THE STUDY OF THE HOMOLOGY OF KUGA VARIETIES

1981 Math. USSR Izv. 16 399

(http://iopscience.iop.org/0025-5726/16/2/A08)

Изв. Акад. Наук СССР
Сер. Матем. Том 44(1980), Вып. 2

Math. USSR Izvestija
Vol. 16(1981), No. 2

THE STUDY OF THE HOMOLOGY OF KUGA VARIETIES

UDC 517.4

V. V. ŠOKUROV

ABSTRACT. The homology of Kuga varieties is studied. A nondegenerate pairing is constructed between certain homology spaces and modular forms.
Bibliography: 10 titles.

This article continuous the proof, begun in [7], of a series of results announced in [6] on periodic cusp forms on Kuga varieties. The author thanks Professor Ju. I. Manin, during the course of whose seminar this work was completed.

§0. Main results

Let $\Gamma \subset \mathrm{SL}(2, \mathbf{Z})$ be a subgroup of finite index. We denote by (Γ, w) a pair such that either the integer w is even or the following condition on Γ holds:

$$-E \notin \Gamma \qquad (*)$$

(see $(*)$ of §4 of [5] and §0 of [7]). This article continues [7] and uses its notation. In particular Δ_Γ and B_Γ are the modular curve and elliptic modular surface for Γ (see §5 of [7]). The corresponding canonical projection is $\Phi_\Gamma : B_\Gamma \to \Delta_\Gamma$. In the sequel we will sometimes omit the index Γ for simplicity.

0.1. Let $S_{w+2}(\Gamma)$ be the space of Γ-cusp forms of weight $w + 2$ (see §2.1 of [3]).

The main goal of this article is to define a canonical pairing

$$(\ ,\ ) : H_1\left(\Delta_\Gamma, \Sigma, \left(R_1, \Phi_* \mathbf{Q}\right)^w\right) \times S_{w+2}(\Gamma) \oplus \overline{S_{w+2}(\Gamma)} \to \mathbf{C},$$

where $\Sigma \subset \Delta_\Gamma$ is any finite subset.

0.2. THEOREM. *The canonical pairing* $(\ ,\ )$ *is nondegenerate on*

$$H_1(\Delta_\Gamma, (R_1\Phi_*\mathbf{Q})^w) \times S_{w+2}(\Gamma) \oplus \overline{S_{w+2}(\Gamma)}.$$

The proof of this theorem is given in §6.

0.3. The construction of the pairing $(\ ,\ )$ is based on the existence of (i) a canonical isomorphism

$$H^0(B_\Gamma^w, \Omega^{w+1} \oplus \overline{\Omega}^{w+1}) \simeq S_{w+2}(\Gamma) \oplus \overline{S_{w+2}(\Gamma)},$$

the proof and construction of which are given in [7]; (ii) a canonical homomorphism

$$GR_{1,w} : H_1\left(\Delta_\Gamma, \Sigma, \left(R_1\Phi_*\mathbf{Q}\right)^w\right) \to H_{1+w}(B_\Gamma^w, B_\Gamma^w|_\Sigma, \mathbf{Q}),$$

1980 *Mathematics Subject Classification.* Primary 10D12; Secondary 14E15, 14M99, 10D15.

where Σ is a finite set of points of Δ_Γ containing the points of singular type (see §1 of [7]), the construction of which is given in §3; and (iii) a canonical pairing

$$H_{w+1}(B_\Gamma^w, B_\Gamma^w|_\Sigma, \mathbf{Q}) \times H^0(B_\Gamma^w, \Omega^{w+1} \oplus \bar{\Omega}^{w+1}) \mapsto \mathbf{C},$$

$$(\text{homology class } \sigma, \omega) \to \int_\sigma \omega.$$

0.4. In §3 we carry out the construction of the "geometrical realization" homomorphisms $GR_{i,j}$ ($0 \leqslant i \leqslant 2$, $0 \leqslant j \leqslant 2w$). Theorem 1 of [6] corresponds to 3.2, and this result may easily be proved over $\mathbf{Z}$ by the same methods. We make the change to $\mathbf{Q}$ for consistency, since in the sequel symmetrization will frequently occur, where division by w is needed! Theorem 2 of [6] is a simple corollary of Theorem 1 of [6].

Theorem 3 of [6] corresponds to Theorem 4.2 in this article, and Theorem 4 of [6] is a slight variation of Theorem 2.5. Finally, Theorem 6 of [6] corresponds to the special case of Corollary 6.1 with $K = \mathbf{R}$.

§1. Neighborhood retracts

Let X be an analytic variety, $D \subset \mathbf{C}$ the disk with center at 0, $D^* = D - \{0\}$, and $\Phi : X \to D$ a proper morphism. In addition we assume that the fiber $\Phi^{-1}(0)$ has *normal type*. This means that for any point $x \in \Phi^{-1}(0)$ there exist a neighborhood $U \subset X$ and coordinates $X_1, \ldots, X_n$ ($n = \dim X$) in this neighborhood in which the canonical projection takes monomial form, i.e. $\Phi|_U = X_1^{m_1} \cdots X_n^{m_n}$ for some positive integers m_i ($1 \leqslant i \leqslant n$). Then by Thom's isotopy theorem $X' = \Phi^{-1}(E_1^*)$ is a topological fiber space over $E_1^* = E_1 - \{0\}$, where $E_1 = \{z \in \mathbf{C} | \, |z| < \varepsilon\} \subset D$ for suitable $0 < \varepsilon$.

1.1. Lemma. *For any sufficiently small ε there exists a deformation retract* (*see* [1], *p.* 28) *of $\Phi^{-1}(E_1)$ onto $\Phi^{-1}(0)$.*

Corollary 1.2 is obtained from this lemma. Let B^w be Kuga's variety corresponding to the elliptic surface B. Consider a pair of topological subvarieties $\Delta \supset F \supset F'$ with smooth boundary. Then to the mapping of pairs $(\Delta, F') \hookrightarrow (\Delta, F)$ there corresponds the homomorphism in homology

$$H_i(B^w, B^w|_{F'}, \mathbf{Q}) \to H_i(B^w, B^w|_F, \mathbf{Q}). \tag{1.1}$$

In particular, those F consisting of small closed disks around points of the set Σ give rise to a projective system of vector spaces $H_i(B^w, B^w|_F, \mathbf{Q})$ with morphisms (1.1).

1.2. Corollary. *There is a canonical isomorphism*

$$H_i(B^w, B^w|_\Sigma, \mathbf{Q}) \xrightarrow{\sim} \varprojlim H_i(B^w, B^w|_F, \mathbf{Q}).$$

Proof. If $N \subset M$ and N is a deformation retract of M, then $H_i(M, N, \mathbf{Q}) = 0$. Therefore, by Lemma 1.1 and 3.4 of [7], $H_i(B^w|_F, B^w|_\Sigma, \mathbf{Q}) = 0$ for F consisting of suifficiently small disks. Then from the exact sequence

$$\to H_i(B^w|_F, B^w|_\Sigma, \mathbf{Q}) \to H_i(B^w, B^w|_\Sigma, \mathbf{Q}) \to H_i(B^w, B^w|_F, \mathbf{Q}) \xrightarrow{\partial}$$

of the triple $(B^w, B^w|_F, B^w|_\Sigma)$ it follows that (1.1) is an isomorphism for sufficiently small F and $F' = \Sigma$. ■

Proof of Lemma 1.1. The only condition on $\varepsilon + 0$ is the condition preceding Lemma 1.1, i.e. the local triviality of $X' = \Phi^{-1}(E_1^*)$ over E_1^*. Indeed, one can easily show, because of the normality of the fiber $\Phi^{-1}(0)$, that it is a neighborhood deformation retract, i.e. there exists a neighborhood $X' \supset U \supset \Phi^{-1}(0)$ which admits a deformation retract onto $\Phi^{-1}(0)$. On the other hand, clearly there exists $0 < \varepsilon' < \varepsilon$ such that $V = \Phi^{-1}(\{z \in \mathbf{C} \mid |z| \leq \varepsilon'\}) \subset U$. Also it is easy to construct a deformation retract of X' onto V. Combining the latter deformation with the restriction to V of the first deformation, we obtain the desired one. ■

§2. Homology with coefficients in the sheaf $R_j\Phi^W_*\mathbf{Q}$

2.1. The sheaf $R_j\Phi^w_*\mathbf{Q}$ is obtained by extending from $\Delta' = \Delta - \Sigma$ (see §1 of [7]) over Δ the sheaf of local coefficients $\bigcup_{v\in\Delta'} H_j(B^w_v, \mathbf{Q})$ in the following way: for a small disk E around $v \in \Sigma$ and $E' = E - v$

$$\Gamma(E, R_j\Phi^w_*\mathbf{Q}) = \Gamma(E', R_j\Phi^w_*\mathbf{Q}).$$

For example, $R_1\Phi^1_*\mathbf{Q} = R_1\Phi_*\mathbf{Q} = G \otimes \mathbf{Q}$, where G is the homological invariant of the elliptic surface B.

2.2. Fix a basis in the lattice $G|_{u_0} \subset R_1\Phi_*\mathbf{Q}|_{u_0}$. Then a representation of the group SL(2, $\mathbf{Q}$) in $R_1\Phi_*\mathbf{Q}|_{u_0}$ is determined. For any integer $w > 0$ the representation of SL(2, $\mathbf{Q}$) in the tensor power $(R_1\Phi_*\mathbf{Q})^{\otimes w}|_{u_0}$ decomposes into a direct sum of irreducible representations of SL(2, $\mathbf{Q}$). Each irreducible representation of SL(2, $\mathbf{Q}$) is a representation in a symmetric power $(R_1\Phi_*\mathbf{Q})^m|_{u_0}$; the positive integer m usually is called the *order* of the irreducible representation. The identification of the subspace which is the sum of all irreducible representations of order m in $(R_1\Phi_*\mathbf{Q})^{\otimes w}|_{u_0}$ does not depend on the choice of basis in the lattice $G|_{u_0}$. The dimension r^w_m of this subspace also is independent of the choice of the point $u_0 \in \Delta'$. The group A_w of permutations of w elements acts naturally on the space $(R_1\Phi_*\mathbf{Q})^{\otimes w}|_{u_0}$:

$$a : x_1 \otimes \ldots \oplus x_w \mapsto x_{a(1)} \otimes \ldots \otimes x_{a(w)},$$

where $x_i \in R_1\Phi_*\mathbf{Q}|_{u_0}$ and $a \in A_w$. The space $(R_1\Phi_*\mathbf{Q})^w|_{u_0}$ admits a canonical embedding into $(R_1\Phi_*\mathbf{Q})^{\otimes w}|_{u_0}$:

$$x_1 \cdots x_w \mapsto \frac{1}{w!} \sum_{a\in A_w} a\,(x_1 \otimes \ldots \otimes x_w).$$

In the sequel $(R_1\Phi_*\mathbf{Q})^w|_{u_0}$ will be identified with its canonical image in $(R_1\Phi_*\mathbf{Q})^{\otimes w}|_{u_0}$. $(R_1\Phi_*\mathbf{Q})^w|_{u_0}$ is an invariant subspace of the representation of SL(2, $\mathbf{Q}$).

2.3. PROPOSITION. a. $r^w_m = 0$ *if* $m \not\equiv w \pmod 2$.

b. $r^w_w = 1$.

c. *There is the following direct sum decomposition into subspaces invariant for* SL(2, $\mathbf{Q}$):

$$(R_1\Phi_*\mathbf{Q})^{\otimes w}|_{u_0} = (R_1\Phi_*\mathbf{Q})^w|_{u_0} \oplus \left(\sum_{a\in A_w} a\,((e_1 \otimes e_2 - e_2 \otimes e_1) \otimes (R_1\Phi_*\mathbf{Q})^{\otimes w-2}|_{u_0})\right)$$

(Σ *is not a direct sum*).

d. *The space of invariant vectors of* $(R_1\Phi_*\mathbf{Q})^{\otimes w}|_{u_0}$, *i.e. the sum of irreducible subspaces of order* 0, *is generated by the vectors*

$$a\,((e_1 \otimes e_2 - e_2 \otimes e_1)^{\otimes \frac{w}{2}}),$$

where $a \in A_w$ (*by* a, *w is even in this case*); e_1, e_2 *are a basis of the lattice* $G|_{u_0}$.

2.4. By the Künneth formula, since $B_{u_0}^w = B_{u_0} \times \cdots \times B_{u_0}$ (w terms), we have

$$R_j\Phi_*^w\mathbf{Q}\,|_{u_0} = \bigoplus_{j_1+\cdots+j_w=j} \bigotimes_{m=1}^{w} R_{j_m}\Phi_*\mathbf{Q}\,|_{u_0}, \tag{2.1}$$

where $0 \leqslant j_m \leqslant 2$. The representation S ([7], 1.4) and the trivial representation $\pi_1(\Delta')$ in $R_0\Phi_*\mathbf{Q}|_{u_0}$ and $R_2\Phi_*\mathbf{Q}|_{u_0}$ give a representation of the fundamental group $\pi_1(\Delta)$ in $R_j\Phi_*^w\mathbf{Q}$. This representation, which will also be denoted by S, is uniquely defined by the sheaf $R_j\Phi_*^w\mathbf{Q}$. Since $\dim R_0\Phi_*\mathbf{Q}|_{u_0} = \dim R_2\Phi_*\mathbf{Q}|_{u_0} = 1$, there is a noncanonical isomorphism

$$\bigotimes_{m=1}^{w} R_{j_m}\Phi_*\mathbf{Q}\,|_{u_0} \simeq (R_1\Phi_*\mathbf{Q})^{\otimes w'}\,|_{u_0}, \tag{2.2}$$

where w' is the number of $j_m = 1$, $w' \equiv j_1 + \cdots + j_w \equiv j \pmod 2$. We have that $S(\pi_1(\Delta')) \subset \mathrm{SL}(2, \mathbf{Q})$, so we may consider the representations of $\pi_1(\Delta')$ on the subspace $(R_1\Phi_*\mathbf{Q})^{\otimes w'}|_{u_0}$ invariant with respect to $\mathrm{SL}(2, \mathbf{Q})$. Below (see Lemma 2.7) we will prove their irreducibility with respect to $\pi_1(\Delta')$. The decomposition of the space $(R_1\Phi_*\mathbf{Q})^{\otimes w'}|_{u_0}$ into irreducible subspaces corresponds to a decomposition of the sheaf $(R_1\Phi_*\mathbf{Q})^{\otimes w'}$ into a direct sum of symmetric sheaves $(R_1\Phi_*\mathbf{Q})^m$, which we will also denote by S_m. We obtain from (2.1), (2.2), and 2.2 a canonical decomposition into a direct sum

$$R_j\Phi_*^w\mathbf{Q} = \bigoplus_m S_m^{r_{j,m}^w}, \text{(}^1\text{)} \tag{2.3}$$

where $r_{j,m}^w$ is the number of irreducible representations of order m in $R_j\Phi_*^w\mathbf{Q}|_{u_0}$, this number not depending on the choice of $u_0 \in \Delta'$. The decomposition of $S_m^{r_{j,m}^w}$ into a sum of sheaves S_m is not canonical.

2.5. THEOREM. a. $H_i(\Delta, R_j\Phi_*^w\mathbf{Q}) = \bigoplus_m H_j(\Delta, S_m)^{r_{j,m}^w}$.

b. $\dim H_0(\Delta, S_m) = \dim H_2(\Delta, S_m) = \begin{cases} 0 \text{ for } m \geqslant 1, \\ 1 \text{ for } m = 0. \end{cases}$

c. *For even* $m > 0$

$$\begin{aligned}\dim H_1(\Delta, S_m) = {}& 2(g-1)(m+1) + \sum_{b \geqslant 1} m(\nu(\mathrm{I}_b) + \nu(\mathrm{I}_b^*)) \\ &+ 2\left[\frac{m+2}{3}\right](\nu(\mathrm{II}) + \nu(\mathrm{II}^*) + \nu(\mathrm{IV}) + \nu(\mathrm{IV}^*)) \\ &+ 2\left[\frac{m+2}{4}\right](\nu(\mathrm{III}) + \nu(\mathrm{III}^*)).\end{aligned}$$

For odd $m > 0$

$$\begin{aligned}\dim H_1(\Delta, S_m) = {}& 2(g-1)(m+1) + \sum_{b \geqslant 1} m\nu(\mathrm{I}_b) \\ &+ (m+1)\sum_{b \geqslant 0}(\nu(\mathrm{I}_b^*) + \nu(\mathrm{II}^*) + \nu(\mathrm{II}) + \nu(\mathrm{III}) + \nu(\mathrm{III}^*)) \\ &+ 2\left[\frac{m+2}{3}\right](\nu(\mathrm{IV}) + \nu(\mathrm{IV}^*)).\end{aligned}$$

For $m = 0$

$$\dim H_1(\Delta, S_m) = 2g.$$

(¹) In this article V^m denotes a direct power, and $(V)^m$ the tensor symmetric power over $\mathbf{Q}$.

Here $\nu(*)$ is the number of fibers of type $*$ of the elliptic surface B, and $[\ ]$ as usual denotes the integer part.

d. $r^{w}_{j,m} = 0$ *for* $j \not\equiv m \pmod 2$.

2.6. Corollary. $H_0(\Delta, R_j\Phi^w_*\mathbf{Q}) = H_2(\Delta, R_j\Phi^w_*\mathbf{Q}) = 0$ *for odd* j. ■

2.7. Lemma. *The representation S of the fundamental group $\pi_1(\Delta')$ in $(R_1\Phi_*\mathbf{Q})^w|_{u_0}$ is irreducible also with respect to this representation*:

a. $((R_1\Phi_*\mathbf{Q})^w|_{u_0})^{\mathrm{inv}} = 0$.

b. $((R_1\Phi_*\mathbf{Q})^w|_{u_0})^{\mathrm{coinv}} = 0$ *for* $w > 1$.

c. *The following table shows the dimension of the space of sections of the sheaf S_m over the point v depending on the type of point.*

type of point v	I_0	$\overset{*}{I_0}$	I_b, $b \geq 1$	$\overset{*}{I_b}$, $b \geq 1$	II, II*	III, III*	IV, IV*
$m > 0$ *even*	$m+1$	$m+1$	1	1	$m+1-2\left[\frac{m+2}{3}\right]$	$m+1-2\left[\frac{m+2}{4}\right]$	$m+1-2\left[\frac{m+2}{3}\right]$
$m > 0$ *odd*	$m+1$	0	1	0	0	0	$m+1-2\left[\frac{m+2}{3}\right]$

2.8. Following Shioda [5], we construct a complex M which allows us to compute the dimension of the homology spaces $H_i(\Delta, S_m)$ (we remark that these spaces are isomorphic to the cohomology spaces $H^{2-i}(\Delta, S_m)$; see for example §7 of [5]). Fix a point $u_0 \in \Delta'$. Let $\Sigma = \{v_1, \ldots, v_t\}$. As in the proof of Lemma 1.5 of [7], we choose the following system of generators α_k, β_k ($1 \leq k \leq g$, where g is the genus of the curve Δ) and γ_l ($1 \leq l \leq t$) of the fundamental group $\pi_1(\Delta') = \pi_1(u_0, \Delta')$ with the single relation

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdots \alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}\gamma_1 \cdots \gamma_t = 1. \tag{2.4}$$

We consider a small positively oriented disk E_l around each point $v_l \in \Sigma$. Set $\gamma_l' = -\partial E_l$. In each oriented circle γ_l' we fix a point u_l, and then we choose a path δ_l from u_0 to u_l such that $\delta_l\gamma_l'\delta_l^{-1}$ is homotopic to γ_l. We consider the following complex Δ: the 0-cells are u_l ($0 \leq l \leq t$), the 1-cells are α_k, β_k ($1 \leq k \leq g$), δ_l and γ_l' ($1 \leq l \leq t$), and the 2-cells are E_l ($1 \leq l \leq t$) and $\Delta_0 = \Delta - \bigcup E_l$.

The i-chains σ_i with coefficients in the sheaf $(R_1\Phi_*\mathbf{Q})^m = S_m$ have the following form:

$$\sigma_0 = \sum_{l=0}^{t} m_l u_l,$$

$$\sigma_1 = \sum_{k=1}^{g} (a_k\alpha_k + b_k\beta_k) + \sum_{l=1}^{t} (c_l\gamma_l' + d_l\delta_l), \tag{2.5}$$

$$\sigma_2 = e\Delta_0 + \sum_{l=1}^{t} e_l E_l,$$

where the coefficients $m_l, a_k, \ldots, e \in (R_1\Phi_*\mathbf{Q})^m|_{u_0}$, and $e_l \in ((R_1\Phi_*\mathbf{Q})^m|_{u_0})^{S_{\gamma_l}}$, i.e. $e_l = e_l S_{\gamma_l}$. Let $\mathfrak{A}_k = S_{\alpha_k}$, $\mathfrak{B}_k = S_{\beta_k}$, $\mathfrak{C}_k = S_{\gamma_k}$, and $\mathfrak{L}_k = \mathfrak{A}_k\mathfrak{B}_k\mathfrak{A}_k^{-1}\mathfrak{B}_k^{-1}$, $\mathfrak{L}^{(k)} = \mathfrak{L}_1 \cdots \mathfrak{L}_k$ and

$\mathcal{C}^{(l)} = \mathcal{C}_1 \cdots \mathcal{C}_l$ ($\mathfrak{L}^{(0)} = \mathcal{C}^{(0)} = 1$, $\mathfrak{L}^{(g)} = \mathfrak{L}$). The boundary operator is then rewritten in the following form:

$$\begin{gathered}\partial(a_k\alpha_k) = a_k(\mathcal{A}_k - 1)u_0, \qquad \partial(b_k\beta_k) = b_k(\mathcal{B}_k - 1)u_0,\\ \partial(c_l\gamma_l') = c_l(\mathcal{C}_l - 1)u_l, \qquad \partial(d_l\delta_l) = d_lu_l - d_lu_0,\\ \partial(e\Delta_0) = \sum_{k=1}^{g} e\mathfrak{L}^{(k-1)}((1 - \mathcal{A}_k\mathcal{B}_k\mathcal{A}_k^{-1})\alpha_k + (\mathcal{A}_k - \mathfrak{L}_k)\beta_k)\\ + \sum_{l=1}^{t} e\mathfrak{L}((\mathcal{C}^{(l-1)} - \mathcal{C}^{(l)})\delta_l + \mathcal{C}^{(l-1)}\gamma_l'), \ \partial(e_lE_l) = -e_l\gamma_l'.\end{gathered} \tag{2.6}$$

Therefore a complex M of vector spaces over $\mathbf{Q}$

$$M_0 \xrightarrow{\partial_1} M_1 \xrightarrow{\partial_2} M_2, \tag{2.7}$$

is determined, where

$$M_0 = S_m|_{u_0} \oplus \bigoplus_{l=1}^{t} S_m|_{v_l}, \qquad M_1 = S_m^{2g+t}|_{u_0}, \qquad M_2 = S_m|_{u_0},$$

and $\partial_1(e, e_1, \ldots, e_t) = (a_k, b_k, c_l)$ for

$$\begin{aligned} a_k &= e\mathfrak{L}^{(k-1)}(1 - \mathcal{A}_k\mathcal{B}_k\mathcal{A}_k^{-1}),\\ b_k &= e\mathfrak{L}^{(k-1)}(\mathcal{A}_k - \mathfrak{L}_k),\\ c_l &= e\mathfrak{L}\,\mathcal{C}^{(l-1)} - e_l,\end{aligned}$$

and ∂_2 is given by

$$\partial_2(a_k, b_k, c_l) = \sum_{k=1}^{g} (a_k(\mathcal{A}_k - 1) + b_k(\mathcal{B}_k - 1)) + \sum_{l=1}^{t} c_l(\mathcal{C}_l - 1).$$

From (2.5)–(2.7) it is easy to obtain an isomorphism of the homology spaces $H_i(\Delta, S_m)$ with the cohomology spaces $H^{2-i}(M)$ of the complex (2.7).

PROOF OF THEOREM 2.5. Part a is an obvious corollary of (2.3).

b. The case $m = 0$ is obtained from the fact that $S_0 = \mathbf{Q}$, the constant sheaf of vector spaces of dimension 1, i.e. there is a canonical isomorphism $H_i(\Delta, S_m) \simeq H_i(\Delta, \mathbf{Q})$. The case $m = 0$ of part c follows obviously from this isomorphism.

By 2.8 there are isomorphisms

$$\begin{aligned} H_0(\Delta, S_m) &\simeq H^2(M) = \operatorname{Coker}\partial_2 = ((R_1\Phi_*\mathbf{Q})^m|_{u_0})^{\text{coinv}},\\ H_2(\Delta, S_m) &\simeq H^0(M) = \operatorname{Ker}\partial_1 = ((R_1\Phi_*\mathbf{Q})^m|_{u_0})^{\text{inv}}.\end{aligned}$$

Then by Lemma 2.7 a and b we obtain the proof of part b for $m > 1$.

c. Let $m > 1$. By the previous part there is an exact sequence

$$0 \to M_0 \xrightarrow{\partial_1} M_1 \xrightarrow{\partial_2} M_2 \to 0.$$

Then the direct calculation

$$\dim H_1(\Delta, S_m) = \dim H^1(M) = \dim M_1 - \dim M_0 - \dim M_2$$

(see 2.8), using the dimension of the space $S_m|v_l$ given in the table of Lemma 2.7, proves part c.

Part d follows from part a of the theorem, (2.2), and part a of Proposition 2.3. ■

PROOF OF PROPOSITION 2.3. Part a is proved by induction on w using Theorem 2 in §18.2 of [8]. Similarly we obtain b.

c. The action of SL(2, **Q**) commutes with the action of A_w. Moreover,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(e_1 \otimes e_2 - e_2 \otimes e_1) = (ad - bc)(e_1 \otimes e_2 - e_2 \otimes e_1) = e_1 \otimes e_2 - e_2 \otimes e_1,$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbf{Q})$. Therefore the spaces in the decomposition are invariant for the action of SL(2, **Q**). The exactness of the sequence

$$0 \to \sum_{a \in A_w} a\left((e_1 \otimes e_2 - e_2 \otimes e_1) \otimes (R_1\Phi_*\mathbf{Q})^{\otimes w-2}|_{u_0}\right)$$
$$\to (R_1\Phi_*\mathbf{Q})^{\otimes w}|_{u_0} \to (R_1\Phi_*\mathbf{Q})^{w}|_{u_0} \to 0$$

is obvious, which proves part c.

Part d is proved by induction for even w; the case of odd w is trivial by a. The case $w = 0$ follows because $(R_1\Phi_*\mathbf{Q})^{\otimes 0} = \mathbf{Q}$ and $a((e_1 \otimes e_2 - e_2 \otimes e_1)^0) = 1$. Further inductive steps are obtained from part c and Lemma 2.7a. ∎

PROOF OF LEMMA 2.7. c. Consider a point $u_0 \in \Delta'$ sufficiently close to v, and a small positive circuit $\beta \subset \Delta'$ around v beginning and ending at u_0. In the lattice $G|_{u_0} = H_1(B_{u_0}, \mathbf{Z})$ choose a basis e_1, e_2 in which the monodromy s_β ([7], (1.3)) has the normal form (see §1 of [7]) $\mathcal{A}_v$. Then

$$S_m|_v = (S_m|_{u_0})^{s_\beta} \simeq ((\mathbf{Q}e_1 \oplus \mathbf{Q}e_2)^m)^{\mathcal{A}_v}. \tag{2.8}$$

From Table 1 of [7] we obtain the following form of the monodromy in the basis $\varepsilon_\alpha = e_1^\alpha e_2^{m-\alpha}$, $0 \leqslant \alpha \leqslant m$, of the space $(\mathbf{Q}e_1 \oplus \mathbf{Q}e_2)^m$ for points v of type I_b or I_b^* $(b \geqslant 0)$:

$$\varepsilon_\alpha \mapsto (\pm 1)^m (e_1 + be_2)^\alpha e_2^{m-\alpha} = (\pm 1)^m \left(\varepsilon_\alpha + \alpha b \varepsilon_{\alpha-1} + \sum_{i \leqslant \alpha-2} * \cdot \varepsilon_i\right).$$

Therefore the monodromy matrix is $(\pm 1)^m$ for $b = 0$ and

$$(\pm 1)^m \begin{pmatrix} 1 & & & & 0 \\ b & \cdot & & & \\ & 2b & \cdot & & \\ * & & \cdot & & \\ & & & mb & 1 \end{pmatrix}, \tag{2.9}$$

for $b \geqslant 1$, the action being on the right, with the + sign corresponding to I_b and the − sign corresponding to I_b^*. Then by (2.8) we obtain the first four columns of our table.

To compute our table at points with finite monodromy we use the relation

$$\dim_{\mathbf{Q}}((\mathbf{Q}e_1 \oplus \mathbf{Q}e_2)^m)^{\mathcal{A}_v} = \dim_{\mathbf{C}}((\mathbf{C}e_1 \oplus \mathbf{C}e_2)^m)^{\mathcal{A}_v}.$$

For a given point in $\mathbf{C}e_1 \oplus \mathbf{C}e_2$ there exists a new basis in which $\mathcal{A}_v$ is diagonal. Depending on the type II, II*; III, III*; IV, IV* of the point in Table 1 of [7], we obtain a corresponding diagonal matrix:

$$\begin{pmatrix} e^{\frac{2\pi i}{6}} & 0 \\ 0 & e^{\frac{-2\pi i}{6}} \end{pmatrix}; \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; \quad \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix}, \quad \eta = e^{\frac{2\pi i}{3}}.$$

Therefore in some basis for the space $\mathbf{C}e_1 \oplus \mathbf{C}e_2$ the monodromy $\mathcal{C}_v$ has the matrix

$$\begin{pmatrix} e_\varkappa^0\ e_\varkappa^{-m} & & 0 \\ & \ddots & \\ & e_\varkappa^{\alpha}\ e_\varkappa^{-(m-\alpha)} & \\ & & \ddots \\ 0 & & e_\varkappa^{m}\cdot e_\varkappa^{0} \end{pmatrix},$$

where κ corresponds to the type of the point v in Table 2 of [7]. Consequently we obtain by (2.8) that

$$\dim_{\mathbf{Q}} S_m|_v = \#\{0\leqslant\alpha\leqslant m \mid e_\varkappa^{2\alpha}=e_\varkappa^m\},$$

from which the last three columns of our table follow by an easy computation.

a. The irreducibility of the representation S is obvious for $w = 0$. Suppose $w \geqslant 1$. Then to prove part a it suffices to establish the irreducibility of the representation S of the fundamental group $\pi_1(\Delta')$ in $(R_1\Phi_*\mathbf{Q})^w|_{u_0}$. Recall that the matrix of the representation S acts on the right. Since the functional invariant $J \not\equiv$ const, there exists a point $v \in \Sigma$ of type I_b or I_b^* $(b \geqslant 1)$ (see the values of $J(v)$ in Table 2 of [7]). Choose a point $u_0 \in \Delta'$ and a basis e_1, e_2 of the lattice $G|_{u_0}$, as was done in part c. Then in the basis $\varepsilon_0, \dots, \varepsilon_w$ (see c) there is a matrix of the representation S of form (2.9). The invariant subspaces for the group generated by the matrix (2.9) have the form $\bigoplus_0^m \mathbf{Q}\varepsilon_\alpha$, $0 \leqslant m \leqslant w$. Suppose the representation S is reducible. In this case the subspace $\bigoplus_0^m \mathbf{Q}\varepsilon_\alpha$ is invariant for $\pi_1(\Delta')$ for some $0 \leqslant m < w$. Consider the matrix $S_\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbf{Z})$ for any arbitrary $\gamma \in \pi_1(\Delta')$. By the invariance we have

$$\varepsilon_0\begin{pmatrix} a & b \\ c & d\end{pmatrix} = (ce_1+de_2)^w = c^w\varepsilon_w + \sum_{0\leqslant\alpha<w} *\cdot\varepsilon_\alpha \in \bigoplus_{\alpha=0}^{m}\mathbf{Q}\varepsilon_\alpha,$$

i.e. $c = 0$. It follows that all points of Δ have either type I_b or type I_b^*, and

$$S_{\gamma_l} = \pm\begin{pmatrix} 1 & b_l \\ 0 & 1\end{pmatrix}^{-1} = \pm\begin{pmatrix} 1 & -b_l \\ 0 & 1\end{pmatrix}, \qquad b_l \geqslant 0$$

(see 2.8). The relation (2.4) then leads to a contradiction, since $\sum_1^t b_l > 0$ for $J \not\equiv$ const.

b. We use the notation and concepts of the preceding part. Since the coinvariant space for the group generated by the matrix (2.9) is

$$\bigoplus_{\alpha=0}^{w}\mathbf{Q}\varepsilon_\alpha / \bigoplus_{\alpha=0}^{w-1}\mathbf{Q}\varepsilon_\alpha \tag{2.10}$$

or 0, if b were false then (2.10) would be the coinvariant space of the representation S. Suppose that this were so. Then for the matrix

$$S_\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z})$$

for an arbitrary $\gamma \in \pi_1(\Delta')$ we would have

$$\varepsilon_w\left(\operatorname{mod}\bigoplus_{\alpha=0}^{w-1}\mathbf{Q}\varepsilon_\alpha\right)\begin{pmatrix} a & b \\ c & d\end{pmatrix} = a^w\varepsilon_w\left(\operatorname{mod}\bigoplus_{\alpha=0}^{w-1}\mathbf{Q}\varepsilon_\alpha\right),$$

i.e. $a = \pm 1$. Iterating the matrix $\mathcal{Q}_v$ if necessary, we may assume that the representation S determines some matrix $\pm \left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right)$ with $b > 2$. Let

$$S_\gamma = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

be any other matrix of the representation S. Then, since the matrix

$$\pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \pm \begin{pmatrix} a_1 + bc_1 & * \\ * & * \end{pmatrix}$$

is also determined by the representation, we have $a_1 + bc_1 = \pm 1$. Consequently $c_1 = 0$ and $d_1 = a_1 = \pm 1$. As in the proof of part a, this leads to a contradiction. ■

§3. Geometric realizations

3.1. Let $\mathcal{F}$ be a locally constant sheaf of vector spaces over Δ'. As in 2.1, this sheaf extends to a sheaf $\mathcal{F}$ over Δ. In this section Π denotes an arbitrary subset of Σ. Let F and F' be topological subvarieties of Δ with smooth boundary such that $\Delta \supset F \supset F'$ and $(\dot{F} \cup \dot{F}') \cap \Sigma = \varnothing$. Then the mapping of pairs $(\Delta, F') \hookrightarrow (\Delta, F)$ induces a homomorphism

$$H_i(\Delta, F', \mathcal{F}) \to H_i(\Delta, F, \mathcal{F}) \tag{3.1}$$

in homology. In particular consider $F = \bigcup_{p \in \Pi} E_p$ consisting of small closed disks E_p around the points $p \in \Pi \subset \Delta$. Then a projective system of spaces $H_i(\Delta, F, \mathcal{F})$ with morphisms (3.1) is determined. We set

$$H_i(\Delta, \Pi, \mathcal{F}) = \varprojlim H_i(\Delta, F, \mathcal{F}).$$

For sufficiently small E_p this projective limit stabilizes and we have the isomorphism

$$H_i(\Delta, \Pi, \mathcal{F}) \simeq H_i(\Delta, F, \mathcal{F}). \tag{3.2}$$

Let $\Pi \supset \Pi'$. Then the exact sequence of the triple $\Delta \supset F \supset F'$ induces in the limit the following exact sequence:

$$\begin{aligned} &0 \to H_1(\Delta, \Pi', \mathcal{F}) \to H_1(\Delta, \Pi, \mathcal{F}) \\ &\xrightarrow{\partial} H_0(\Pi, \Pi', \mathcal{F}|_{\Pi}) \to H_0(\Delta, \Pi', \mathcal{F}) \to H_0(\Delta, \Pi, \mathcal{F}) \to 0, \end{aligned} \tag{3.3}$$

since $H_1(F, F', \mathcal{F}|_F) = 0$ (in the future the restriction $\mathcal{F}|_F$ of the coefficients for the homology of a subvariety will not be indicated). We identify $H_1(\Delta, \Pi, \mathcal{F})$ with its image in $H_1(\Delta, \Sigma, \mathcal{F})$ under the embedding of 1-dimensional homology from the exact sequence (3.3) for the pair $\Pi \subset \Sigma$. Then by the functoriality of homology we have the inclusion $H_1(\Delta, \Pi', \mathcal{F}) \subset H_1(\Delta, \Pi, \mathcal{F})$ for $\Pi' \subset \Pi$. In the following considerations the role of the sheaf $\mathcal{F}$ will be played by a subsheaf of $R_j\Phi^w_*\mathbf{Q}$. In contrast to §1 of [7], we will require (unless the contrary is stated) only one Σ, namely the finite set consisting of all singular points of Δ.

We denote by $\overline{H}_j(B^w, \mathbf{Q})$ the image of the homology space $H_j(B^w, \mathbf{Q})$ in $H_j(B^w, B^w|_\Sigma, \mathbf{Q})$ under the natural homomorphism of the pair $(B^w, B^w|_\Sigma)$. The aim of this section is to define natural homomorphisms

$$GR_{0,j}: H_0(\Sigma, R_j\Phi_*^w \mathbf{Q}) \to H_j(B^w|_\Sigma, \mathbf{Q}),$$
$$GR_{1,j}: H_1(\Delta, \Sigma, R_j\Phi_*^w\mathbf{Q}) \to H_{1+j}(B^w, B^w|_\Sigma, \mathbf{Q}),$$
$$GR_{2,j}: H_2(\Delta, R_j\Phi_*^w\mathbf{Q}) \to H_{2+j}(B^w, B^w|_\Sigma, \mathbf{Q})$$

and to apply them to describe the spaces $\overline{H}_j(B^w, \mathbf{Q})$. These homomorphisms will be called *geometric realizations*. Their definition is given in 3.5, 3.4 and 3.10, and a discussion of the "geometry" in 3.6, 3.7 and 3.10.

From the decomposition into a direct sum of subsheaves $R_j\Phi_*^w\mathbf{Q} = \mathcal{F} \oplus \mathcal{F}'$ we have a decomposition of homology spaces $H_i(\ , R_j\Phi_*^w\mathbf{Q}) = H_i(\ , \mathcal{F}) \oplus H_i(\ , \mathcal{F}')$. In such a situation we will in what follows identify $H_i(\ , \mathcal{F})$ with the corresponding subspace of $H_i(\ , R_j\Phi_*^w\mathbf{Q})$.

3.2. THEOREM. a. *$GR_{0,j}$, $GR_{1,j}$ and $GR_{2,j}$ are monomorphisms.*

b. *The following diagram is commutative*:

$$\begin{array}{ccc} H_1(\Delta, \Sigma, R_j\Phi_*^w\mathbf{Q}) & \xrightarrow{\partial} & H_0(\Sigma, R_j\Phi_*^w\mathbf{Q}) \\ \downarrow GR_{1,j} & & \downarrow GR_{0,j} \\ H_{1+j}(B^w, B^w|_\Sigma, \mathbf{Q}) & \xrightarrow{\partial} & H_j(B^w|_\Sigma, \mathbf{Q}) \end{array}$$

c.

$$GR_{1,j-1}(H_1(\Delta, R_{j-1}\Phi_*^w\mathbf{Q})) \subset \overline{H}_j(B^w, \mathbf{Q}), \qquad GR_{2,j-2}(H_2(\Delta, R_{j-2}\Phi_*^w\mathbf{Q})) \subset \overline{H}_j(B^w, \mathbf{Q})$$

and

$$\overline{H}_j(B^w, \mathbf{Q}) = GR_{1,j-1}(H_1(\Delta, R_{j-1}\Phi_*^w\mathbf{Q})) \oplus GR_{2,j-2}(H_2(\Delta, R_{j-2}\Phi_*^w\mathbf{Q})).$$

d. $\overline{H}_{w+1}(B^w, \mathbf{Q}) = GR_{1,w}(H_1(\Delta, (R_1\Phi_*\mathbf{Q})^w)) \oplus H'$, *where each homology class of the subspace H' decomposes into a sum of classes having some representation as a cyclic proper subvariety of B^w.*

Part c of the theorem and Corollary 2.6 imply

3.3. COROLLARY. *For odd j there is an isomorphism*

$$\overline{H}_j(B^w, \mathbf{Q}) \simeq H_1(\Delta, R_{j-1}\Phi_*^w\mathbf{Q}). \quad \blacksquare$$

3.4. Let $F = \bigcup_1^t E_l$ and $\Delta_0 = \Delta - \operatorname{Int} F$, where the E_l are sufficiently small disks around the points $v_l \in \Sigma$. Δ_0 and $B^w(2) = B^w|_{\Delta_0}$ are compact real varieties with smooth boundary. $B^w(2)$ is a fiber space over Δ_0 with fibers homeomorphic to the product of $2w$ circles. Consider a cell decomposition of the pair $(\Delta_0, \partial\Delta_0)$. To each decomposition corresponds a filtration of cell complexes over the base Δ_0:

$$(\Delta_0, \partial\Delta_0), \qquad (\Delta_0(1), \partial\Delta_0), \qquad (\Delta_0(0), \partial\Delta_0(0)),$$

and this means also a filtration of complexes of the bundle $B^w(2)$:

$$(B^w(2), \partial B^w(2)), \qquad (B^w(1), \partial B^w(1)), \qquad (B^w(0), \partial B^w(0)).$$

Let $E^r_{i,j}$ ($r \geqslant 0$) be the corresponding spectral sequence (see Chapter 9 of [9]). This sequence reduces to the term $E^r_{1,j}$ for $r \geqslant 2$, since $E^r_{1-r,j+r-1} = 0$ and $E^r_{1+r,j-r+1} = 0$ for such r. From the assumption $J \not\equiv$ const (see the proof of Lemma 2.7a) it follows that $\Sigma \neq \varnothing$, and this means $\partial\Delta_0 \neq \varnothing$. Therefore

$$\mathrm{Im}\,(H_{1+j}(B^w(0), \partial B^w(0), \mathbf{Q}) \to H_{1+j}(B^w(2), \partial B^w(2), \mathbf{Q})) = 0.$$

Then we obtain the isomorphisms

$$H_1(\Delta_0, \partial\Delta_0, R_j\Phi^w_*\mathbf{Q}) \simeq E^2_{1,j} \simeq E^\infty_{1,j}$$
$$\simeq \mathrm{Im}\,(H_{1+j}(B^w(1), \partial B^w(2), \mathbf{Q}) \to H_{1+j}(B^w(2), \partial B^w(2), \mathbf{Q})).$$

Consequently, there is a natural homomorphic embedding

$$H_1(\Delta_0, \partial\Delta_0, R_j\Phi^w_*\mathbf{Q}) \subsetneq H_{1+j}(B^w|_{\Delta_0}, B^w|_{\partial\Delta_0}, \mathbf{Q}). \tag{3.4}$$

Moreover, there are isomorphisms

$$H_1(\Delta_0, \partial\Delta_0, R_j\Phi^w_*\mathbf{Q}) \simeq H_1(\Delta, F, R_j\Phi^w_*\mathbf{Q}),$$
$$H_{1+j}(B^w|_{\Delta_0}, B^w|_{\partial\Delta_0}, \mathbf{Q}) \simeq H_{1+j}(B^w, B^w|_F, \mathbf{Q})$$

by the excision theorem. Then the monomorphism (3.4) determines the canonical monomorphism

$$H_1(\Delta, F, R_j\Phi^w_*\mathbf{Q}) \subsetneq H_{1+j}(B^w, B^w|_F, \mathbf{Q}). \tag{3.5}$$

Passing to the projective limit on both sides of (3.5), we obtain by Lemma 1.2 a canonical mapping $GR_{1,j}$. Obviously $GR_{1,j}$ is injective.

3.5. In analogy with 3.4, the spectral sequence of the filtration of the bundle $B^w|_{\partial\Delta_0}$, induced by the filtration of the skeletons of the base $\partial\Delta_0$, reduces to the term $E^r_{0,j}$ for $r \geqslant 0$. Therefore there is a canonical monomorphism

$$H_0(\partial\Delta_0 = \partial F, R_j\Phi^w_*\mathbf{Q}) \subsetneq H_j(B^w|_{\partial\Delta_0=\partial F}, \mathbf{Q}). \tag{3.6}$$

It is to establish the isomorphism $H_0(\partial F, R_j\Phi^w_*\mathbf{Q}) \simeq H_0(F, R_j\Phi^w_*\mathbf{Q})$ for the natural mapping of the pair $(F, \partial F)$. For the proof it suffices to consider a simple cell decomposition of the pair ($F = \bigcup_1^t E_l$, $\partial F = \bigcup_1^t \partial E_l$); for example, 0-cells u_l ($1 \leqslant l \leqslant t$), 1-cells γ'_l ($1 \leqslant l \leqslant t$) and 2-cells E_l ($1 \leqslant l \leqslant t$) (see (2.8)). The composition of this isomorphism, the mapping (3.6), and the natural homomorphism $H_j(B^w|_{\partial F}, \mathbf{Q}) \to H_j(B^w|_F, \mathbf{Q})$ of the pair $(B^w|_F, B^w|_{\partial F})$ determines the canonical homomorphism

$$H_0(F, R_j\Phi^w_*\mathbf{Q}) \to H_j(B^w|_F, \mathbf{Q}). \tag{3.7}$$

Passing to the projective limit, we obtain the homomorphism $GR_{0,j}$, since

$$\varprojlim H_j(B^w|_F, \mathbf{Q}) = H_j(B^w|_\Sigma, \mathbf{Q}).$$

Indeed, $H_j(B^w|_F, B^w|_\Sigma, \mathbf{Q}) = 0$ for sufficiently small F (see the proof of Corollary 1.2). Then from the exact sequence of the pair $(B^w|_F, B^w|_\Sigma)$ we get the isomorphism

$$H_j(B^w|_F, \mathbf{Q}) \simeq H_j(B^w|_\Sigma, \mathbf{Q}),$$

i.e.

$$\varprojlim H_j(B^w|_F, \mathbf{Q}) = H_j(B^w|_\Sigma, \mathbf{Q}). \tag{3.8}$$

3.6. We will give an explicit geometric description of the mapping $GR_{0,j}$. First we describe (3.7). Fix a cell decomposition of ∂F. The 2-cells E_l augment this complex to a decomposition of F. Let u_i be the 0-cells of the given complex. Then a 0-cycle with coefficients in $R_j\Phi^w_*\mathbf{Q}$ has the following form:

$$\sigma_0 = \sum m_i u_i,$$

where $m_i \in R_j\Phi^w_*\mathbf{Q}|_{u_i} = H_j(B^w_{u_i}, \mathbf{Q})$. Consider an arbitrary representative $[m_i]$ of the homology class m_i in the fiber $B^w_{u_i}$. Then the homology class $\Sigma\,[m_i]$ in $B^w|_F$ is the image of the homology class of the 0-cycle σ_0 under the mapping (3.7). Further, for sufficiently small F the retraction of the cycle $\Sigma\,[m_i]$ in the fiber $B^w|_\Sigma$ and the isomorphism (3.8) describe the mapping $GR_{0,j}$.

3.7. Consider a cell decomposition of the pair (Δ, F) for sufficiently small F. We require that the intersection of this complex with F provide F with a cell decomposition of the type described in 3.6. Let $\Delta_0(1)$ be the 1-skeleton of the cell complex of (Δ, F). We denote one-dimensional cells by γ. We construct a cell decomposition of the bundle $B_w|_{\Delta_0(1)}$ over the cell complex $\Delta_0(1)$. To do this, fix in each one-dimensional cell an arbitrary point u_0 and a basis e_1, e_2 in the lattice $G|_{u_0} = H_1(B_{u_0}, \mathbf{Z})$, as in §1 of [7]. Then canonical periods z and 1, $z \in H$, are determined, and $B_{u_0} \simeq \mathbf{C}/z\mathbf{Z} + \mathbf{Z}$. The lattice $z\mathbf{Z} + \mathbf{Z}$ determines a cell decomposition of the elliptic curve B_{u_0}: the 0-cell e is the image of 0, the 1-cells e_1 and e_2 are the images of $z \times [0, 1]$ and $[0, 1]$ respectively, and the 2-cell ε is the image of $z \times [0, 1] \oplus [0, 1]$. We will call the dimension of the cells e, e_i and ε their *degree*. Then the concept of degree is defined in the free tensor algebra over $\mathbf{Q}$ for these cells. The cell complex e, e_i, ε induces a cell decomposition of $B^w_{u_0}$, since $u_0 \in \Delta'$, and consequently

$$B^w_{u_0} = \underbrace{B_{u_0} \times \cdots \times B_{u_0}}_{w}.$$

We will call this cell decomposition of $B^w_{u_0}$ the *cell decomposition corresponding to the choice of basis in the lattice* $G|_{u_0}$ (note that the basis must be chosen with negative orientation). The cells of this decomposition will be written as w-fold free tensor products of the cells e, e_i and ε. The dimension of the cell coincides with the degree of the corresponding tensor product. To each homology class $m \in H_j(B^w_{u_0}, \mathbf{Q})$ there corresponds a unique representation $[mu_0]$, a cycle in the cell decomposition corresponding to the choice of basis in $G|_{u_0}$. In the future by the representative $[m_i]$ in 3.6 we will mean the cycle described in this form. Continuation of the cell decomposition of $B^w_{u_0}$ along γ in both directions by the linear connection gives a cell decomposition of $B^w|_\gamma$ over γ, and continuation of the representative $[mu_0]$ gives the representative $[c\gamma]$ of the chain $c\gamma$, $c \in R_j\Phi^w_*\mathbf{Q}|_{\mathrm{Int}\,\gamma} \simeq H_j(B^w_{u_0}, \mathbf{Q})$. "Sections" of the cell decomposition over each point $u_0' \in \gamma$ are also cell decompositions corresponding to a choice of basis in $G|_{u_0'}$. Each cell lies either over γ or over one of the ends $\partial\gamma$. A complete cell decomposition of $B^w|_{\Delta_0(1)}$ is obtained by taking the union of the cell complexes formed over γ by 1-cells and the intersection of terminal cell decompositions over each 0-cell of $\Delta_0(1)$. For an arbitrary 1-chain $\sigma_1 = \Sigma\, c\gamma$ with coefficients in $R_j\Phi^w_*\mathbf{Q}$ we set $[\sigma_1] = \Sigma[c\gamma]$. The geometric realization $[c\gamma]$ is a relative cycle of the pair $(B^w|_\gamma, B^w|_{\partial\gamma})$. Therefore $[\sigma_1]$ is a relative cycle of the pair $(B^w|_{\Delta_0(1)}, B^w|_{\Delta_0(0)})$. If σ_1 is a cycle of the pair (Δ, F), then the boundary of the chain $[\sigma_1]$ is homologous to 0 over the interior 0-cells of $\Delta_0 = \Delta - F$. Therefore in this case the chain $[\sigma_1]$ may be completed to a relative cycle (σ_1) of the pair $(B^w|_{\Delta_0(1)}, B^w|_{\partial\Delta_0})$ over the interior 0-cells of Δ_0. The mapping (σ_1) induces the mapping (3.5). From this description of the mapping (3.5) and the description 3.6 of the mapping (3.7) we obtain the commutativity of the diagram

$$\begin{array}{ccc} H_1(\Delta, F, R_j\Phi_*^w\mathbf{Q}) & \xrightarrow{\partial} & H_0(F, R_j\Phi_*^w\mathbf{Q}) \\ {\scriptstyle (3.5)}\downarrow & & \downarrow{\scriptstyle (3.7)} \\ H_{1+j}(B^w, B^w|_F, \mathbf{Q}) & \xrightarrow{\partial} & H_j(B^w|_F, \mathbf{Q}). \end{array} \tag{3.9}$$

The boundaries of (σ_1) are situated over F. Retracting the boundaries of (σ_1) to $B^w|_\Sigma$, we obtain a description of the mapping $GR_{1,j}$, thanks to the isomorphism (3.2) for $\Pi = \Sigma$.

The isogeny of multiplication of the fiber $B^w_{u_0}$, $u_0 \in \Delta'$, by any integer n induces an analytic mapping of the pair $B^w|_{\Delta_0}, B^w|_{\partial\Delta_0}$. The corresponding mapping in homology we denote by n_*. We easily get the following result from the explicit description of the mapping (3.5), which of course applies also to (3.4).

3.8. LEMMA. $n_*|_{\mathrm{Im}(3.4)} = n^j$.

For the proof it suffices to take a cell decomposition of the pair (Δ, F) such that all the 0-cells lie in F. ■

Fix a point $u_0 \in \Delta'$ and a basis e_1, e_2 in the lattice $G|_{u_0}$. Then $B^2_{u_0}$ has a cell decomposition corresponding to the choice of a negative basis. We denote by D the homology class of the diagonal of $B^2_{u_0}$ with the natural analytic orientation.

3.9. LEMMA. $D = \varepsilon \otimes e + e \otimes \varepsilon - (e_1 \otimes e_2 - e_2 \otimes e_1)$.

3.10. Consider an F such that $u_0 \in \Delta_0$. Then by Theorem 2.5b

$$H_2(\Delta, R_j\Phi_*^w\mathbf{Q}) = H_2(\Delta, S_0^{w\,r\,j,0}) \simeq (R_j\Phi_*^w\mathbf{Q}\,|_{u_0})^{\mathrm{inv}},$$

where the invariant subspace is taken relative to the representation of SL(2, $\mathbf{Q}$) analogous to the representation (2.4) of the fundamental group $\pi_1(\Delta')$ (the action of the matrix is on the left in this case). This representation is defined by a componentwise Künneth decomposition (2.1) in the following way: it is induced by the choice of basis $G|_{u_0}$ for $R_1\Phi_*\mathbf{Q}|_{u_0}$ (see (2.2)) and it is trivial for $R_0\Phi_*\mathbf{Q}|_{u_0}$ and $R_2\Phi_*\mathbf{Q}|_{u_0}$. Fix generators e and ε in the spaces $R_0\Phi_*\mathbf{Q}|_{u_0}$ and $R_2\Phi_*\mathbf{Q}|_{u_0}$ respectively. Let $a \in A_w$ be a permutation. It determines the analytic mapping $a : B^w|_{\Delta_0} \to B^w|_{\Delta_0}$ which permutes the components of the fiber, the ith component mapping to the $a(i)$th. We denote by a_* the corresponding mapping in homology. We denote the corresponding action on the sheaf $R_j\Phi_*^w\mathbf{Q}$ the same way. This mapping is connected as follows with the mapping a defined in 2.2 of the space of sections of $(R_1\Phi_*\mathbf{Q})^{\otimes w}|_{u_0}$: $a = \mathrm{sign}(a)a_*$. Then from the decomposition (2.1), the isomorphism (2.2), and Lemma 2.3d we find that the space $(R_j\Phi_*^w\mathbf{Q}|_{u_0})^{\mathrm{inv}}$ has the following generators: the vectors

$$a_*(e^{\otimes k} \otimes \varepsilon^{\otimes l} \otimes (e_1 \otimes e_2 - e_2 \otimes e_1)^{\otimes m})$$

of degree j, where $a \in A_w$, k, l and m are positive integers, and $k + l + 2m = w$, $l + m = j/2$. We put in correspondence with the vector

$$e^{\otimes k} \otimes \varepsilon^{\otimes l} \otimes (e_1 \otimes e_2 - e_2 \otimes e_1)^{\otimes m}$$

of degree j the relative algebraic cycle $D_{k,l,m}$ of dimension $j + 2$ for the pair $(B^w, B^w|_\Sigma)$.

This cycle is uniquely determined by the following property:

$$D_{k,l,m}|_{u\in\Delta'} = \underbrace{e\times\ \cdots\ \times e}_{k}\times\underbrace{B_{u_0}\times\cdots\times B_{u_0}}_{l}$$
$$\times\underbrace{(B_{u_0}\times e+e\times B_{u_0}-D)\times\ \cdots\ \times(B_{u_0}\times e+e\times B_{u_0}-D)}_{m}.$$

It is easy to verify, using the symmetric compactification of B^w, that the mapping a extends to a regular morphism $a : B^w \to B^w$ (for the sequel its birationality and regularity over Δ' suffice, and they are obvious). We obtain the mapping $GR_{2,j}$ by putting the relative cycle $a_*(D_{k,l,m})$ in correspondence with the vector

$$a_*(e^{\otimes k}\otimes\varepsilon^{\otimes l}\otimes(e_1\otimes e_2-e_2\otimes e_1)^{\otimes m}.$$

We show that it is well defined. For odd j, the mapping $GR_{2,j}$ is trivial by Corollary 2.6. Therefore we assume that j is even, unless the contrary is stated. Consider the spectral sequence of (3.4). This sequence reduces to the term $E^r_{2,j}$ for $r > 2$. For $r > 3$ this is obvious. For $r = 2$ we have

$$E^3_{2,j} = \operatorname{Ker} d^2_{2,j}/\operatorname{Im} d^2_{0,j+1} = \operatorname{Ker} d^2_{2,j} = E^2_{2,j}.$$

Since $H_0(\Delta, R_{j+1}\Phi^w_*) = 0$ by 2.6, we have

$$E^2_{0,j+1} = H_0(\Delta_0, \partial\Delta_0, R_{j+1}\Phi^w_*\mathbf{Q}) \simeq H_0(\Delta, F, R_{j+1}\Phi^w_*\mathbf{Q}) \simeq H_0(\Delta, R_{j+1}\Phi^w_*\mathbf{Q}) = 0.$$

Since the spectral sequence reduces to the term $E^2_{2,j}$, by the excision theorem we obtain the isomorphisms

$$\begin{aligned}H_2(\Delta, F, R_j\Phi^w_*\mathbf{Q}) &\simeq H_2(\Delta_0, \partial\Delta_0, R_j\Phi^w_*\mathbf{Q}) \simeq E^2_{2,j} \simeq E^\infty_{2,j}\\ &\simeq H_{2+j}(B^w(2), \partial B^w(2), \mathbf{Q})/\operatorname{Im}(H_{2+j}(B^w(1), \partial B^w(2), \mathbf{Q})\\ &\to H_{2+j}(B^w(2), \partial B^w(2), \mathbf{Q})) \simeq H_{2+j}(B^w, B^w|_F, \mathbf{Q})/\operatorname{Im}(3.5).\end{aligned}$$

Consequently there is a natural isomorphism

$$H_2(\Delta, F, R_j\Phi^w_*\mathbf{Q}) \simeq H_{2+j}(B^w, B^w|_F, \mathbf{Q})/\operatorname{Im}(3.5). \qquad (3.10)$$

Passing to the projective limit on both sides of (3.10), we obtain the natural isomorphism

$$H_2(\Delta, \Sigma, R_j\Phi^w_*\mathbf{Q}) \simeq H_{2+j}(B^w, B^w|_\Sigma, \mathbf{Q})/\operatorname{Im} GR_{1,j+1}. \qquad (3.11)$$

By Lemma 3.9 and the geometric description of the mapping $GR_{2,j}$ given above we obtain the congruence $(3.11) \equiv GR_{2,j}$ (mod Im $GR_{1,j+1}$). Therefore to prove that $GR_{2,j}$ is well defined it suffices to establish the triviality of the intersection $H'' \cap \operatorname{Im} GR_{1,j+1} = 0$, where H'' is the subspace of $H_{2+j}(B^w, B^w|_\Sigma, \mathbf{Q})$ generated by the algebraic cycles $a_*(D_{k,l,m})$ of dimension $j + 2$. Using the stability of the projective limits and the excision theorem, this problem may be reduced to proving the triviality of the intersection $H''|_{\Delta_0} \cap (3.4) = 0$ for sufficiently small F, where the subspace $H''|_{\Delta_0} \subset H_{2+j}(B^w|_{\Delta_0}, B^w|_{\partial\Delta_0}, \mathbf{Q})$ is generated by the restrictions of the algebraic cycles $a_*(D_{k,l,m})$ of dimension $j + 2$. The last is obvious from the relation $n_*|_{H''|_{\Delta_0}} = n^j$, and, by Lemma 3.8, $n_*|_{\operatorname{Im}(3.4)} = n^{j+1}$. The operator n_* on the homology space $H_{2+j}(B^w|_{\Delta_0}, B^w|_{\partial\Delta_0}, \mathbf{Q})$ is induced by the fiberwise isogeny of multiplication by n.

Now we assume j arbitrary, not just even.

3.11. LEMMA. a. *$GR_{1,j}$ and $GR_{2,j}$ are monomorphisms.*
b. $H_{2+j}(B^w, B^w|_\Sigma, \mathbf{Q}) = \operatorname{Im} GR_{2,j} \oplus \operatorname{im} GR_{1,j+1}$.

PROOF. The injectivity of $GR_{1,j}$ comes from the process of defining the homomorphism in 3.4. For even j the injectivity of $GR_{2,j}$ and the decomposition b are immediate corollaries of (3.11), since the intersection

$$\operatorname{Im} GR_{2,j} \cap \operatorname{Im} GR_{1,j+1} = H'' \cap \operatorname{Im} GR_{1,j+1} = 0$$

is trivial, and the mapping (3.11) is induced by $GR_{2,j}$. Suppose j is odd. In this case the injectivity is obvious because of the triviality of $GR_{2,j}$ (see 2.6). Since $H_2(\Delta, R_j\Phi^w_*\mathbf{Q}) = 0$ and $H_1(F, R_j\Phi^w_*\mathbf{Q}) = 0$ for sufficiently small F, we get the triviality of $H_2(\Delta, F, R_j\Phi^w_*\mathbf{Q}) = 0$ from the exact sequence of the pair (Δ, F). Then $H_2(\Delta_0, \partial\Delta_0, R_j\Phi^w_*\mathbf{Q}) = 0$ by the excision theorem. Consequently the spectral sequence of (3.4) reduces to the term $E^r_{2,j}$ for $r \geqslant 2$, and

$$\begin{aligned} 0 = H_2(\Delta_0, \partial\Delta_0, R_j\Phi^w_*\mathbf{Q}) &= E^2_{2,j} \simeq E^\infty_{2,j} \\ \simeq H_{2+j}(B^w(2), \partial B^w(2)\,\mathbf{Q}) / \operatorname{Im} &(H_{2+j}(B^w(1), \partial B^w(2), \mathbf{Q}) \\ &\to H_{2+j}(B^w(2), \partial B^w(2), \mathbf{Q})). \end{aligned}$$

This proves the surjectivity of (3.4) for $j + 1$, and similarly the surjectivity of (3.5). Therefore $GR_{1,j+1}$ is an isomorphism for odd j, which together with the triviality of $GR_{2,j}$ proves b. ■

Consider an arbitrary point $v \in \Delta$. Let $u_0 \in \Delta'$ be a point sufficiently close to v, i.e. $u_0 \in E_1$, a small closed disk around v satisfying Lemma 1.1. Then the composition of the embedding $B^w_{u_0} \hookrightarrow B^w|_{E_1} = B^w_1$ and the retraction $B^w_1 \to B(1) = B^w|_v$ determines the following homomorphism:

$$H_j(B^w|_{u_0}, \mathbf{Q}) \to H_j(B^w|_v, \mathbf{Q}). \tag{3.12}$$

We denote by β a single positive circuit around the point v, $\beta \subset \Delta'$, with origin at the point u_0. To this circuit there corresponds an endomorphism s_β of the space $H_j(B^w_{u_0}, \mathbf{Q})$ defined as in (1.3) of [7] by the natural connection on $B^w|_{\Delta'}$. Then (3.12) determines the specialization homomorphism

$$\mathrm{Sp} : (H_j(B^w_{u_0}, \mathbf{Q}))^{\mathrm{coinv}} \to H_j(B^w|_v, \mathbf{Q}),$$

where the space of coinvariant vectors is taken with respect to the endomorphism s_β.

3.12. PROPOSITION. Sp *is a monomorphism.*

PROOF OF THEOREM 3.2. a. The injectivity of $GR_{1,j}$ and $GR_{2,j}$ was proved in Lemma 3.11a. We prove injectivity for $GR_{0,j}$. Because of the stability of the projective limit it suffices to prove this for (3.7) for sufficiently small F. In this case $F = \bigcup_1^t E_l$ decomposes into the connected components E_l. Consequently, (3.7) also decomposes into a direct sum of natural homomorphisms

$$H_0(E_l, R_j\Phi^w_*\mathbf{Q}) \to H_j(B^w|_{E_l}, \mathbf{Q}) \tag{3.13}$$

and it suffices to establish their injectivity for small E_l. Consider one of the disks, say E_1, and assume it is so small that Lemma 1.1 holds. Let v be the center of the disk E_1. Then we get a natural isomorphism $H_j(B^w_1, \mathbf{Q}) \simeq H_j(B^w|_v, \mathbf{Q})$ as in the proof of the isomorphism (3.8). Proving the injectivity of $GR_{0,j}$ reduces to checking the injectivity of the composition

$$H_0(E_1, R_j\Phi^w_*\mathbf{Q}) \to H_j(B^w|_v, \mathbf{Q}) \tag{3.14}$$

of this isomorphism and the mapping (3.13) for $l = 1$. Consider the point $u_0 \in \partial E_1$ in the boundary of E_1. This last choice determines a cell decomposition of E_1: 0-cell u_0, 1-cell ∂E_1 and 2-cell E_1. It follows immediately from 3.6 that for this cell decomposition the mapping (3.14) assumes the form Sp. Therefore the injectivity of (3.14) follows from Proposition 3.12.

Part b follows from the commutative diagram (3.9) by passing to the projective limit.

c. From the construction 3.10 of the mapping $GR_{2,j-2}$ we have the inclusion $\operatorname{Im} GR_{2,j-2} \subset \overline{H}_j(B^w, \mathbf{Q})$. Therefore this part of the theorem is an immediate corollary of 3.4 and 3.2a,b, since

$$\operatorname{Ker}(H_j(B^w, B^w|_\Sigma, \mathbf{Q}) \xrightarrow{\partial} H_{j-1}(B^w|_\Sigma, \mathbf{Q})) = \overline{H}_j(B^w, \mathbf{Q})$$

from the exact sequence of the pair $(B^w, B^w|_\Sigma)$.

d. By the construction of the mapping $GR_{2,w-1}$ we have $\operatorname{Im} GR_{2,w-1} \subset H'$. Therefore by 3.2c it suffices to establish the analogous decomposition for $GR_{1,w}(H_1(\Delta, R_w\Phi_*^w\mathbf{Q}))$. The Künneth formula (2.1) for $j = w$ reduces this problem to the decomposition of $GR_{1,w}(H_1(\Delta, (R_1\Phi_*\mathbf{Q})^{\otimes w}))$. If even one $j_m \neq 1$, then by the description 3.7 of the mapping $GR_{1,w}$ we have

$$GR_{1,w}\left(H_1\left(\Delta, \bigotimes_{m=1}^{w} R_{j_m}\Phi_*\mathbf{Q}\right)\right) \subset H'.$$

The decomposition

$$GR_{1,w}(H_1(\Delta, (R_1\Phi_*\mathbf{Q})^{\otimes w}) = GR_{1,w}(H_1(\Delta, (R_1\Phi_*\mathbf{Q})^{w}) \oplus H_1',$$

where $H_1 \subset H'$, is an immediate corollary of 2.3c, 3.9 and 3.7. ■

PROOF OF LEMMA 3.9. We have

$$D = (e_2 \otimes e + e \otimes e_2) \otimes (e_1 \otimes e + e \otimes e_1) = \varepsilon \otimes e - e_1 \otimes e_2 + e_2 \otimes e_1 + e \otimes \varepsilon. \quad ■$$

PROOF OF PROPOSITION 3.12. a. We denote by

$$\overline{\mathrm{Sp}}: H_j(B^w_{u_0}, \mathbf{Q})^{\mathrm{coinv}} \to H_j(\overline{B}^w|_v, \mathbf{Q})$$

the composition of Sp with the natural homomorphism in homology induced by the projection $B^w|_v \to \overline{B}^w|_v$. We note that $B^w_{u_0} = \overline{B}^w_{u_0}$, since $u_0 \in \Delta'$. We will show below that $\overline{\mathrm{Sp}}$ is a monomorphism, from which the injectivity of Sp follows immediately. The projection Ψ^w (see §3 of [7]) of the deformation of Lemma 1.1 determines a deformation retract of $\overline{B}^w_1 = \overline{B}^w|_{E_1}$ onto $\overline{B}^w|_v$. Therefore we have the canonical isomorphism

$$H_j(\overline{B}^w_1, \mathbf{Q}) \simeq H_j(\overline{B}^w_v, \mathbf{Q}).$$

This isomorphism shows the equivalence of the injectivity of $\overline{\mathrm{Sp}}$ and

$$H_j(\overline{B}^w_{u_0}, \mathbf{Q})^{\mathrm{coinv}} \hookrightarrow H_j(\overline{B}^w_1 \mathbf{Q}), \tag{3.15}$$

where the last homomorphism is induced by the natural mapping of the pair $(\overline{B}^w_1, \overline{B}^w_{u_0})$.

b. *Reduction to the case* I_b $(b > 1)$. We know that $\overline{B}^w_1 \simeq C \setminus F^w$, where C is a finite cyclic group of order κ, with action compatible with the projection of σ on the base D (see [7], 2.2). In the case under consideration $D = \{|\sigma|^\kappa < \varepsilon\}$ is a closed disk. In the base

D the generator $e_\kappa = e^{2\pi i/\kappa}$ of the group C acts by multiplication. Therefore there is an isomorphism

$$(H_j(F^w_{\sqrt[\kappa]{\varepsilon}}, \mathbf{Q})^{\mathrm{coinv}\,(D)})^{\mathrm{coinv}\,(C)} = H_j(B^w_{u_0}, \mathbf{Q})^{\mathrm{coinv}\,(E_1)},$$

where $\sqrt[\kappa]{\varepsilon}$ is the arithmetic root, and coinv(D) and coinv(E_1) denote the coinvariants of the circuits around the boundaries ∂D and ∂E_1; coinv(C) the coinvariants of the group C; and $\tau(u_0) = \sigma^\kappa(u_0) = \varepsilon$. On the other hand,

$$H_j(\overline{B}^w_1, \mathbf{Q}) \simeq H_j(F^w, \mathbf{Q})^{\mathrm{inv}} \simeq H_j(F^w, \mathbf{Q})^{\mathrm{coinv}}.$$

The last isomorphism follows from the semisimplicity of the representation of the finite cyclic group C of automorphisms in the homology space $H_j(F^w, \mathbf{Q})$. Consequently, to prove the injectivity of (3.15) it suffices to show the injectivity of

$$H_j(F^w_d, \mathbf{Q}) \hookrightarrow H_j(F^w, \mathbf{Q})$$

for the natural mapping of the pair (F^w, F^w_d), where $d \in \partial D$. Then from 2.2 of [7] and Chapter 8 of [10] it follows that F^w_0 is the only singular fiber of F^w of type I_b $(b > 0)$. This concludes the reduction to the case I_b.

c. If v has type I_0, then $s_\beta = \mathrm{id}$ and Sp is an isomorphism, since the bundle B^w_1 is topologically trivial for sufficiently small E_1.

d. Suppose the point v has type I_b $(b > 1)$. It is easy to check that

$$H_0(B_{u_0}, \mathbf{Q}) \simeq H_0(B_v, \mathbf{Q}), \quad H_1(B_{u_0}, \mathbf{Q})^{\mathrm{coinv}} \simeq H_1(B_v, \mathbf{Q}),$$
$$H_2(B_{u_0}, \mathbf{Q}) \hookrightarrow H_2(B_v, \mathbf{Q}),$$

where in the first and last cases $s_\beta = \mathrm{id}$. Hence by 2.2(ii) of [7] and by the Künneth decomposition (2.1) at the points u_0 and v, we obtain the injectivity of $\overline{\mathrm{Sp}}$, since by (2.10)

$$(H_1(B_{u_0}, \mathbf{Q})^{\otimes m})^{\mathrm{coinv}} \simeq (H_1(B_{u_0}, \mathbf{Q})^{\mathrm{coinv}})^{\otimes m}. \quad \blacksquare$$

§4. Nondegeneracy conditions of the canonical pairing

4.1. If $\omega \in H^0(B^w, \Omega^{w+1})$ is a first order differential form on Kuga's variety B^w, then its integrals are trivial along every chain of fibers over points of the base. Therefore a pairing

$$(\ ,\ ): H_{w+1}(B^w, B^w|_\Sigma, \mathbf{Q}) \times H^0(B^w, \Omega^{w+1} \oplus \overline{\Omega}^{w+1}) \to \mathbf{C},$$

(homology class σ, ω) $= \int_\sigma \omega$, is defined, where σ represents a homology class of the space $H_{w+1}(B^w, B^w|_\Sigma, \mathbf{Q})$ for some cell decomposition, and $\omega \in H^0(B^w, \Omega^{w+1} \oplus \overline{\Omega}^{w+1})$. The pairing (,) and the monomorphism $GR_{1,w}$ determine the pairing

$$\langle\ ,\ \rangle: H_1(\Delta, \Sigma, R_w\Phi^w_* \mathbf{Q}) \times H^0(B^w, \Omega^{w+1} \oplus \overline{\Omega}^{w+1}),$$
$$\langle\ ,\ \rangle = (GR_{1,w},\ ).$$

The pairing $\langle\ ,\ \rangle$ is always nondegenerate on the right.

4.2. THEOREM. *$H_1(\Delta, (R_1\Phi_*\mathbf{Q})^w)^\perp = 0$, i.e. $\langle H_1(\Delta, (R_1\Phi_*\mathbf{Q})^w), \omega\rangle = 0$ implies $\omega = 0$ for any $\omega \in H^0(B^w, \Omega^{w+1} \oplus \overline{\Omega}^{w+1})$.*

The proof follows immediately from de Rham's theorem, Theorem 3.2d, and the triviality of the pairing of $H^0(B^w, \Omega^{w+1} \oplus \overline{\Omega}^{w+1})$ with H'. $\blacksquare$

4.3. Corollary. *If*

$$\dim H^0(B^w, \Omega^{w+1}) = (g-1)(w+1) + \sum_{b\geqslant 1} \frac{w}{2}(\nu(\mathrm{I}_b) + \nu(\mathrm{I}_b^*)) + \left[\frac{w+2}{3}\right](\nu(\mathrm{II}) + \nu(\mathrm{II}^*) + \nu(\mathrm{IV}) + \nu(\mathrm{IV}^*)) + \left[\frac{w+2}{4}\right](\nu(\mathrm{III}) + \nu(\mathrm{III}^*))$$

for even $w > 0$,

$$\dim H^0(B^w, \Omega^{w+1}) = (g-1)(w+1) + \sum_{b\geqslant 1} \frac{w}{2}\nu(\mathrm{I}_b) + \left(\frac{w+1}{2}\right)\Big(\sum_{b\geqslant 0}\nu(\mathrm{I}_b^*) + \nu(\mathrm{II}^*) + \nu(\mathrm{II}) + \nu(\mathrm{III}) + \nu(\mathrm{III}^*)\Big) + \left[\frac{w+2}{3}\right](\nu(\mathrm{IV}) + \nu(\mathrm{IV}^*))$$

for odd $w > 0$, *or*

$$\dim H^0(B^w, \Omega^{w+1}) = g$$

for $w = 0$, *then the pairing*

$$\langle,\rangle \,|\, H_1(\Delta, (R_1\Phi_*\mathbf{Q})^w) \times H^0(B^w, \Omega^{w+1} \oplus \overline{\Omega}^{w+1})$$

is nondegenerate.

The proof follows directly from Theorems 3.2a, 2.5c, and 4.2. ■

§5. Application. The Shimura torus

Consider the Hodge decomposition of the $(w+1)$-cohomology of B^w:

$$H^{w+1}(B^w, \mathbf{Q}) \otimes \mathbf{C} = H^{w+1,0}(B^w) \oplus \ldots \oplus H^{p,q}(B^w) \oplus \ldots \oplus H^{0,w+1}(B^w).$$

We project $H^{w+1}(B^w, \mathbf{Q})$ onto $H^{0,w+1}(B^w)$ and denote the resulting $\mathbf{Q}$-subspace by Q. On the other hand, we have

$$H_{w+1}(B^w, \mathbf{Q}) \xrightarrow{D} H^{w+1}(B^w, \mathbf{Q}) \xrightarrow{\mathrm{pr}_{0,w+1}} H^{0,w+1}(B^w), \tag{5.1}$$

where the mapping D comes from Poincaré duality. The mapping (5.1) may be realized as follows. An element $c \in H_{w+1}(B^w, \mathbf{Q})$ determines on $H^{w+1,0}(B^w)$ the $\mathbf{C}$-functional $\int_c \omega$, $\omega \in H^{w+1,0}(B^w)$, $(H^{w+1,0}(B^w))^* = H^{0,w+1}(B^w)$. Hence an element corresponding to $\int_c$ is determined in $H^{0,w+1}(B^w)$. This element is the image of c under the homomorphism (5.1). Since $\int_c \omega = 0$ for any ω of type $(w+1, 0)$ and

$$c \in \mathrm{Im}(H_{w+1}(B^w|_\Sigma, \mathbf{Q}) \to H_{w+1}(B^w, \mathbf{Q})),$$

(5.1) determines the mapping

$$\overline{H}_{w+1}(B^w, \mathbf{Q}) \to H^{0,w+1}(B^w). \tag{5.2}$$

By 3.2c, we may compose the mapping (5.2) and $GR_{1,w}$ to obtain a mapping

$$H_1(\Delta, (R_1\Phi_*\mathbf{Q})^w) \to H^{0,w+1}(B^w). \tag{5.3}$$

5.1. Proposition. Im (5.3) = Q.

The proof follows directly from 3.2d and the fact that D in (5.1) is an isomorphism. ■

5.2. Definition. If $\dim_{\mathbf{Q}} Q = \dim_{\mathbf{R}} H^{0,w+1}(B^w)$, then the torus

$$T(B^w) = H^{0,w+1}(B^w)/Z$$

is determined up to isogeny, where $Z \subset \mathbf{Q} \subset H^{0,w+1}(B^w)$ is some lattice. $T(B^w)$ is called the *Shimura torus*. A Kuga variety for which this condition in the definition of the Shimura torus is satisfied will be called a *special Kuga variety*.

5.3. THEOREM. *If w is even and B^w is a special Kuga variety, then $T(B^w)$ is an abelian variety.*

This theorem is a generalization of Theorem 2 of [4] (see Theorem 7 of [6]).

PROOF. We interpret cohomology in terms of harmonic forms. Then $(\alpha, \beta)_{B^w} = \int_{B^w} \alpha \wedge \beta$. Consider the hermitian form

$$H(\omega_1, \omega_2) = 2i \int_{B^w} \omega_1 \wedge \bar{\omega}_2$$

on $H^{0,w+1}(B^w)$. The form H is real, hermitian, positive definite for $w + 2 \equiv 2 \pmod 4$ and negative definite for $w + 2 \equiv 0 \pmod 4$. Therefore by the Riemann-Frobenius condition (see §6 of [2]) it remains to verify the rationality of Im H in Q. If $\alpha, \beta \in Q$, then

$$A = \alpha + \bar{\alpha}, \quad B = \beta + \bar{\beta}$$

are rational cohomology classes. Hence

$$A \wedge B = 2\mathrm{Re}(\alpha \wedge \bar{\beta})$$

and therefore

$$\mathrm{Im}\, H(\alpha, \beta) = \mathrm{Im}\, 2i \int_{B^w} \alpha \wedge \bar{\beta} = (A, B)_{B^w} \in \mathbf{Q}. \quad \blacksquare$$

5.4. REMARKS. 1. The Jacobi variety $\Theta_{w/2}(B^w)$ ([2], §6) admits a canonical projection onto $T(B^w)$ in the categroy of complex tori up to isogeny.

2. Corollary 4.3 gives a sufficient condition for B^w to be a special Kuga variety.

§6. Application. The modular case

PROOF OF THEOREM 0.2. This assertion is a direct corollary of the definition from (0.1) of the pairing (,):

$$(\sigma, (\varphi, \bar{\psi})) = \langle \sigma, \omega_\varphi + \bar{\omega}_\psi \rangle,$$

where $\sigma \in H_1(\Delta, \Sigma, (R_1\Phi_*\mathbf{Q})^w)$, $\varphi, \psi \in S_{w+2}(\Gamma)$, and $w_\varphi, w_\psi \in H^0(B^w_\Gamma, \Omega^{w+1})$ are the corresponding regular differentials of Theorem 0.3 of [7], and also Corollary 4.3 of this paper and Corollary 5.3 and Theorem 0.3 of [7]. $\blacksquare$

Let $\mathbf{C} \supset K \supset \mathbf{Q}$ be an arbitrary field. Then we may define a canonical pairing

$$(\, , \,) : H_1(\Delta, \Sigma, (R_1\Phi_*K)^w) \times S_{w+2}(\Gamma) \oplus \overline{S_{w+2}(\Gamma)} \to \mathbf{C},$$

since

$$H_1(\Delta, \Sigma, (R_1\Phi_*K)^w) \simeq H_1(\Delta, \Sigma, (R_1\Phi_*\mathbf{Q})^w) \otimes_{\mathbf{Q}} K.$$

6.1. COROLLARY. *The pairing* (,) *on*

$$H_1(\Delta, (R_1\Phi_*K)^w) \times S_{w+2}(\Gamma) \oplus \overline{S_{w+2}(\Gamma)}$$

is nondegenerate. $\blacksquare$

Moreover, we have

6.2. COROLLARY. i) *B_Γ^w is a special Kuga variety.*

ii) *$T(B_\Gamma^w)$ is an abelian variety for even w.*

PROOF. ii) follows from i) and 5.3. The proof of i) follows from the fact that

$$\dim_{\mathbf{Q}} Q = \dim_{\mathbf{R}} H^{0,w+1}(B_\Gamma^w),$$

since (5.3) is an embedding: by the nondegeneracy of (,) and the relation $(\, , \bar{\omega}) = \overline{(\, , \omega)}$, and also from the equations

$$\dim_{\mathbf{Q}} H_1(\Delta, (R_1\Phi_*\mathbf{Q})^w) = 2\dim_{\mathbf{C}} H^{0,w+1}(B_\Gamma^w) = 2\dim_{\mathbf{C}} H^0(B_\Gamma^w, \Omega^{w+1}) = 2\dim_{\mathbf{C}} S_{w+2}(\Gamma)$$

(see Theorem 0.3 and Corollary 5.3 of [7], and Theorem 2.5c of this paper). ■

Received 26/SEPT/79

BIBLIOGRAPHY

1. Sze-Tsen Hu, *Homotopy theory*, Academic Press, 1959.

2. S. S. Chern, *Complex manifolds*, Inst. Fis. e Mat., Univ. Recife, Recife, 1959.

3. Gorô Shimura, *Introduction to the arithmetic theory of automorphic functions*, Iwanami Shoten, Tokyo, and Princeton Univ. Press, Princeton, N. J., 1971.

4. ______, *Sur les intégrales attachées aux formes automorphes*, J. Math. Soc. Japan **11** (1959), 291–311.

5. T. Shioda, *On elliptic modular surfaces*, J. Math. Soc. Japan **24** (1972), 20–59.

6. V. V. Šokurov, *Periods of cusp forms and Kuga varieties*, Uspehi Mat. Nauk **30** (1975), no. 3(183), 183–184. (Russian)

7. ______, *Holomorphic differential forms of higher degree on Kuga's modular varieties*, Mat. Sb. **101 (143)** (1976), 131–157; English transl. in Math. USSR Sb. **30** (1976).

8. A. A. Kirillov, *Elements of the theory of representations*, "Nauka", Moscow, 1972; English transl., Springer-Verlag, 1976.

9. Edwin H. Spanier, *Algebraic topology*, McGraw-Hill, 1966.

10. K. Kodaira, *On compact analytic surfaces*. II, Ann. of Math. (2) **77** (1963), 563–626.

Translated by H. P. BOAS