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SMOOTHNESS OF THE GENERAL ANTICANONICAL DIVISOR ON A FANO 3-FOLD

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Abstract. Smoothness of the general anticanonical divisor of a Fano 3-fold is proved. In addition, an analogous result is established for the linear system $|\mathcal{H}|$, where $r\mathcal{H} \sim -K_V$ for some natural number r. The results obtained in the paper can be used to investigate projective imbeddings of Fano 3-folds.

Bibliography: 6 titles.

Following [4], we call a smooth complete irreducible algebraic variety V of dimension 3 over a field k which has an ample anticanonical class $-K_V$ a *Fano 3-fold*. In [4] projective embeddings of such varieties were considered under the following hypothesis:

HYPOTHESIS (1.14) [4]. *There exist an invertible sheaf* $\mathcal{H} \in \operatorname{Pic} V$ *and a natural number r such that* $r\mathcal{H} \simeq -K_V$ *and the linear system* $|\mathcal{H}|$ *contains a smooth surface H* (*the greatest such r is called the index of V*).

The purpose of the present work is to show that this hypothesis is satisfied for every Fano 3-fold over an algebraically closed field of characteristic 0. Thus all the results of [4] where Hypothesis (1.14) is assumed remain true also without that assumption.

The question considered in this paper can be given the following more general formulation. Let V be a complete nonsingular smooth irreducible algebraic variety of dimension n with an ample anticanonical class $-K_V$. Does there exist a smooth divisor in the linear system $|-K_V|$? This problem naturally arises in considering the mapping defined by the linear system $|-K_V|$. The answer to this question is affirmative in the case of an algebraically closed field k of any characteristic if $n \leqslant 2$ and in characteristic 0 for $n \leqslant 3$. In the remaining cases the answer is unknown. In connection with the notion of the index of a variety there arises also an analogous question for $-K_V/r \in \operatorname{Pic} V$.

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§1. The main result

1.1. All the algebraic varieties considered in this paper are defined over an algebraically closed field k of characteristic zero.

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1.2. THEOREM. *Let V be a Fano 3-fold, and let $\mathscr{A}$ be an invertible sheaf such that $r\mathscr{A} \simeq -K_V$ for some natural r. Then in the linear system $|\mathscr{A}|$ there is a smooth surface D.*

Theorem 1.2 is proved in §3 for the case $r = 1$, and in §4 for $r \geqslant 2$. §2 is devoted to auxiliary propositions. The general plan of the proof is the following. First we prove that the linear system $|\mathscr{A}|$ is not composite with a pencil. Then using Bertini's theorem we bring the general element of $|\mathscr{A}|$ to the form $D + D_0$ with fixed part D_0 and irreducible reduced movable divisor D. The dimension of the space $H^0(V, \mathscr{A})$ is known to us from [4]. On the other hand, $h^0(V, \mathscr{O}_V(D)) = h^0(V, \mathscr{A})$. The presence of fixed components or of singularities in the general divisor D reduces the last equality to a contradiction either with the Riemann-Roch theorem on the surface $\widetilde{D}$, which resolves the singularities of D, or with Lemma 2.3. If $r \geqslant 2$ one shows that the base locus of $|\mathscr{A}|$ consists of no more than a finite number of points. Further, one uses Theorem 4.1 of [3].

§2. Auxiliary lemmas

2.1. LEMMA. *If V is a Fano 3-fold, then every effective divisor D from the linear system $|-K_V|$ is connected.*

PROOF. According to (1.4) (i) of [4], $h^0(D, \mathscr{O}_D) = 1$ for $D \in |-K_V|$. Therefore, D is connected. ∎

2.2. LEMMA. *Let D be an effective divisor on a $K3$ surface X such that some multiple of D gives a linear system without fixed components and*

$$h^0(X, \mathscr{O}_X(D)) = \frac{D^2}{2} + 2.$$

Then the fixed components of D have multiplicity 1.

PROOF. By Bertini's theorem [1] we may assume that the movable components of D have multiplicity one. We denote by $D_1, \ldots, D_n$ the connected components of the multiplicity one part of the general D. Then we have the following representation of D as the sum of effective divisors: $D = \Sigma_0^n D_i$, where D_0 denotes a multiple of the fixed component of D. We need to show that $D_0 = 0$. Let us assume the contrary: $D_0 \neq 0$. By duality and the Riemann-Roch theorem we have

$$h^2(X, \mathscr{O}_X(D)) = h^1(X, \mathscr{O}_X(D)) = 0.$$

The latter, using duality and the Ramanujan vanishing theorem for a regular surface (see the remark on page 180 in [2]) implies that

$$h^0(D, \mathscr{O}_D) = h^1(X, \mathscr{O}_X(-D)) + 1 = h^1(X, \mathscr{O}_X(D)) + 1 = 1.$$

Therefore D is connected. Consequently, by the nontriviality of D_0, $(D_i, D_0) \geqslant 2$ for $n \geqslant i \geqslant 1$. Hence $(\Sigma_1^n D_i, D_0) \geqslant 2n$.

Using Ramanujan's theorem and duality for the divisor $\Sigma_1^n D_i$, we obtain

$$h^1\left(X, \mathscr{O}_X\left(\sum_{i=1}^n D_i\right)\right) = h^0\left(\bigcup_{i=1}^n D_i, \mathscr{O}_{\bigcup_{i=1}^n D_i}\right) - 1 = n - 1.$$

By duality and the nontriviality of $\Sigma_1^n D_i$ (since there exists a movable part), we have $h^2(X, \mathscr{O}_X(\Sigma_1^n D_i)) = 0$. Consequently by the Riemann-Roch theorem

$$h^0\left(X, \mathcal{O}_X\left(\sum_{i=1}^n D_i\right)\right) = \frac{\left(\sum_{i=1}^n D_i\right)^2}{2} + n + 1.$$

By construction, the movable part of D is contained in the components of multiplicity one. Therefore

$$h^0\left(X, \mathcal{O}_X\left(\sum_{i=1}^n D_i\right)\right) = h^0(X, \mathcal{O}_X(D)) = \frac{D^2}{2} + 2,$$

whence we obtain the relation

$$\frac{D^2}{2} + 2 = \frac{\left(\sum_{i=1}^n D_i\right)^2}{2} + n + 1,$$

i.e.

$$\frac{\left(D_0, \sum_{i=1}^n D_i\right) + (D_0, D)}{2} = n - 1.$$

But $(D, D_0) \geqslant 0$ because of the absence of fixed components in a multiple of the divisor D. The latter contradicts the inequality $(\Sigma_1^n D_i, D_0) \geqslant 2n$. ■

2.3. LEMMA. *Let D be an effective divisor on a K3 surface X such that some multiple lD, l a natural number, gives a linear system $|lD|$ without base points and such that the image of the corresponding morphism is two-dimensional. Then D can have at most one fixed component, which is a smooth rational curve.*

PROOF. The linear system $|D|$ satisfies the assumptions of Mumford's theorem about degeneration. Hence by duality and the Rieman-Roch theorem we have

$$h^0(X, \mathcal{O}_X(D)) = \frac{D^2}{2} + 2,$$

but then by Lemma 2.2 the fixed part D_0 of D has multiplicity one. Every irreducible component of D_0 is a smooth rational curve C with $C^2 = -2$. We will show that every connected component D_0' of D_0 is a tree such that at every vertex two curves meet and $(D_0')^2 = -2$. The proof will proceed by induction starting with some curve C_1 in D_0 and adding curves $C_2, \ldots, C_n$ so that the divisor $\Sigma_1^n C_i$ should be connected and contained in D_0'. The first step of the induction is trivial. Therefore we assume that $\Sigma_1^n C_i$ is a connected tree of the kind described above and that $(\Sigma_1^n C_i)^2 = -2$. We also assume that in D_0' there is a curve C_{n+1} which intersects $\Sigma_1^n C_i$; in the contrary case everything is proven. By Ramanujan's theorem, since $\Sigma_1^{n+1} C_i$ is connected and of multiplicity one, and by the Riemann-Roch theorem,

$$h^0\left(X, \mathcal{O}_X\left(\sum_{i=1}^{n+1} C_i\right)\right) = \frac{\left(\sum_{i=1}^{n+1} C_i\right)^2}{2} + 2 = \left(\sum_{i=1}^{n} C_i, C_{n+1}\right).$$

Then, because $\Sigma_1^{n+1} C_i$ is fixed,

$$\left(\sum_{i=1}^{n+1} C_i\right)^2 = -2, \quad \left(\sum_{i=1}^{n} C_i, C_{n+1}\right) = 1.$$

This completes the induction. Let us now consider the movable part D_1 of D. If D_1 is not a pencil, then its general element is irreducible and reduced. Hence, again using Ramanujan's theorem and the Riemann-Roch theorem, we obtain

$$h^0(X, \mathcal{O}_X(D_1)) = \frac{D_1^2}{2} + 2.$$

Let D_0' be a connected component of the fixed part. By the assumption of the lemma on the divisor D we have $(D, D_0') \geqslant 0$. On the other hand, $(D, D_0') = (D_1 + D_0', D_0') = (D_1, D_0') + (D_0')^2$. Then $(D_1, D_0') \geqslant 2$. The divisor $D_1 + D_0'$ is connected and of multiplicity one. Therefore, as above,

$$h^0(X, \mathcal{O}_X(D_1 + D_0')) = \frac{(D_1 + D_0')^2}{2} + 2 = \frac{D_1^2}{2} + 2 + (D_1, D_0') + \frac{(D_0')^2}{2},$$

whence $h^0(X, \mathcal{O}_X(D_1 + D_0')) > h^0(X, \mathcal{O}_X(D_1))$. Consequently in this case D has no fixed components. If D_1 is a pencil, then $|D_1| = |nE|$, where $|E|$ is an elliptic pencil on the $K3$ surface X. In this case because D is connected there must exist at least one fixed component. We will prove that it is unique and that it is a nonsingular rational curve which is a section of $|E|$. Because D is connected there exists a curve C in D_0 such that C does not lie in the fibers of $|E|$, i.e. $C \cdot E > 0$. Because $C + E$ is connected and of multiplicity one, we have

$$h^0(X, \mathcal{O}_X(C + E)) = \frac{(C + E)^2}{2} + 2 = h^0(X, \mathcal{O}_X(E)) + (C, E) + \frac{C^2}{2};$$

hence $(C, E) = 1$. Consequently C is a section. If in D_0 there are two sections C_1 and C_2, and $n \geqslant 2$, then

$$h^0(X, \mathcal{O}_X(C_1 + C_2 + 2E)) = \frac{(C_1 + C_2 + 2E)^2}{2} + 2 = h^0(X, \mathcal{O}_X(2E))$$

$$+ \frac{C_1^2}{2} + \frac{C_2^2}{2} + 2(C_1, E) + 2(C_2, E) + (C_1, C_2) - 1 \geqslant h^0(X, \mathcal{O}_X(2E)) + 1.$$

The latter contradicts the choice of C_1 and C_2 from the fixed part of D. Therefore if D_0 has two sections then $n \leqslant 1$. But $|D| = |nE + D_0|$ and $D^2 = \Sigma_{i=1}^m (2nE, D_0^{(i)}) + (D_0^{(i)})^2 > 0$, where $D_0^{(i)}$ is a connected component of D_0. Hence it follows that $n = 1$ and that there exists a connected component $D_0^{(i)}$ with $(D_0^{(i)}, E) \geqslant 2$. From this, as above in the nonpencil case, we derive the inequality

$$h^0(X, \mathcal{O}_X(E + D_0^{(i)}) > h^0(X, \mathcal{O}_X(E)),$$

which leads to a contradiction. Consequently in D_0 there exists exactly one section C, and the remaining curves D_0 lie in the fibers. We will assume that the last set of curves is nonempty. Then there exists a curve C' in D_0 extreme in some tree, i.e. $(C', D_0) = -1$ and $(C', E) = 0$. Then $(D, C') = (nE + D_0, C') = -1$, which contradicts the choice of $|D|$. Consequently C is the only fixed component of $|D|$ and $|D| = |nE + C|$. ■

REMARK. Lemma 2.3 in the case of an ample D was proved in [6].

§3. Proof of the theorem in the case $r = 1$

3.1. We denote by W the image of the rational map $V \dashrightarrow \mathbf{P}^{\dim|-K_V|}$ defined by the linear system $|-K_V|$.

3.2. LEMMA. $\dim W \geqslant 2$.

PROOF. Let the linear system $|-K_V|$ define a mapping onto a curve W in $\mathbf{P}^{q+1}$, $g = (-K_V)^3/2 + 1$. We denote by D_0 the fixed component of the system $|-K_V|$ and by D the general divisor of the movable part. The curve W is rational since $h^1(V, \mathcal{O}_V) = 0$ (see (1.3) in [4]). From linear normality it follows that W is a smooth rational curve of degree $g + 1$ which generates $\mathbf{P}^{g+1}$. Therefore $D \sim (g + 1)E$ and the (projectively) one-dimensional system $|E|$ defines a rational map π: $V \dashrightarrow W \simeq \mathbf{P}^1$. We have $((D_0 + (g + 1)E)^2, -K_V) = 2g - 2$ from the definition of g, since $-K_V \sim D_0 + (g + 1)E$. The following relation is evident:

$$((D_0 + (g+1)E)^2, -K_V) = ((g+1)^2E^2 + (g+1)(E, D_0) + (D_0, -K_V), -K_V).$$

The movability of E and the ampleness of $-K_V$ implies the inequalities

$$(E^2, -K_V) \geqslant 0, \quad (D_0, (-K_V)^2) \geqslant 0, \quad (E, D_0, -K_V) \geqslant 0.$$

If $(E^2, -K_V) > 0$, then

$$2g - 2 = ((D_0 + (g+1)E)^2, -K_V) \geqslant (g+1)^2.$$

The latter leads to a contradiction. Therefore $(E^2, -K_V) = 0$. Then by the ampleness of $-K_V$ the general members of $|E|$ do not intersect and the linear system $|E|$ defines a morphism π: $V \longrightarrow W \simeq \mathbf{P}^1$ whose fibers give $|E|$. By Lemma 2.1 every divisor in $|-K_V|$ is connected. Consequently $D_0 \neq 0$ and intersects the general member of $|E|$ along a nontrivial effective one-dimensional algebraic cycle. In addition,

$$2g - 2 = (g + 1)(E, D_0, -K_V) + (D_0, K_V^2),$$

where $(E, D_0, -K_V) > 0$ and $(D_0, K_V^2) > 0$. That means that $(E, D_0, -K_V) = 1$ and

$(D_0, K_V^2) = g - 3$. Then obviously $(E, K_V, K_V) = 1$. This last equality together with the ampleness of $-K_V$ implies that any fiber (i.e. an element of $|E|$) is irreducible and reduced. Therefore D_0 does not have components contained in the fibers of π. Since $(E, D_0, -K_V) = 1$ and $-K_V$ is ample, it follows that D_0 is an irreducible reduced divisor and the fibers of the morphism $\pi\colon D_0 \longrightarrow W$, which are irreducible and reduced curves, define a linear system $|(E, D_0)|_{D_0}$ on D_0 whose elements we will call the fibers of D_0. Also, the relation $(E, D_0, -K_V) = 1$ implies the smoothness of the general point of all the fibers of D_0. By the Bertini-Zariski theorem the general fiber E of π is a smooth irreducible surface. The given surface E is a del Pezzo surface of degree 1, and $(E, D_0) = (E, -K_V)$ gives an ample anticanonical class of degree 1 on E, $(E, D_0^2) = 1$. Therefore there exists on D_0 a pencil of irreducible reduced curves of arithmetic genus one consisting of the fibers of D_0. Consequently $h^1(D_0, \mathcal{O}_{D_0}) \leqslant 1$.

On the other hand, from the long exact cohomology sequence for the triple $0 \longrightarrow \mathcal{O}_V(-D_0) \longrightarrow \mathcal{O}_V \longrightarrow \mathcal{O}_{D_0} \longrightarrow 0$ we find that $h^1(D_0, \mathcal{O}_{D_0}) = h^2(V, \mathcal{O}_V(-D_0))$. By duality

$$h^2(V, \mathcal{O}_V(-D_0)) = h^1(V, \mathcal{O}_V(-(g+1)E)).$$

From the exact sequence corresponding to

$$0 \to \mathcal{O}_V(-(g+1)E) \to \mathcal{O}_V \to \mathcal{O}_{(g+1)E} \to 0,$$

it follows that

$$h^1(V, \mathcal{O}_V(-(g+1)E)) = h^0((g+1)E, \mathcal{O}_{(g+1)E}) - 1.$$

Hence, since the general member of the pencil $|E|$ is irreducible and reduced, we have $h^1(V, \mathcal{O}_V(-(g+1)E)) = g$. This means that $h^1(D_0, \mathcal{O}_{D_0}) = g$. Then because of the above we obtain the inequality $1 \geqslant h^1(D_0, \mathcal{O}_{D_0}) = g$. But $(-K_V)^3 = 2g - 2 > 0$ because of the ampleness of $-K_V$. This contradiction completes the proof of the lemma. ∎

Proof of Theorem 1.2 (*case* $r = 1$). By Lemma 3.2, $\dim W \geqslant 2$. Then by Bertini's theorem [1] the general element of the linear system $|-K_V|$ is of the form $D + D_0$, where D_0 is the fixed component of $|-K_V|$ and D is the movable irreducible reduced divisor normally intersecting D_0 ($\dim D \cap D_0 \leqslant 1$) and having singular points only at the base points of the linear system $|D|$. We will resolve the points of indeterminacy of $|D|$ (in the Hironaka-Zariski sense) by monodial transformations with smooth centers in the base locus. We denote a general resolution by $\sigma\colon \widetilde{V} \longrightarrow V$. By Bertini's theorem the strict transform $\widetilde{D}$ of a generic D is nonsingular and $\sigma^*(D) = \widetilde{D} + \Sigma_1^m n_i E_i$, where E_i is the surface corresponding to the ith monoidal transformation (the strict transform on $\widetilde{V}$ of the ith center of blowing up). Also $\widetilde{D}$ is the maximal movable part of the linear system $|\sigma^*(D)|$. Consequently

$$h^0(\widetilde{V}, \mathcal{O}_{\widetilde{V}}(\widetilde{D})) = h^0(\widetilde{V}, \mathcal{O}_{\widetilde{V}}(\sigma^*(D)) = h^0(V, \mathcal{O}_V(D)) = \frac{-K_V^3}{2} + 3$$

(the last part because of (1.3) (ii) of [4]). The canonical class of $\widetilde{V}$ is computed from the formula

$$K_{\widetilde{V}} \sim \sigma^*(K_V) + \sum_{i=1}^{m} \alpha_i E_i,$$

where $\alpha_i \geqslant 1$. Under blowing up a curve the canonical class changes according to the formula $K_{\widetilde{V}} \sim \sigma^*(K_V) + E$. In our case the blowing up is carried out only at the base curves and points. Hence by induction we obtain $n_i \geqslant \alpha_i$, if $\sigma(E_i)$ is a curve on V. Then

$$-K_{\widetilde{V}} \sim \widetilde{D} + \sigma^*(D_0) + \sum_{i=1}^{m} (n_i - \alpha_i) E_i.$$

By the adjunction formula

$$K_{\widetilde{D}} \sim -\left(\widetilde{D}, \sum_{i=1}^{m} (n_i - \alpha_i) E_i + \sigma^*(D_0)\right).$$

Let us consider on $\widetilde{D}$ the divisors $F = (\widetilde{D}, \sigma^*(D + D_0))$ and $L = (\widetilde{D}, \widetilde{D} + \Sigma_1^m \alpha_i E_i)$. Then $K_{\widetilde{D}} + F \sim L$. A multiple of F comes from a hyperplane section because of the ampleness of $-K_V$. The sheaf $\mathcal{O}_{\widetilde{D}}(F)$ satisfies the conditions of Mumford's vanishing theorem [5], $H^1(\widetilde{D}, \mathcal{O}_{\widetilde{D}}(-F)) = 0$, since $\sigma_* \mathcal{O}_{\widetilde{D}}(F)$ is ample on D. Consequently

$$h^1(\widetilde{D}, \mathcal{O}_{\widetilde{D}}(L)) = h^1(\widetilde{D}, \mathcal{O}_{\widetilde{D}}(K_{\widetilde{D}} - L)) = h^1(\widetilde{D}, \mathcal{O}_{\widetilde{D}}(-F)) = 0.$$

Also it is obvious that $h^2(\widetilde{D}, \mathcal{O}_{\widetilde{D}}(L)) = 0$ since $h^0(\widetilde{D}, \mathcal{O}_{\widetilde{D}}(-F)) = 0$, whence by the Riemann-Roch theorem we obtain

$$h^0(\widetilde{D}, \mathcal{O}_{\widetilde{D}}(L)) = \frac{L(L - K_{\widetilde{D}})}{2} + 1 - q + p_g.$$

Using the zero part of the cohomology sequence corresponding to the short exact sequence $0 \to \mathcal{O}_{\widetilde{V}} \to \mathcal{O}_{\widetilde{V}}(\widetilde{D}) \to \mathcal{O}_{\widetilde{D}}((\widetilde{D}, \widetilde{D})) \to 0$, we obtain the inequality

$$h^0(\widetilde{D}, \mathcal{O}_{\widetilde{D}}(L)) \geqslant h^0(\widetilde{D}, \mathcal{O}_{\widetilde{D}}((\widetilde{D}, \widetilde{D}))) \geqslant \frac{-(K_V)^3}{2} + 2.$$

The latter together with the previous computations gives

$$\frac{-(K_V)^3}{2} + 2 \leqslant \frac{L(L - K_{\widetilde{D}})}{2} + 1 - q + p_g. \tag{3.3}$$

We now prove that $p_g - q - 1 \geqslant 0$. We have $L - K_{\widetilde{D}} \sim F$, and by (3.3)

$$\frac{\sigma^*(-K_V)^3}{2} - \frac{\left(\sigma^*(-K_V), \widetilde{D}, \widetilde{D} + \sum_{i=1}^{m} \alpha_i E_i\right)}{2} \leqslant p_g - q - 1. \tag{3.4}$$

The left-hand side of (3.4) can be written in the form

$$\left(\frac{\sigma^*(-K_V)}{2}, \left(\widetilde{D}, \sum_{i=1}^{m} (n_i - \alpha_i) E_i + \sigma^*(D_0)\right) + \left(\sum_{i=1}^{m} n_i E_i + \sigma^*(D_0), \sigma^*(-K_V)\right)\right).$$

Here $\sigma^*(-K_V)$ is the lifting of the ample divisor, and $\widetilde{D}$ is movable, irreducible and reduced. Hence the left-hand side of (3.4) is obviously positive in all its terms except perhaps $(\sigma^*(-K_V)/2, \widetilde{D}, (n_i - \alpha_i)E_i)$ in the case when $\sigma(E_i)$ is a point of V, since in the opposite case $n_i \geqslant \alpha_i$. But then in this case the corresponding term is equal to 0 by the projection formula. From (3.4) we obtain the desired inequality. Let us now consider a divisor D such that for some smooth model of it the inequality $p_g - q - 1 \geqslant 0$ is satisfied, and in addition let D be chosen so general that its singularities lie only in the base locus of its complete linear system. We resolve the singularities of D by monoidal transformations centered in singular sets of D.

We will denote the new resolution by σ': $V' \longrightarrow V$. Accordingly $\sigma'^*(D) = D' + \Sigma_1^{m'} n'_i E'_i$ and $K_{V'} \sim \sigma'^*(K_V) + \Sigma_1^{m'} \alpha'_i E'_i$, where $\alpha'_i \geqslant 1$, since this time we perform monoidal transformations only in singular sets $n'_i \geqslant \alpha'_i$. Therefore by the adjunction formula $K_{D'} \leqslant 0$. Consequently $p_g \leqslant 1$, whence because $p_g - q - 1 \geqslant 0$ we have $p_g = 1, q = 0$ and $K_{D'} = 0$. This means that D' is a $K3$ surface. From the latter one easily concludes by Lemma 2.1 that $D_0 = 0$ and $K_{V'} \sim -D'$. Consequently,

$$\sigma'^*(-K_V) \sim D' + \sum_{i=1}^{m} \alpha'_i E'_i.$$

We have

$$h^0\left(V', \mathcal{O}_{V'}\left(D' + \sum_{i=1}^{m'} \beta_i E'_i\right)\right) = h^0(V, \mathcal{O}_V(-K_V)) = \frac{(-K_V)^3}{2} + 3$$

for the maximal movable part $|D' + \Sigma_1^{m'} \beta_i E'_i|$ in $|D' + \Sigma_1^{m'} \alpha'_i E'_i|$, where $\beta_i \leqslant \alpha'_i$. For $L' = (D', D' + \Sigma_1^{m'} \alpha'_i E'_i)$ we have

$$h^0(D', \mathcal{O}_{D'}(L')) \geqslant h^0\left(D', \mathcal{O}_{D'}\left(\left(D', D' + \sum_{i=1}^{m'} \beta_i E'_i\right)\right)\right) \geqslant \frac{-(K_V)^3}{2} + 2.$$

Hence, as above, using Mumford's theorem about degeneration and the Riemann-Roch theorem we obtain the inequality

$$\frac{-(K_V)^3}{2} + 2 \leqslant h^0(D', \mathcal{O}_{D'}(L')) \leqslant \frac{(L')^2}{2} + 2, \tag{3.5}$$

since in the last case $K_{D'} = 0$, $q = 0$ and $p_g = 1$. Considering the difference in the left-hand side of the corresponding inequality analogous to (3.4), we find that it is positive, which means that our inequality (3.5) becomes an equality. Hence the linear system $|L'|$ on D' has a fixed component $\Sigma_1^{m'}(\alpha'_i - \beta_i)(E'_i, D')$. Obviously the first resolution in σ' as well as all the others resolve an isolated quadratic singularity, i.e. $\alpha'_i = 2$ according to the formula $-D' \sim K_{V'}$ for the canonical class of V'. Hence, by Lemma 2.3, $\beta_1 \geqslant 1$. This means that the first resolved singularity is movable. By the requirement that singularities should be at the base points we obtain that $\beta_1 = 1, \alpha'_1 - \beta_1 = 1$, and $|D|$ and $|D' + \Sigma_1^{m'} \alpha'_i E'_i|$ have a fixed curve outside of E'_i. Hence $|L'|$ has at least two distinct fixed curves: (E'_1, D') and one lying outside E_i'. The latter is impossible by Lemma 2.3. Consequently the general element $D = D'$, and it is nonsingular. This completes the proof of the theorem for the case $r = 1$. ∎

§4. Proof of the theorem for $r \geqslant 2$

4.1. We denote by W the image of the rational map $V \dashrightarrow \mathbf{P}^{\dim |H|}$ defined by the linear system $|H|$, where H is an effective divisor in $|\mathcal{H}|$.

4.2. Lemma. $\dim W \geqslant 2$.

Proof. Let us assume the contrary; then, as in the proof of Lemma 3.2, we obtain the decomposition $|H| = |D_0 + nE|$, $n = h^0(V, \mathcal{O}_V(H)) - 1$, and the one-dimensional linear system $|E|$ without fixed components gives a rational mapping $\pi\colon V \dashrightarrow W \simeq \mathbf{P}^1$. According to (1.9) (ii) of [4],

$$n = \frac{(r+1)(r+2)}{2} H^3 + \frac{2}{r} \geqslant 2;$$

hence

$$H^3 = \frac{12n}{(r+1)(r+2)} - \frac{24}{r(r+1)(r+2)} < n$$

for $r \geqslant 2$. Using the relation $H^3 = (H, n^2E^2 + nED_0 + D_0H)$, the ampleness of H and the absence of fixed components in $|E|$, we show as in the case $r = 1$ that $(H, E^2) = 0$. Because of the connectedness of the divisors in $|H|$ (a simple consequence of 2.1) we have $(H, E, D_0) \geqslant 1$ and $(H^2, D_0) \geqslant 1$. Therefore $n > H^3 \geqslant n + 1$, a contradiction. ∎

4.3. Lemma. *For $r \geqslant 2$ the linear system $|H|$ can only have base points in the absence of a fixed component.*

Proof. By Theorems 1.2 ($r = 1$) and 1.5 of [4] the general surface D of the linear system $|-K_V|$ is a smooth $K3$ surface. Let us assume that the linear system $|H|$ has a fixed curve. Then by the ampleness of D we obtain fixed points of the restricted system $|(H, D)|_D$. After restricting to D one obtains a complete linear system. The latter follows from the exact cohomology sequence of the short exact sequence

$$0 \to \mathcal{O}_V((1-r)H) \to \mathcal{O}_V(H) \to \mathcal{O}_D((D, H)) \to 0,$$

since $h^1(V, \mathcal{O}_V((1-r)H)) = 0$ by (1.9) (i) of [4]. The restricted linear system is ample. In [6] it is shown that for every ample sheaf $\mathcal{L}$ on a $K3$ surface D the linear system $|\mathcal{L}|$ has no base points if it has no fixed components. Therefore the linear system $|(H, D)|_D$ has a fixed component. Consequently by Lemma 2.3 the linear system $|(H, D)|_D = |nE + Z|$, with Z a fixed curve. Then either $|H|$ has a fixed component or $|-K_V|$ has the fixed curve Z. We will show that the last case is impossible. Indeed, assuming the contrary we obtain for the restricted linear system $|(-K_V, D)|_D$ on D a representation of the form $|Z + n'E'|$, where E' is a fiber of the elliptic pencil $|E'|$ on D. Consequently $rZ + rnE \sim Z + n'E'$. Z is a section of both pencils. Intersecting both sides of the last equivalence with E', we obtain a contradiction for $r \geqslant 2$. ∎

4.4. LEMMA. *Let the linear system* $|H|$ (4.1) *have only fixed points and* $H^3 < 8$. *Then the general element of* $|H|$ *is smooth.*

PROOF. By Bertini's theorem [1] singular points of the general surface H can only be among the fixed base points. If there exists a singular base point, then $H^3 \geqslant 8$ since at that singular point the general surfaces from $|H|$ have intersection index $\geqslant 8$. ■

PROOF OF THEOREM 1.2 (*case* $r \geqslant 2$). By Lemma 4.2 and Bertini's theorem [1] the general element of the linear system $|H|$ has the form $D + D_0$, where D_0 is the fixed component of $|H|$ and D is a movable irreducible and reduced divisor normally intersecting D_0 and having singular points only at the base points of the linear system $|D|$, $K_V \sim -rD - rD_0$. We resolve by monoidal transformations the points of indeterminacy of $|D|$. We denote the general resolution by $\sigma\colon \widetilde{V} \longrightarrow V$. The strict transform for the general divisor D, by Bertini's theorem, will be a smooth divisor $\widetilde{D} \subset \widetilde{V}$, and $\sigma^*(D) = \widetilde{D} + \Sigma_1^m n_iE_i$, where E_i is the surface corresponding to the ith transform and $n_i \geqslant 1$. We may assume that $\widetilde{D}$ is the maximal movable part in $\sigma^*(D)$. Hence by (1.9) (ii) of [4] we have

$$h^0(\widetilde{V}, \mathcal{O}_{\widetilde{V}}(\widetilde{D})) = h^0(V, \mathcal{O}_V(H)) = \frac{(r+1)(r+2)}{12}H^3 + \frac{2}{r} + 1.$$

The canonical classes that we need have the form

$$-K_{\widetilde{V}} \sim r\widetilde{D} + \sum_{i=1}^{m} (rn_i - \alpha_i)E_i + r\sigma^*(D_0),$$

and

$$K_{\widetilde{D}} \sim -\left(\widetilde{D},\ (r-1)\widetilde{D} + \sum_{i=1}^{m} (rn_i - \alpha_i)E_i + r\sigma^*(D_0)\right),$$

where $n_i \geqslant \alpha_i$ if $\sigma(E_i)$ is not a point of V and all $\alpha_i \geqslant 1$. We consider on the surface $\widetilde{D}$ the following divisors:

$$F = (\widetilde{D},\ \sigma^*(D + D_0)) \quad \text{and} \quad L = \left(\widetilde{D},\ \widetilde{D} + \sum_{i=1}^{m} \alpha_iE_i\right).$$

Then $K_{\widetilde{D}} + rF \sim L$.

Further using the degeneration theorem as in §3 for the sheaf $\mathcal{O}_{\widetilde{D}}(F)$, we obtain the inequalities

$$\frac{(r+1)(r+2)}{12}H^3 + \frac{2}{r} \leqslant h^0(\widetilde{D}, \mathcal{O}_{\widetilde{D}}((\widetilde{D}, \widetilde{D}))) = \frac{L(L - K_{\widetilde{D}})}{2} + 1 - q. \qquad (4.5)$$

In contrast to §3, in (4.5) we have $p_g = 0$, as it is easy to check that $K_{\widetilde{D}} < 0$. The extreme terms of (4.5) give the inequality

$$\left(\sigma^*(H),\ \frac{(r+1)(r+2)}{12}\sigma^*(H)^2 - \frac{r}{2}\left(\widetilde{D},\ \widetilde{D} + \sum_{i=1}^{m} \alpha_iE_i\right)\right) \leqslant 1 - q - \frac{r}{2}. \qquad (4.6)$$

Substituting in (4.6) the expression for $\sigma^*(H) = \sigma^*(D + D_0)$ and collecting like terms, we obtain

$$\frac{(r-1)(r-2)}{12}\left(\sigma^*(H), \tilde{D}, \tilde{D} + \sum_{i=1}^{m} \alpha_i E_i\right) + \frac{(r+1)(r+2)}{12} \times \left(\sigma^*(H)^2, \sum_{i=1}^{m} n_i E_i + \sigma^*(D_0)\right) + \frac{(r+1)(r+2)}{12}\left(\sigma^*(H), \tilde{D}, \sum_{i=1}^{m} (n_i - \alpha_i) E_i + \sigma^*(D_0)\right) \leqslant 1 - q - \frac{2}{r}. \tag{4.7}$$

As in §3, one proves the positivity of the left-hand side of (4.7). Therefore $q = 0$ and $D_0 = 0$. The latter follows from the fact that $(\sigma^*(H), \tilde{D}, \sigma^*(D_0)) = (H, D, D_0) \geqslant 1$ by the connectedness of H. We now note that if $D_0 = 0$ then by Lemma 4.3 $|H|$ has only base points. Then by Lemma 4.4 we may assume that $H^3 \geqslant 8$. Let $d = H^3 > 0$ and $\Delta = 3 + d - h^0(V, \mathcal{O}_V(H))$, and let g be defined by the relation $2g - 2 = (K_V + 2H)H^2 = (2 - r)H^3$, i.e. $g = ((2 - r)d + 2/2$. Knowing $h^0(V, \mathcal{O}_V(H))$ from (1.9) in [4], we can easily check that $\Delta \leqslant g$ for $d = H^3 \geqslant 2$. Therefore, by Theorem 4.1 of [3], there are no base points in $|H|$ if $d \geqslant 2\Delta$. This inequality fails to be satisfied only for $r = 2, d = 1$ and $r = 3, d = 1$. In our case $d = H^3 \geqslant 8$. Consequently there are no base points in this case. Therefore by Bertini's theorem the general member of $|H|$ is smooth. ■

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