct points lying in H in general position.*

Lemma 1.5. *There is a hyperplane H in P^n such that $H \cdot C = \Sigma_{i=1}^{\deg C} x_i$, where the x_i are distinct isolated points of H among which are $n-1$ points such that $(E \cdot C) = n-1$, E being the projective subspace of codimension 2 in P^n generated by these $n-1$ points.*

Deduction of the lemma on the choice of a "good" hyperplane from Lemma 1.5. We shall denote by $C(n-1)$ the $(n-1)$-fold symmetric product of the curve C; then $C(n-1)$ is a variety, since C is an irreducible reduced curve, with $\dim C(n-1) = n-1$. Let us consider the reduced subscheme $\widetilde{C} \subset C(n-1) \times \hat{P}^n$, whose isolated points are pairs (d, h), where d is an effective divisor of degree $n-1$ on the curve C and h is a hyperplane in P^n such that $d \subset h \cdot C$. We shall consider the two natural projections of the scheme $\widetilde{C}$:

$$\begin{array}{ccc} & \pi_1\ \widetilde{C}\ \pi_2 & \\ \swarrow & & \searrow \\ C(n-1) & & \hat{P}^n \end{array}$$

Let $U(n-1)$ be an open set in $C(n-1)$ whose isolated points coincide with those divisors $x_1 + \cdots + x_{n-1}$ on C for which the points x_i are $(n-1)$ distinct points in general position in P^n. The proof of the existence of $U(n-1)$ presents no difficulty. It is also easy to prove that there is an open set $W(n-1) \subset U(n-1)$ such that the isolated points of $W(n-1)$ are those divisors $x_1 + \cdots + x_{n-1}$ for which the points x_i generate a projective space E of codimension 2 in P^n, and $(C \cdot E) = n-1$. It follows from Lemma 1.5 that $W(n-1) \neq \emptyset$, and so the dimension of the scheme $U(n-1) \backslash W(n-1)$ is not greater than $n-2$, since $C(n-1)$ is an irreducible scheme.

Let us consider a set $\mathcal{H}$ in $\hat{P}^n$ which is open in the Zariski topology and whose isolated points consist of hyperplanes h such that h is transversal to C. The projection π_2 is quasi-finite. It projects open sets of C into open sets of $\hat{P}^n$. Hence there exists a closed subscheme $\mathcal{H}'$ of $\hat{P}^n$ such that its isolated points contain the images of the isolated points $U(n-1)\backslash W(n-1))$ under the transformation $\pi_2 \circ \pi_1^{-1}$. Since the fiber of π_1 over the points of $U(n-1)$ has dimension 1, $\dim \mathcal{H}' \leq n-2+1 = n-1$. Let h be a hyperplane of $\mathcal{H}$ which is not "good"; then there is a divisor $d \in U(n-1)\backslash W(n-1)$ such that $(d, h) \in \widetilde{C}$, because the transversal hyperplane cuts out points of C which generate this hyperplane, since C generates P^n. Thus $h \in \mathcal{H}'$, and so the hyperplanes of $\mathcal{H}\backslash\mathcal{H}'$ are "good". The lemma on the choice of a "good" hyperplane is proved, since $\dim \mathcal{H}' \leq n-1$ and $\mathcal{H}$ is open in the Zariski topology on $\hat{P}^n$.

Lemma 1.6. *For $n \geq 3$ there is an isolated point x on C such that there are only finitely many lines l passing through x for which $(C, l) > 2$.*

Deduction of Lemma 1.5 from Lemma 1.6. We will prove Lemma 1.5 by induction on $n \geq 2$. For $n = 2$ the assertion of Lemma 1.5 follows from the fact that, for any plane reduced curve, there is a line transversal to it.

Let us assume that Lemma 1.5 is proved for $n \leq k$, and let C be a reduced irreducible curve generating P^{k+1}, $k \geq 2$. By Lemma 1.6 there is a point x on C such that there are only finitely many lines passing through x and having an intersection with C of index not less than 3. Let us consider the projection π of the curve C from the point x onto some hyperplane H', $x \notin H'$. Let C' be the closure of the image of C under π. C' is a reduced irreducible curve of degree $\deg C - 1$ which generates H'. By the inductive assumption there is a hyperplane E' in H' such that $E' \cdot C' = \sum_{i=1}^{\deg C-1} y_i$, where the y_i are $\deg C'$ distinct isolated points of E' among which are $k-1$ points such that $(E'' \cdot C') = k-1$, E'' being the projective subspace of codimension 2 in H' generated by these $k-1$ points. Consider the hyperplane H passing through x and E'. It is easily shown that this hyperplane is of the desired type. Lemma 1.5 is proved.

Proof of Lemma 1.6. For $n \geq 4$ there is a point $x \in P^n$ such that the projection of C from this point is an isomorphism. If Lemma 1.6 were false for C, it is also false for its image under projection from the point x. In order to prove Lemma 1.6, it is thus sufficient to prove it for $n=3$. The proof for this case can be found in [8] (page 289). Lemma 1.6 is proved.

§2. Computation of the cohomology of the twisted sheaves of the sheaf of ideals of a canonically immersed curve

Let us consider a canonical immersion of a curve X

$$\varkappa : X \xrightarrow{\sim} C \subset \mathbf{P}^{g-1}.$$

The mapping κ has the following properties:

i) By the definition of κ, $\kappa^*(O_{\mathbf{P}}(1)) = \Omega_X$, and so $\deg C = 2g-2$.

ii) C generates the space $\mathbf{P}^{g-1}$.

We shall denote by I the sheaf of ideals of C. This section is devoted to computing the cohomology of the twisted sheaf $I(n)$. The result obtained in this connection yields the following assertion.

Theorem 2.1

$$\dim H^0(\mathbf{P}^{g-1}, I(n)) = \begin{cases} \binom{g+n-1}{n} - (2n-1)(g-1) & \text{for } n \geqslant 2, \\ 0 & \text{for } n \leqslant 1; \end{cases}$$

$$\dim H^2(\mathbf{P}^{g-1}, I(n)) = \begin{cases} 0 & \text{for } n \geqslant 2, \\ 1 & \text{for } n = 1, \\ g & \text{for } n = 0, \\ (1-2n)(g-1) & \text{for } n \leqslant -1; \end{cases}$$

$$\dim H^{g-1}(\mathbf{P}^{g-1}, I(n)) = \begin{cases} 0 & \text{for } n \geqslant -g+1, \\ \binom{-n-1}{-n-g} & \text{for } n \leqslant -g; \end{cases}$$

while $H^i(\mathbf{P}^{g-1}, I(n)) = 0$ *for all other values of* i.

Proposition 2.1. *For any natural number* $n \geq 2$ *we have*

a) $$\dim H^0(\mathbf{P}^{g-1}, I(n)) = \binom{g+n-1}{n} - (2n-1)(g-1),$$

b) $\dim H^1(\mathbf{P}^{g-1}, I(n)) = 0,$

c) $\dim H^2(\mathbf{P}^{g-1}, I(n)) = 0.$

Lemma 2.1. $H^0(\mathbf{P}^{g-1}, I(1)) = H^1(\mathbf{P}^{g-1}, I(1)) = 0$, *and* $\dim H^2(\mathbf{P}^{g-1}, I(n)) = 1$.

Deduction of Theorem 2.1 from Proposition 2.1 and Lemma 2.1. For arbitrary $n \in \mathbf{Z}$ consider the exact triple

$$0 \to I(n) \to O_{\mathbf{P}}(n) \to \varkappa_*(\Omega_X^{\otimes n}) \to 0$$

and the cohomology sequence corresponding to it

$$\begin{aligned} 0 &\to H^0(\mathbf{P}^{g-1}, I(n)) \to H^0(\mathbf{P}^{g-1}, O_{\mathbf{P}}(n)) \to H^0(X, \Omega_X^{\otimes n}) \\ &\to H^1(\mathbf{P}^{g-1}, I(n)) \to H^1(\mathbf{P}^{g-1}, O_{\mathbf{P}}(n)) \to H^1(X, \Omega_X^{\otimes n}) \\ &\to \cdots \\ &\to H^{g-1}(\mathbf{P}^{g-1}, I(n)) \to H^{g-1}(\mathbf{P}^{g-1}, O_{\mathbf{P}}(n)) \to H^{g-1}(X, \Omega_X^{\otimes n}) \\ &\to \cdots \end{aligned} \tag{2.1}$$

Since $H^i(X, \Omega_X^{\otimes n}) = 0$ for $i \geq 2$ and $H^j(\mathbf{P}^{g-1}, O_{\mathbf{P}}(n)) = 0$ for $j \neq 0, g-1$, we obtain from this sequence that $H^i(\mathbf{P}^{g-1}, I(n)) = 0$ for $i \neq 0, 1, 2, g-1$. We also obtain that

$$\dim H^{g-1}(\mathbf{P}^{g-1}, I(n)) = \dim H^{g-1}(\mathbf{P}^{g-1}, O_{\mathbf{P}}(n)) = \begin{cases} 0 & \text{for } n \geqslant -g+1, \\ \binom{-n-1}{-n-g} & \text{for } n \leqslant -g. \end{cases}$$

Since $H^1(\mathbf{P}^{g-1}, O_{\mathbf{P}}(n)) = H^2(\mathbf{P}^{g-1}, O_{\mathbf{P}}(n)) = 0$, we have

$$\dim H^2(\mathbf{P}^{g-1}, I(n)) = \dim H^1(X, \Omega_X^{\otimes n}) = \dim H^0(X, \Omega_X^{\otimes(1-n)})$$

$$= \begin{cases} 0 & \text{for } n \geqslant 2, \\ 1 & \text{for } n = 1, \\ g & \text{for } n = 0, \\ (1-2n)(g-1) & \text{for } n \leqslant -1. \end{cases} \tag{2.2}$$

To complete the proof of Theorem 2.1, it remains for us to compute the zeroth and first cohomology groups, this being the most difficult part of the proof. We remark that all the other cohomology groups have been calculated without using Proposition 2.1 or Lemma 2.1, with the result that a part of Lemma 2.1 and a part c) of Proposition 2.1 have been proved in the process.

Let $n = 0$ in the sequence (2.1); then

$$\dim H^0(\mathbf{P}^{g-1}, I(0)) = \dim H^1(\mathbf{P}^{g-1}, I(0)) = 0,$$

whence $H^1(\mathbf{P}^{g-1}, O_{\mathbf{P}}(0)) = 0$, while

$$\dim H^0(\mathbf{P}^{g-1}, O_{\mathbf{P}}(0)) = \dim H^0(X, \Omega_X^{\otimes 0}) = 1$$

and the mapping $H^0(\mathbf{P}^{g-1}, O_{\mathbf{P}}(0)) \to H^0(X, \Omega_X^{\otimes 0})$ is evidently an isomorphism. For $n < 0$ we have

$$H^0(\mathbf{P}^{g-1}, O_{\mathbf{P}}(n)) = H^0(X, \Omega_X^{\otimes n}) = H^1(\mathbf{P}^{g-1}, O_{\mathbf{P}}(n)) = 0,$$

and we thus obtain from the exact sequence (2.1) that

$$\dim H^0(\mathbf{P}^{g-1}, I(n)) = \dim H^1(\mathbf{P}^{g-1}, I(n)) = 0.$$

Lemma 2.1 and Proposition 2.1 in conjunction with what we have proved complete the proof of Theorem 2.1.

Proof of Lemma 2.1. It follows from property (ii) of the canonical immersion κ that $H^0(\mathbf{P}^{g-1}, I(1)) = 0$ and, since

$$\dim H^0(\mathbf{P}^{g-1}, O_{\mathbf{P}}(1)) = \dim H^0(X, \Omega_X) = g \text{ and } H^1(\mathbf{P}^{g-1}, O_{\mathbf{P}}(1)) = 0,$$

we have $\dim H^1(\mathbf{P}^{g-1}, I(1)) = 0$. The lemma is proved.

$H^1(\mathbf{P}^{g-1}, O_{\mathbf{P}}(n)) = 0$. Taking the alternating sum of the dimensions of the first four terms of the exact sequence (2.1), we obtain that

$$\dim H^0(\mathbf{P}^{g-1})I(n)) - \dim H^0(\mathbf{P}^{g-1}, O_{\mathbf{P}}(n)) + \dim H^0(X, \Omega_X^{\otimes n}) - \dim H^1(\mathbf{P}^{g-1}, I(n)) = 0.$$

For $n \geq 2$, this equation yields the following identity:

$$\dim H^0(\mathbf{P}^{g-1}, I(n)) = \binom{g+n-1}{n} - (2n-1)(g-1) + \dim H^1(\mathbf{P}^{p-1}, I(n)), \quad (2.3)$$

because $\dim H^0(\mathbf{P}^{g-1}, O_{\mathbf{P}}(n)) = \binom{g+n-1}{n}$ for $n \geq 0$, while $\dim H^0(X, \Omega_X^{\otimes n}) = (2n-1) \times (g-1)$ for $n \geq 2$.

With the aid of (2.3) and the following lemma, we will prove Proposition 2.1 by induction on $n \geq 2$.

Lemma 2.2. I. *Let M be a closed subscheme in P^{g-2} which contains $2g-3$ isolated points lying in general position; then*

$$\dim H^0(P^{g-2}, I_M(2)) \leqslant \frac{(g-2)(g-3)}{2}. \quad (2.4)$$

II. *Let M be a closed subscheme in P^{g-2} which contains $2g-2$ isolated points lying in general position; then, for $n \geq 3$,*

$$\dim H^0(P^{g-2}, I_M(n)) \leqslant \binom{g+n-2}{n} - (2g-2). \quad (2.5)$$

Deduction of Proposition 2.1 from Lemma 2.2. Part c) was proved in Theorem 2.1, and so it remains to prove parts a) and b); these will be proved simultaneously by induction on n.

Let $n = 2$. It follows from the lemma on the choice of a "good" hyperplane that there exists a hyperplane H such that $H \cdot C = \sum_{i=1}^{\deg C} x_i$, where the x_i

are $\deg C$ distinct points situated in general position on H. The sheaf of ideals I_H is defined by the Corollary in §1 to be the restriction of the sheaf of ideals I. Let M be the closed subscheme of H which corresponds to the sheaf of ideals I_H. By the choice of the hyperplane H it is clear that the conditions of Lemma 2.2 are fulfilled, and so inequalities (2.4) and (2.5) hold for $n > 3$. By the proposition on the restriction in §1 we have the immersion (1.5), and thus

$$\dim H^0(\mathbf{P}^{g-1}, I(2)) \leqslant \dim H^0(H, I_H(2)).$$

By inequality (2.4)

$$\dim H^0(H, I_H(2)) = \dim H^0(H, I_M(2)) \leqslant \frac{(g-2)(g-3)}{2}.$$

According to (2.3) for $n = 2$, we obtain that

$$\dim H^0(\mathbf{P}^{g-1}, I(2)) = \frac{(g-2)(g-3)}{2} + \dim H^1(\mathbf{P}^{g-1}, I(2)) \geqslant \frac{(g-2)(g-3)}{2}.$$

Hence

$$\dim H^1(\mathbf{P}^{g-1}, I(2)) = 0 \quad \text{and} \quad \dim H^0(\mathbf{P}^{g-1}, I(2)) = \frac{(g-2)(g-3)}{2}.$$

This proves the first step of the induction ($n = 2$).

Let us assume that parts a) and b) of Proposition 2.1 hold for $n \leq k$. By applying the proposition on the restriction for $n = k$, we obtain an exact sequence

$$0 \to H^0(\mathbf{P}^{g-1}, I(k)) \to H^0(\mathbf{P}^{g-1}, I(k+1)) \to H^0(H, I_H(k+1)) \to H^1(\mathbf{P}^{g-1}, I(k)) \to \dots .$$

By the inductive assumption, $\dim H^1(\mathbf{P}^{g-1}, I(k)) = 0$. Hence

$$\dim H^0(\mathbf{P}^{g-1}, I(k+1)) = \dim H^0(H, I_H(k+1)) + \dim H^0(\mathbf{P}^{g-1}, I(k)).$$

We remarked earlier that inequality (2.5) is satisfied for $n \geq 3$, and since $k + 1 \geq 3$ we can use the inductive assumption for the zeroth cohomology groups and inequality (2.5) to obtain that

$$\dim H^0(\mathbf{P}^{g-1}, I(k+1)) \leqslant \binom{g+k-1}{k} - (2k-1)(g-1)$$
$$+ \binom{g+k-1}{k+1} - (2g-2) = \binom{g+k}{k+1} - (2k+1)(g-1).$$

The reverse inequality is contained in (2.3) for $n = k + 1$. This completes the induction, and the proposition is proved.

Proof of Lemma 2.2. Let X be some reduced scheme which is a closed subscheme of M; then we have an exact sequence

$$0 \to I_M \to I_X \to \mathcal{P} \to 0,$$

which induces the following immersion:

$$H^0(P^{g-2}, I_M(n)) \subsetneq H^0(P^{g-2}, I_X(n)).$$

This allows us to reduce the lemma in the first case to a reduced scheme X_{2g-3} consisting, as a topological space, of $2g-3$ isolated points lying in general position, and in the second to X_{2g-2}. For any point of X_{2g-3} there exists a quadric Q which contains the remaining $2g-4$ points and does not contain this point. (To see this, it is sufficient to break up the $2g-4$ points arbitrarily into two groups of $g-2$ points and to draw hyperplanes H and H' through each of these groups, whereupon we obtain from the fact that the $2g-3$ points of X_{2g-3} are in general position that $Q = H \cup H'$ is the desired quadric.) This means that for any point $x \in X_{2g-3}$ there exists among the global sections $\Gamma(X_{2g-3}, O_{X_{2g-3}}(2))$ a section f_x such that f_x is the image of image of some quadric under the restriction isomorphism to X_{2g-3}:

$$\text{Res}: \Gamma(P^{g-2}, O_P(2)) \to \Gamma(X_{2g-3}, O_{X_{2g-3}}(2)).$$

Then $\dim \operatorname{Im}(\text{Res}) \geq 2g-3$, but clearly, $\dim \Gamma(X_{2g-3}, O_{X_{2g-3}}(2)) \leq 2g-3$. Hence

$$\operatorname{Im}(\text{Res}) = \Gamma(X_{2g-3}, O_{X_{2g-3}}(2)).$$

It follows from the last equation that

$$\dim \operatorname{Ker}(\text{Res}) + \dim H^0(X_{2g-3}, O_{X_{2g-3}}(2)) = \dim H^0(P^{g-2}, O_P(2)) = \frac{g(g-1)}{2}.$$

Also, $\operatorname{Ker}(\text{Res}) = \Gamma(P^{g-2}, I_{X_{2g-3}}(2))$, since X_{2g-3} is a reduced scheme. Hence

$$\dim H^0(P^{g-2}, I_{X_{2g-3}}(2)) = \frac{(g-1)g}{2} - \dim H^0(X_{2g-3}, O_{X_{2g-3}}(2))$$

$$= \frac{(g-1)g}{2} - \dim \operatorname{Im}(\text{Res}) \leqslant \frac{(g-1)g}{2} - 2g + 3 = \frac{(g-2)(g-3)}{2}.$$

Inequality (2.5) is proved analogously; to do this it is first of all necessary to show that the image of the space $\Gamma(P^{g-2}, O_P(n))$ under restriction to X_{2g-2} coincides with the space $\Gamma(X_{2g-2}, O_{X_{2g-2}}(n))$ for all $n \geq 3$. Lemma 2.2 is proved.

§3. Some properties of schemes which are intersections of $(g-2)(g-3)/2$ quadrics in P^{g-2}

Let M be a closed subscheme of the projective space P^{g-2} having the following properties:

α) There exist $2g-2$ isolated points in M which lie in general position in P^{g-2}.

β) The ideal of the scheme M is generated by quadratic forms and

$$\dim H^0(P^{g-2}, I_M(2)) \geqslant (g-2)(g-3)/2,$$

where I_M denotes the sheaf of ideals of M.

In this section we will study the properties of such schemes M which contain not less than $2g-1$ isolated points; the result of this study is expressed in

Theorem 3.1. *If* $\#(M) \geq 2g - 1$, *then* M *is a reduced, irreducible, nonsingular, projective curve of degree* $g - 2$ *which clearly generates* P^{g-2}.

We will denote by G a projective irreducible reduced curve of degree $g - 2$ which generates P^{g-2}. To prove Theorem 3.1, we will need certain properties of G contained in the following lemma.

Lemma 3.1. a) *There is one such curve* G *passing through any* $g + 1$ *isolated points of* P^{g-2} *lying in general position.*

b) G *is a smooth curve of genus* 0.

c) $\dim H^0(P^{g-2}, I_G(2)) = (g-2)(g-3)/2$ *and* G *is the intersection of the quadrics which pass through it; that is, the ideal of the scheme* G *is generated by the* $(g-2)(g-3)/2$*-dimensional space of quadratic forms of this ideal.*

Proof. a) Let $x_0, \cdots, x_g$ be isolated points of P^{g-2} lying in general position. We choose homogeneous coordinates X_j $(j = 0, 1, \cdots, g-2)$ in P^{g-2} in such a way that $x_i = (\underbrace{0, \cdots, 1}_{i}, \cdots, 0)$ for $i \leq g-2$, $x_{g-1} = (1, \cdots, 1)$ and $x_g = (a_0, \cdots, a_{g-2})$.

Since the points x_i lie in general position, it is clear that $a_i \neq 0$ and $a_i \neq a_j$ for $0 \leq i < j \leq g-2$. Letting $t_j = \Pi_{i \neq j} a_i$, we will specify the curve G in parametric form. Let $X_i(t) = \Pi_{j \neq i}(t - t_j)$, where $0 \leq i \leq g-2$ and $t \in k \cup \{\infty\}$ is a parameter. It is easy to show that for the values $t_0, \cdots, t_{g-2}, \infty$ and 0 of the parameter we obtain the points $x_0, \cdots, x_{g-2}, x_{g-1}$ and x_g respectively. This demonstrates the existence of a curve G passing through the points $x_0, \cdots, x_g$.

Before proving the uniqueness of this G, let us prove part b).

b) Since G is irreducible and generates P^{g-2}, we can define for any hyperplane H the index of its intersection with G; $(G \cdot H) = \deg G = g - 2$. Consider an arbitrary point $x \in G$. It is easy to prove by induction on k that for any $0 \leq k \leq g-3$ there exists a k-dimensional projective subspace $H \subset P^{g-2}$ passing through x and containing $k+1$ points of G. Hence there is a hyperplane H such that $(H \cdot G)_x \geq 1$ and $(H \cdot G)_{y_i} \geq 1$ for $i = 1, \cdots, g-3$, where the y_i are distinct isolated points different from x. It follows from this that $(H \cdot G)_x = 1$; that is, x is a simple point. This shows that G is smooth.

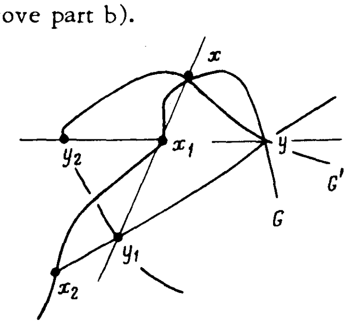

Figure 1

The proof of the following property of G presents no difficulty.

(SG) For an arbitrary k-dimensional projective subspace $H \subset P^{g-2}$, it follows from the fact that $k \leq g-3$ that $\mathrm{supp}(H \cdot G)$ consists of not more than $k+1$ points which lie in general position in H.

a) (continued). We will carry out the proof of uniqueness by induction on $g \geq 4$. For $g = 4$, this is a well-known fact from analytic geometry. Assume the assertion to be proved for $g \leq k$; we will prove uniqueness for $g = k+1$. Let us suppose that there exist two curves G and G' passing through $k+2$ isolated points and let x and y be two of these points. Consider the projections of G' and G from x onto a hyperplane H not passing through x. It is evident from the inductive assumption that the images of these curves must coincide. Let $G \neq G'$; then there is an isolated point $x_1 \in G$ through which G' does not pass; that is, $x_1 \notin G'$. Consider the line through x and x_1. Since the images of G and G' under projection coincide, there is an isolated point $y_1 \in G' \cap \overline{xx_1}$ different from both x and x_1. By property (SG), $y_1 \notin G$. Drawing the lines through x_1, y and y_1, y, we obtain isolated points $y_2 \in G' \cap \overline{x_1 y}$ and $x_2 \in G' \cap \overline{y_1 y}$. It is clear that x, y, x_1, x_2 lie in one plane and belong to G. Since $k-2 \geq 2$, this contradicts property (SG). The proof of part a) of Lemma 3.1 is complete.

c) The proof of this part is based on the following lemma.

Lemma 3.2. *Let M' be a closed subscheme in P^n which contains $n+1$ isolated points lying in general position and such that* $\dim H^0(P^n, I_{M'}(2)) \geq n(n+1)/2$. *Then M' is a reduced* 0*-dimensional scheme coinciding with the $n+1$ isolated points, which are in M' by hypothesis, and* $\dim H^0(P^n, I_{M'}(2)) = n(n+1)/2$.

Deduction of part c) of Lemma 3.1 from Lemma 3.2. Consider the exact triple

$$0 \to I_G(2) \to O_P(2) \to i_*(O_G(2D)) \to 0,$$

where $D = H \cdot G$ is an effective divisor of degree $g-2$ on the curve G. This short exact sequence induces the following exact cohomology sequence:

$$0 \to H^0(P^{g-2}, I_G(2)) \to H^0(P^{g-2}, O_P(2)) \to H^0(G, O_G(2D)) \to H^1(P^{g-2}, I_G(2)) \to 0, \tag{3.1}$$

because $\dim H^1(P^{g-2}, O_P(2)) = 0$ for $g-2 \geq 2$. From the exact sequence (3.1) we obtain that

$$\dim H^0(P^{g-2}, I_G(2)) = \dim H^0(P^{g-2}, O_P(2)) - \dim H^0(G, O_G(2D)) + \dim H^1(P^{g-2}, I_G(2)). \tag{3.2}$$

We know that the genus of G is equal to 0 (this can be proved, for example, from the property (SG) for hyperplanes). Then, by the Riemann-Roch Theorem, $\dim H^0(G, O_G(2D)) = 2\deg G - 0 + 1 = 2g-3$. From formula (3.2) we thus obtain the inequality

$$\dim H^0(P^{g-2}, I_G(2)) \geqslant \frac{(g-1)g}{2} - (2g-3) = \frac{(g-2)(g-3)}{2}. \tag{3.3}$$

Consider the closed subscheme $G' \subset P^{g-2}$ whose ideal is generated by the forms of degree 2 contained in the ideal of the scheme G. By the lemma on the choice of a "good" hyperplane in §1, there exists a hyperplane H such that $H \cdot G = \sum_{i=1}^{g-2} x_i$, where the x_i are distinct isolated points lying in general position in H. Since G is irreducible and generates P^{g-2}, it it clear that

$$\dim H^0(P^{g-2}, I_{G'}(2)) = \dim H^0(H, I_{G'\cap H}(2)),$$

Hence $\dim H^0(H, I_{G'\cap H}(2)) \geq (g-2)(g-3)/2$. By Lemma 3.2 $G' \cap H$ is the reduced scheme M', and so $G \cap H = G' \cap H$. From the equality of these schemes we obtain that

$$\dim H^0(H, I_{G\cap H}(2)) = \dim H^0(H, I_{G'\cap H}(2)) = \frac{(g-2)(g-3)}{2}.$$

By the proposition on the restriction (see §1), this yields the inequality

$$\dim H^0(P^{g-2}, I_G(2)) \leqslant \frac{(g-2)(g-3)}{2}. \tag{3.4}$$

It follows from inequalities (3.3) and (3.4) that

$$\dim H^0(P^{g-2}, I_G(2)) = \frac{(g-2)(g-3)}{2}. \tag{3.5}$$

The next step in the proof of part c) will be to prove that $G'_{\mathrm{red}} = G$. Since this is obvious for $g = 4$, we can assume that $g \geq 5$ for this step. Proceeding by the method of contradiction, let us assume that there exists an isolated point $x \in G'$ with $x \notin G$. We assert first of all that there is a hyperplane H_1 passing through x which does not touch G; that is, $H_1 \cdot G = \Sigma_{i=1}^{g-2} x_i$, where the x_i are distinct isolated points, lying in general position by property (SG) Consider the lines passing through x and a point of G. Let one of these touch G or intersect G in two points. Then G' contains isolated points of G and this line L. Let us consider then the generic hyperplane, for which $H \cap (G \cup L) = \{x_i\}_{i=1,\cdots,g-2} \cup \{y\}$, where the x_i are isolated points of H lying in general position, and y is an isolated point not coinciding with any of the x_i. Consider the scheme $G' \cap H = G \cap H$; it consists of $g-2$ isolated points, which leads to a contradiction. We have shown that any line passing through x intersects G in exactly one point. The projection π of G from x onto some hyperplane H^1 is thus a regular immersion. By the lemma on the choice of a "good" hyperplane there exists a hyperplane H_1^1 not tangent to $\pi(G)$. There corresponds to it a hyperplane H_1 passing through x and not tangent to G. This proves the existence of a hyperplane H_1 transversal to G and passing through x. By Lemma 3.2 the scheme $G' \cap H_1$ consists of $g-2$ isolated points, contradicting the supposition that $x \notin G$. Thus $G'_{\mathrm{red}} = G$.

It was shown earlier that there is a hyperplane H such that $G' \cap H = G \cap H$, and so $G = G'$ by the lemma on reducibility (see §1). The proof of Lemma 3.1 is complete.

Deduction of Theorem 3.1 from Lemma 3.1. By hypothesis P^{g-2} contains $2g-2$ isolated points $x_i \in M$ $(i = 1, 2, \cdots, 2g-2)$ lying in general position, and also a certain isolated point x distinct from the x_i.

We show first of all that the points $\{x_i\} \cup \{x\}$ lie in general position in P^{g-2}. Let us assume the opposite; then there is a hyperplane H in P^{g-2} which contains the points x_i $(i = 1, 2, \cdots, g-2)$ and x. Considering the scheme $M \cap H$ and using Lemma 3.2, we obtain a contradiction since $\dim H^0(H, I_{M\cap H}(2)) \geq (g-2)(g-3)/2$, and so the restriction of quadrics to this hyperplane has no kernel, by which we mean that among the quadratic forms of the scheme M there is no form q for which $q|_H \equiv 0$. In fact, if we assume that there is a $q \in \Gamma(P^{g-2}, I_M(2))$ such that $q|_H \equiv 0$, then $q = h \cdot h'$ is a splitting quadric, h being the equation of H and h' giving some hyperplane H', distinct from H since $M \not\subset H$. It is clear that H' contains the points x_i for $i \geq g-1$; that is, H' contains g points of $\{x_i\}$, contradicting the fact that the points of $\{x_i\}$ lie in general position. Indeed, it would be sufficient to assume the existence of $2g-3$ points of M lying in general position in order to prove that any finite set of isolated points of M lay in general position. In comparison with what is stated above, the proof of this fact contains no new ideas.

We will denote by $\mathcal{J}$ a reduced closed subscheme of the space P^{g-2} which, as a topological space, consists of $2g-1$ isolated points. These lie in general position in P^{g-2}, as has been shown. Consider any $g-3$ isolated points of $\mathcal{J}$. Denote by $\mathcal{J}'$ the reduced subscheme of $\mathcal{J}$ containing these $g-3$ isolated points, and by $\mathcal{J}''$ the "complementary" reduced subscheme. The subscheme $\mathcal{J}'$ determines a projective subspace $W \subset P^{g-2}$ of codimension 2, since it consists of $g-3$ points lying in general position. It is easy to show that the sequence

$$0 \to H^0(P^{g-2}, I_{\mathcal{J}''\cup W}(2)) \to H^0(P^{g-2}, I_{\mathcal{J}}(2)) \xrightarrow{\text{(restriction to } W)} H^0(W, I_{\mathcal{J}'}(2)), \tag{3.6}$$

is exact, where by $\mathcal{J}'' \cup W$ we mean the reduced scheme whose points comprise the set-theoretic union of those of the scheme $\mathcal{J}''$ and of W. From Lemma 3.2 we obtain the inequality $\dim H^0(W, I_{\mathcal{J}'}(2)) \leq (g-2)(g-3)/2 - (g-3)$. It then follows from the exact sequence (3.6) that

$$\dim H^0(P^{g-2}, I_{\mathcal{J}''\cup W}(2)) \geqslant g-3;$$

that is, there are $g-3$ linearly independent quadrics $Q_1, \cdots, Q_{g-3}$ such that Q contains W and $\mathcal{J}$ as closed subschemes. Let us denote by G' an irreducible component of the scheme $G'' = \bigcap_{i=1}^{g-3} Q_i$ which contains some point of $\mathcal{J}''$. We will prove in the sequel that $G = G'_{\text{red}}$; that is, we will prove that G'_{red} is an irreducible curve of degree $g-2$ generating P^{g-2}. By definition, G is a reduced irreducible scheme of dimension not less than 1 not contained in W. Let H be some hyperplane passing through W such that $\text{red}(H \cap G'') \neq W$; that is, there is an isolated point $x \in H \cap G''$ lying outside W. It is clear that the restriction of the quadrics Q_i to H has no kernel, as for all quadrics in $H^0(P^{g-2}, I_{\mathcal{J}}(2))$. The $Q_i|_H = h_i \cdot h$ are splitting quadrics in H, h being the equation of the hyperplane W in H, and the h_i being independent linear forms since the restriction of quadrics to H has no kernel. Hence $x \in \bigcap_{i=1}^{g-3} H_i$ and $\bigcap_{i=1}^{g-3} H_i$ consists of not more than one point. Summarizing what has been proved in this paragraph, we have

$$\operatorname{red}(G'' \cap H) = \begin{cases} W \\ \text{or} \\ W \cup \{x\}, \text{ where } x \text{ is an isolated point of } H, \\ \qquad\text{lying outside } W. \end{cases} \tag{3.7}$$

On the isolated points of the variety $G \subset G''$ not lying in W, we define the mapping $\phi: G \to P^1$ which associates with an isolated point $x \in G$ lying outside W the hyperplane determined by the space W and the point x; that is, ϕ is a rational morphism. By property (3.7) of the intersection $G'' \cap H$ it is clear that we have $\dim G \leq 1$; that is, G is a projective curve lying outside W. For a generic hyperplane passing through W, $H \cap G$ contains a point lying outside W. Since the dimension of each irreducible component of G'' is not less than 1, $\operatorname{red} G'' = W \cup G$. Hence $G \supset \mathcal{T}''$, and so G generates P^{g-2} and $\deg G \geq g-2$.

We will prove that $\deg G = g-2$. Let us assume the contrary; that is, let $\deg G \geq g-1$. Applying the lemma on the choice of a "good" hyperplane, it is easy to prove that there is a hyperplane H with the following properties: H does not pass through W, and $H \cap G$ contains $g-1$ isolated points lying in general position in H and lying outside W. Consider a hyperplane E in H which passes through $W \cap H$ and one of the points x_i. Let this point be x_{g-1}, where $x_1, \cdots, x_{g-1}$ are the isolated points of $H \cap G$. From property (3.7) for the intersection $G'' \cap H$ we obtain that $\operatorname{red}(E \cap G'') = (W \cap H) \cup \{x_{g-1}\}$. It is clear that the quadrics $Q_i|_E = h_i \cdot h$ are splitting ($i = 1, 2, \cdots, g-3$). The restriction of the quadrics Q_i to E has a kernel, since otherwise we would have $\operatorname{red}(E \cap G'') = W \cap H$. Hence there is a quadric Q, generated by the Q_i, for which $Q|_E \equiv 0$. The restriction to any hyperplane has no kernel, and so $Q|_H = l \cdot l' \not\equiv 0$, where l is the equation of the hyperplane E and l' gives the hyperplane E' in H. E' contains the points $x_1, \cdots, x_{g-2}$, since $\operatorname{red}(G'' \cap E) = (W \cap H) \cup \{x_{g-1}\}$ and the points $x_i \notin W$. This contradicts the fact that the x_i lie in general position in H. Hence $\deg G = g-2$.

We have proved that through any $g+2 = [(2g-1)-(g-3)]$ points of $\mathcal{T}$ there passes a reduced irreducible curve G of degree $g-2$. By part a) of Lemma 3.1 we know that there is a curve G passing through $2g-1$ isolated points of $\mathcal{T}$. Since $\mathcal{T}$ is a closed subscheme of G, we have an immersion

$$H^0(P^{g-2}, I_G(2)) \subsetneq H^0(P^{g-2}, I_{\mathcal{T}}(2)). \tag{3.8}$$

By Lemma 2.2

$$\dim H^0(P^{g-2}, I_{\mathcal{T}}(2)) \leqslant (g-2)(g-3)/2,$$

and by Lemma 3.1

$$\dim H^0(P^{g-2}, I_G(2)) = (g-2)(g-3)/2.$$

The immersion (3.8) is thus an isomorphism; that is,

$$H^0(P^{g-2}, I_{\mathcal{J}}(2)) = H^0(P^{g-2}, I_G(2)) = H^0(P^{g-2}, I_M(2)).$$

By property β) of the scheme M we have $M = \bigcap Q_i$, where $Q_i \in H^0(P^{g-2}, I_M(2))$. By part c) of Lemma 3.1 we thus have that $M = G$ is a reduced irreducible nonsingular curve of degree $g-2$ generating P^{g-2}. The theorem is proved.

Proof of Lemma 3.2. By hypothesis there are isolated points $\{x_i\}_{i=1}^{n+1}$ lying in general position in P^n. We can choose in P^n homogeneous coordinates X_i such that the points x_i form a basis; that is, $X_i(x_j) = \delta^i_j$. Then the quadrics Q defining the scheme M' pass through the points x_i. $Q(x_i)$ is thus the coefficient in $x_i^2 = 0$; that is, $Q = \sum_{i\neq j} \alpha_{ij} X_i \cdot X_j$. It is easy to compute that in this case the number of independent quadrics defining M' is not more than $(n+1)\,n/2$. M' is thus defined by the $(n+1)\,n/2$ quadrics $Q_{ij} = X_i \cdot X_j$ $(i \neq j)$.

Let $y = (x_1, \ldots, x_{n+1})$ be an isolated point of M'. If it has two nonzero coordinates x_i and x_j, then $y \notin M$, since $Q_{ij}(y) = x_i \cdot x_j \neq 0$. Then, as a topological space, M' consists of $n+1$ isolated points which represent basis points. We will prove that M' is a reduced scheme. To do this it is clearly sufficient to show that any form which vanishes on the basis points is generated by the quadrics Q_{ij}. The verification of this fact does not present any difficulty. The lemma is proved.

§4. $Q_{\rm red}$

In this section we investigate the structure of the scheme $Q_{\rm red}$. The definitive result is the following.

Theorem 4.1. a) $\dim Q \leq 2$.

b) *If* $\dim Q = 1$ *then* $Q_{\rm red} = C$.

c) *If* $\dim Q = 2$, *then* $Q = Q_{\rm red}$ *is an irreducible reduced surface of degree* $g-2$ *which generates* $\mathbf{P}^{g-1}$.

The proof of this theorem in the case $g = 4$ is obvious, and so it will be assumed in the rest of this section that $g \geq 5$.

Proposition 4.1. *Let* Q *contain, in addition to* C, *one other isolated point* O *not lying on* C; *then there is an irreducible reduced surface* S *such that* $Q \supset S \supset O \cup C$.

Deduction of Theorem 4.1 from Proposition 4.1. a. By the lemma on the choice of a "good" hyperplane, there is a hyperplane H such that $H \cdot C = \sum_{i=1}^{2g-2} x_i$, where the x_i are isolated points lying in general position. The scheme $Q \cap H$ thus has the properties α) and β) indicated at the beginning of §3, since $\dim H^0(\mathbf{P}^{g-1}, I_Q(2)) = (g-2)(g-3)/2$ and the restriction of quadrics to any hyperplane is without kernel. We obtain from Theorem 3.1 that $\dim Q \cap H \leq 1$, and so $\dim Q \leq 2$.

c. If $\dim Q = 2$, then $\dim Q \cap H = 1$ by the above inequality. Hence, by Theorem 3.1, $Q \cap H = G$ is a reduced irreducible curve of degree $g-2$ generating H. This holds for a generic hyperplane, and so $Q_{\rm red}$ has only one component of dimension 2.

Using Proposition 4.1, it is easy to prove that the curve C lies in the irreducible component of dimension 2 and that Q_{red} is an irreducible surface. Q_{red} has degree $g-2$, since for the generic hyperplane $Q_{red} \cap H = G$ is a curve of degree $g-2$. By the lemma on reducibility, it is evident that $Q = Q_{red}$.

b follows immediately from Proposition 4.1. Theorem 4.1 is proved.

Proposition 4.1 will be proved with the aid of the following lemmas.

Lemma 4.1. *If Q contains a reduced and irreducible curve $L \neq C$, then there is an irreducible surface S such that $Q \supset S \supset L \cup C$.*

Lemma 4.2. *Let M be a closed subscheme in P^n which contains $k \leq 2n+1$ isolated points lying in general position in P^n. Then the following assertions are true.*

a) $\dim H^0(P^n, I_M(2)) \leq (n+1)(n+2)/2 - k$.

b) *If $M_{red} = M$ and $\#(M) = k \leq 2n+1$, then* $\dim H^0(P^n, I_M(2)) = (n+1)(n+2)/2 - k$.

Proof of Proposition 4.1. If there exists a line L passing through the point O which either touches or intersects C in at least two points, then $L \subset Q$ (since otherwise $(L \cdot K) \geq 3$ for some quadric $K \supset C$). It follows then from Lemma 4.1 that the desired surface S exists; we can therefore assume further that $(L \cdot C) \leq 1$ for any line passing through O.

Let H be a hyperplane not passing through O. We consider the morphism $\pi: C \to H$, where $\pi(x) = H \cap \overline{Ox}$ for isolated points x of C. It follows from the above agreement that π is a biregular morphism. $\pi(C)$ is thus a nonsingular algebraic curve of degree $2g-2$ which clearly generates H. By the lemma on the choice of a "good" hyperplane, the generic hyperplane E' of H intersects $\pi(C)$ in $2g-2$ isolated points which lie in general position in E'.

Consider the hyperplane E in P^{g-1} passing through O and the space E'. We will prove that $\dim Q \cap E \geq 1$. We have $Q \cap E \supset (C \cap E) \cup \{O\}$, and obviously $C \cap E = \{x_i\}_{i=1,\dots,2g-2}$, where the x_i are isolated points of E not coinciding with O. If the points $\{x_i\}$ lay in general position in E, by Theorem 3.1 we would have that $Q \cap E$ is a complete, reduced, irreducible, algebraic curve. The proof of the proposition would then follow from Lemma 4.1, and so we will assume in what follows that the points of $C \cap E$ do not lie in general position.

We will show first of all that any $g-2$ points of $C \cap E$ lie in general position in E. In fact, if this were not so, there would exist a projective space $E'' \subset P^{g-1}$ having codimension 2 in E and containing $g-2$ points of $C \cap E$. $\pi(E'')$ would thus contain $g-2$ points of $\pi(C)$ and would have codimension not less than 1 in E', and this would contradict the generality of the points of $\pi(C) \cap E$. Similarly, it is easy to prove that the point O and any $g-2$ points of $C \cap E$ lie in general position in E.

Since, by the assumption made above, the points $\{x_i\}$ do not lie in general position, there are $g-1$ points in $\{x_i\}$ which generate a hyperplane (since $g-2$ points are already in general position). Let us denote this hyperplane by H'. Hence the restriction of quadrics cutting $Q \supset C$ to the projective subspace $H' \subset \mathbf{P}^{g-1}$ has a kernel (by Lemma 3.2). This means that there is a quadric $q \in \Gamma(\mathbf{P}^{g-1}, I_Q(2)) = \Gamma(\mathbf{P}^{g-1}, I_C(2))$ for which $q|_{H'} \equiv 0$. But since $q \in \Gamma(\mathbf{P}^{g-1}, I_C(2))$ and C is an irreducible curve generating $\mathbf{P}^{g-1}$, we have $q|_E \not\equiv 0$. Then $q|_E = h'' \cdot h'$ is a splitting quadric, where h' is the equation of H' and h'' defines a hyperplane $H'' \neq H'$. The isolated points $\{x_i\}$ and the point O lie in the subspaces H' and H'' since $(Q \cap E)_{\mathrm{red}} \subset H'' \cup H'$. Denote by M' the reduced 0-dimensional scheme consisting of the points x_i lying in the hyperplane H', and by M'' the reduced scheme consisting of the remaining points x_i and the point O. M' is obviously a closed subscheme of H'. Further, M'' is a closed subscheme of H'' since O lies in H'', for O and any $g-2$ points of $C \cap E$ lie in general position in E.

By its choice, the hyperplane E is not a tangent, and so it is generated by the isolated points $\{x_i\} = C \cap E$. Hence $\#(M'') \geq 2$.

The exactness of the sequence

$$0 \to \Gamma(E, I_{M''}(1)) \xrightarrow{\times h'} \Gamma(E, I_M(2)) \xrightarrow{\text{restriction to } H'} \Gamma(H', I_{M'}(2)), \tag{4.1}$$

where M is a reduced scheme whose isolated points are precisely the points of $M' \cup M''$, is obvious. The exact sequence (4.1) yields the inequality

$$\dim H^0(H', I_{M'}(2)) \geqslant \dim H^0(E, I_M(2)) - \dim H^0(E, I_{M''}(1)). \tag{4.2}$$

Consider the case when $\#(M'') = 2$ and $\#(M') = 2g-3$. Then

$$\dim H^0(E', I_{M''}(1)) = [(g-1)-2] = g-3,$$

and

$$\dim H^0(E, I_M(2)) \geqslant \dim H^0(E, I_{E\cap Q}(2)) = \frac{(g-2)(g-3)}{2},$$

since $M \subset E \cap Q$. From inequality (4.2) we have that

$$\dim H^0(H', I_{M'}(2)) \geqslant \frac{(g-2)(g-3)}{2} - (g-3) = \frac{(g-4)(g-3)}{2}; \ \#(M') = 2g-3.$$

Hence, by Theorem 3.1, $\dim H^0(H', I_{M'}(2)) = (g-3)(g-4)/2$; that is,

$$\dim H^0(E, I_M(2)) = \dim H^0(E, I_{E\cap Q}(2)).$$

The last equation means that $\Gamma(E, I_M(2)) = \Gamma(E, I_{E\cap Q}(2))$, and so

$$\dim H^0(H', I_{Q\cap H'}(2)) \geqslant (g-3)(g-4)/2.$$

It then follows from Theorem 3.1 that $\dim H' \cap Q \geq 1$, whence $\dim E \cap Q \geq 1$.

We will now prove that the cases $\#(M'') = 3, 4$ are impossible. In fact, in these cases we have $\dim H^0(E, I_{M''}(1)) = g-4$ or $g-5$, since O and any $g-2$ points of $C \cap E$ lie in general position. By Lemma 2.2

$$\dim H^0(H', I_{M'}(2)) \leqslant (g-3)(g-4)/2,$$

since $\#(M') \geq 2g-5$ and the isolated points of M' lie in general position in H'. We thus obtain from inequality (4.2) that

$$\dim H^0(E, I_M(2)) \leqslant \frac{(g-3)(g-4)}{2} + g-4, g-5$$
$$\leqslant \frac{(g-3)(g-4)}{2} + (g-3) - 1 = \frac{(g-2)(g-3)}{2} - 1.$$

The inequality $\dim H^0(E, I_M(2)) \leq (g-3)(g-2)/2 - 1$ contradicts Theorem 2.1, since we have the immersion

$$\Gamma(\mathbf{P}^{g-1}, I_Q(2)) \simeq \Gamma(E, I_{Q\cap E}(2)) \subsetneq \Gamma(E, I_M(2)).$$

We now consider the case when $5 \leq \#(M'') \leq g-2$ (clearly, $\#(M'') \leq g-2$ in any case). Then $g+1 \leq \#(M') \leq 2g-6$. In this case, by Lemma 4.2,

$$\dim H^0(H', I_{M'}(2)) = \left[\frac{(g-2)(g-1)}{2} - \#(M')\right] \text{and} \dim H^0(E, I_{M''}(1))$$
$$= [(g-1) - \#(M'')]$$

since the isolated points of M'' are in general position in E. We thus obtain from inequality (4.2) that

$$\dim H^0(E, I_M(2)) \leqslant \left[\frac{(g-2)(g-1)}{2} - \#(M')\right] + [(g-1) - \#(M'')]$$
$$= \frac{(g-2)(g-1)}{2} + (g-1) - \#(M) = \frac{(g-2)(g-1)}{2}$$
$$+ (g-1) - (2g-1) = \frac{(g-2)(g-1)}{2} - g$$
$$= \frac{(g-2)(g-3)}{2} - 2 < \dim \Gamma(\mathbf{P}^{g-1}, I_Q(2))|_E = \frac{(g-2)(g-3)}{2},$$

which also leads to a contradiction. Therefore these cases are also impossible.

Summarizing what has been said above, we obtain that $\dim Q \cap E \geq 1$ in all possible cases, and, more precisely, there is an irreducible reduced curve $L \subset Q \cap E$ passing through O. Hence $L \neq C$; that is (by Lemma 4.1.), there is an irreducible reduced surface $S \subset Q$ passing through L and C, and so also through O. The proposition is proved.

Proof of Lemma 4.1. By the lemma on the choice of a "good" hyperplane, the following holds for the generic hyperplane H: $H \cap C \supset \{x_i\}_{i=1,\cdots,2g-2}$, where the x_i are isolated points of $\mathbf{P}^{g-1}$ lying in general position in H. In addition, $H \cap L$ contains one other point y distinct from the x_i. Hence for the generic hyperplane H we have $\#(Q \cap H) \geq 2g-1$ and $Q \cap H$ contains $2g-2$ isolated points lying in general position. The space of quadrics through C restricts to any hyperplane H without kernel, and so

$$\dim \Gamma(H, I_{Q\cap H}(2)) = \dim \Gamma(\mathbf{P}^{g-1}, I_Q(2)) = (g-2)(g-3)/2,$$

whence, by Theorem 3.1, for the generic hyperplane H the scheme $Q \cap H$ is a reduced irreducible curve; consequently $\dim Q = 2$. The scheme Q_{red} can obviously have only one component of dimension 2, and no component of dimension 1, and so there is an irreducible reduced surface S such that $Q \supset S \supset L \cup C$. Lemma 4.1 is proved.

Proof of Lemma 4.2. a) Let X be a reduced 0-dimensional closed subscheme of M whose points are k isolated points lying in general position in P^n. Then we have an immersion $\Gamma(P^n, I_M(2)) \hookrightarrow \Gamma(P^n, I_X(2))$, and so the proof of part a) is contained in the proof of part b).

b) Let $x \in X$ be some point, the remaining $k-1$ points being divided into two groups each of which contains not more than n points. There are thus hyperplanes H and H' containing $k-1$ points of X. We can obviously assume that H and H' do not contain the point x, because the points of X lie in general position in P^n. This means that for any point $x \in X$ there is a quadric q for which $q(y) = \delta^x_y$ for $y \in X$. Hence for the restriction homomorphism

$$\mathrm{Res} : \Gamma(P^n, O_P(2)) \to \Gamma(X, O_X(2))$$

$\dim \mathrm{Im}(\mathrm{Res}) \geq k$; but $\dim \Gamma(X, O_X(2)) \leq k$, and so $\mathrm{Im}(\mathrm{Res}) = \Gamma(X, O_X(2))$ and $\dim(\mathrm{Res}) = k$.

It is then obvious that we have the equation

$$\dim \Gamma(P^n, O_P(2)) = \dim \mathrm{Ker}\ (\mathrm{Res}) + \dim \mathrm{Im}\ (\mathrm{Res}) = \dim \mathrm{Ker}\ (\mathrm{Res}) + k. \tag{4.3}$$

$\mathrm{Ker}(\mathrm{Res}) = \Gamma(P^n, I_X(2))$, since X is a reduced scheme. Hence, from equation (4.3),

$$\dim H^0(P^n, I_X(2)) = \dim H^0(P^n, O_P(2)) - k = \frac{(n+1)(n+2)}{2} - k.$$

Lemma 4.2 is proved.

§5. Some properties of schemes which are intersections of $(g-2)(g-3)/2$ quadrics in P^{g-2} (conclusion)

As in §3, let M be a closed subscheme of the projective space P^{g-2} possessing properties α) and β) (see §3). The cases when $\#(M) \geq 2g-1$ were investigated in §3 (see Theorem 3.1). The result of the present section is

Theorem 5.1. $M_{red} = M$.

If $\#(M) \geq 2g-1$, then M is a reduced irreducible curve, by Theorem 3.1. This proves Theorem 5.1 when $\#(M) \neq 2g-2$, and so we will assume for the remainder of this section that $\#(M) = 2g-2$ unless otherwise stated. It is then clear that $g \geq 5$.

Before entering upon the proof of Theorem 5.1, let us choose a system of homogeneous coordinates X_i in P^{g-2} in such a way that its basis points are contained in M. Consider the hyperplane $H \subset P^{g-2}$ given by the equation $X_{g-1} = 0$. The restriction of quadrics cutting M to the hyperplane H is without kernel, because M contains $2g-2$ isolated points lying in general position, by property α). Hence

$$\dim \Gamma(P^{g-2}, I_M(2)) = \dim \Gamma(H, I_{M\cap H}(2)) \geqslant (g-2)(g-3)/2.$$

By the choice of the homogeneous coordinates X_i the scheme $M \cap H$ contains the basis points of the hyperplane H, and so the general form of a quadric $q \in \Gamma(H, I_{M\cap H}(2))$ will be

$$q = \sum_{1\leqslant i<j\leqslant g-2} \alpha_{ij}X_iX_j.$$

Hence $\dim \Gamma(H, I_{M\cap H}(2)) \leq (g-2)(g-3)/2$. Summarizing what has been said in this paragraph, we have that

$$\Gamma(P^{g-2}, I_M(2)) \overset{\text{res to } H}{=\!=\!=} \Gamma(H, I_{M\cap H}(2))$$

$$= \left\{ q \in \Gamma(H, O_H(2)) \mid q = \sum_{1\leqslant i<j\leqslant g-2} \alpha_{ij}X_i \cdot X_j, \text{ where } \alpha_{ij} \in k \right\}.$$

Then the space $\Gamma(P^{g-2}, I_M(2))$ has a natural basis of quadrics

$$Q_{ji} = Q_{ij} = X_i \cdot X_j + X_{g-1}\left(\sum_{k=1}^{g-1} \alpha_{ij}^k X_k\right),$$

where $i \neq j$ and $1 \leq i, j \leq g-2$. By the choice of the homogeneous coordinates, the isolated point $(0, 0, \cdots, 0, 1) \in M$. Hence $\alpha_{ij}^{g-1} = Q_{ij}(0, \cdots, 0, 1) = 0$.

Denote by $b_1, \cdots, b_{g-1}$ the basis points $(1, 0, \cdots, 0), \cdots, (0, 0, \cdots, 0, 1, 0), (0, \cdots, 0, 0, 1)$ of the space P^{g-2}. We will call the points b_i and b_j *M*-connected if there is a cubic

$$K \in \Gamma(P^{g-2}, O_P(1)) \otimes \Gamma(P^{g-2}, I_M(2))$$

such that $K = X_{g-1} \cdot Q'$, where $Q'(b_i) \neq 0$, $Q'(b_j) \neq 0$ and Q' vanishes on the remaining points of M. Let $\mathfrak{T}'$ be the graph whose vertices are the points $b_1, \cdots, b_{g-2}$ and such that two of its vertices b_i and b_j are joined if and only if b_i and b_j are *M*-connected.

Proposition 5.1. *The graph $\mathfrak{T}'$ is trivial; that is, any two of its vertices are M-connected.*

Lemma 5.1. *If a quadric q vanishes on $2g-3$ points of M, then $q \in \Gamma(P^{g-2}, I_M(2))$ and vanishes on all the points of M.*

Deduction of Theorem 5.1 from Proposition 5.1 and Lemma 5.1. To prove Theorem 5.1, it is sufficient to prove the equation

$$\Gamma(P^{g-2}, I_{M_{\text{red}}}(n)) = \Gamma(P^{g-2}, O_P(n-2)) \otimes \Gamma(P^{g-2}, I_M(2)) \tag{5.1}$$

for all $n \geq 2$, since $\Gamma(P^{g-2}, I_{M_{red}}(1)) = 0$. For $n = 2$, equation (5.1) clearly follows from Lemma 2.2 and condition β for the scheme M. We will prove equation (5.1) for $n \geq 3$ by induction on n.

$n = 3$. Let $S \in \Gamma(P^{g-2}, I_{M_{red}}(3))$. Denote by $S|_H$ the restriction of S to the hyperplane H. Then $S|_H(b_i) = 0$ for $i = 1, \cdots, g-2$. Clearly, by Lemma 3.2, $M_{red} \cap H = M \cap H$, and so

$$S|_H \in \Gamma(H, O_H(1)) \otimes \Gamma(H, I_{M\cap H}(2)).$$

For the proof of equation (5.1) in the case $n = 3$, we can thus assume that $S|_H \equiv 0$; that is, $S = X_{g-1} \cdot q$. Consider the cubic

$$K = X_{g-1} \cdot \left[q - \sum_{j=2}^{g-2} \lambda_j Q'_{1j}\right] = X_{g-1} \cdot Q',$$

where

$$X_{g-1} \cdot Q'_{1j} \in \Gamma(P^{g-2}, O_P(1)) \otimes \Gamma(P^{g-2}, I_M(2))$$

and Q'_{1j} is a quadric not equal to zero only at the points b_1 and b_j; such quadrics Q'_{1j} exist by Proposition 5.1. We can clearly select $\lambda_j \in k$ such that Q' vanishes on all the points of M except for the point b_1. Hence, by Lemma 5.1, $Q' \in \Gamma(P^{g-2}, I_M(2))$, which proves equation (5.1) for $n = 3$.

Let $k \geq 3$ and assume (5.1) to be proved for $n \leq k$. Consider $S \in \Gamma(P^{g-2}, I_{M_{red}}(k+1))$. As above, we can assume for the proof of (5.1) that $S|_H \equiv 0$; that is, $S = X_{g-1} \cdot f$, where f is a form of degree k. Consider the form

$$\Phi = X_{g-1} \cdot \left[f - \sum_{j=2}^{g-2} \lambda_j X_j^{k-2} Q'_{1j} - \lambda_1 \cdot X_1^{k-2} Q'_{12}\right] = X_{g-1} \cdot f'.$$

Since the form $X_j^{k-2} Q'_{1j}$ vanishes on all the points of M except for b_j, there are $\lambda_j \in k$ $(j = 1, \cdots, g-2)$ such that $\Phi = X_{g-1} \cdot f'$ and f' vanishes on the points b_i $(i = 1, \cdots, g-2)$. It is evident that f' vanishes on the remaining points of M lying outside H. Hence $f' \in \Gamma(P^{g-2}, I_{M_{red}}(k))$; that is, by the inductive assumption,

$$f' \in \Gamma(P^{g-2}, O_P(k-2)) \otimes \Gamma(P^{g-2}, I_M(2)),$$

and so

$$\Phi \in \Gamma(P^{g-2}, O_P(k-1)) \otimes \Gamma(P^{g-2}, I_M(2)).$$

By choice, the quadrics Q'_{ij} are such that

$$\lambda_j X_i^{k-2} \cdot X_{g-1} \cdot Q'_{ij} \in \Gamma(P^{g-2}, O_P(k-1)) \otimes \Gamma(P^{g-2}, I_M(2)).$$

This proves (5.1) for $n = k+1$. Theorem 5.1 is proved.

Proof of Lemma 5.1. Let M' be the closed reduced subscheme of P^{g-2} consisting of the $2g-3$ points on which q vanishes. It is clear that $q \in \Gamma(P^{g-2}, I_{M'}(2))$.

By Lemma 2.2 $\dim \Gamma(P^{g-2}, I_{M'}(2)) \leq (g-2)(g-3)/2$. Hence, by condition β) for the scheme M, the immersion

$$\Gamma(P^{g-2}, I_M(2)) \subsetneq \Gamma(P^{g-2}, I_{M'}(2))$$

will be an isomorphism, and so $g \in \Gamma(P^{g-2}, I_M(2))$ and vanishes on all the points of M. The lemma is proved.

Consider three distinct integers i, j, k in the interval $[1, g-2]$; such members exist for $g \geq 5$. Let

$$K^j_{kl} = -K^j_{ik} = X_k \cdot Q_{ij} - X_i \cdot Q_{kj}$$

$$= X_{g-1} \cdot \left(\sum_{l=1}^{g-2} \alpha^l_{ij} \cdot X_l \cdot X_k - \sum_{m=1}^{g-2} \alpha^m_{kj} \cdot X_m \cdot X_i \right) = X_{g-1} \cdot Q^j_{kl} = -X_{g-1} \cdot Q^j_{ik}.$$

The quadric Q^j_{ki} clearly vanishes on all the points of M except for b_i and b_k. Denote by $\mathfrak{T}$ the subgraph of $\mathfrak{T}'$ in which two distinct vertices b_i and b_k are joined if and only if there exists j $(1 \leq j \leq g-2)$ distinct from i and k such that the quadric Q^j_{ki} does not vanish on the points b_i and b_k.

Proposition 5.2. *The graph $\mathfrak{T}$ is connected.*

Deduction of Proposition 5.1 from Proposition 5.2. $\mathfrak{T}$ is a subgraph of $\mathfrak{T}'$, and so $\mathfrak{T}'$ is connected because both graphs have the same number of vertices.

Let the distinct vertices b_i, b_k and b_j, b_k be M-connected and $i \neq j$. By the definition of M-connectedness there are cubics

$$K_{ik}, K_{jk} \in \Gamma(P^{g-2}, O_P(1)) \otimes \Gamma(P^{g-2}, I_M(2))$$

such that $K_{ik} = X_{g-1} \cdot Q'_{ik}$ and $K_{jk} = X_{g-1} \cdot Q'_{jk}$, where Q'_{ik} does not vanish on the two points b_i, b_k, nor does Q'_{jk} on b_j, b_k. Consider the cubic

$$K = \alpha K_{ik} + \beta K_{jk} = X_{g-1} \cdot (\alpha Q'_{ik} + \beta Q'_{jk}).$$

Since $Q'_{ik}(b_k) \neq 0$ and $Q'_{jk}(b_k) \neq 0$, there exist nonzero $\alpha, \beta, \in k$ such that $Q' = \alpha Q'_{ik} + \beta Q'_{jk}$ vanishes on b_k, and so it is clear that b_i and b_j are M-connected. Hence $\mathfrak{T}'$ is trivial, since it is connected.

Later in this section we will thoroughly investigate the properties of the coefficients α^k_{ij} of the quadrics Q_{ij} forming a basis in the space of quadrics through M. To this end, we will introduce the notions of subbasis, real subbasis, exponent of a subbasis, and exponent of complexity of a subbasis.

Definition 5.1. A *subbasis* of the basis $\{Q_{ij}\}$ is a system of quadrics $\{Q_{i_\alpha i_\beta}\}_{\alpha,\beta}$, where $\alpha \neq \beta$ and $1 \leq \alpha, \beta \leq n \leq g-2$, and the indices $i_1, \cdots, i_n$ form a subset of the integers $1, 2, \cdots, g-2$ such that $\alpha^k_{i_\alpha i_\beta} = 0$ for $k \neq i_1, \cdots, i_n$. The number n is called the exponent of the subbasis $\{Q_{i_\alpha i_\beta}\}$ and is denoted by $P(Q_{i_\alpha i_\beta})$.

Examples. a) $\{Q_{ij}\}$ is a subbasis of itself with $P(Q_{ij}) = g-2$, since $\alpha^{g-1}_{ij} = 0$.

b) If there is a quadric

$$Q_{i_0 j_0} = X_{i_0} \cdot X_{j_0} + X_{g-1} \cdot (\alpha^{j_0}_{i_0 j_0} X_{j_0} + \alpha^{i_0}_{i_0 j_0} X_{i_0}),$$

then $Q_{i_0 j_0}$ is a subbasis and $P(Q_{i_0 j_0}) = 2$.

Lemma 5.2. *If the vertices $b_{i_1}, \cdots, b_{i_n}$ are only connected among themselves in the graph $\mathfrak{T}$, then $\{Q_{i_\alpha i_\beta}\}$ is a subbasis.*

Proof. The lemma is obvious for $n = 1$. Let $n \geq 2$ and $i_\alpha \neq i_\beta$, where $1 \leq \alpha, \beta \leq n$. Consider an integer $k \in [1, g-2]$ not equal to $i_1, \cdots, i_n$; since the points b_k and b_{i_α} are not connected in the graph $\mathfrak{T}$, the quadric

$$Q_{i_\alpha k}^{i_\beta} = \sum_{l=1}^{g-2} \alpha_{k i_\beta}^{l} \cdot X_l \cdot X_{i_\alpha} - \sum_{m=1}^{g-2} \alpha_{i_\alpha i_\beta}^{m} X_m \cdot X_k$$

vanishes on b_k and b_{i_α}; hence $\alpha_{i_\alpha i_\beta}^{k} = Q_{i_\alpha k}^{i_\beta}(b_k) = 0$; that is, $\alpha_{i_\beta i_\alpha}^{k} = 0$. The lemma is proved.

Definition 5.2. A subbasis $\{Q_{i_\alpha i_\beta}\}$ is called *real* if it is not possible to make up a smaller subbasis using its elements, and the number

$$SP\{Q_{ij}\} = \max_{\{Q_{i_\alpha i_\beta}\} \text{ a real subbasis}} \{P(Q_{i_\alpha i_\beta})\}$$

is called the *exponent of complexity* of the basis $\{Q_{ij}\}$.

Lemma 5.3. *$SP\{Q_{ij}\} = g-2$, or $SP\{Q_{ij}\} = 2$ and every element $Q_{i_0 j_0}$ of a proper subbasis will be a subbasis, and also there is a Q_{ij} which is not a subbasis.*

Deduction of Proposition 5.2 from Lemma 5.3. The proposition will be proved by contradiction; suppose $\mathfrak{T}$ is not connected and consider the following cases.

Case I. There exist vertices $b_{i_1}, \cdots, b_{i_n}$, where $2 \leq n \leq g-3$, forming the vertices of some connected component of $\mathfrak{T}$. To the vertices $b_{i_1}, \cdots, b_{i_n}$ there corresponds, by Lemma 5.2, a subbasis $\{Q_{i_\alpha i_\beta}\}$ which is proper since $n \leq g-3$. Hence every quadric $Q_{i_\alpha i_\beta}$ is a subbasis; that is, $\alpha_{i_\alpha i_\beta}^{k} = 0$ for $k \neq i_\alpha, i_\beta$. Since the vertices $b_{i_1}, \cdots, b_{i_n}$ form a connected component and $n \geq 2$, there are two connected vertices b_{i_α}, b_{i_β}; that is, there exists a p $(1 \leq p \leq g-2)$, not equal to i_α or i_β, such that the quadric

$$Q_{i_\alpha i_\beta}^{p} = \sum_{l=1}^{g-2} \alpha_{i_\beta p}^{l} \cdot X_l \cdot X_{i_\alpha} - \sum_{m=1}^{g-2} \alpha_{i_\alpha p}^{m} \cdot X_m \cdot X_{i_\beta}$$

does not pass through the points b_{i_α} and b_{i_β}. Hence $\alpha_{i_\beta p}^{i_\alpha} \neq 0$ and $\alpha_{i_\alpha p}^{i_\beta} \neq 0$.

Consider the quadric

$$Q_{i_\alpha p}^{i_\beta} = \sum_{l=1}^{g-2} \alpha_{p i_\beta}^{l} \cdot X_l \cdot X_{i_\alpha} - \sum_{m=1}^{g-2} \alpha_{i_\alpha i_\beta}^{m} \cdot X_m \cdot X_p,$$

which by definition vanishes on all the points of M except possibly for b_{i_α} and b_p. The coefficient $\alpha^{i_\alpha}_{p i_\beta} = \alpha^{i_\alpha}_{p i_\beta} \neq 0$, and so $Q^{i_\beta}_{i_\alpha p}(b_{i_\alpha}) \neq 0$; that is, by Lemma 5.1, $Q^{i_\beta}_{i_\alpha p}(b_p) = \alpha^p_{i_\alpha i_\beta} \neq 0$. It has been shown above that $\alpha^k_{i_\alpha i_\beta} = 0$ for $k \neq i_\alpha, i_\beta$. This means that Case I is impossible.

Case II. There do not exist connected vertices; that is, for any distinct integers i, j, k in the interval $[1, g-2]$ the quadric Q^j_{ik} contains points of M. Hence $\alpha^k_{ij} = 0$, which means that every quadric Q_{ij} is a subbasis. This contradicts Lemma 5.3, and so this case is also impossible. The proposition is proved.

Lemma 5.4. *If* $\#(M) \geq 2g-1$, *then* $SP\{Q_{ij}\} = 2$ *and every quadric* Q_{ij} *will be a subbasis.*

Proof. By Theorem 3.1, M is a reduced irreducible curve which generates P^{g-2}. Since Q^j_{ik} is equal to zero on all points of M lying outside H, we have $Q^j_{ik} = 0$ on almost all points of M, and so $Q^j_{ik} \in \Gamma(P^{g-2}, I_M(2))$. Hence $\alpha^k_{ij} = 0$ for $k \neq i, j$; that is, every quadric Q_{ij} will be a subbasis and $SP\{Q_{ij}\} = 2$. The lemma is proved.

Lemma 5.5. *If* $SP\{Q_{ij}\} \neq g-2$, *then* $SP\{Q_{ij}\} = 2$ *and every element* $Q_{i_0 j_0}$ *of a a proper subbasis will be a subbasis.*

Proof. We can assume without loss of generality that the proper subbasis is de- defined by the indices $1, 2, \cdots, n$, where $2 \leq n \leq g-3$. Consider the projective subspace P^n of P^{g-2} given by the equations $X_{n+1} = \cdots = X_{g-2} = 0$. To every point $x_i = (\alpha_1, \cdots, \alpha_{g-1})$ there corresponds the point $\overline{x_i} = (\alpha_1, \cdots, \alpha_n, \alpha_{g-1}) \in P^n$, provided that x_i does not have the form $(0, \cdots, 0, \alpha_{n+1}, \cdots, \alpha_{g-2}, 0)$. The only points of M having this form are the points $b_i = \{\underbrace{0, \cdots, 0, 1}_{i}, 0, \cdots, 0\}$ for $i = n+1, \cdots, g-2$. Consider $M'' = \overline{M' \setminus \{b_{n+1}, \cdots, b_{g-2}\}}$, where M' consists of $(2g-2)$ isolated points of M lying in general position; these exist by hypothesis. We will prove that M'' consists of these $g+n$ isolated points lying in general position. In fact, if this were not so, there would exist $n+1$ points $x_i \in M' \setminus \{b_{n+1}, \cdots, b_{g-2}\}$ such that the vectors $\overline{x_i} = \{\alpha^i_1, \cdots, \alpha^i_n, \alpha^i_{g-1}\}$ are linearly dependent. It is then obvious that the points $\tilde{x}_i = \{\alpha^i_1, \cdots, \alpha^i_n, 0, \cdots, \alpha^i_{g-1}\}$ and the points $b_{n+1}, \cdots, b_{g-2}$ will in conjunction be linearly dependent in P^{g-2}. It would then follow that the points x_i and $b_{n+1}, \cdots, b_{g-2}$ of M are dependent, but their number is equal to $n+1+g-2-n = g-1$. This is of course impossible. Hence $P^n \supset M''$ contains $g+n$ isolated points lying in general position in P^n; from the definition of a subbasis we obtain that

$$Q_{ij}\{\alpha_1, \ldots, \alpha_{n+1}, \ldots, \alpha_{g-1}\} = Q_{ij}\{\alpha_1, \ldots, \alpha_n, 0, \ldots, 0, \alpha_{g-1}\}$$

for $i \neq j$ and $1 \leq i, j \leq n$; that is, through the points of M'' there pass $n(n-1)/2$ quadrics of the subbasis and the number of points in M'' lying in general position in P^n is equal to $g+n \geq n+3+n \geq 2n+3$. By Theorem 3.1 the quadrics Q_{ij} for $1 \leq i < j \leq n$ define in P^n a reduced irreducible curve which generates P^n. Then, by

Lemma 5.4, every quadric in the given proper subbasis will be a subbasis. The proof of Lemma 5.5 is then obvious.

Lemma 5.6. *If* Q_{ij} *has a subbasis of order* $g-3$, *then* $\#(M) \geq 2g-1$.

Deduction of Lemma 5.3 from Lemmas 5.5 and 5.6. By Lemma 5.5, it is clear that if $SP\{Q_{ij}\} \neq g-2$, then $SP\{Q_{ij}\} = 2$ and every element $Q_{i_0 j_0}$ of a proper subbasis will be a subbasis; but by Lemma 5.6 there is a quadric Q_{ij} which is not a subbasis. Lemma 5.3 is proved.

Proof of Lemma 5.6. We can assume without loss of generality that a subbasis of order $g-3$ is defined by the indices $1, \cdots, g-3$. We denote by O the point $(0, \cdots, 1, 0)$ and by H the hyperplane defined by the equation $X_{g-2} = 0$. Consider the projection $\pi\colon P^{g-2}\setminus O \to H$. There are $2g-2$ isolated points in M lying in general position. Under projection to H, they give rise to $2g-3$ isolated points in general position, on which vanishes the $(g-3)(g-4)/2$-dimensional space of quadrics generated by the Q_{ij} for $1 \leq i < j \leq g-3$. Hence $H \cap K = G'$, where K is the scheme defined by the quadrics Q_{ij} for $1 \leq i < j \leq g-3$, and G' is a reduced irreducible curve of degree $g-3$ generating H (by Theorem 3.1). K is thus an irreducible cone with vertex at the point O, since the Q_{ij} do not depend on the variable X_{g-2} $(1 \leq i < j \leq g-3)$.

The remaining $g-3$ quadrics

$$Q_{g-2,i} = X_{g-2} \cdot X_i + X_{g-1}\left(\sum_{j=1}^{g-2} \alpha^j_{g-2,i} X_j\right),$$

where $i = 1, \cdots, g-3$, vanish on the projective subspace H' defined by the equations $X_{g-1} = X_{g-2} = 0$, clearly of codimension 2 in P^{g-2}. Consider the closed subscheme K' of P^{g-2} defined as the intersection of the quadrics $Q_{g-2,i}$ $(i = 1, \cdots, g-3)$. $K' \supset M$, and so it contains $2g-2$ isolated points lying in general position; in proving Theorem 3.1 (see the deduction of Theorem 3.1 from Lemma 3.1) it was shown that $K'_{\text{red}} = G \cup H'$, where G is an irreducible curve of degree $g-2$ generating P^{g-2}. It is evident that G passes through all the points of M lying outside H'; that is, it passes through at least $(2g-2)-(g-3) = g+1$ points. Under projection of the curve G onto the hyperplane H, it is easy to prove that its image $\pi(G)$ is an irreducible curve of degree $g-3$ generating H. The curve $\pi(G)$ passes through g points lying among the $2g-3$ isolated points situated in general position in H. $\pi(G)$ and G' thus have g common points, and so $\pi(G) = G'$ by Lemma 3.1. This equality shows that $\dim K \cap G \geq 1$; that is, $\dim K \cap K' = \dim M \geq 1$. Lemma 5.6 then follows from Theorem 3.1.

§6. Reducibility and smoothness of Q

Theorem 6.1. a. $Q_{\text{red}} = Q$.

b. *If* $\dim Q = 1$, *then* $Q = C$.

c. *If* $\dim Q = 2$ *and* $g \neq 4$, *then* Q *is a smooth surface of degree* $g - 2$.

d. *If* $g = 4$, *then* $\dim Q = 2$ *and* Q *is either a nonsingular quadric or a cone with nonsingular basis curve of degree* 2.

Proof of parts a and b of Theorem 6.1. a. By Theorem 4.1 we have $\dim Q \leq 2$; if $\dim Q = 2$, then $Q_{\mathrm{red}} = Q$ by the same theorem, and so for the proof of part a it is sufficient to prove part b.

b. Let $\dim Q = 1$; then by Theorem 4.1, $Q_{\mathrm{red}} = C$, and by the lemma on the choice of a "good" hyperplane there exists a hyperplane H such that $H \cdot C = \Sigma_{i=1}^{2g-2} x_i$, where the x_i are distinct points lying in H in general position. The scheme $Q \cap H$ thus contains $2g - 2$ isolated points lying in general position in H, and $\#(Q \cap H) = 2g - 2$. The scheme $Q \cap H$ also satisfies property β), since the quadrics through C restrict without kernel to any hyperplane. Hence by Theorem 5.1, $(Q \cap H)_{\mathrm{red}} = Q \cap H$, and so $Q_{\mathrm{red}} = Q$ by the lemma on reducibility (see b, §1).

The proof of part c of Theorem 6.1 will be based on the following proposition which will not be proved in the present article (its proof can be found in [2], §10, Theorem 7).

Proposition 6.1. *If* F *is a singular irreducible reduced surface of degree* $g - 2$ *lying in* P^{g-1} *and generating* P^{g-1}, *then* F *is a cone with nonsingular rational basis curve of degree* $g - 2$ *generating a hyperplane in* P^{g-1}.

Lemma 6.1. *For any point* $O \in \mathbf{P}^{g-1}$ *the generic hyperplane passing through* O *does not touch the curve* C.

Deduction of the remainder of Theorem 6.1 from Proposition 6.1 and Lemma 6.1. c. Let $\dim Q = 2$ and $g \neq 4$. We will assume that Q is a singular variety. Then, by Proposition 6.1, Q is a cone with nonsingular rational basis curve G of degree $g - 2$ generating a hyperplane H. We consider the projection π from the point 0 (the vertex of the cone Q) of C onto H. This projection is defined at all points of the curve C, except perhaps at O, if $O \in C$. The preimage of each generic point of G obviously consists of the same number of points of C. We denote this number by k ($k \geq 1$); that is, a generic generator of the cone Q intersects C in k isolated points, ignoring the point O if $O \in C$. Since the generic hyperplane through O intersects Q in $g - 2$ distinct generic generators and does not touch C, by Lemma 6.1 it follows that $2g - 2 = k(g - 2)$ for $O \notin C$ and $2g - 2 = k(g - 2) + 1$ for $O \in C$. It is easy to show that both equations are impossible for $g > 4$. This leads to a contradicition of the assumption that Q is not smooth for $g \geq 5$.

d. If $g = 4$, then $\dim H^0(\mathbf{P}^{g-1}, I_C(2)) = (g - 2)(g - 3)/2 = 1$, and therefore $\dim Q = 2$ and Q is a quadric in $\mathbf{P}^3$. If Q is a singular surface, by Proposition 6.1 Q will be a cone with nonsingular basis curve of degree 2. The proof of Theorem 6.1 is complete.

Remark. The case of a singular quadric for $g = 4$ can actually be realized, for example, by the curve defined by the equations

$$X_0^2+X_1^2+X_2^2=0, \qquad X_0^3+X_1^3+X_3^3=0$$

in P^3. For char $k \neq 2, 3$ it is easily proved that this curve is not hyperelliptic and that its image under the canonical immersion in Q is the singular quadric defined by the equation

$$X_0^2+X_1^2+X_2^2=0.$$

In case $k=2, 3$ it is also easy to construct a corresponding example. It will be proved in §9 that the case of a nonsingular quadric is realizable.

Proof of Lemma 6.1. The hyperplanes passing through O and touching C form a closed subset in $\mathbf{P}^{g-1}$ which we denote by P. Consider the lines passing through O and some point $x \in C$. Since C is a curve generating P^{g-1} and $g-1 \geq 3$, only a finite number of lines passing through O touch the curve C (see [3]), or a tangent line at the generic point of C does not pass through O. Let $x \in C$; then $P(x)$, the space of planes passing through O and touching C at x, will coincide with the space of planes passing through O and a tangent line at x. Hence $\dim P(x) = g-4$ for the generic point $x \in C$, because a tangent line at the generic point does not pass through O. At the remaining points x, which form a finite set, $\dim P(x) = g-5$. Hence $\dim P \leq g-3$, and the dimension of the space of hyperplanes passing through O in $\mathbf{P}^{g-1}$ is equal to $g-2$, and so the generic hyperplane passing through O does not touch C. The lemma is proved.

We recall some properties of rational ruled surfaces which will be needed in the sequel for the investigation of the structure of Q in the case when Q is a smooth surface. A rational ruled surface is a rational surface F for which there exists a morphism $f\colon F \to P^1$ each of whose fibers is isomorphic to the projective line. It is well known (see [1] or [2], for example) that every such surface is isomorphic to one of the surfaces F_n, $n \geq 0$, defined in the following way: $F_0 \simeq P^1 \times P^1$; F_n for $n \geq 1$ has a canonical section $b\colon P^1 \to F_n$ whose image b_n is a unique irreducible curve on F_n with negative index of self-intersection and $(b_n \cdot b_n) = -n$. It is easy to show that the fibers of the projection $f\colon F_n \to P^1$ form a linear equivalence class, which will be denoted by s_n. For each F_n we have an isomorphism of groups

$$\operatorname{Pic} F_n = \operatorname{cl}(F_n) \simeq \mathbf{Z} \oplus \mathbf{Z}. \tag{6.1}$$

The generators for $n \geq 1$ are the classes s_n and b_n, and for $n=0$ they are classes of coefficients, one of which will be denoted by s_0 and the other by b_0. It is easy to compute the canonical class of the surface

$$\omega_{F_n} = -(2+n)s_n - 2b_n. \tag{6.2}$$

§7. Q is a smooth surface

In this section we will define a set of surfaces, to one of which Q is isomorphic if it is nonsingular, and also we will calculate the class of the curve C in $\operatorname{Pic} Q$.

Theorem 7.1. *If Q is a smooth surface, then the following possibilities arise.*

1. *$Q \simeq P^2$; then $g = 6$, Q will be a Veronese image of the plane P^2 and the curve C, immersed in Q, will be a curve of degree 5.*
2. *$Q \simeq F_n$, where n satisfies the following relations:*

$$0 \leqslant n \leqslant \min\left\{\frac{g+2}{3}, g-4\right\}; \qquad n \equiv g \pmod 2) \tag{7.1}$$

and

$$O_Q(1) = O_{F_n}\left(b_n + \frac{g+n-2}{2} s_n\right); \quad O_Q(C) = O_Q(1) \otimes \Omega_Q^{-1}$$
$$= O_{F_n}\left(3b_n + \frac{g+3n+2}{2} s_n\right). \tag{7.2}$$

Before proceeding to the proof of Theorem 7.1, let us recall a known result from the theory of rational surfaces; its proof can be found in [2] (§10, Theorem 7), for example.

Proposition 7.1. *If F is a nonsingular irreducible reduced surface of degree $g - 2$ lying in P^{g-1} and generating this space, then F is either the surface F_n for some $0 \leq n \leq g - 4$ with $n \equiv g \pmod 2$, and the generators (that is, divisors from the class s_n) are lines in P^{g-1}, or $g = 6$ and then F is isomorphic to P^2; in the latter case F is a Veronese image of the plane P^2.*

Deduction of Theorem 7.1 from Proposition 7.1. Q is an irreducible reduced surface of degree $g - 2$; let us assume that it is regular, so that we obtain from Proposition 7.1 the following possibilities:

1. $Q \simeq P^2$; then $g = 6$ and Q is a Veronese image of the plane P^2. Let n be the degree of C, immersed in the plane P^2; then $g = 6 = (n-2)(n-1)/2$ by the formula for the genus of a curve of degree n. Hence $n = 5$, because $n \geq 1$; that is, the curve C, immersed in P^2, will be a curve of degree 5.

2. $Q \simeq F_n$. We denote by h the linear equivalence class of divisors of the hyperplanar sections $F_n \simeq Q \subset \mathbf{P}^{g-1}$. From the isomorphism (6.1) we have $h = b \cdot b_n + s \cdot s_n$, and we compute b and s for the given immersion $F_n \simeq Q \subset \mathbf{P}^{g-1}$. For the generic hyperplane H, $Q \cap H$ is an irreducible reduced curve of degree $g - 2$ generating H (see the proof of part c of Theorem 4.1). Hence

$$(h \cdot h) = g - 2. \tag{7.3}$$

Since the curve G is rational, by the formula for the genus of a curve on a surface we have

$$(h\cdot(h+\omega_{F_n}))=2g(G)-2=-2, \tag{7.4}$$

where ω_{F_n} denotes the canonical class of the surface $F_n \simeq Q$. Equations (7.3) and (7.4) yield the following system of two equations for b and s:

$$\begin{gathered}((b\cdot b_n+s\cdot s_n)\cdot(b\cdot b_n+s\cdot s_n))=+2bs-nb^2=g-2,\\ ((b\cdot b_n+s\cdot s_n)((b-2)\cdot b_n+(s-n-2)s_n))\\ =(b-2)s+(s-n-2)b-b(b-2)n=-2,\end{gathered} \tag{7.5}$$

since $b_n^2=-n$, $s_n^2=0$ and $(b_n\cdot s_n)=1$. We transform the second equation of (7.5) making use of the first:

$$\begin{gathered}0=2+(b-2)\cdot s+(s-n-2)b-b(b-2)n=2-2s-nb\\ +2nb-2b+2bs-nb^2\\ =2+nb-2b-2s+g-2=g+nb-2b-2s,\end{gathered}$$

that is,

$$g+nb-2b-2s=0.$$

Multiplying the last equation by b and once again using the first equation of (7.5), we obtain

$$0=gb+nb^2-2bs-2b^2=gb-g+2-2b^2,$$

that is, b satisfies the quadratic equation

$$2b^2-gb+g-2=0.$$

Hence $b=1$ or $(g-2)/2$. In the latter case the first equation of (7.5) implies that $s=1+(g-2)n/4$ and $h=((g-2)/2)b_n+[(g-2)n/4+1]\cdot s_n$. It is evident that $(l\cdot h)\geq 1$ for the linear equivalence class l of any irreducible curve, and so $(h\cdot b_n)\geq 1$ or

$$\left(\left(\frac{g-2}{2}b_n+\left[\frac{g-2}{4}\cdot n+1\right]\cdot s_n\right)\cdot b_n\right)=\frac{-n(g-2)}{2}+\frac{(g-2)n}{4}+1\geqslant 1,$$
$$\frac{n(g-2)}{4}\leqslant 0.$$

Hence $n=0$, $s=1$ and $b=(g-2)/2$, but $b\neq(g-2)/2$ for $n\geq 1$. There does not exist a canonical choice of the class b_0 on the surface F_0, and the classes b_0 and s_0, being the classes of the generators, can always be interchanged. Then we obtain that b can be assumed equal to 1 for $n = 0$, while b will always be equal to 1 for $n \geq 1$. Since $b = 1$, we have from the first equation of (7.5) that $s = (g + n - 2)/2$ and $h = b_n + ((g + n - 2)/2)s_n$, and hence

$$O_Q(1) = O_{F_n}\left(b_n + \frac{g+n-2}{2} s_n\right).$$

We will compute the linear equivalence class of the curve C immersed in $F_n \simeq Q$. The curve C is a nonsingular irreducible reduced curve of degree $2g - 2$ lying in P^{g-1}, and $g(C) = g$, the genus of C, and so

$$(\mathrm{cl}\ (C) \cdot h) = 2g-2, \tag{7.6}$$

and, by the formula for the genus of a nonsingular curve lying on a regular surface,

$$(\mathrm{cl}\ (C) \cdot (\mathrm{cl}\ (C) + \omega_{F_n})) = 2g-2. \tag{7.7}$$

Let $\mathrm{cl}(C) = b \cdot b_n + s \cdot s_n$ for the isomorphism (6.1). We then obtain from equations (7.6) and (7.7) a system of two equations for the unknowns s and b:

$$\left((b \cdot b_n + s \cdot s_n) \ \left(b_n + \frac{g+n-2}{2} s_n\right)\right) = -bn + s + b\left(\frac{g+n-2}{2}\right) = 2g-2,$$

$$((b \cdot b_n + s \cdot s_n) \cdot ((b-2) \cdot b_n + (s-n-2) s_n)) = -b(b-2)n \tag{7.8}$$
$$+ s(b-2) + (s-n-2) \cdot b = 2g-2.$$

We obtain from the first equation of this system that $s = (2g - 2) - b(g - n - 2)/2$ or

$$2bs = (4g-4)b - b^2(g-n-2).$$

Hence

$$2g-2 = -b(b-2)n + s(b-2) + (s-n-2)b = -b(b-1) \cdot n + 2bs$$
$$-2s - 2b = -b(b-1)n + (4g-4)b - b^2(g-n-2)$$
$$-2s - 2b = bn + (4g-4)b - b^2(g-2) - (4g-4) + b(g+n-2) - 2b$$
$$= b^2(2-g) + 4 - 4g + b(5g+8);$$

that is, b is a root of the quadratic equation

$$b^2(2-g) + 6 - 6g + b(5g-8) = 0, \tag{7.9}$$

and so $b = 3$ or $b = 2 + 2/(g - 2)$. The last value for b is possible only when $g = 4$, since $g \geq 4$ and b is an integer. For $g = 4$ the quadratic equation (7.9) has the double root $b = 3$. Hence

$$b = 3,\ s = 2g - 2 - b\frac{(g-n-2)}{2} = \frac{g+3n+2}{2}$$

and

$$\operatorname{cl}(C) = 3b_n + \frac{g+3n+2}{2}s_n = h - \omega_{F_n}. \tag{7.10}$$

It follows from the last equation that

$$O_Q(C) = O_Q(1) \otimes \Omega_Q^{-1} = O_{F_n}\left(3b_n + \frac{g+3n+2}{2}s_n\right).$$

To conclude the proof of Theorem 7.1, we will show that n satisfies the relations (7.1). The second relation follows from Proposition 7.1. It is clear that $(b_n \cdot \operatorname{cl}(C)) \geq 0$, and so

$$\left(b_n \cdot \left(3b_n + \frac{g+3n+2}{2}s_n\right)\right) = \frac{g+3n+2}{2} - 3n \geqslant 0,$$

that is,

$$\frac{g+2}{3} \geqslant n. \tag{7.11}$$

From the inequality (7.11) and the fact that $0 \leq n \leq g-4$ by Proposition 7.1, we obtain the first relation of (7.1):

$$0 \leqslant n \leqslant \min\left\{\frac{g+2}{3},\ g-4\right\}.$$

Theorem 7.1 is proved.

§8. Proof of Theorem 2

Theorem 8.1. a. *$Q \simeq F_n$ for $g \geq 5$ if and only if there exists on the curve C an effective divisor D of degree 3 such that* $\dim H^0(C, O_C(D)) = 2$.

b. *$Q \simeq P^2$ if and only if $g = 6$ and there exists on the curve C an effective divisor of degree 5 such that its carrier consists of five distinct points and* $\dim H^0(C, O_C(D)) = 3$.

Proof. a. **Necessity.** For $Q \simeq F_n$ it was shown in the proof of Theorem 7.1 that

$$\operatorname{cl}(C) = 3b_n + \frac{g+3n+2}{2}s_n.$$

Let l be a divisor from the class s_n. By Proposition 7.1, l is a line in $\mathbf{P}^{g-1}$ under the immersion $F_n \simeq Q \subset \mathbf{P}^{g-1}$, because $\deg Q = g-2$. Consider the divisor $D = l \cdot C$ on the curve C. It is clear that D is an effective divisor and

$$\deg D = (\operatorname{cl}(C) \cdot \operatorname{cl}(l)) = \left(\left(3b_n + \frac{g+3n+2}{2}s_n\right) \cdot s_n\right) = 3.$$

Let ω be a divisor from the canonical class of the curve C which is determined by a hyperplane H not passing through points of the divisor D and having equation h. Then, from the definition of the canonical immersion,

$$H^0(C, O_C(\omega)) = \left\{ \frac{h'}{h} \,\middle|\, \begin{array}{l} h' \text{ is a linear form from the coordinate ring of the} \\ \text{space } \mathbf{P}^{g-1} \end{array} \right\}. \quad (8.1)$$

It is easy to show that

$$H^0(C, O_C(\omega - D)) = \left\{ \frac{h'}{h} \,\middle|\, \begin{array}{l} h' \text{ is a linear form on } \mathbf{P}^{g-1} \text{ which defines} \\ \text{a hyperplane } H' \subset l \end{array} \right\}. \quad (8.2)$$

Hence $\dim H^0(C, O_C(\omega - D)) = g - 2$. By the duality theorem, $\dim H^1(C, O_C(D)) = g - 2$, and so, by the Riemann-Roch Theorem for curves,

$$\dim H^0(C, O_C(D)) = \dim H^1(C, O_C(D)) + \deg D - g + 1 = 2.$$

The necessity is proved.

Sufficiency. Let D be an effective divisor of degree 3 on the curve C such that $\dim H^0(C, O_C(D)) = 2$; then by the Riemann-Roch Theorem,

$$\dim H^0(C, \Omega_C \otimes O_C(-D)) = g - 2. \quad (8.3)$$

We choose a divisor ω from the canonical class of the curve C in such a way that $(\omega \cdot D) = 0$; it then follows from formula (8.1) that $H^0(C, O_C(\omega - D)) = \{h'/h \mid h'$ is a linear form from the coordinate ring of the space $\mathbf{P}^{g-1}$ such that, for the hyperplane H' corresponding to it, the divisor $H' \cdot C \geq D\}$, and hence the "projectivization" of the space $H^0(C, O_C(\omega - D))$ is isomorphic to the projective space of hyperplanes intersecting the curve C in a divisor not less than D, and the dimension of this space is equal to $g - 3$ by formula (8.3); on intersection these hyperplanes therefore define a line l. Obviously $(l \cdot c) \geq 3$. Hence any quadric passing through C contains l, and therefore $l \subset Q$ and $\dim Q = 2$; by Theorem 6.1, Q is a smooth surface of degree $g - 2$, since it was assumed in part a of Theorem 8.1 that $g \geq 5$. Q cannot be a Veronese image of the plane P^2 because, as is well known, a Veronese image of a plane does not contain lines. Hence $Q \simeq F_n$ by Proposition 7.1.

b. **Necessity.** Let $Q \simeq P^2$; then by Theorem 7.1 we have $g - 6$ and Q is a Veronese image of the plane P^2, while the curve C, immersed in Q, will be a curve of degree 5 on P^2. Under a Veronese transformation the line l of the plane P^2 passes to a curve of degree 2 in $\mathbf{P}^5$ which generates the plane π. Since the degree of C, immersed in $P^2 \simeq Q$, is equal to 5, there exists a line l intersecting C in five distinct points which form a divisor D. The corresponding plane π then contains the divisor D and is generated by it. As in the proof of necessity in part a, it is easy to show that

$$\dim H^0(C, \Omega_C \otimes O_C(-D)) = g - 3 = 3,$$

and, by the Riemann-Roch Theorem,

$$\dim H^1(C,\ O_C(D))=3,\qquad \dim H^0(C,\ O_C(D))$$
$$=\dim H^1(C,\ O_C(D))+\deg D-g+1=3.$$

The necessity is proved.

Sufficiency. Let C be a curve of genus 6 and D an effective divisor of degree 5 whose carrier consists of 5 points, and $\dim H^0(C,\ O_C(D)) = 3$; then, by the Riemann-Roch Theorem,

$$\dim H^0(C,\ \Omega_C\otimes O_C(-D)) = g-3=3. \tag{8.4}$$

We can choose on C a divisor ω from the canonical class in such a way that $(D\cdot\omega) = 0$. Hence, as in the proof of sufficiency in part a, we obtain that the "projectivization" of the space $H^0(C,\ O_C(\omega - D))$ is isomorphic to the projective space of hyperplanes which intersect the curve C in a divisor not less than D. By (8.4) the dimension of this space is equal to 2, and so, on intersection, these hyperplanes define a plane π. It is easy to show that $C\cdot\pi \geq D$. If Q were equal to C, there would exist two quadrics q_1 and q_2 whose restrictions to π would give a finite number of points, and we would clearly have $\deg D \leq (C\cdot\pi) \leq (\pi\cdot q_1\cdot q_2) = 4$, contradicting the fact that $\deg D = 5$. Hence $\dim Q = 2$ and $Q \cap \pi = q$ is a quadric lying in the plane π, since Q clearly does not contain π.

We will show that Q cannot be a ruled surface. To this end we examine the following cases.

I. $q_{\rm red} = l$ is a line which contains the divisor D. Since $\#(\operatorname{Supp} D) = 5$, $(l\cdot C) \geq 5$. If Q were a ruled surface, then a divisor of a hyperplanar section on $Q \simeq F_n$ would lie in the ruled system $b_n + ((g+n-2)/2)s_n$, as was shown in §7. It is easy to prove from the last equation that if l is a line lying on $Q \simeq F_n$, then either $l\in s_n$, or $l \in b_n$ and $n = g-4$. Therefore $q_{\rm red} = l \in b_n$ and $n = g-4$, because

$$(s_n\cdot C)=\left(s_n\cdot\left(3b_n+\frac{g+3n+2}{2}s_n\right)\right)=3,$$

while $(l\cdot C)\geq 5$. Hence

$$(l\cdot C)=(b_n\cdot C)=-3n+\frac{g+3n+2}{2}=\frac{g-3n+2}{2}=1,$$

since $g = 6$ and $n = g-4 = 2$. This contradicts the fact that $(l\cdot C)\geq 5$, and so Q cannot be a ruled surface in this case.

II. q is a singular reduced quadric, and so it splits into two distinct lines, l and l'. If Q were the ruled surface F_n, then, by what has been said in the analysis of case I, it would follow that $n = g-4$ and the two lines l and l' lie in the classes s_n and b_n. Therefore $(C\cdot q) = (C\cdot(s_n+b_n)) = (C\cdot s_n) + (C\cdot b_n) = 4$, since $g = 6$

and $n = g - 4 = 2$. This contradicts the fact that q contains 5 distinct points of C, and so Q cannot be a ruled surface in this case.

III. q is a smooth reduced quadric. We will assume that $Q \simeq F_n$, $n = 0, 2$, by Proposition 7.1. Let $\mathrm{cl}(q) = b \cdot b_n + s \cdot s_n$. Using the fact that a divisor of a hyperplanar section under the immersion $F_n = Q \subset \mathbf{P}^{g-1}$ lies in the class $b_n + ((g+n-2)/2)s_n$, it is easy to show that for $n = 2$ a nonsingular quadric cannot lie on F_n, while $\mathrm{cl}(q) = b_0$ for $n = 0$. Hence

$$(C \cdot q) = (C \cdot b_0) = \left(\left(3b_0 + \frac{g+3n+2}{2} s_0\right) \cdot b_0\right) = 4,$$

since $n = 0$ and $g = 6$. It was noted earlier that $(C \cdot q) \geq 5$, and so $Q \not\simeq F_n$. From Theorems 6.1 and 7.1 we then obtain $Q \simeq P^2$.

Theorem 8.1 is proved.

Remark. For the proof of sufficiency in part b it was shown that, if on a curve C of genus b there exists an effective divisor of degree 5 for which $\dim H^0(C, O_C(D)) = 3$, then $\dim Q = 2$; that is, the curve C, the image under the canonical immersion of the curve X, will be a special curve. The author does not know whether Q is isomorphic to P^2 or possibly to F_n in this case. It was assumed in part b of Theorem 8.1 that $\#(\mathrm{supp}\, D) = 5$.

§9. Proof of Theorem 3

Theorem 9.1. a. *Let n satisfy the following relations*:

$$0 \leqslant n \leqslant \min\left\{\frac{g+2}{3},\ g-4\right\};\quad n \equiv g \pmod 2, \tag{9.1}$$

then there exists a special curve X of genus g such that $Q \simeq F_n$ for it.

b. $\dim H^0(Q, O_Q - X) = 2g + 8$ *if* $Q \simeq F_n$.

The proof of Theorem 9.1 will be based on the following assertions.

Lemma 9.1. *If C is a smooth curve of genus g and degree $2g-2$ lying in the space $\mathbf{P}^{g-1}$ and generating it, then C is a nonhyperelliptic curve which is its own image under the canonical immersion.*

Proposition 9.1. a. *The sheaf $O_{F_n}(b_n + ms_n)$ is very ample for $m > n$, and* $\dim H^0(F_n, O_{F_n}(b_n + ms_n)) = 2m - n + 2$.

b. *In the linear system $3b_n + ks_n$, where $k \geq 3n$, there exists for $n \geq 1$ a reduced irreducible smooth curve; the same holds for the linear system $3b_0 + ks_0$ for $k \geq 1$.*

c. $\dim H^0(F_n, O_{F_n}(3b_n + ks_n)) = 4k - bn + 4$, *where* $k \geq 3n$.

Deduction of Theorem 9.1 from Proposition 9.1 and Lemma 9.1. Let n satisfy the relations (9.1); then $(g + n - 2)/2$ is an integer greater than n, and so the sheaf $O_{F_n}(b_n + ((g+n-2)/2)s_n)$ is very ample, and the linear system $b_n + ((g+n-2)/2)s_n$

defines an immersion of the surface F_n into projective space of dimension equal to

$$\dim H^0\left(F_n, O_{F_n}\left(b_n+\frac{n+g-2}{2}s_n\right)\right)-1=g+n-2-n+2-1=g-1$$

(by Proposition 9.1). Denote the image of the surface F_n in P^{g-1} by Q'. It is evident that Q' is a reduced irreducible smooth surface generating $\mathbf{P}^{g-1}$. We have

$$\deg Q'=\left(b_n+\frac{g+n-2}{2}s_n,\ b_n+\frac{g+n-2}{2}s_n\right)=g-2.$$

Consider the linear system $3b_n+((g+3n+2)/2)s_n$; then $(g+3n+2)/2\geq 3n$, since $(g+2)/3\geq n$ for $n\geq 1$, and $(g+3n+2)/2\geq 3$ for $n=0$, as $g\geq 4$. Hence by Proposition 9.1 there exists in the class $3b_n+((g+3n+2)/2)s_n$ a reduced irreducible smooth curve X, and

$$\dim H^0(Q', O_{Q'}(X))=4\left(\frac{g+3n+2}{2}\right)-6n+4=2g+8.$$

The genus of the curve X is equal to

$$\frac{(\mathrm{cl}\,(X),\,\mathrm{cl}\,(X)+\omega_{F_n})}{2}+1=\frac{\left(3b_n+\frac{g+3n+2}{2}s_n,\ b_n+\frac{g+n-2}{2}s_n\right)}{2}+1=g,$$

Denote by C the image of X lying in F_n under the immersion defined by the linear system $b_n+((g+n-2)/2)s_n$. We have

$$\deg C=\left(\mathrm{cl}(X)\cdot b_n+\frac{g+n-2}{2}s_n\right)=2g-2,$$

whence, by Lemma 9.1, C is a nonhyperelliptic curve, being a canonical image of X. Let l be a divisor of the class s_n; then l is a reduced irreducible smooth curve isomorphic to P^1. Furthermore,

$$\left(\mathrm{cl}\,(l)\cdot b_n+\frac{g+n-2}{2}s_n\right)=1,$$

and so l is a line lying on $Q'\subset \mathbf{P}^{g-1}$. Also,

$$(l\cdot C)=\left(\mathrm{cl}\,(l)\cdot 3b_n+\frac{g+3n+2}{2}s_n\right)=3.$$

The line l is thus contained in every quadric passing through C; that is, $l\subset Q$, and so $Q'\subset Q$ since the generators l mark the whole surface $Q'\simeq F_n$. Hence $\dim Q=2$ and $Q=Q'\simeq F_n$ by Theorem 4.1, and so there exists a special curve X of genus g such that $Q=Q'\simeq F_n$ for it. Theorem 9.1 is proved.

Proof of Lemma 9.1. Let H be some hyperplane in $\mathbf{P}^{g-1}$. Since C generates $\mathbf{P}^{g-1}$, the effective divisor $D=H\cdot C$ is defined and $\deg D=\deg C=2g-2$. Clearly $\dim H^0(C, O_C(D))\geq g$. Hence by the Riemann-Roch Theorem

$$\dim H^0(C, O_C(-D)\otimes\Omega_C)=\dim H^0(C, O_C(D))-\deg D+g-1\geqslant 1;$$

thus, since $\deg(\Omega_C)=\deg(D)=2g-2$, there is a function f on C such that $(f)_\infty$

lies in the canonical class of C and $(f)_0 = D$. D thus lies in the canonical class, and $\dim H^0(C, O_C(D)) = g$, and so C is a proper image under the canonical mapping of the curve C. The lemma is proved.

Lemma 9.2. *The linear systems $b_n + ns_n$ and $3b_n + 3b_n s_n$ have reduced irreducible regular curves for $n \geq 1$.*

Proof. It is well known that for the ruled surface F_n the arithmetic genus is equal to 0. Hence for any divisor D on the surface F_n we have

$$\dim H^0(F_n, O_{F_n}(D)) - \dim H^1(F_n, O_{F_n}(D)) + \dim H^2(F_n, O_{F_n}(D)) = 1 + \frac{(D, D - \omega_{F_n})}{2},$$

where ω_{F_n} is the canonical class of F_n. For any effective divisor D on F_n it is easy to show, from the duality theorem and the fact that $\omega_{F_n} = -2b_n + (n+2)s_n$, that $\dim H^2(F_n, O_{F_n}(D)) = 0$, and thus, for $k \geq 0$,

$$\begin{aligned} &\dim H^0(F_n, O_{F_n}(b_n + ks_n)) - \dim H^1(F_n, O_{F_n}(b_n + ks_n)) \\ &= 1 + \frac{(bn + ks_n \cdot 3b_n + (k+n+2)s_n)}{2} = 2k - n + 2 \end{aligned} \tag{9.2}$$

and

$$\begin{aligned} &\dim H^0(F_n, O_{F_n}(3b_n + ks_n)) - \dim H^1(F_n, O_{F_n}(3b_n + ks_n)) \\ &= 1 + \frac{(3b_n + ks_n \cdot 3b_n + (k+n+2)s_n)}{2} = 4k - 6n + 4. \end{aligned} \tag{9.3}$$

Let $n \geq 1$; then b_n is the unique reduced irreducible curve with negative index of self-intersection (see [1]). As is well known, b_n is a regular curve of genus 0. Let $\phi \in \Gamma(F_n, O_{F_n}(b_n + ns_n))$ and let ϕ be a function which is constant along generators; that is, a divisor of the class s_n; it is then obvious that $\phi \in \Gamma(F_n, O_{F_n}(ns_n))$. Consider the restriction ϕ' of ϕ to b_n; then $\phi' \in \Gamma(b_n, O_{b_n}(D))$, where D is an effective divisor of degree n. The restriction to b_n of functions constant along generators takes place without kernel, and for any function $\phi' \in \Gamma(b_n, O_{b_n}(D))$ there is a function ϕ, constant on generators, whose restriction to b_n is equal to ϕ'. There thus exists on the spaces $\Gamma(F_n, O_{F_n}(b_n + ns_n))$ a set of linearly independent functions $\phi_0, \phi_1, \ldots, \phi_n, f$, where the ϕ_i are functions constant on generators, and f is a function not constant on generators, because $\dim \Gamma(b_n, O_{b_n}(D)) = \deg D + 1 = n + 1$ (the curve b_n is rational), and equation (9.2) yields the inequality $\dim \Gamma(F_n, O_{F_n}(b_n + ns_n)) \geq n + 2$. Since f is not constant along generators, there exists on a divisor of its zeros an irreducible reduced curve G which does not coincide with s_n or b_n. Hence $(G \cdot s_n) \geq 1$ and $(G \cdot b_n) \geq 0$, and therefore $\mathrm{cl}(G) = b \cdot b_n + s \cdot s_n$, where $b \geq 1$

and $s \geq n$; but clearly $\operatorname{cl}(G) \leq b_n + ns_n$. We obtain from this inequality that $G \in b_n + ns_n$. It is easy to show that the curve G is smooth and rational. Consider the morphism μ from F_n into the projective space P^{n+1} defined by the functions $(\phi_0, \phi_1, \cdots, \phi_n, f)$. Under this morphism, b_n passes to the point O with coordinates $(0, \cdots, 0, 1)$, while generators pass into lines passing through the point O. Under μ, G is immersed in the plane $f = 0$. Thus $\mu(F_n)$ is a cone with vertex at O and nonsingular basis curve G, whose degree is equal to $\deg D = n$. Under μ, the generators F_n and the curve G are mapped isomorphically onto generators of the cone $\mu(F_n)$ and the basis curve of the cone $\mu(F_n)$ respectively, and thus μ is an isomorphism of F_n minus b_n. Let f' be some nonconstant function in $\Gamma(F_n, O_{F_n}(b_n + ns_n))$. It was shown above that $(f')_0 = G'$ is an irreducible reduced smooth curve. Since $G' \in b_n + ns_n$, it is obvious that $\deg \mu(G') = (G' \cdot ns_n) = b_n + ns_n \cdot ns_n) = n$. G' thus lies in some hyperplane of P^{n+1}; that is, $G' = (\lambda f + \Sigma\lambda_i \rho_i)_0$, and thus $f' = \lambda f + \Sigma\lambda_i f_i$, and the functions $\phi_0, \phi_1, \cdots, \phi_n, f$ form a basis of the space $\Gamma(F_n, O_{F_n}(b_n + ns_n))$, $\dim H^0(F_n, O_{F_n}(b_n + ns_n)) = n + 2$.

Consider the morphism

$$\mu_3 : F_n \to P^{(\dim H^0(F_n, O_{F_n}(3b_n + 3ns_n)) - 1)} = P^{N-1},$$

defined by the linear system $3b_n + 3ns_n$. It follows from properties of the morphism μ that $\mu_3(F_n)$ is an irreducible reduced surface having one singular point, and on $\mu_3(F_n)$ the generic hyperplane cuts out divisors of the class $3b_n + 3ns_n$. By Bertini's theorem (see [4] or [5]) the generic hyperplane H in the space P^{N-1} cuts out a reduced irreducible smooth curve. There thus exists a reduced irreducible smooth curve in the class $3b_n + ns_n$. The lemma is proved.

Lemma 9.3. *The sheaf $O_{F_n}(b_n + (n+1)s_n$ is very ample.*

Proof. This is obvious for $n = 0$, since on a regular quadric lying in P^3 hyperplanar sections cut out elements of the class $b_0 + s_0$, and so we can assume $n \geq 1$ for the proof of this lemma. Let $\phi_0, \phi_1, \cdots, \phi_{n+1}$ be linearly independent functions in $\Gamma(F_n, O_{F_n}(b_n + (n+1)s_n))$, constant along generators. Consider the morphism η defined by the functions $\phi_0, \cdots, \phi_n, \phi_{n+1}$(3) f and $f \cdot \phi_{n+1}$, where f is the function defined in the proof of Lemma 9.2. The morphism η is an isomorphism on $F_n \setminus b_n$, since the morphism μ defined by the functions $\phi_0, \cdots, \phi_n$ and f is an isomorphism on this space. Under η generators pass into lines, and b_n passes isomorphically into a line which is given parametrically in the following way:

$$l : (0, \ldots, 0, 1, X),$$

where $X \in k \cup \{\infty\}$. Thus η is a one-to-one mapping. In order to prove that η is an isomorphism, it is necessary to show that the tangent mapping at the points of b_n is an isomorphism. This holds because the line $\eta(b_n)$ is transversal to the lines which

(3)It is assumed that $\phi_{n+1} \in \Gamma(F_n, O_{F_n}(s_n))$ is a nonconstant function are images of the generators F_n. Thus η is an isomorphic immersion of F_n, and, since there is a function $f \cdot \phi_{n+1} \in (F_n, O_{F_n}(b_n + (n+1)s_n))$ at which $(f\phi_{n+1})_\infty \in b_n + (n+1)s_n$, the sheaf $O_{F_n}(b_n + (n+1)s_n)$ is very ample. The lemma is proved.

Deduction of parts a and b of Proposition 9.1 from Lemmas 9.2 and 9.3.

a. The sheaf $O_{F_n}(b_n + (n+1)s_n)$ is very ample, by Lemma 9.3. The sheaf $O_{F_n}(b_n + ms_n)$ is thus very ample for $m > n$, since the system $(m-n-1)s_n$ has no fixed components.

Let Q be the image of F_n under the immersion defined by the very ample sheaf $O_{F_n}(b_n + ms_n)$. Then Q is an irreducible reduced surface generating P^N, where

$$N = \dim H^0(F_n, O_{F_n}(b_n + ms_n)) - 1.$$

Hence $\deg Q \geq N - 1$; that is,

$$\deg Q = (b_n + ms_n, b_n + ns_n) = 2n - n \geqslant \dim H^0(F_n, O_{F_n}(b_n + ms_n) - 2$$

and

$$\dim H^0(F_n, O_{F_n}(b_n + ms_n)) \leqslant 2m - n + 2.$$

The latter inequality and (9.2) imply that for $m > n$

$$\dim H^0(F_n, O_{F_n}(b_n + ms_n)) = 2m - n + 2, \qquad \dim H^1(F_n, O_{F_n}(b_n + ms_n)) = 0.$$

b. It was proved in Lemma 9.2 that for $n \geq 1$ there exists an irreducible reduced smooth curve in the linear system $3b_n + 3ns_n$, and so we can assume for the proof of part b of Proposition 9.1 that $k > 3n$ on the linear system $3b_n + ks_n$. Hence $k - 2n > n$, and the sheaf $O_{F_n}(b_n + (k-2n)s_n)$ is very ample; the class $b_n + ns_n$ does not have fixed components (this is obvious for $n = 0$ and is proved in Lemma 9.2 for $n \geq 1$), and so the sheaf $O_{F_n}(3b_n + ks_n)$ is very ample. It follows from Bertini's theorem (see [5]) that there exists a reduced irreducible smooth curve in the full linear system $3b_n + ks_n$ for $k > 3n$.

Parts a and b of Proposition 9.1 are proved.

Lemma 9.4. a. $\dim H^0(F_n, O_{F_n}(2b_n + 2ns_n)) \leq 3n + 3$.
b. $\dim H^0(F_n, O_{F_n}(3b_n + 3ns_n)) \leq 6n + 4$.

Proof. a. It was shown in Lemma 9.2 that there exists a reduced irreducible smooth curve G of genus 0 in the class $b_n + ns_n$. Let G' be another such curve in $b_n + ns_n$, not equal to G. Consider the restriction of functions in $\Gamma(F_n, O_{F_n}(2G'))$ to the curve G; functions in $\Gamma(F_n, O_{F_n}(2G'))$ thus pass to functions in $\Gamma(G, O_G(D))$, where $D = 2(G \cdot G')$ is an effective divisor of degree $2n$. The kernel of this restriction consists of functions lying in $\Gamma(F_n, O_{F_n}(2G' - G))$. Hence

$$\dim H^0(F_n, O_{F_n}(2b_n + 2ns_n)) \leqslant \dim H^0(G, O_G(D)) + \dim H^0(F_n, O_{F_n}(b_n + ns_n)) =$$

$$= 2n + 1 + n + 2 = 3n + 3.$$

b. As in part a, it is easy to show that

$$\dim \Gamma(F_n, O_{F_n}(3G')) \leqslant \dim \Gamma(G, O_G(3G \cdot G')) + \dim \Gamma(F_n, O_{F_n}(3G' - G)).$$

Hence

$$\dim H^0(F_n, O_{F_n}(3b_n + 3ns_n)) \leqslant \dim H^0(G, O_G(3(G \cdot G')))$$
$$+ \dim H^0(F_n, O_{F_n}(2b_n + 2ns_n)) \leqslant 3n + 1 + 3n + 3 = 6n + 4.$$

The lemma is proved.

Lemma 9.5.

$$\dim H^0(F_n, O_{F_n}(3b_n + (k+1)s_n)) - \dim H^0(F_n, O_{F_n}(3b_n + ks_n)) \leqslant 4.$$

Proof. Let l and l' be two distinct generators from s_n. Restricting functions from $\Gamma(F_n, O_{F_n}(3b_n + (k+1)l))$ to l' as in the proof of Lemma 9.4, we obtain the inequality

$$\dim H^0(F_n, O_{F_n}(3b_n + (k+1)l)) \leqslant \dim H^0(F_n, O_{F_n}(3b_n + (k+1)l - l'))$$
$$+ \dim H^0(l', O_{l'}(l' \cdot (3b_n + (k+1)l))).$$

Furthermore,

$$\dim H^0(l', O_{l'}(l' \cdot (3b_n + (k+1)l))) = (s_n \cdot (3b_n + (k+1)s_n)) + 1 = 4.$$

Hence

$$\dim H^0(F_n, O_{F_n}(3b_n + (k+1)s_n)) - \dim H^0(F_n, O_{F_n}(3b_n + ks_n)) \leqslant 4.$$

The lemma is proved.

Proof of part c of Proposition 9.1. The inequality

$$\dim H^0(F_n, O_{F_n}(3b_n + ks_n)) \leqslant 4k - 6n + 4,$$

where $k \geq 3n$, follows from Lemmas 9.4 and 9.5. We then obtain from equation (9.3) that for $k \geq 3n$

$$\dim H^0(F_n, O_{F_n}(3b_n + ks_n)) = 4k - 6n + 4, \qquad \dim H^1(F_n, O_{F_n}(3b_n + ks_n)) = 0.$$

Proposition 9.1 is proved.

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