ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2009, Vol. 264, pp. 178–200. © Pleiades Publishing, Ltd., 2009.
Published in Russian in Trudy Matematicheskogo Instituta imeni V.A. Steklova, 2009, Vol. 264, pp. 184–208.

Letters of a Bi-rationalist. VII
Ordered Termination

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Received August 2008

To V.A. Iskovskikh,
who has greatly shaped my vision of mathematics

Abstract—To construct a resulting model in the LMMP, it is sufficient to prove the existence of log flips and their termination for some sequences. We prove that the LMMP in dimension $d-1$ and the termination of terminal log flips in dimension d imply, for any log pair of dimension d, the existence of a *resulting* model: a strictly log minimal model or a strictly log terminal Mori log fibration, and imply the existence of log flips in dimension $d+1$. As a consequence, we prove the existence of a resulting model of 4-fold log pairs, the existence of log flips in dimension 5, and Geography of log models in dimension 4.

DOI: 10.1134/S00815438090010192

Теперь кончаемо, бо числа не знаемо . . .
Из письма запорожских казаков турецкому султану[1]

This paper is mainly about the termination of flips[2] but not about their existence. It shows that under certain inductive assumptions and the termination of terminal flips, we can construct either a log minimal model or a Mori log fibration for any log pair. This amounts to a weaker form of the Log Minimal Model Program (LMMP) in which the termination of any sequence of log flips is replaced by the termination of some sequences. This idea polishes the reduction to prelimiting (pl) flips [8, 4.5 and Section 6; 12] and also appeared recently in [1] (cf. Definition 2 below). It seems that this weaker form is sufficient for most applications of the LMMP. Up to dimension 4, we can omit the inductive assumptions, including the termination. The results in dimension ≤ 4 indicate that the terminal termination is much easier than the nonterminal one, e.g., the klt termination, and does not imply automatically the latter. Moreover, we do not use any classification of singularities for this weak LMMP in dimension ≤ 4. However, the results should not be exaggerated. They only demonstrate that progress in the LMMP hinges upon that in termination.

We work over a base field k of characteristic 0; in some instances, k is algebraically closed or we need to slightly change the meaning of an extremal curve (see Definition 1). We use standard facts and notation of the LMMP, as in [4, 5, 12]. In particular, we use standard abbreviations: *lt* for log

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[1] It is difficult to preserve all flavors in a translation of this phrase and, in particular, an ambiguity of the word *число* in Ukranian: *Now we will conclude, for we do not know the date* (*number*) . . . . From the reply of Zaporozhian Cossacks to Sultan Mehmed IV of Turkey.

[2] Their existence was announced in C. Birkar, P. Cascini, C. Hacon, and J. McKernan, "Existence of Minimal Models for Varieties of Log General Type," arXiv: math.AG/0610203. *Added by the author at the proofreading stage.*

terminal; *dlt* for divisorially lt; *klt* for Kawamata lt; *lc* for log canonical; *wlc* for weakly log canonical. Let us briefly recall some terminology and notation: a birational *rational* 1-contraction does not blow up any divisor [12, Definition 3.1]; $\mathcal{D}_B$ is the $\mathbb{R}$-vector space of Weil $\mathbb{R}$-divisors D supported in $\operatorname{Supp} B$ with the maximal absolute value norm $\|D\|$ [12, text before Definition 5.17]; an FT (Fano type) relative variety means a relative variety X/Z for which there exists an $\mathbb{R}$-boundary B such that $(X/Z, B)$ is a klt Fano pair; for the cone of curves, contractions on an FT X/Z, and more details, see [6, Section 2]; $K = K_X$ denotes a canonical divisor on a variety X; $\mathrm{LCS}(X, B)$ is the subvariety of nonklt points.

The LMMP means that of [11, Section 5] under the dlt condition. Flips are supposed to be klt or dlt $\mathbb{Q}$-factorial and extremal in most situations. However, the lc LMMP holds up to dimension 4, and dimension 5 is the first one when the existence of lc (even dlt) flips is still unknown. By a terminal log flip we mean a log flip or a divisorial contraction having only terminal points in the flipping or exceptional locus, respectively; that is, the minimal log discrepancy (mld) for these points is >1. Terminal termination: any sequence of terminal log flips is finite. Usually we apply this termination to extremal and $\mathbb{Q}$-factorial contractions at least near the flipping locus (cf. Caution 2 in the proof of Theorem 2).

Theorem 1. *We assume the LMMP in dimension $d-1$ and the termination of terminal log flips in dimension d. Then any pair $(X/Z, B)$ of dimension d with an $\mathbb{R}$-boundary B has a* resulting *model. More precisely, $(X/Z, B)$ has either a strictly log minimal model or a strictly log terminal Mori log fibration.*

Moreover, $(X/Z, B)$ has a strictly log minimal model if and only if its numerical log Kodaira dimension is nonnegative [11, p. 2673], *or equivalently, for lc (X, B), $K + B$ is pseudo-effective$/Z$.*

The LMMP can be actually replaced by the terminal termination in dimension $\le d-1$ (cf. Corollary 3 and its proof).

Question. However, if B is a $\mathbb{Q}$-boundary, are the LMMP with $\mathbb{Q}$-boundaries and the terminal termination with $\mathbb{Q}$-boundaries enough?

Addendum 1. *If an initial pair $(X/Z, B)$ of dimension d is strictly log terminal or dlt, a resulting model can be constructed by a terminated sequence of (extremal) log flips.*

Perhaps, for a more general initial pair, e.g., lc $(X/Z, B)$, this part of the LMMP also works: resulting models exist and Geography of log models holds in dimension d. Similarly, under the same assumptions, directed klt flops terminate in dimension d. In what follows, we mention some of these results without the assumptions for 4-folds; we can add to this the termination of 4-fold klt directed flops.

Corollary 1. *Under the assumptions of Theorem 1, the nonnegative numerical log Kodaira dimension is a closed condition with respect to boundaries, or equivalently, the existence of a Mori log fibration is an open condition on B.*

To obtain the same result for the usual log Kodaira dimension, one needs semiampleness [11, Conjecture 2.6].

Proof. By definition the numerical dimension is defined for a wlc model [11, Proposition 2.4], and the wlc property is closed. Moreover, for a limiting boundary, a resulting model is wlc. Otherwise, by Theorem 1 and Addendum 1, we get a Mori log fibration on a birational model, for which the existence condition is open with respect to boundaries. □

Corollary 2. *Any 4-fold pair $(X/Z, B)$ with an $\mathbb{R}$-boundary B has a resulting model. The closed and open properties of Corollary 1 hold for 4-fold pairs.*

Proof. Immediate by the LMMP up to dimension 3 [11, Theorem 5.2] and the terminal termination up to dimension 4 [13, Example 9 and Lemma 2]. □

Note that semiampleness [11, Conjecture 2.6], special termination [13, Example 8], the Kawamata–Matsuki finiteness [4, Conjecture 3.16], and [1, Theorem 2.15] improve the last result: any sequence of log flips for a 4-fold lc pair $(X/Z, B)$ with a pseudo-effective/Z divisor $K+B$ terminates (see a remark before Definition 2). Actually, semiampleness can be replaced by the effective property of $(X/Z, B)$ in the sense of Birkar, that is, the effectiveness of $K+B$ up to numerical equivalence/Z. By the special termination we can consider only klt pairs $(X/Z, B)$. Assuming that $K+B$ is pseudo-effective/Z, we obtain an effective divisor $E \sim_{\mathbb{R}} K+B/Z$ by the corollary and semiampleness. Thus, the pairs $(X/Z, B)$ and $(X/Z, B+\varepsilon E)$ with a sufficiently small real number $\varepsilon > 0$ have log minimal models $(Y/Z, B_Y)$ and $(Y/Z, (B+\varepsilon E)_Y)$, and the varieties Y/Z are the same. The rank of the homology groups $H_4(Y/Z, \mathbb{Z})$ is bounded for all these Y/Z due to Kawamata–Matsuki. Hence, by [1, Theorem 2.15] for $(X/Z, B+\varepsilon E)$, any sequence of log flips terminates: after finitely many flips, each next one blows down only curves and blows up a surface, which increases the rank of algebraic cycles of dimension 2. The rank is bounded according to the above, since we can always apply Addendum 1. Under the conditions of [1, Theorem 3.4], both the semiampleness conjecture and the Kawamata–Matsuki conjecture hold according to [7, Theorem 2.1] (cf. Corollary 10 below) and to Geography of log models (see Corollary 5 below). Actually, we can omit the $\mathbb{Q}$-factorial property of X. As in the proof of [1, Theorem 3.14], we can suppose that $B \geq A$, where A is an ample effective $\mathbb{R}$-divisor. Then we apply the geography to $\sum D_i = \operatorname{Supp} B$ near B. The varieties Y/Z of the log minimal models for $(X/Z, B)$ correspond to countries near B. For the log Kodaira dimension $-\infty$, the log termination is more difficult and is still out of our grasp in general.

Revised Reduction. *The LMMP in dimension $d-1$, the termination of terminal log flips in dimension d, and the existence of pl flips in dimension $d+1$ imply the existence of klt log flips in dimension $d+1$.*

We recall that the flipping contraction is assumed to be extremal and small, and the flipping variety is assumed to be $\mathbb{Q}$-factorial.

Revised Induction. *The LMMP in dimension $d-1$ and the termination of terminal log flips in dimension d imply the existence of pl flips in dimension $d+1$.*

Actually, it is sufficient to assume the termination in the birational situation, *that is, when $X \to Z$ is a birational contraction.*

In the same way as we prove it below [12, 3], we can prove that any restricted divisorial algebra on the reduced component of a pl contraction in dimension $d+1$ is finitely generated. Similarly we can establish the existence of pl flips in dimension n with the core dimension $n-s \leq d$ [12, 1.1]. So, we get a "more terminal" version of the main theorem in [3].

Corollary 3. *The LMMP in dimension $d-1$ and the termination of terminal log flips in dimension d imply the existence of klt log flips in dimension $d+1$.*

As in Revised Reduction, the flipping contraction is assumed to be extremal and small, and the flipping variety is assumed to be $\mathbb{Q}$-factorial. For d-dimensional log flips, these conditions can be omitted.

Proof. Immediate by Revised Reduction and Revised Induction.

Notice that in dimension d we can obtain more general log flips, as in the usual reduction [12, Theorem 1.2]. □

Corollary 4. *Log flips as in Corollary 3 exist up to dimension 5.*

For a different proof see [1, Theorem 4.3]. Note that for 4-fold log flips we need only the terminal termination in dimension 3 [4, Theorem 3.5].

Proof. Immediate by the LMMP up to dimension 3 [11, Theorem 5.2] and the terminal termination up to dimension 4 [13, Example 9 and Lemma 2]. □

Other applications are as follows.

Corollary 5 (see [11, Theorem 6.20; 4, Conjecture 2.10]). *The geography conjecture holds for relative klt* 4*-folds. This gives a bounded termination for D-flips of any relative* FT 4*-fold.*

This can be established for any lc relative pair after a slight generalization of Proposition 1 and its addenda and corollaries for lc singularities instead of dlt (see a remark after Proposition 1).

Proof. The geography can be obtained from Theorem 1 and Theorem 2 below (cf. the proof of Corollary 3).

In the case of FT varieties, this gives a universal bound for the D-termination. □

A detailed treatment will be done elsewhere.

Corollary 6 (see [12, Theorem 3.33]). *Zariski decompositions exist on relative* FT 4*-folds. In particular, the divisorial algebra of any $\mathbb{Q}$-divisor is finitely generated on such a variety.*

Proof. Immediate by Corollary 5. □

Corollary 7. *If a complete algebraic space of dimension* 4 *has only klt singularities and has no rational curves over an algebraic closure of the base field, then it is projective.*

Proof. We can use the methods of [10] and Addendum 1. □

Definition 1. An irreducible curve C on X/Z is called *extremal* if it generates an extremal ray $R = \mathbb{R}_+[C]$ of the Kleiman–Mori cone $\overline{\mathrm{NE}}(X/Z)$ and has the minimal degree among the curves in this ray (with respect to any ample divisor). We also suppose that R is contractible.

If the base field is not algebraically closed, then the contraction may be nonextremal over its algebraic closure and we take, as an extremal curve, that of a (partial) extremal subcontraction or a sum of conjugations of such a curve divided by the number of curves in the orbit.

If $(X/Z, B)$ is a dlt log pair with a boundary B such that $K+B$ has index m, then, for any extremal curve C/Z,

$$(K+B, C) \in \Big\{\frac{n}{m} \mid n \in \mathbb{Z} \text{ and } n \geq -2dm\Big\},$$

where $d = \dim X$. This follows from the anticanonical boundedness [9, Theorem]. In particular, $(K+C, B) \geq 1/m$ if $(K+B, C) > 0$ (cf. [11, Lemma 6.19]).

We can generalize these results to $\mathbb{R}$-boundaries.

Proposition 1. *Let $(X/Z, B)$ be an lc pair with an $\mathbb{R}$-boundary B. Then there exist a finite set of real positive numbers r_i and a positive integer m such that, for any extremal curve C/Z near the generic point of which (X, B) is dlt,*

$$(K+B, C) \in \Big\{\sum \frac{r_i n_i}{m} \mid n_i \in \mathbb{Z} \textit{ and } n_i \geq -2dm\Big\},$$

where $d = \dim X$.

If (X, B) is dlt everywhere, we can take any extremal curve. It is expected that actually we can relax dlt to lc in the proposition and in its corollaries and addenda: The LMMP is sufficient for this in dimension d [9, Conjecture and Heuristic Arguments]. More precisely, the existence of a strictly log minimal model over any lc pair is sufficient. For this, the existence of log flips and special termination in dimension d are enough, which follows from the LMMP in dimension $d-1$ by Corollary 3, [3], and [12, Theorem 2.3]. In addition, the lc property is better than the dlt one: the former is closed (see the example and the proof of Corollary 9 below). However, in our applications we only need the dlt case.

Addendum 2. *The numbers r_i, m, and d depend on the pair $(X/Z, B)$, but they are the same after a* (*generalized*) *log flop outside* $\mathrm{LCS}(X, B)$, *that is, only in curves C with $C \cap \mathrm{LCS}(X, B) = \varnothing$.*

To determine these numbers, we use the following.

Lemma 1. *Under the assumptions of Proposition* 1, *there exists a decomposition* $B = \sum r_i B_i$, *where* r_i *are positive real numbers and* B_i *are* (*Weil*) $\mathbb{Q}$*-boundaries such that*

(1) $\sum r_i = 1$;

(2) *each* $\operatorname{Supp} B_i \subseteq \operatorname{Supp} B$;

(3) *each* $K + B_i$ *is a* $\mathbb{Q}$*-Cartier divisor that has the trivial intersection* $(K + B_i, C) = 0$ *for any curve* C/Z *with* $(K + B, C) = 0$;

(4)
$$K + B = \sum r_i (K + B_i);$$

(5) *each* (X, B_i) *is lc,* $\operatorname{LCS}(X, B_i) \subseteq \operatorname{LCS}(X, B)$, *and* (X, B_i) *is dlt in the locus where* (X, B) *is dlt.*

The last assumption is meaningful because there exists a maximal dlt set in X and it is open: the complement to the closure of log canonical centers that are not dlt.

Proof. The main problem here is the possible real multiplicities of B. Property (4) is immediate by (1). To satisfy (2) and (3), we consider an affine $\mathbb{R}$-space of $\mathbb{R}$-divisors

$$\mathcal{D}_B^0 = \{D \mid \operatorname{Supp} D \subseteq \operatorname{Supp} B, \text{ and } K + D \text{ satisfies the intersection condition of (3)}\}.$$

The last condition means that $K + D$ is $\mathbb{R}$-Cartier and $(K + D, C) = 0$ for any curve C/Z with $(K+B, C) = 0$. This space is actually finite-dimensional and is defined over $\mathbb{Q}$. More precisely, this is a finite-dimensional $\mathbb{R}$-space/$\mathbb{Q}$ in the finite-dimensional $\mathbb{R}$-space $\mathcal{D}_B$ of Weil $\mathbb{R}$-divisors supported in $\operatorname{Supp} B$. Note for this that the $\mathbb{R}$-Cartier condition gives a linear subspace over $\mathbb{Q}$, and any canonical divisor K is integral. Thus, the condition that $K + D$ is $\mathbb{R}$-Cartier gives a finite-dimensional affine subspace over $\mathbb{Q}$. Each condition $(K + D, C) = 0$ is also rational linear because each intersection $(K + B_i, C) = m_i$ is rational. Finally, any $D \in \mathcal{D}_B^0$ is an affine (weighted) linear combination of $\mathbb{Q}$-Cartier divisors $K + B_i$ with B_i (not necessarily boundaries for the present) supported in $\operatorname{Supp} B$. Note that $B \in \mathcal{D}_B^0$.

On the other hand, the $\mathbb{R}$-boundaries $D \in \mathcal{D}_B^0$ with lc (X, D) form a convex closed rational polyhedron [8, 1.3.2], and B belongs to this polyhedron. Any small perturbation inside the polyhedron preserves the klt property outside $\operatorname{LCS}(X, B)$ and the dlt property of (5). Therefore, $K + B$ has a required decomposition (cf. Step 1 in the proof of Corollary 9 below). □

Proof of Proposition 1. The numbers r_i were introduced in Lemma 1. The positive integer m is an index for all divisors $K + B_i$; that is, each $m(K + B_i)$ is Cartier. By (5) of Lemma 1 and the anticanonical boundedness [9, Theorem], each $(K + B_i, C) \geq -2d$. Thus,

$$(K + B, C) = \sum r_i (K + B_i, C) = \sum r_i \frac{n_i}{m},$$

where $n_i \in \mathbb{Z}$ and $n_i \geq -2dm$.

Finally, we prove Addendum 2. For simplicity, we consider only usual log flops (for some remarks about more general flops see below). By these we mean birational rational 1-contractions $X \dashrightarrow X'/Z$ that, just as their inverses, are indetermined only in curves C/Z with $(K + B, C) = 0$. Note that the decomposition $B = \sum r_i B_i$ with all its properties is preserved under log flops in these curves [4, Definition 3.2]. The same holds for the dimension d and index m [7, 2.9.1]. The space $\mathcal{D}_B^0$ and the boundary polyhedron are also preserved (for small flops), or are surjective on the corresponding space and the polyhedron for any log flop. For (5), it is enough that the log flop is outside $\operatorname{LCS}(X, B)$.

A generalized log flop is a crepant modification that can blow up some exceptional divisors with log discrepancies ≤ 1 and >0 outside $\operatorname{LCS}(X, B)$, that is, with centers not in $\operatorname{LCS}(X, B)$.

Kawamata says that such a modification of log pairs is a *log $(K+B)$-equivalence*, and the pairs are *log $(K+B)$-equivalent*. For example, it can be a crepant blowup or its composition with subsequent log flops. Under a certain assumption (see Lemma 3 below), such a blowup exists according to the finiteness of the set of exceptional divisors with log discrepancies ≤ 1 outside $\mathrm{LCS}(X,B)$. Then the $\mathbb{Q}$-Cartier property of $K+B_i$ is enough on a blowup or even on a log resolution. This condition is preserved under log flops (even generalized) because the intersection numbers can be computed on any common resolution by the projection formula. □

Corollary 8 (on an interval; cf. [11, Lemma 6.19]). *Let $(X/Z,B)$ be an lc pair with an $\mathbb{R}$-boundary B. Then there exists a real number $\hbar>0$ such that, for any extremal curve C for which (X,B) is dlt near the generic point of C,*

$$\text{either}\quad (K+B,C)\geq\hbar\qquad\text{or}\quad (K+B,C)\leq 0.$$

Thus, the intersection numbers $(K+B,C)$ do not belong to the interval $(0,\hbar)$.

Addendum 3. *The number $\hbar$ depends on a pair $(X/Z,B)$, but it is the same after a (generalized) log flop outside* $\mathrm{LCS}(X,B)$.

Note that extremal curves may not be preserved under flops!

Proof. By Proposition 1, the intersection numbers $(K+B,C)=\sum r_in_i/m$ with extremal curves C/Z under the assumptions of the proposition satisfy the descending chain condition. Moreover, for any real number A the set of these numbers $\leq A$ is finite. Thus,

$$\hbar=\min\Bigl\{\sum\frac{r_in_i}{m}>0\;\Big|\;n_i\in\mathbb{Z}\ \text{and}\ n_i\geq-2dm\Bigr\}$$

is the required positive number.

By Addendum 2 we can take the same $\hbar$ after any log flop outside $\mathrm{LCS}(X,B)$. □

Example. Let L_i, $i=1,2,3$, be three distinct lines in the plane $\mathbb{P}^2$ that pass through a point P. Then, for $F=L_1+L_2+L_3$,

$$\mathcal{P}=\bigl\{D\in\mathcal{D}_F\mid(\mathbb{P}^2,D)\ \text{is an lc pair with an}\ \mathbb{R}\text{-boundary}\ D\bigr\}$$

is a convex closed rational polyhedron. The face of boundaries $D=\sum b_iL_i$ with $\sum b_i=2$ gives nondlt pairs $(\mathbb{P}^2,D)$ with three exceptions $(\mathbb{P}^2,F-L_i)$, $i=1,2,3$. Thus, the dlt property is not closed and not convex. However, the dlt property holds exactly in $(\mathbb{P}^2\setminus P,D)$ for any interior point D of the face.

Corollary 9 (stability of extremal rays). *Let $(X/Z,B)$ be an lc pair with an $\mathbb{R}$-boundary B and F be a reduced divisor on X. Then there exists a real number $\varepsilon>0$ such that if another $\mathbb{R}$-boundary $B'\in\mathcal{D}_F$ and an extremal contractible ray $R\subset\overline{\mathrm{NE}}(X/Z)$ satisfy the following*:

(1) $\|B'-B\|<\varepsilon$;

(2) *$K+B'$ is $\mathbb{R}$-Cartier, and $(K+B',R)<0$;*

(3) *for some extremal curve C in R, the pair (X,B) is dlt near the generic point of C, and so is (X,B'), possibly for a different extremal curve,*

then the seminegativity $(K+B,R)\leq 0$ holds.

If $(X,B+E)$ is a dlt pair with an effective $\mathbb{R}$-divisor E and $\mathrm{Supp}(B+E)=F$, *then we can omit condition* (3).

Addendum 4. *For a fixed B', a log flop of $(X/Z,B)$ outside* $\mathrm{LCS}(X,B)$ *in any R as in the corollary preserves ε in the direction B'; that is, the stability holds again for any D in the segment $[B_Y,B'_Y]$ on flopped $(Y/Z,B_Y)$, where B_Y and B'_Y denote the birational transforms on Y of the* *corresponding boundaries* B *and* B' *from* X. *More precisely,* $D \in \mathcal{D}_{F_Y}$ *is an* $\mathbb{R}$*-boundary*; *the log flop projects* $\mathcal{D}_F$ *onto* $\mathcal{D}_{F_Y}$ (*some components of* F *are contracted*), *where* F_Y *denotes the birational transform of* F *on* Y;

(1) $\|B_Y - D\| < \varepsilon$.

For any extremal contractible ray $R \subset \overline{\mathrm{NE}}(Y/Z)$ *such that*

(2) $(K_Y + D, R) < 0$ *and*

(3) *for some extremal curve* C *in* R, *the pair* (Y, B_Y) *is dlt near the generic point of* C, *and so is* (Y, B'_Y), *possibly for a different extremal curve,*

$(K_Y + D, R) \le 0$ *holds.*

Caution 1. In other directions, log flops can spoil the singularities of (Y, D).

Addendum 5. *The same applies to any log flop that is a composition of log flops as in Addendum* 4 *and log flops with a weaker condition*:

(2′) $(K + B, R) = (K + B', R) = 0$.

Proof. The main idea is that the dlt property is *conical.*

Step 1. *Choice of* δ. We can suppose that $B \in \mathcal{D}_F$. There exists a real number $\delta > 0$ such that

(3′) for any $\mathbb{R}$-boundary $B' \in \mathcal{D}_F$ with $\mathbb{R}$-Cartier $K + B'$, $\|B' - B\| \le \delta$, and any D in the *open* ray $\overrightarrow{BB'}$ with $\|D - B\| \le \delta$, D is an $\mathbb{R}$-boundary and the pair (X, D) is dlt (exactly) in the locus where so is (X, B'); in particular, (X, D) satisfies (3) for the same extremal curve as (X, B').

By [8, 1.3.2]

$$\mathcal{P}_B = \bigl\{D \in \mathcal{D}_F \mid (X, D) \text{ is an lc pair with an } \mathbb{R}\text{-boundary } D\bigr\}$$

is a convex closed (rational) polyhedral cone in some δ-neighborhood of the vertex B. Unfortunately, a similar set for the dlt property instead of the lc one may not be closed (see the example above). However, the dlt property of (X, D) holds in the same maximal open subset of X for all D in the interior of every face of $\mathcal{P}_B$. Indeed, according to the linear behavior of discrepancies with respect to D, all D in the interior have the same support on X and the same log canonical centers in X. Thus, a dlt resolution of (X, D) over the maximal dlt open subset in X gives the same for any other divisor in the interior. Now we obtain (3′) from the dlt property in the interior of a (minimal) face of $\mathcal{P}_B$ with $D = B' \in \mathcal{P}_B$.

By monotonicity and stability [8, 1.3.3, 1.3.4], under the last assumption in Corollary 9, the dlt property is open and closed in $\mathcal{P}_B$ near B. Moreover, in the definition of $\mathcal{P}_B$, we can replace the lc property by the $\mathbb{R}$-Cartier one near B. Then we do not need (3). Indeed, no prime component of E passes through any log canonical center of (X, B).

Step 2. *The required number is*

$$\varepsilon = \frac{\delta}{N + 1},$$

where N is any positive number $\ge 2d/\hbar$ and $d = \dim X$.

Indeed, if B' is an $\mathbb{R}$-boundary and C is an extremal curve in R as in (1)–(3) of Corollary 9 but $(K + B, C) > 0$, then, by Corollary 8, $(K + B, C) \ge \hbar$ and

$$(B - B', C) = (K + B, C) - (K + B', C) > \hbar.$$

Hence

$$(K + B' + N(B' - B), C) = (K + B', C) + N(B' - B, C) < -N\hbar \le -2d,$$

which contradicts the anticanonical boundedness [9, Theorem]. Indeed, by our choice of ε and (3′), $D = B' + N(B' - B) = B + (N+1)(B' - B)$ is an $\mathbb{R}$-boundary, $\|B - D\| = (N+1)\|B' - B\| < (N+1)\varepsilon \le \delta$, and D satisfies (3′). Thus, $(K + B, R) \le 0$.

Step 3. *Addenda.* In both addenda we consider a log flop in R, that is, $(K + B, R) = 0$. Then we can replace B' by a maximal B' in the direction B', that is, by a (possibly) new boundary B' on the ray $\overrightarrow{BB'}$ such that the quantity $\|B' - B\| = \varepsilon$ or is infinitely close to ε; in the former case we need to slightly decrease ε. (Since we use the maximal absolute value norm, actually in most directions we can take B' with a larger Euclidean distance.) Then B' and any $D' \in (B, B']$ will satisfy properties (1)–(3) of Corollary 9 and (3′) of its proof. In particular, (2) holds because $(K + B, R) = 0$. The same applies to (2′) in Addenda 5.

Property (1) of D in Addendum 4 follows almost by definition. The distance in (1) is $<\varepsilon$ and is less than the length of $[B_Y, B'_Y]$, which may be shorter and even vanish if $B'_Y = B_Y$. If the birational rational 1-contraction $X \dashrightarrow Y$ contracts divisors, then F_Y is smaller than F, and we need to change ε to $\varepsilon_Y = \|B'_Y\|$, which is the maximal absolute value norm with respect to the (noncontracted) prime components of F_Y. (Since F has finitely many components, ε_Y stabilizes after finitely many log flops.) Property (3′) on Y in the direction D'_Y with $\delta_Y = (N+1)\varepsilon_Y$ can be obtained from the fact that log flips and log flops preserve or improve log singularities. Note that log flops outside $\mathrm{LCS}(X, B)$ preserve klt and dlt singularities and $\mathrm{LCS}(X, B)$ itself. In Addendum 4, the log flop is a log flip with respect to $K + D'$ if $D' \ne B$. Such a log flip with respect to $K + D'$ improves the singularities of (X, D'); actually, (3′) holds after the log flop for any D in the segment $[B_Y, (N+1)B'_Y]$; $D' \to D'_Y = D$ is a surjective projection. Thus, by Addendum 2, the constants $\hbar$, N, and d are the same, ε_Y and δ_Y are as above, and under the conditions (2) and (3) of Addendum 4 the required seminegativity holds. This gives the same constants for log flops with $(K + D', R) = 0$.

Addendum 5 follows by induction on the composition. □

Corollary 10 (cf. [11, Corollary 6.18]). *Let $(X/Z, B)$ be a klt wlc model with big $K + B/Z$. Then it has an lc model*; *that is, $K + B$ is semiample.*

Proof. This is well-known when B is a $\mathbb{Q}$-divisor [7, Theorem 2.1; 5, Theorem 3-1-1 and Remark 3-1-2]. By Lemma 1 we have a decomposition $B = \sum r_i B_i$ satisfying (1)–(5) of the lemma. Moreover, if $\|B_i - B\| < \varepsilon$ for a sufficiently small real number $\varepsilon > 0$, we can suppose that $(X/Z, B_i)$ is klt with big $K + B_i/Z$, and by Corollary 9, $K + B_i$ is nef/Z. According to the construction of the decomposition, we can always find boundaries B_i in the ε-neighborhood of B. Therefore, each $K + B_i$ is semiample/Z, and so is $K + B$. Moreover, $\big(X/Z, B' = \sum r'_i B_i\big)$ with any $0 < r'_i \in \mathbb{Q}$, $\sum r'_i = 1$, is a klt minimal model with big $K + B'$ and with the same lc model as $(X/Z, B)$; all these models are *equivalent* in the sense of [11, Definition 6.1]. □

Corollary 11 (stability of wlc models). *Let $(X/Z, B)$ be a dlt wlc model with an $\mathbb{R}$-boundary B and F be a reduced divisor such that, near B for any $B' \in \mathcal{P}_B$, (X, B') is a dlt pair and* $\operatorname{Supp} B \subseteq F$. *Then there exists a real number $\varepsilon > 0$ such that, for any other $\mathbb{R}$-boundary $B' \in \mathcal{D}_F$ with $\|B' - B\| < \varepsilon$, the following statements are equivalent*:

(1) *$H = B' - B$ is nef on any* (*irreducible*) *curve C/Z with $(K + B, C) = 0$*;

(2) *for some real number $0 < \delta < \varepsilon/\|H\|$, $(X/Z, B + \delta H)$ is a dlt wlc model*; *and*

(3) *for any real number $0 < \delta < \varepsilon/\|H\|$, $(X/Z, B + \delta H)$ is a dlt wlc model.*

The dlt property near B holds if there exists an effective $\mathbb{R}$-divisor E such that $(X, B + E)$ is a dlt pair and $\operatorname{Supp}(B + E) = F$.

Addendum 6. *The models in statement* (3) *of Corollary* 11 *are equivalent.*

Proof. We choose the same ε as in the proof of Corollary 9. By our assumption the dlt property near B coincides with the lc one: if, in Step 1 of the proof of Corollary 9, we replace the lc condition by the dlt one, and even by the $\mathbb{R}$-Cartier one, we obtain the same cone $\mathcal{P}_B$ in the ε-neighborhood of the vertex B.

(1) $\Rightarrow$ (3): The nef property in (1) includes the $\mathbb{R}$-Cartier property of H. By our choice of ε, δ, and B', $D = B + \delta H \in \mathcal{P}_B$, and D is an $\mathbb{R}$-boundary. Thus, by the definition of $\mathcal{P}_B$, the pair $(X/Z, D)$ is a dlt pair. If it is not wlc, then $K + D$ is not nef/Z, and by [2, Theorem 2] there exists an extremal contractible ray $R \subset \overline{\mathrm{NE}}(X/Z)$ satisfying conditions (2) and (3) of Corollary 9 with $B' = D$. Hence by the corollary $(K + B, R) \leq 0$, and so $(K + B, R) = 0$ by the wlc property of $(X/Z, B)$. But then $(H, R) < 0$, which contradicts (1).

(3) $\Rightarrow$ (2): Immediate by assumptions.

(2) $\Rightarrow$ (1): Suppose that $(H, C) < 0$ for some (irreducible) curve C/Z with $(K + B, C) = 0$. Then $(K + B + \delta H, C) = \delta(H, C) < 0$, which contradicts the wlc property in (2).

The equivalence of the addendum follows from the linear property of intersections. If $(K + B + \delta H, C) = 0$ for some δ in (3), then $(K + B + \delta H, C) = 0$ for any δ in (3). Otherwise, $(K + B + \delta H, C) < 0$ for some δ in (3). Similarly, if $(K + B + \delta H, C) > 0$ for some δ in (3), then $(K + B + \delta H, C) > 0$ for any δ in (3). $\square$

Lemma 2 (convexity of equivalence)**.** *If two wlc models are equivalent, then all resulting models between them exist and are wlc equivalent to each of the two models.*

For simplicity, one can assume that the models are isomorphic in codimension 1, which is enough for our applications. To explain the more general case, one needs b-divisors and/or Geography.

Proof. Suppose that wlc models $(X/Z, B)$ and $(X/Z, B')$ are equivalent; by definition we can suppose that they have the same variety X/Z. We verify that any model $(X/Z, B'')$ between them, that is, for any $B'' \in [B, B']$, is wlc and equivalent to each of the above models; again we take the same variety X/Z. This also gives the existence of a resulting model for $(X/Z, B'')$.

Indeed, for some real numbers $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, $B'' = \alpha B + \beta B'$. Let C/Z be a curve with $(K + B, C) = 0$. Then $(K + B', C) = 0$ because the models $(X/Z, B)$ and $(X/Z, B')$ are equivalent. Hence, by the linear property of intersection,

$$(K + B'', C) = \alpha(K + B, C) + \beta(K + B', C) = 0.$$

Similarly, if $(K + B, C) > 0$, then $(K + B', C), (K + B'', C) > 0$. Thus, $(X/Z, B'')$ is a wlc model equivalent to $(X/Z, B)$ and $(X/Z, B')$. Notice also that (X, B'') is lc (cf. [8, 1.3.2]) and each log discrepancy $a(E, X, B'') = \alpha a(E, X, B) + \beta a(E, X, B')$. $\square$

The following concept is a formalization of the well-known method from [8, 4.5] for reducing log flips to pl flips, and it is extremely important in our proofs. The same concept under a different name, directed flips, appeared in [1]. However, it is redundant there: in [1, Theorem 3.4 and Corollaries 3.5, 3.6] any sequence of log flips terminates (see comments after Corollary 2).

Definition 2 (H-termination)**.** Let $(X/Z, B)$ be an lc pair and H be an $\mathbb{R}$-divisor. A sequence of log flips (not necessarily extremal)

$$(X_1 = X/Z, B_1 = B) \dashrightarrow (X_2 = X_1^+/Z, B_2 = B_1^+) \dashrightarrow \dots$$

is called *H-ordered* if we can associate a real number $\lambda_i > 0$ with each flip $X_i \dashrightarrow X_{i+1}/Z$ so that

(1) the numbers decrease: $\lambda_1 \geq \lambda_2 \geq \dots$;

(2) each flip $X_i \dashrightarrow X_{i+1}/Z$ is a log flop with respect to $K_{X_i} + B_i + \lambda_i H_i$, where H_i is the birational image of H on X_i; and

(3) each pair $(X_i/Z, B_i + \lambda_i H_i)$ is a wlc model.

We say that the flip $X_i \dashrightarrow X_{i+1}/Z$ has the *level* λ_i with respect to H. So the H-*termination* of a given sequence of H-ordered log flips means that it terminates, i.e., is finite.

It is clear that the termination of any sequence of nontrivial log flips implies the H-termination for its every H-ordering. On the other hand, the H-termination for a sequence of log flips is sufficient for its termination and allows one to construct a resulting model.

Proposition 2. *For any initial klt wlc model* $(X/Z, B+\lambda_1 H)$, *the* H*-termination implies the existence of a resulting model for* $(X/Z, B)$, *in particular, the termination of the corresponding log flips of* $(X/Z, B)$.

Moreover, for a given initial model, an H*-ordered sequence of log flips exists if the log flips exist in dimension* $d = \dim X$.

Proof. Let $X_n \dashrightarrow X_{n+1}/Z$ be the last flip of level λ_n. By definition $(X_{n+1}/Z, B_{n+1}+\lambda_n H_{n+1})$ is a wlc model, and $\lambda = \lambda_n > 0$. If $K_{X_{n+1}} + B_{n+1}$ is nef/Z, then $(X_{n+1}/Z, B_{n+1})$ is a wlc model, a resulting model of $(X/Z, B)$.

Otherwise the divisor $K_{X_{n+1}} + B_{n+1}$ is not nef/Z. By induction we can suppose that $(X_{n+1}/Z, B_{n+1} + \lambda_n H_{n+1})$ is klt (see below the proof of the existence of H-ordered flips). Thus, by Corollary 9 with $(X, B) = (X_{n+1}, B_{n+1} + \lambda_n H_{n+1})$, either $\overline{\mathrm{NE}}(X_{n+1}/Z)$ has an extremal ray R such that $(K_{X_{n+1}} + B_{n+1} + \lambda_n H_{n+1}, R) = 0$, $(H_{n+1}, R) > 0$, and $(K_{X_{n+1}} + B_{n+1}, R) < 0$, or by Corollary 11 there exists $0 < \lambda_{n+1} < \lambda_n$ such that $(X_{n+1}/Z, B_{n+1} + \lambda_{n+1} H_{n+1})$ is a wlc model. In the former case, by our assumptions, R gives a Mori log fibration $X_{n+1} \to Y/Z$, a resulting model with the boundary B_{n+1} for $(X/Z, B)$, because H_{n+1} is numerically ample/Y and $\lambda_n > 0$. In the latter case, we proceed as follows. We can assume that λ_{n+1} is minimal in our construction, that is, with nef $K_{X_{n+1}} + B_{n+1} + \lambda_{n+1} H_{n+1}/Z$; the klt property is preserved by monotonicity [8, 1.3.3]. Then we obtain a Mori log fibration as in the former case.

Now we explain how to extend the sequence of log flips if $X_n \dashrightarrow X_{n+1}$ is not the last one. If the above contraction for R is birational, it has a log flip $X_{n+1} \dashrightarrow X_{n+2} = X_{n+1}^+/Z$ (possibly a divisorial contraction). It is a log flop of the wlc model $(X_{n+1}/Z, B_{n+1} + \lambda_n H_{n+1})$ and thus satisfies (1)–(3) of Definition 2 with $\lambda_{n+1} = \lambda_n$. Note that it preserves the klt property of the pair. Otherwise we consider a similar construction for a minimal $\lambda_{n+1} < \lambda_n$ as above. Again it extends the H-ordered sequence if R corresponds to a birational contraction. Since (X_{n+1}, B_{n+1}) is lc, $(X_{n+1}, B_{n+1} + \lambda_n H_{n+1})$ is klt, and $0 < \lambda_{n+1} < \lambda_n$, by monotonicity [8, 1.3.3] the pair $(X_{n+1}, B_{n+1} + \lambda_{n+1} H_{n+1})$ is klt. Any flop preserves the klt property. Hence $(X_{n+2}, B_{n+2} + \lambda_{n+1} H_{n+2})$ is also klt, which completes the induction.

Usually we include divisorial contractions in log flips. Thus, either we consider not only small modifications as log flops, or we use the fact that, after finitely many log flips, all the next ones are small and so are log flops. □

It is easy to give an example of a sequence of log flips that cannot be H-ordered at least for some divisor H. Take two disjoint birational contractions one of which is positive and the other is negative with respect to H.

In what follows, all isomorphisms of models, e.g., local ones, are induced by their birational isomorphisms.

Theorem 2. *We assume the LMMP in dimension* $d-1$ *and the termination of terminal log flips in dimension* d. *Let* $(X_i/Z, B_i)$ *be a sequence of* d*-dimensional dlt wlc models that converges to a dlt pair* $(X/Z, B)$ *in the following sense*:

(1) *each* X_i *is isomorphic to* X (*and the models are isomorphic to each other*) *in codimension* 1; *all divisors* B_i *and* B *are finitely supported*; *that is, there exists a reduced divisor* F *such that* B *and each* B_i *belong to* $\mathcal{D}_F$;

(2) *each* X_i *is isomorphic to* X *near* $\mathrm{LCS}(X,B) = S = \lfloor B \rfloor$ *and* $\mathrm{LCS}(X_i,B_i) = \mathrm{LCS}(X,B)$: *there exist neighborhoods* U_i *of* $\mathrm{LCS}(X_i,B_i)$ *and* V_i *of* $\mathrm{LCS}(X,B)$ *that are isomorphic and are identified under the birational isomorphism of* (1);

(3) *there exist finitely many prime b-divisors* (*exceptional and nonexceptional*) D_j outside (*possibly not* disjoint *from*) $\mathrm{LCS}(X,B)$, *that is,* $\mathrm{center}_X D_j \not\subseteq \mathrm{LCS}(X,B)$, *that contain all positive* codiscrepancies $b(D_j,X_i,B_i) = 1 - a(D_j,X_i,B_i)$ *outside* $\mathrm{LCS}(X,B)$; *that is, if* $\mathrm{center}_{X_i} D_j \not\subseteq \mathrm{LCS}(X,B)$ *and* $b(D_j,X_i,B_i) > 0$ *for some* i, *then* D_j *is one of these b-divisors*;

(4) *there exists a limit of* b-$\mathbb{R}$-divisors $\overline{B}_i = S + \sum b(D_j,X_i,B_i)D_j$:

$$\overline{B}_{\mathrm{lim}} = S + \sum b_j D_j = S + \sum \lim_{i\to\infty} b(D_j,X_i,B_i)D_j;$$

(5) $B = B_{\mathrm{lim}} = S + \sum b_j D_j$, *where the summation is only over nonexceptional* D_j *on* X, *and* $\overline{B} \geq \overline{B}_{\mathrm{lim}}$, *where* $\overline{B} = S + \sum b(D_j,X,B)D_j$ *is the crepant b-subboundary for* (X,B) *extended in the b-divisors* D_j.

Then the sequence is finite in the model sense; *that is, the set of equivalence classes of models* $(X_i/Z,B_i)$ *is finite.*

Notice that actually (3) implies the existence of a finite support in (1).

Corollary 12. *We assume the LMMP in dimension* $d-1$ *and the termination of terminal log flips in dimension* d. *Let* $(X_i/Z,B_i)$ *be a sequence of* d*-dimensional dlt log pairs such that*

(1) *each* $(X_i/Z,B_i)$ *is a wlc model*;

(2) *the models are isomorphic in codimension* 1 *and isomorphic near* $\mathrm{LCS}(X_i,B_i)$;

(3) *for some* $\mathbb{R}$*-boundaries* B *and* B', *each* $B_i \in [B,B']$ *and the models are ordered in the segment*: $B_i = B + \lambda_i H$, $H = B' - B$, $\lambda_1 \geq \lambda_2 \geq \dots$, $\lambda_i \in (0,1]$; *and*

(4) *for some* i, $(X_i/Z,B)$ *and* $(X_i/Z,B')$ *are dlt log pairs with* $\mathrm{LCS}(X_i,B) = \mathrm{LCS}(X_i,B') = \mathrm{LCS}(X_i,B_i)$.

Then the models stabilize: *the models are equivalent for* $i \gg 0$.

Note that (3) is meaningful because the Weil divisors on each model are the same by the first statement of (2).

Proof. *Step* 1. *Nonequivalence of models.* By Lemma 2 we can suppose that the numbers λ_i form an infinite sequence, $\lambda_0 = \lim_{i\to\infty}\lambda_i$, and the models (X_i,B_i) are pairwise nonequivalent. Otherwise the stabilization holds.

Step 2. *Conditions of Theorem* 2 *hold for an appropriate subsequence.* We can suppose $i = 1$ in our assumption (4). Take $(X = X_1/Z, B := B_{\mathrm{lim}})$, where $B := B_{\mathrm{lim}} = \lim_{i\to\infty} B_i = B + \lambda_0 H$, or $\lambda_0 = 0$ for the new B. Conditions (1) and (2) of Theorem 2 hold by assumptions (2)–(4). We can satisfy condition (3) in Theorem 2 by taking a subset of b-divisors D_j with $b(D_j,X,B) \geq 0$ and $\mathrm{center}_X D_j \not\subseteq \mathrm{LCS}(X,B)$, or equivalently, $\mathrm{center}_X D_j \in X \setminus \mathrm{LCS}(X,B)$; if D_j is also nonexceptional, it is assumed that D_j is supported in $\mathrm{Supp}\, B$ or in $\mathrm{Supp}\, B'$. The set of D_j is finite by [11, Corollary 1.7]. Then the condition holds for (X,B_i) with all B_i sufficiently close to B by assumptions (2) and (4), the stability of the klt property, and the continuity of log discrepancies with respect to the multiplicities in D_j, where B_i on X is its birational transform from X_i. Hence assumption (1) and monotonicity [4, Lemma 2.4] imply (3) in Theorem 2 for all $i \gg 0$.

Up to convergence in (4) of Theorem 2, the conditions in (5) of Theorem 2 follow from the construction and monotonicity [4, Lemma 2.4]. Indeed, $B = B_{\mathrm{lim}}$ by construction, $b(D_j,X,B_i) \geq$

$b(D_j, X_i, B_i)$ by the monotonicity and the wlc property of $(X_i/Z, B_i)$. Thus,

$$b(D_j, X, B) = b(D_j, X, B_{\lim}) = \lim_{i\to\infty} b(D_j, X, B_i) \geq \lim_{i\to\infty} b(D_j, X_i, B_i) = b_j$$

and $\overline{B} \geq \overline{B}_{\lim}$, which gives (5) of Theorem 2.

If, for a fixed D_j, $b(D_j, X_i, B_i)$ is not bounded from below, we can drop such D_j and take a subsequence with $\lim_{i\to\infty} b(D_j, X_i, B_i) = -\infty$. In the bounded case, we finally have a convergent subsequence in (4).

Now the stabilization follows from the finiteness in Theorem 2, which contradicts Step 1. □

Lemma 3 (canonical blowup)**.** *We assume the LMMP in dimension $d-1$ and the termination of terminal log flips in dimension d. Let (X, B) be a klt log pair of dimension d and $Z \subset X$ be a closed subvariety of codimension ≥ 2. Then there exists a* (*unique under the algorithm in the proof*) *crepant blowup $Y \to X$ such that*

(1) *Y is isomorphic to X over $X \setminus Z$;*
(2) *(Y, B_Y) is cn in codimension ≥ 2 over Z;*
(3) *if, in addition, $K_Y + B$ with the birational transform of the boundary B on Y is ample/X, a blowup is unique.*

Proof–construction. *Step* 1. Consider a log resolution $(Y/X, B^+)$ with a boundary $B^+ = \sum b_j^+ D_j$ consisting of codiscrepancies: $b_j^+ = \max\{b(D_j, X, B), 0\}$. We can suppose that the prime components of $\operatorname{Supp} B^+$ are disjoint and there exist only finitely many exceptional b-divisors E/X with $b(E, X, B) \geq 0$, or equivalently with $a(E, X, B) \leq 1$, by the klt property [11, Corollary 1.7]. Moreover, all $b_j^+ < 1$, and (Y, B^+) is terminal in codimension ≥ 2. We use a slightly different boundary $B_Z \leq B^+$ on Y: 0 in all exceptional divisors/Z and B^+ elsewhere.

Step 2. We apply the LMMP to $(Y/X, B_Z)$. Log flips exist by [3, Theorem 1.1] or induction of Corollary 3. Each flip is terminal because we never contract nonzero components E of B_Z. Indeed, this holds over Z by assumption because the corresponding boundary multiplicities are zero. Otherwise we get a component E with $P = \operatorname{center}_X E \not\subset Z$, and $(X/X, B)$ near P is the lc model of $(Y/X, B_Z)$, even after the divisorial contraction. This contraction decreases the codiscrepancy and increases the discrepancy in E, which contradicts [4, Lemma 2.4]. Thus, the termination holds by our assumptions. Since the resulting model $(Y/X, B_Z)$ is birational/X, it is terminal in codimension ≥ 2 and is a strictly log minimal model.

Step 3. Using semiampleness in the big klt case, we obtain the lc model $(Y/X, B_Z)$; the previous model Y in Step 2 and this model Y are FT/X (by Step 4 below). The model satisfies (1) and (3). Indeed, (1) follows from the uniqueness of an lc model. By construction B_Z on Y is the birational transform of B and $K_Y + B$ is ample, and such a model is also unique.

Step 4. By the negativity [8, 1.1], for the crepant model (Y, B_Y) of (X, B), the subboundary B_Y is a boundary. It is still possible that B_Y have noncanonical singularities of codimension ≥ 2. Then we can apply Steps 1 and 2 to (Y, B_Y) with Z_Y being the union of all noncanonical centers/Z of (Y, B_Y). This process is terminated, and we finally obtain a crepant model $(Y/X, B_Y)$. Indeed, each time the construction blows up at least one exceptional divisor E with $a(E, X, B) > 0$ over a noncanonical center, and there exist only finitely many such divisors. However, in general, $K_Y + B$ may not be ample over X. The above algorithm gives a unique blowup by (3). □

Lemma 4 (D-flip)**.** *We assume the LMMP in dimension $d-1$ and the termination of terminal log flips in dimension d. Let (X, B) be a klt pair of dimension d and D be a prime divisor on X such that (X, B) is terminal in codimension ≥ 2 at D; that is, if E is an exceptional prime divisor with $a(E, X, B) \leq 1$, then* $\operatorname{center}_X E \not\subseteq D$. *Then a D-flip of X/X exists* [11, p. 2684].

Proof–construction. The construction is quite standard (cf. terminalization [4, Theorem 6.5] and $\mathbb{Q}$-factorialization [4, Lemma 7.8]). By the uniqueness of D-flips, they can be constructed locally/X [12, Corollary 3.6].

Step 1. As in Steps 1 and 2 of the proof of Lemma 3 with $Z = \varnothing$, that is, the initial $B_Z = B^+$, we obtain a crepant blowup $(Y/X, B_Y)$ that is terminal in codimension ≥ 2 and strictly log minimal/X. By our assumptions there are no exceptional divisors/D.

Step 2. Let D be its birational transform on Y. Now we apply the D-MMP in order to construct a nef D/X. For a sufficiently small real number $\varepsilon > 0$, $(Y/X, B_Y + \varepsilon D)$ is terminal in codimension ≥ 2, and the D-MMP is the LMMP for the pair. Again log flips exist. The termination is terminal and holds: D-flips do not contract any divisor. Thus, we can suppose that D is nef/X.

Step 3. The contraction given by D or $K_Y + B_Y + \varepsilon D$ is the required model for D, a D-flip. The contraction exists as in Step 3 of the proof of Lemma 3 because Y is FT/X. Note that the model is small over D, because Y does not have exceptional divisors/D. On the other fibers/X, D is trivial, and Y is isomorphic to X over $X \setminus D$. □

Main lemma. *Let $(X/Z, B)$ be a dlt pair, $(X'/Z, B_{X'})$ be its wlc model isomorphic to (X, B) near* $\mathrm{LCS}(X, B) = \mathrm{LCS}(X', B_{X'})$, *and $X \to Y/Z$ be an (extremal) contraction negative with respect to $K + B$. Then the contraction is birational, with the exceptional locus disjoint from* $\mathrm{LCS}(X, B)$, *and contracts only b-divisors D with $b(D, X, B) > b(D, X', B_{X'})$.*

A contracted b-divisor D has $\mathrm{center}_X D$ in the exceptional locus.

Proof. By our assumptions $B^{\log}_{X'} = B_{X'}$, that is, $X \dashrightarrow X'$ is a rational birational 1-contraction. Thus, by [4, Proposition 2.5(ii)] (cf. [11, Proposition 2.4.1]) $X \to Y/Z$ is not fibered. In addition, for any irreducible curve C/Z intersecting $\mathrm{LCS}(X, B)$, $(K + B, C) \geq (K_{X'} + B_{X'}, C') \geq 0$, where C' is the birational image of C on X'; the latter is well defined by our assumptions. Indeed, the log discrepancies for prime b-divisors with centers near $\mathrm{LCS}(X, B)$, in particular, for centers intersecting $\mathrm{LCS}(X, B)$, are the same for $(X', B_{X'})$ and (X, B), and by monotonicity [4, Lemma 2.4] $b(D_i, X, B) \geq b(D_i, X', B_{X'})$ for the other b-divisors D_i. This and the projection formula for a common resolution of X and X' imply the inequality (cf. the proof of [4, Proposition 2.5(i)]). In particular, the exceptional locus of X/Y is disjoint from $\mathrm{LCS}(X, B)$.

Now for simplicity suppose that there exists a log flip $(X^+/Y/Z, B^+)$ of X/Y: in our applications we always have it. (Otherwise one can use the last statement of [4, Lemma 2.4].) Then $(X'/Z, B_{X'})$ is also a wlc model of the divisorial contraction or of the log flip; this is the basic fact of the LMMP. Thus, $b(D, X, B) > b(D, X^+, B^+) \geq b(D, X', B_{X'})$, or equivalently, $a(D, X, B) < a(D, X^+, B^+) \leq a(D, X', B_{X'})$ by monotonicities [4, Lemmas 3.4, 2.4]. □

Proof of Theorem 2. Taking a subsequence, we can suppose that the models $(X_i/Z, B_i)$ are pairwise nonequivalent. Then we need to verify that the sequence is finite.

We care only about models outside $\mathrm{LCS}(X, B)$. Near the $\mathrm{LCS}(X, B)$, the models are dlt by (2), and we will keep this: assuming F is minimal in (1),

(6) in the proof below, all models $(Y/Z, D)$ with an $\mathbb{R}$-boundary D are isomorphic to X near $\mathrm{LCS}(Y, D) = \mathrm{LCS}(X, B)$, $D \in \mathcal{D}_F$ near $\mathrm{LCS}(X, B)$, and thus there exists a real number $\varepsilon > 0$ such that (Y, D) is dlt near $\mathrm{LCS}(X, B)$ if, in addition, $K_Y + D$ is $\mathbb{R}$-Cartier and $\|D - B'\| < \varepsilon$.

Notice also that in our construction below B, B_i, and similar $\mathbb{R}$-boundaries D will have the same reduced part: $\mathrm{LCS}(Y, D) = \lfloor D \rfloor = \mathrm{LCS}(X, B) = S$.

Step 1. *Terminal limit.* We construct a dlt model $(\overline{X}/Z, B_{\overline{X}})$ such that

(7) $\overline{X}$, X, and each X_i are isomorphic near $\mathrm{LCS}(X, B) = \mathrm{LCS}(\overline{X}, B_{\overline{X}})$;

(8) $\overline{X} \dashrightarrow X$ and each $\overline{X} \dashrightarrow X_i$ are birational rational 1-contractions; $\overline{X}$ blows up all D_j of (3) with $b_j \geq 0$ and with $\operatorname{center}_X D_j \cap \operatorname{LCS}(X,B) = \varnothing$;

(9) every $\mathbb{R}$-divisor D on $\overline{X}$ that is $\mathbb{R}$-Cartier near $\operatorname{LCS}(X,B)$ is $\mathbb{R}$-Cartier everywhere on $\overline{X}$; in particular, each divisor D on $\overline{X}$ with $\operatorname{Supp} D \cap \operatorname{LCS}(X,B) = \varnothing$ is $\mathbb{Q}$-Cartier;

(10) $B_{\overline{X}} \geq \overline{B}_{\lim}$ as b-divisors but divisors on $\overline{X}$, that is, for multiplicities in the prime divisors on $\overline{X}$; in particular, in D_j with nonnegative multiplicities b_j and with $\operatorname{center}_X D_j \cap \operatorname{LCS}(X,B) = \varnothing$ (see (8)); and

(11) the pair $(\overline{X}, B_{\overline{X}})$ is dlt and is terminal *completely* outside $\operatorname{LCS}(X,B)$ in the following sense: $a(E,\overline{X},B_{\overline{X}}) > 1$, or equivalently $b(E,\overline{X},B_{\overline{X}}) < 0$, for each exceptional prime divisor E with $\operatorname{center}_{\overline{X}} E \cap \operatorname{LCS}(X,B) = \varnothing$.

By Lemma 3 we can construct a slightly weaker version with properties (7) and (8) for $b_j > 0$ by (5), because $b(D_j,X,B) \geq b_j > 0$, and with (10). We apply the lemma to (X,B) with the closed subvariety that is the union of $\operatorname{center}_X E$ for exceptional divisors E with $a(E,X,B) < 1$, or equivalently $b(E,X,B) > 0$, and with $\operatorname{center}_X E \cap \operatorname{LCS}(X,B) = \varnothing$, in particular, of all $\operatorname{center}_X D_j$ in (8) with $b_j > 0$.

Since $\overline{X}$ may not be $\mathbb{Q}$-factorial, we slightly modify $\overline{X}$ to make it sufficiently $\mathbb{Q}$-factorial and terminal. To blow up the canonical centers lying completely outside or *disjoint* from $\operatorname{LCS}(X,B)$, we can use an increased divisor (with the boundary outside $\operatorname{LCS}(X,B)$) $B_{\overline{X}} + \varepsilon H$, where H is a general ample Cartier divisor passing through such centers. For a sufficiently small real number $\varepsilon > 0$, the noncanonical centers of $(\overline{X}, B_{\overline{X}} + \varepsilon H)$ are only the canonical $\operatorname{center}_{\overline{X}} E$'s with $a(E,\overline{X},B_{\overline{X}}) = 0$ and $\operatorname{center}_{\overline{X}} E \cap \operatorname{LCS}(X,B) = \varnothing$ (cf. [8, 1.3.4]). This gives (8) for $b_j = 0$ by (5) and (11) and preserves (7) and (10). To satisfy (9), it is enough to perform this for one divisor D that is sufficiently general near $\operatorname{LCS}(X,B)$. Indeed, by the rationality of klt singularities, the Weil $\mathbb{R}$-divisors modulo $\sim_{\mathbb{R}}/X \setminus \operatorname{LCS}(X,B)$ have finitely many generators. Since the $\mathbb{R}$-Cartier property defines a linear $\mathbb{R}$-subspace over $\mathbb{Q}$ among $\mathbb{R}$-divisors, we can suppose that its generators D are Cartier near $\operatorname{LCS}(X,B)$ and integral. Adding ample divisors, we can suppose that they are prime and free near $\operatorname{LCS}(X,B)$ and thus by (11) do not pass through the canonical (i.e., nonterminal) centers outside $\operatorname{LCS}(X,B)$ (even everywhere). We can make each D $\mathbb{Q}$-Cartier one by one. According to Lemma 4, there exists a small modification (D-flip) over $\overline{X}$ such that D is $\mathbb{Q}$-Cartier on the modification. This gives (9) and concludes the step. The dlt property of (11) near $\operatorname{LCS}(\overline{X}, B_{\overline{X}})$ holds by (6) and (7).

Step 2. *Limit of boundaries.* For each i, let B_i^+ be an $\mathbb{R}$-boundary on $\overline{X}$ with multiplicities $\max\{b(D,X_i,B_i),0\}$ in the prime divisors D on $\overline{X}$. We can replace (10) by a more precise version:

(10′) $B_{\overline{X}} = \overline{B}^+_{\lim} = S + \sum b_j^+ D_j$, $b_j^+ = \max\{b_j,0\}$, as b-boundaries, including exceptional D_j on X with the nonnegative multiplicities $b_j = b_j^+$ and with $\operatorname{center}_X D_j \cap \operatorname{LCS}(X,B) = \varnothing$ (cf. (3), (8), and (10) above), and $0 = b_j^+$ for all other D_j with $\operatorname{center}_X D_j \cap \operatorname{LCS}(X,B) = \varnothing$ (and with $b_j < 0$; and such D_j are possible); or, equivalently,

$$B_{\overline{X}} = \lim_{i\to\infty} B_i^+.$$

By monotonicity [8, 1.3.3] property (11) is preserved; (6) and the other properties of $(\overline{X}/Z, B_{\overline{X}})$ are also preserved. By (9) the $\mathbb{R}$-Cartier property holds for all adjoint divisors $K_{\overline{X}} + B_i^+$ and $K_{\overline{X}} + B_{\overline{X}}$; $B_{\overline{X}} = B = B_{\lim}$ near $\operatorname{LCS}(X,B)$.

In addition, by (6), (7), (9), and (11),

(11′) each $(\overline{X}, B_i^+)$ is a dlt pair terminal in the sense of (11); $\operatorname{LCS}(\overline{X}, B_i^+) = \operatorname{LCS}(\overline{X}, B_{\overline{X}}) = \operatorname{LCS}(X,B)$; and $B_i^+ = B_i$ near $\operatorname{LCS}(X,B)$.

This is true by the stability of terminal and klt singularities (cf. [8, 1.3.4]) after taking a subsequence of models $(X_i/Z, B_i)$ for all $i \gg 0$. The last statement in (11′) allows one to use the properties of B_i for B_i^+ near $\operatorname{LCS}(X,B)$, e.g., (2).

Step 3. *Wlc terminal limit.* We can suppose that $(\overline{X}/Z, B_{\overline{X}})$ is a wlc model terminal in the sense of (11). Otherwise by (11) there exists an extremal contraction $\overline{X} \to Y/Z$ negative with respect to $K_{\overline{X}} + B_{\overline{X}}$ [2, Theorem 2]. We claim that the contraction is birational, does not contract components D of $B_{\overline{X}}$ with positive multiplicities, and does not touch $\mathrm{LCS}(X, B)$, i.e., is an isomorphism in a neighborhood of $\mathrm{LCS}(X, B)$. Indeed, such a contraction is stable for a small perturbation of the divisor $B_{\overline{X}}$: for any $\mathbb{R}$-boundary $B' \in \mathcal{D}_{F+\sum D_j}$ sufficiently close to $B_{\overline{X}}$ and with $\mathbb{R}$-Cartier $K_{\overline{X}} + B'$, the contraction will be negative with respect to $K_{\overline{X}} + B'$. By (10′) and (11′), for all $i \gg 0$, the contraction is negative with respect to $K_{\overline{X}} + B_i^+$. By construction and definition $(X_i/Z, B_i)$ is a wlc model of $(\overline{X}/Z, B_i^+)$. Notice that $B_i = (B_i^+)_{X_i}^{\log}$ because $\overline{X} \dashrightarrow X_i$ is a birational rational 1-contraction by (1) and (8). Therefore, the contraction is not fibered by the main lemma; or, equivalently, it is birational. The contraction is disjoint from $\mathrm{LCS}(X, B)$ by (6), (7), and the same lemma. Finally, the contraction does not contract prime divisors D with positive multiplicities in $B_{\overline{X}}$ because by (10′) they are positive for B_i^+ with $i \gg 0$ and $D = D_j$ with $b_j > 0$. This is impossible by the main lemma again: $b(D_j, \overline{X}, B_i^+) = b(D_j, X_i, B_i)$. Moreover, the contraction does not contract D_j with $b_j = 0$. Indeed, after such a contraction $a+1 = a(D_j, Y, B_Y) > a(D_j, \overline{X}, B_{\overline{X}}) = 1$, $a > 0$, and since the boundaries B_{Yi}^+ for all $i \gg 0$ are small perturbations of B_Y, it follows that $a(D_j, Y, B_{Yi}^+) \ge 1 + a/2$ for all $i \gg 0$, where B_Y and B_{Yi}^+ are the images of $B_{\overline{X}}$ and B_i^+, respectively, on Y. Each $(X_i/Z, B_i)$ is also a wlc model of $(Y/Z, B_{Yi}^+)$, and $a(D_j, X_i, B_i) \ge a(D_j, Y, B_{Yi}^+) \ge 1 + a/2$ by [4, Lemma 2.4], or equivalently, $b(D_j, X_i, B_i) \le -a/2$ and

$$0 = b_j = \lim_{i\to\infty} b(D_j, X_i, B_i) \le -a/2 < 0,$$

a contradiction.

Therefore, either the contraction $\overline{X}/Y$, if it is divisorial, or otherwise its log flip preserve properties (7)–(11), (10′), and (11′) with the images of the corresponding boundaries. Note that by (9) the divisorial extremal contraction blows down a $\mathbb{Q}$-Cartier divisor disjoint from $\mathrm{LCS}(X, B)$ and, in particular, preserves (9). The log flip exists by [3, Theorem 1.1] or induction of Corollary 3, since it does not touch $\mathrm{LCS}(X, B)$ and thus it is klt. It preserves (9) by its extremal property (cf. [7, 2.13.5]). Of course, property (11′) holds after taking models for all $i \gg 0$.

Since each log flip is extremal by construction and terminal by (11), the flips terminate and we obtain a wlc $(\overline{X}/Z, B_{\overline{X}})$ with the required terminal property.

Caution 2. A log flip may be non-$\mathbb{Q}$-factorial; that is, $\overline{X}$ may be non-$\mathbb{Q}$-factorial. However, according to the usual reduction [12, Theorem 1.2], such a log flip exists (cf. also the proof of Corollary 3 above).

Termination may also be non-$\mathbb{Q}$-factorial. We can reduce it to the usual $\mathbb{Q}$-factorial terminal termination as in the proof of special termination [4, Theorem 4.8] by taking a strictly log terminal blowup of $(\overline{X}, B_{\overline{X}})$; for a dlt pair, a $\mathbb{Q}$-factorialization can be constructed in this way. To construct such a model for any lc pair in dimension d, the existence of $\mathbb{Q}$-factorial log flips and special termination in this dimension are sufficient. The existence of terminal non-$\mathbb{Q}$-factorial log flips follows from the same construction and Corollary 10 (cf. the proof of Lemma 4).

Step 4. *Equivalence intervals.* The intervals belong to the affine space $\mathcal{B}$ of $\mathbb{R}$-divisors on $\overline{X}$ generated by the divisors $B_{\overline{X}}$ and B_i^+. It is a finite-dimensional subspace in the linear space of $\mathbb{R}$-divisors having the support in divisors D_j and the birational transform of F by (1), (10′), (11′), and (6)–(8). In this affine space $B_{\overline{X}} = \lim_{i\to\infty} B_i^+$. Geography of log models [11, Section 6; 4, 2.9] gives an expectation that near $B_{\overline{X}}$, that is, for boundaries in $\mathcal{B}$ close to $B_{\overline{X}}$, there are only finitely many equivalence classes of wlc models satisfying (6). We prove this partially: there exists a real number $\varepsilon > 0$ such that, in each direction B_i^+, the wlc models are equivalent in the interval of length ε. Of course, we can assume that each $B_i^+ \ne B_{\overline{X}}$: otherwise the model $(X_i/Z, B_i)$ is equivalent to $(\overline{X}/Z, B_{\overline{X}})$ (cf. the proof below). Hence each direction is well-defined. More precisely, there exists an $\mathbb{R}$-boundary $B_i' \in \mathcal{B}$ such that

(12) $\|B_i' - B_{\overline{X}}\| = \varepsilon$;

(13) $B_i^+ \in (B_{\overline{X}}, B_i')$; and

(14) all wlc models in the interval $(B_{\overline{X}}, B_i')$ are equivalent: for any $D \in (B_{\overline{X}}, B_i')$, $D \in \mathcal{B}$ and is an $\mathbb{R}$-boundary, $\lfloor D \rfloor = S$, and $(X_i/Z, D_{X_i})$ is a dlt wlc model equivalent to $(X_i/Z, B_i)$ and having nonnegative codiscrepancies $b(D_j, X_i, D_{X_i}) \geq 0$ only in the above generators D_j of the linear space of $\mathbb{R}$-divisors (see the beginning of this step), where D_{X_i} is the image of D on X_i; D, just as B_i^+, also satisfies (6); D_{X_i}, just as B_i, satisfies (2), (3), and (6).

We can suppose that the $\mathbb{R}$-boundaries $D \in \mathcal{B}$ form a cone with the vertex $B_{\overline{X}}$ in the ε-neighborhood of $B_{\overline{X}}$ (cf. [8, 1.3.2]). To establish (14), we use ε from Corollaries 9 and 11 under the additional assumption

(11″) for each $\mathbb{R}$-boundary $D \in \mathcal{B}$ with $\|D - B_{\overline{X}}\| \leq \varepsilon$, with $\mathbb{R}$-Cartier $K_{\overline{X}} + D$, and with $\lfloor D \rfloor = S$, $(\overline{X}, D)$ is a dlt pair satisfying (6) and the terminal property of (11).

This follows from the stability of terminal and klt singularities under small perturbations of $\mathbb{R}$-boundaries [8, 1.3.4].

Indeed, take $D = B_i'$ satisfying (12), (13), and thus (11″). Then it is an $\mathbb{R}$-boundary and by (10′) property (13) holds for all $i \gg 0$. Now we apply the LMMP to $(\overline{X}/Z, B_i^+)$. Again by (6) as in Step 3 above, if $K_{\overline{X}} + B_i^+$ is not nef, there exists an extremal contraction negative with respect to $K_{\overline{X}} + B_i^+$ [2, Theorem 2]. Its log flip exists by [3, Theorem 1.1] or induction of Corollary 3. The termination holds as in Step 3 (see also Caution 2 above). As in that step all extremal contractions are birational, disjoint from $\mathrm{LCS}(X, B)$, terminal, and do not contract any prime component D_j disjoint from $\mathrm{LCS}(X, B)$ with $b(D_j, X_i, B_i) \geq 0$ being equal to the multiplicity of B_i^+ in D_j; other divisors (even D_j) with multiplicity 0 of B_i^+ can be contracted. According to the terminal termination we obtain a dlt wlc model $(\overline{X}_i/Z, B_i^+)$, and $(X_i/Z, B_i)$ is its wlc model. They are equivalent; in particular, $(\overline{X}_i/Z, B_i^+)$ is a crepant pair of $(X_i/Z, B_i)$ and $B_i^+ = \overline{B}_i$ on all divisors of $\overline{X}_i$, in particular, on D_j blown up on $\overline{X}_i$. By Corollary 11 and its Addendum 6, the same holds for any D in the interval $(B_{\overline{X}}, B_i')$ with the corresponding model $(X_i/Z, D_{X_i})$. The dlt property holds near $B_{\overline{X}}$ by (11″) for F being the minimal reduced divisor on $\overline{X}$ supporting $\mathcal{B}$. The models are equivalent to each other by Addendum 6. Notice also that by Corollary 9 the above log flips are log flops with respect to $K_{\overline{X}} + B_{\overline{X}}$. Thus, in any direction, ε is preserved by Addenda 4 and 5. By (1), (7), and (8) we can suppose that all $\operatorname{Supp} B_i^+$ are the same. Then in the addenda ε and δ are preserved for log flops with respect to $K_{\overline{X}} + B_{\overline{X}}$: the components of $\operatorname{Supp} B_i^+$ are not contracted (see Step 3 in the proof of Corollary 9).

To fulfill the last statement in (14), one should add some b-divisors D_j as generators of the linear space of $\mathbb{R}$-divisors, namely, all D_j with $b_j \geq 0$ and, in particular, those D_j that intersect $\mathrm{LCS}(X, B)$.

Step 5. 1-*dimensional case.* If the real affine space $\mathcal{B}$ has dimension 1, there are at most two intervals $(B_{\overline{X}}, \pm B_i')$ and at most two types of models. For a higher dimension we use

Step 6. *Induction, or limit of equivalence intervals.* The intervals $[B_{\overline{X}}, B_i']$ have a convergent subsequence $\lim_{l\to\infty} B_{i_l}' = B' \in \mathcal{B}$. Otherwise we have finitely many intervals and models as in Step 5. The limit B' is also an $\mathbb{R}$-boundary on $\overline{X}$ and by construction satisfies (11″). Now we cut the limit by an affine *rational* hyperplane: there exists an affine rational hyperplane $\mathcal{B}' \subset \mathcal{B}$ such that it intersects $(B_{\overline{X}}, B')$ in B_s and the intervals $(B_{\overline{X}}, B_{i_l}')$ in B_{sl}^+. The new boundaries B_s and B_{sl}^+ on $\overline{X}$ and the images B_{sl} of the latter on X_{i_l} instead of $B_{\overline{X}}$, $B_{i_l}^+$, and B_{i_l}, respectively, satisfy the same conditions (1)–(11), (10′), and (11′) after taking a subsequence. Thus, the space of divisors corresponding to $\mathcal{B}$ is a subspace of $\mathcal{B}'$. (Actually we do not need $(X/Z, B)$ and the corresponding properties; $(\overline{X}, B_{\overline{X}} := B_s)$ is sufficient.) Properties (1), (2), (6), (7), (9), (11), and even (11′) are immediate by construction. In (3) we can keep the same D_j by (14). If the set of codiscrepancies $b(D_j, X_{i_l}, B_{sl})$ is unbounded from below for some D_j, we take a corresponding subsequence and discard this D_j. Therefore, we can find a subsequence satisfying (4). Then (5), even (10), (10′), and (8) hold by construction. For (10′), notice that $B_{sl}^+ = \overline{B}_{sl}$ for all D_j with nonnegative multiplicities in $\overline{B}_{sl}$ by Step 4 and (14); we extend $\overline{B}_{sl}$ by 0 in other components exceptional on X_{i_l} (cf. Step 2).

However, $(\overline{X}/Z, B_s)$ is not necessary wlc. Therefore, we apply again Step 3, etc. This completes the induction on the dimension of $\mathcal{B}$. □

Proof–construction of Theorem 1. We construct strictly log terminal resulting models $(X/Z, B_\lambda)$, $B_\lambda = B^{\log} + \lambda H$, for some effective $\mathbb{R}$-divisor H and all $\lambda \in [0,1]$ and find a real number $\lambda_0 \in [0,1)$ such that $(X/Z, B_\lambda)$ are minimal for $\lambda \geq \lambda_0$ and are Mori log fibrations for $\lambda < \lambda_0$. Thus, we get a required minimal model for $\lambda_0 = 0$ and a Mori log fibration in all other cases.

Step 1. Using a Hironaka resolution, we can suppose that $(X/Z, B^{\log})$ is strictly log terminal.

Step 2. Then, by special termination [12, Theorem 2.3; 13, Corollary 4], we can suppose that in any sequence of log flips of $(X/Z, B^{\log})$ (H-ordered or not), the flips are *nonspecial*, that is, do not intersect $\mathrm{LCS}(X, B^{\log})$. (For $\lambda_0 = 0$, this means that $K + B^{\log}$ is nef on $\mathrm{LCS}(X, B^{\log})/Z$; see Step 4 below.)

Step 3. We can add a rather ample $\mathbb{R}$-boundary $H = \sum h_i H_i$, $h_i \neq 0$, with prime divisors H_i such that

(1) $\lfloor H \rfloor = 0$ and $\operatorname{Supp} H \cap \operatorname{Supp} B^{\log} = \varnothing$ in codimension 1;

(2) $(X/Z, B_1 = B^{\log} + H)$ is a strictly log minimal model;

(3) the prime components H_i of $\operatorname{Supp} H$ generate the numerical classes of all divisors/Z; and

(4) the multiplicities h_i of H are *independent* over $\mathbb{Q}(B)$:

$$\sum a_i h_i = a \quad \text{and} \quad \text{all } a_i, a \in \mathbb{Q}(B) \quad \Rightarrow \quad \text{all } a_i = 0 \quad \text{and} \quad a = 0,$$

where $\mathbb{Q}(B) = \mathbb{Q}(B^{\log}) \subset \mathbb{R}$ is the field generated/$\mathbb{Q}$ by the multiplicities of B and $B^{\log}$, respectively.

Since $\mathbb{Q}(B)$ is countable (small), it is easy to find the required h_i as small perturbations of the multiplicities for a divisor H with ample $K + B^{\log} + H$.

Step 4. If $K + B_0$ is nef, then $\lambda_0 = 0$, and we are done: $(X/Z, B_0 = B^{\log})$ is a strictly log minimal model.

Otherwise there exists

$$0 < \lambda_1 = \min\{\lambda \mid K + B_\lambda \text{ is nef}/Z\}.$$

$(X/Z, B_{\lambda_1})$ is a strictly log minimal model too.

Step 5. *H-ordered flips.* As in the proof of Proposition 2, the construction terminates at the level λ_1 by a Mori log fibration, or one can find a log flip (possibly a divisorial contraction) $X_1 \dashrightarrow X_2/Z$ with respect to $K + B_0$ of level λ_1. To prove the existence of a Mori log fibration or of a flipping contraction, one can use Corollary 9. The flip exists by [3, Theorem 1.1] or induction of Corollary 3. By Step 2 the log flop $X_1 \dashrightarrow X_2/Z$ with respect to $K + B_{\lambda_1}$ does not touch $\mathrm{LCS}(X, B_{\lambda_1}) = \mathrm{LCS}(X, B_0)$ (see property (1) in Step 3). Thus, it preserves the strictly log minimal model property of $(X/Z, B_{\lambda_1})$, that is, $(X^+/Z, B_{\lambda_1})$ is a strictly log minimal model too. By Corollaries 9, 11 and Addenda 4, 5, as in the proof of Proposition 2, we obtain an H-ordered sequence of extremal log flips $X_i \dashrightarrow X_{i+1}/Z$ that are disjoint from $\mathrm{LCS}(X_i, B_{\lambda_i}) = \mathrm{LCS}(X, B_0)$.

Step 6. *Termination of log flops.* For each level $\lambda > 0$, the log flips are log flops with respect to $K + B_\lambda$. We claim that there exists at most one such flop, or equivalently, at most one extremal contractible ray R with $(K + B_\lambda, R) = 0$.

Indeed, let C be a curve/Z in R. Hence $(K + B^{\log} + \lambda H, C) = 0$. Let C' be another curve/Z with $(K + B^{\log} + \lambda H, C') = 0$. We will verify that C' is also in R. By the definition of $\mathbb{Q}(B)$ we have two relations

$$\lambda(H, C) = a \qquad \text{and} \qquad \lambda(H, C') = a'$$

with real numbers $a, a' \in \mathbb{Q}(B)$. Moreover, by (3) and (4) $a, a' \neq 0$. Otherwise, $a = 0$, all $(H_i, C) = 0$, and $C \equiv 0/Z$; the same holds for a' and C'. Therefore,

$$(H, C) = \frac{a}{\lambda}, \qquad (H, C') = \frac{a'}{\lambda}$$

and

$$(H, a'C - aC') = \frac{a'a}{\lambda} - \frac{aa'}{\lambda} = 0.$$

Thus, if all $(H_i, a'C - aC') = 0$, by (3) $C' \equiv \frac{a'C}{a/Z}$ and C' is in R; $a'/a > 0$ by the projectivity of X/Z. Otherwise, the 1-cycle $a'C - aC'$ gives a nontrivial relation/$\mathbb{Q}(B)$, which contradicts (4).

Step 7. *Stabilization of models.* The levels stabilize by Corollary 12 for $(X_i/Z, B_i) = (X/Z, B_{\lambda_i})$; that is, there exist only finitely many levels: $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$. Of course, we use here a birationally changing model X. This means that actually X depends on λ. After finitely many log flips we can suppose that condition (2) of Corollary 12 holds. The second statement in (2) holds by Step 2. The other conditions (1), (3), and (4) of Corollary 12 follow immediately from the construction with $B = B_{\lambda_1}$ and $B' = B_0$ in (3) and with $X_i = X_1 = X$ in (4) of the minimal model $(X/Z, B_{\lambda_1})$. Notice also that models with distinct levels are nonequivalent by Lemma 2. Indeed, by the construction of $\lambda_{i+1} > 0$ and Step 5 (see also Corollary 11 and its Addendum 6), the model $(X/Z, B_{\lambda_{i+1}})$ has an extremal ray R with $(K + B_{\lambda_{i+1}}, R) = 0$ in which the model is not numerically equivalent to $(X/Z, B_{\lambda_i})$: $(K + B_{\lambda_i}, R) > 0$, and thus it is not equivalent to $(X/Z, B_{\lambda_j})$ for $j \leq i$.

Finally, the last statement in the theorem follows from two facts: the numerical log Kodaira dimension of each minimal model $(X/Z, B)$ is ≥ 0, and this is equivalent to the pseudo-effective property of $K + B$ (cf. Corollary 1). $\square$

Proof of Revised Reduction. The main idea of the reduction is to find an ordered sequence of pl flips that, at any level, has at most one nonspecial flip of this level on each reduced component of the boundary. Therefore, the termination of these flips amounts to the special termination and stabilization of models. Perturbing and subtracting a divisor H, we can achieve this.

We use the proof–construction and notation from [4, § 4]. The construction of a strictly log minimal model $(\overline{V}/Y, B_{\overline{V}}^{\log} + H_{\overline{V}})$ uses only pl flips and the special termination onto lc centers of codimension ≥ 2, for which the log termination in dimension $d - 1$ is enough.

We use the following properties of H, an effective reduced (having only multiplicities 1) Cartier divisor on Y:

(1) for any birational contraction $\tau \colon Y' \to Y$ with normal $\mathbb{Q}$-factorial variety Y' such that the components of τ^*H are all prime exceptional divisors E_i of Y'/Y and the components of the proper transform $H_{Y'}$ of H, the components of τ^*H allow one to present any numerical class of divisors/Y [4, p. 63 of the English translation, c)]; and for such a contraction

(2) there exists a linear numerical relation between the components of τ^*H:

$$\tau^*H = H_{Y'} + \sum a_i E_i = \sum H_i + \sum a_i E_i \equiv 0/Y$$

with positive integral numbers a_i; and

(3) the support of τ^*H is the reduced part of the divisor $B_{Y'}^{\log} + H_{Y'}$ [4, p. 62 of the English translation, a)].

Notice that in both appearances in (2) and (3), that is, in the relation and in the boundary, $H_{Y'}$ is reduced. All models in our constructions satisfy the assumption in (1) and thus properties (1)–(3). (Moreover, in what follows, τ is an isomorphism over $Y \setminus \operatorname{Supp} H$.)

The next part of the proof, subtracting $H_{\overline{V}}$, is quite different. It should be modified to use only special termination and the terminal termination in dimension d. We can suppose as usual (for flips) that B and $B_{\overline{V}}^{\log}$ are $\mathbb{Q}$-boundaries (or, in all integral independences below, independence should be with multiplicities of B as well; cf. Step 3, (4) in the proof of Theorem 1).

Step 1. *Perturbation of* H. There exist a log flop (not necessarily elementary) $(\overline{V}/Y, B_{\overline{V}}^{\log} + H_{\overline{V}})$ and a boundary $\Gamma H_{\overline{V}} = \sum \gamma_i H_i$ such that

(4) $(\overline{V}/Y, B_{\overline{V}}^{\log} + \Gamma H_{\overline{V}})$ is a strictly log minimal model; and

(5) the multiplicities γ_i are integrally (or rationally) independent:

$$\sum n_i\gamma_i = n \quad \text{and} \quad \text{all } n_i, n \in \mathbb{Z} \quad \Rightarrow \quad \text{all } n_i = 0 \quad \text{and} \quad n = 0;$$

in particular, each $0 < \gamma_i < 1$. Moreover, we need γ_i arbitrarily close to 1: $0 \ll \gamma_i < 1$. Equivalently, there exists an effective divisor $\Delta = \sum \delta_i H_i$ with $0 < \delta_i \ll 1$ such that

(4′) $(\overline{V}/Y, B_{\overline{V}}^{\log} + H_{\overline{V}} - \Delta)$ is a strictly log minimal model; and

(5′) the multiplicities δ_i are integrally independent.

Indeed, then we can take $\gamma_i = 1 - \delta_i$. Actually, it is enough to construct such a model/W:

(4″) $(\overline{V}/W, B_{\overline{V}}^{\log} + H_{\overline{V}} - \Delta)$ is a strictly log minimal model,

where W/Y is an lc model of $(\overline{V}/Y, B_{\overline{V}}^{\log} + H_{\overline{V}})$; that is, the contraction $\overline{V} \to W$ is given by $K_{\overline{V}} + B_{\overline{V}}^{\log} + H_{\overline{V}}$. This and other similar models exist by the LSEPD trick [8, 4.5.2 and 10.5 of the English translation] and Corollary 10. By construction $K_{\overline{V}} + B_{\overline{V}}^{\log} + H_{\overline{V}}$ is ample on W/Y and $\equiv 0/W$, and $-\Delta$ is nef and actually semiample on $\overline{V}/W$. Hence, for any $0 < \delta \ll 1$, $K_{\overline{V}} + B_{\overline{V}}^{\log} + H_{\overline{V}} - \delta\Delta$ is nef on $\overline{V}/Y$. This gives a model in (4′) with $\Delta := \delta\Delta$. Indeed, $K_{\overline{V}} + B_{\overline{V}}^{\log} + H_{\overline{V}} - \delta\Delta$ is nef/Y by construction. On the other hand, by construction $(\overline{V}, B_{\overline{V}}^{\log} + H_{\overline{V}})$ is lc, and by (4″) and monotonicity [8, 1.3.3] $(\overline{V}, B_{\overline{V}}^{\log})$ is strictly log terminal. Since $\operatorname{Supp}\Delta = \operatorname{Supp} H_{\overline{V}}$, again by monotonicity [8, 1.3.3] $(\overline{V}, B_{\overline{V}}^{\log} + H_{\overline{V}} - \delta\Delta)$ is strictly log terminal. The multiplicities of $\delta\Delta$ are $\delta\delta_i$. They are integrally independent if $\delta \in \mathbb{Q}$. Therefore, (5′) holds.

The construction of a model in (4″) uses induction on the number of irreducible components of $H_{\overline{V}}$. If it is zero, we are done (see Step 5 below). We start by subtracting the first component $D = H_1$ of $H_{\overline{V}}$. This gives a new strictly log minimal model $(\overline{V}/W, B_{\overline{V}}^{\log} + H_{\overline{V}} - D)$. Since $K_{\overline{V}} + B_{\overline{V}}^{\log} + H_{\overline{V}} \equiv 0/W$, we need to consider only log flips in the extremal rays R of $\overline{\mathrm{NE}}(\overline{V}/W)$ with $(D, R) > 0$, or equivalently, to construct a log minimal model for $(\overline{V}, B_{\overline{V}}^{\log} + H_{\overline{V}} - D)$. As in the usual reduction, the flips are pl in dimension $d+1$ and thus exist by our assumptions. The only problem is termination. Note also that in a birational/Y situation a Mori log fibration is impossible. By the special termination and log termination in dimension $d-1$, there remains a problem with termination in dimension d. More precisely, it is enough to consider in addition the case of log flips on a reduced component $F \neq D$ of $B_{\overline{V}}^{\log} + H_{\overline{V}} - D$ with $(F, R) < 0$. Such a component exists by properties (2) and (3). According to the special termination, we can consider only flips outside the reduced components of the adjoint boundary B_F for $B_{\overline{V}}^{\log} + H_{\overline{V}} - D$ on F, which consists of intersections with components of $H_{\overline{V}} - D - F + \sum E_i$ (see the log adjunction [8, 3.2.3]), where $\sum E_i$ is the reduced part of $B_{\overline{V}}^{\log}$. Let C be a flipped curve of one of these log flips, that is,

$C \cap \operatorname{Supp}(H_{\overline{V}} - D - F + \sum E_i) = \varnothing$ and $(D, C) < 0$. Let C' be the next flipping curve. Then $(D, C') > 0$. By construction the support of $H_{\overline{V}}$ and exceptional divisors/Y allow one to present any numerical class of divisors/Y (see (1) and (3) above); in particular, this holds for any numerically ample divisor/Y. Thus, there exists a linear combination $aF + bD$ with real multiplicities a and b that is ample on C and C'. On the other hand, there exists a nontrivial (all coefficients are positive) linear relation between the support of $H_{\overline{V}}$ and exceptional divisors E_i/Y (see (2) and (3) above). Hence near C and C' the ample divisor is given by cD with a real number c, which is impossible by the inequalities $(D, C) < 0$ and $(D, C') > 0$. Therefore, two subsequent flips of this type do not (co)exist and we get the termination. Thus, we have constructed a log pair $(\overline{V}/W, B_{\overline{V}}^{\log} + H_{\overline{V}} - D)$ that is a strictly log minimal model over W, and let W_1/W denote its lc model. Note that in the construction each model $(\overline{V}, B_{\overline{V}}^{\log} + H_{\overline{V}} - D)$ is strictly log terminal because the initial model $(\overline{V}, B_{\overline{V}}^{\log} + H_{\overline{V}} - D)$ is strictly log terminal, and the log flips preserve this.

Now by induction we can suppose that a strictly log minimal model $(\overline{V}/W_1, B_{\overline{V}}^{\log} + H_{\overline{V}} - D - \Delta')$ is constructed, where $\Delta' = \sum_{i \neq 1} \delta_i' H_i$ and the multiplicities $0 < \delta_i' \ll 1$ are integrally independent. Since $K_{\overline{V}} + B_{\overline{V}}^{\log} + H_{\overline{V}} - D \equiv 0/W_1$ and $K_{\overline{V}} + B_{\overline{V}}^{\log} + H_{\overline{V}} \equiv 0/W$, by construction $D = H_1 \equiv 0/W_1$. Hence (1)–(3) hold on all models Y'/W_1 without H_1 that are obtained from $\overline{V}/W_1$ by log flops with respect to $K_{\overline{V}} + B_{\overline{V}}^{\log} + H_{\overline{V}} - D \equiv 0/W_1$ because $H_1 \equiv 0/W_1$ on them too by the assumption $K_{\overline{V}} + B_{\overline{V}}^{\log} + H_{\overline{V}} \equiv 0/W$. This means that all numerical classes and relations are considered over W_1. In (3) we can replace $\tau^* H$ by $\tau^* H - H_1$, and the initial model $(\overline{V}, B_{\overline{V}}^{\log} + H_{\overline{V}} - D)$ is strictly log terminal. Therefore, a required model $\overline{V}/W_1$ exists. By construction $-D$ is ample on W_1/W, and $-D - \Delta'$ is nef and actually semiample on $\overline{V}/W_1$. Hence, for any $0 < \delta' \ll \delta \ll 1$, $-\Delta = -\delta D - \delta'(D + \Delta')$ is nef on $\overline{V}/W$. This gives a model in (4″). Indeed, $K_{\overline{V}} + B_{\overline{V}}^{\log} + H_{\overline{V}} - \Delta$ is nef/W because $K_{\overline{V}} + B_{\overline{V}}^{\log} + H_{\overline{V}} \equiv 0/W$. On the other hand, by construction $(\overline{V}, B_{\overline{V}}^{\log} + H_{\overline{V}})$ is lc, and by monotonicity [8, 1.3.3] $(\overline{V}, B_{\overline{V}}^{\log})$ is strictly log terminal. Since $\operatorname{Supp} \Delta = \operatorname{Supp} H_{\overline{V}}$, again by monotonicity [8, 1.3.3] $(\overline{V}, B_{\overline{V}}^{\log} + H_{\overline{V}} - \Delta)$ is strictly log terminal. The multiplicities of Δ are $\delta_1 = \delta + \delta'$ and $\delta_i = \delta' \delta_i'$, $i \neq 1$. We can take integrally independent δ, δ_i', $i \neq 1$, and $\delta' \in \mathbb{Q}$. Then (5′) holds: δ_i are integrally independent.

Step 2. *H-ordered flips.* Set $B' = B_{\overline{V}}^{\log} + \Gamma H_{\overline{V}} - \lambda_{\max} H_{\overline{V}}$, where $\lambda_{\max}$ is the maximal number λ such that $B_{\overline{V}}^{\log} + \Gamma H_{\overline{V}} - \lambda H_{\overline{V}}$ is a boundary. It is easy to find that $\lambda_{\max} = \min\{\gamma_i\}$, and if, say, $\lambda_{\max} = \gamma_1$, then $B' = B_{\overline{V}}^{\log} + \sum_{i \neq 1} (\gamma_i - \gamma_1) H_i$. By construction and monotonicity [8, 1.3.3], the pairs $(\overline{V}, B')$ and $(\overline{V}, B_{\overline{V}}^{\log} + \Gamma H_{\overline{V}} = B' + \lambda_{\max} H_{\overline{V}})$ are strictly log terminal with reduced components E_i exceptional/Y, and the second one is a log minimal model. Note also that by (5)

(5″) the multiplicities $\gamma_i - \gamma_1$, $i \neq 1$, are integrally independent, and $0 < \gamma_i - \gamma_1 \ll 1$.

To construct a log flip in the reduction, we will find a strictly log minimal model of $(\overline{V}/Y, B')$ subtracting $H_{\overline{V}}$ from $B' + \lambda_{\max} H_{\overline{V}}$ as in the proof of Proposition 2. The existence of $H_{\overline{V}}$-ordered log flips in this situation is more straightforward. Indeed, for each level $\lambda \leq \lambda_{\max}$, we can use log flops over an lc model W/Y of $(\overline{V}/Y, B' + \lambda H_{\overline{V}})$ and the fact that $\overline{V}/Y$ is FT/W. After constructing a log minimal model/W, we can convert it into a log minimal model/Y as in Step 1, or use Corollary 11 with the LSEPD trick. Note also that a Mori log fibration is impossible.

On the other hand, each flip is pl as in the usual reduction. If it is a log flip in an extremal ray R, then by construction $(K_{\overline{V}} + B' + \lambda H_{\overline{V}}, R) = 0$ and $(H_{\overline{V}}, R) > 0$. Hence by (2) there exists E_i with $(E_i, R) < 0$, and E_i is a reduced component of B'. Since $\lambda > 0$, $(\overline{V}, B')$ is strictly log terminal.

Step 3. *Termination of log flops.* For each level $\lambda > 0$, the log flips of this level are log flops with respect to $K_{\overline{V}} + B' + \lambda H_{\overline{V}}$. By the special termination of flips, we can suppose that after finitely many steps all the next flips are *nonspecial* on $\bigcup E_j$; that is, log flips in extremal rays R are such that, for any curve C/Y of R, C intersects only one reduced component, say E_1. Actually, $C \subset E_1$, and $(K_{\overline{V}} + B' + \lambda H_{\overline{V}}, C) = 0$. We claim that there exists only one such extremal ray for E_1. Therefore, such log flips terminate.

Let $C' \subset E_1$ be a curve/Y with $(K_{\overline{V}} + B' + \lambda H_{\overline{V}}, C') = 0$ and disjoint from E_i with $i \neq 1$. Since $K_{\overline{V}} + B_{\overline{V}}^{\log}$ is a $\mathbb{Q}$-divisor, both relations can be transformed into relations for λ and the multiplicities of B':

$$\lambda(H_{\overline{V}}, C) + \sum_{i\neq 1}(\gamma_i - \gamma_1)(H_i, C) = r$$

and

$$\lambda(H_{\overline{V}}, C') + \sum_{i\neq 1}(\gamma_i - \gamma_1)(H_i, C') = r',$$

where $r, r' \in \mathbb{Q}$. Note that $(H_{\overline{V}}, C), (H_{\overline{V}}, C') \neq 0$ because otherwise we get a rational relation, which contradicts (5″). Indeed, if $(H_{\overline{V}}, C) = 0$, then all intersections $(H_i, C) = 0$, $i \neq 1$, and $(H_1, C) = 0$ too since $H_{\overline{V}} = \sum H_i$. This is impossible by (1) and (3) because by construction $(E_i, C) = 0$, $i \neq 1$, and by (2) $(E_1, C) = 0$. The same holds for C'. Similarly we verify that $C \equiv cC'/Y$ with $c = (H_{\overline{V}}, C)/(H_{\overline{V}}, C') \neq 0$, i.e., with $(H_{\overline{V}}, C) = c(H_{\overline{V}}, C')$.

To this end we eliminate λ and obtain one relation:

$$(H_{\overline{V}}, C')\left(\sum_{i\neq 1}(\gamma_i - \gamma_1)(H_i, C)\right) - (H_{\overline{V}}, C)\left(\sum_{i\neq 1}(\gamma_i - \gamma_1)(H_i, C')\right) = r''$$

with $r'' \in \mathbb{Q}$. By (5″) this is possible only if

$$(H_{\overline{V}}, C')(\gamma_i - \gamma_1)(H_i, C) = (H_{\overline{V}}, C)(\gamma_i - \gamma_1)(H_i, C').$$

Again by (5″) each $\gamma_i - \gamma_1 \neq 0$ for $i \neq 1$. Hence $(H_i, C) = c(H_i, C')$ for $i \neq 1$. On the other hand, this implies that $(H_1, C) = c(H_1, C')$ because $H_{\overline{V}} = \sum H_i$. Then we can prove that $(E_i, C) = c(E_i, C')$ for all E_i. Therefore, by (1) $C \equiv cC'/Y$, and we have the only possibility on E_1 for a nonspecial log flip of level λ.

Step 4. *Stabilization of models.* This means that the levels $\lambda_{\max} \geq \lambda_1 \geq \ldots > 0$ stabilize. This follows from Corollary 12. By the special and divisorial termination, after finitely many log flips, we can suppose that the other log flips are nonspecial and are actually log flips on the corresponding reduced components $F = E_j$. Thus, conditions (1) and (2) of Corollary 12 hold for $(X_i/Z, B_i) = (F/Y, B_F)$, where B_F is the adjoint boundary for $B' + \lambda_i H_{\overline{V}}$ on $F = E_j$. For a fixed $F = E_j$, we consider only the corresponding levels λ_i (truncation) and models $(X_i/Z, B_i)$. Then condition (3) of Corollary 12 holds for the adjoint boundary B on $X_i = F$ of the pair $(\overline{V}, B')$ and, respectively, for the adjoint boundary B' of $(\overline{V}, B' + \lambda_1 H_{\overline{V}})$. By Step 2 each $(\overline{V}/Y, B' + \lambda_i H_{\overline{V}})$ is a strictly log minimal model. Hence the construction and adjunction give (4) of Corollary 12 with X_i corresponding to λ_1. The models of F/Y are not equivalent, which implies stabilization: there are only finitely many levels λ_i. Indeed, models with distinct levels are nonequivalent by Lemma 2. By the definition of $\lambda_{i+1} > 0$ and Step 2 (see also Corollary 11 and its Addendum 6), the model $(\overline{V}/Y, B' + \lambda_{i+1} H_{\overline{V}})$ has an extremal ray R with $(K_{\overline{V}} + B' + \lambda_{i+1} H_{\overline{V}}, R) = 0$ in which the model is not numerically equivalent to $(\overline{V}/Y, B' + \lambda_i H_{\overline{V}})$: $(K_{\overline{V}} + B' + \lambda_i H_{\overline{V}}, R) > 0$, and the ray is supported on $F = E_j$. Thus, the model $(X_{i+1}/Z, B_{i+1})$ is not equivalent to $(X_i/Z, B_i)$.

Step 5. *Flip.* We claim that the lc model $(X^+/Y, B^+)$ of $(\overline{V}/Y, B' + \lambda H_{\overline{V}})$ with $0 < \lambda \ll 1$ is a required log flip. A contraction to the lc model exists again by the LSEPD trick and Corollary 10. Since the multiplicities $\gamma_i - \gamma_1$ of $B' + \lambda H_{\overline{V}} = B_{\overline{V}}^{\log} + \lambda H_1 + \sum_{i\neq 1}(\lambda + \gamma_i - \gamma_1)H_i$ are small and X is

$\mathbb{Q}$-factorial, the boundary $B+\lambda H_1+\sum_{i\neq 1}(\lambda+\gamma_i-\gamma_1)H_i$ on X is klt and $K+B+\lambda H_1+\sum_{i\neq 1}(\lambda+\gamma_i-\gamma_1)H_i$ is numerically negative on X/Y. Notice also that $(\overline{V}/Y, B'+\lambda H_{\overline{V}})$ is a strictly log minimal model of $(X/Y, B+\lambda H_1+\sum_{i\neq 1}(\lambda+\gamma_i-\gamma_1)H_i)$. Thus, by monotonicity [4, Lemma 2.4], $B'+\lambda H_{\overline{V}}$ does not have reduced components and by (3) the contraction $\overline{V}\to Y$ is small. Thus, the lc model is the log flip of $(X/Y, B+\lambda H_1+\sum_{i\neq 1}(\lambda+\gamma_i-\gamma_1)H_i)$. Since the contraction $X\to Y$ is extremal, its flip is unique [12, Corollary 3.6], and $(X^+/Y, B^+)$ is a required log flip. □

Proof of Revised Induction. According to the methods of [12, 3] it is sufficient to establish Theorem 7.2 from [3] in dimension d. It can be obtained from Theorems 1 and 2: the former gives the existence of models W_i and the later gives the finiteness of models. Moreover, we can assume that $(X/Z,\Delta)$ is a strictly log terminal pair with a birational contraction X/Z. Thus, we need both theorems only in the birational situation, and the required terminal termination is also needed only for birational pairs. Notice that for klt pairs their $\mathbb{Q}$-factorialization can be obtained by Theorem 1 as a strictly log minimal model with boundary multiplicities 1 for the exceptional divisors; actually, the existence of log flips and special termination in dimension d are enough. The log flips exist by [3, Theorem 1.1] or induction in Corollary 3.

Using the LSEPD trick, we can slightly increase Δ and assume that $\Delta\in V\subseteq\mathcal{D}_F$ for $F=\operatorname{Supp}\Delta$; and F contains all exceptional divisors of X/Z. Any $\Theta\in\mathcal{D}_F$ sufficiently close to Δ is an $\mathbb{R}$-boundary. Thus, $(X/Z,\Theta)$ has a strictly log minimal model $(W_i/Z,\psi_{i*}\Theta)$, where $\psi_i\colon X\dashrightarrow W_i$ is a birational rational 1-contraction. It is a 1-contraction by definition and monotonicity [4, Lemma 2.4]. Since X/Z is birational, Mori log fibrations are impossible. (According to the proof of Theorem 1 after Step 3, we can decompose ψ_i into a composition of log flips, which gives (1) in [3, Theorem 7.2]; cf. also Proposition 2. However, this is not important for pl flips, in particular, for [3, Corollary 7.3].) By construction (2) and (3) of [3, Theorem 7.2] hold, and for (4) see the proof of [3, Theorem 7.2].

Thus, we need to establish the finiteness up to equivalence of models. If this does not hold, we have a convergent sequence of $\mathbb{R}$-boundaries $\Theta_i\in\mathcal{D}_F$: $\lim_{i\to\infty}\Theta_i=\Delta$, with pairwise nonequivalent wlc models $(X_i/Z,B_i)=(W_i/Z,\psi_{i*}\Theta_i)$. This contradicts Theorem 2 (cf. the proof of Corollary 12). For a subsequence of models, assumption (1) of Theorem 2 holds by construction and the birational property of X/Z (still we need to construct $(X/Z,B)$ and verify (1) for it). We construct an appropriate model $(X/Z,B)$ as a modification of a strictly log minimal model $(W/Z,\Delta)$. The latter is birationally larger than X_i for a subsequence with $i\gg 0$: $W\dashrightarrow X_i$ is a birational rational 1-contraction. Indeed, we can construct it from X by a sequence of ordered log flips of $(X/Z,\Delta)$ (see the proof of Theorem 1). By the stability of negative contractions and the main lemma we never contract a component of Δ that is not contracted on the models X_i with all $i\gg 0$ (cf. Step 3 in the proof of Theorem 2). Since the sequence of log flips is finite, we get a required subsequence of models $(X_i/Z,B_i)$. Hence the model $(W/Z,\Delta)$ is between $(X/Z,B)$ and $(\overline{X}/Z,B_{\overline{X}})$ in the proof of Theorem 2, and we can construct $(\overline{X}/Z,B_{\overline{X}})$ as in the proof. Otherwise we consider an lc model $(W/Z,\Delta)$, which exists by Corollary 10. It contracts all prime divisors D on X that are exceptional on the varieties X_i with $i\gg 0$. Otherwise $K+\Delta$ is big on D/Z, and so is $K+\Theta_i$ with $i\gg 0$, which contradicts the construction: each log flip or divisorial contraction of $(X/Z,\Theta_i)$ preserves this property [12, Proposition 3.20]. Thus, we can construct $(X/Z,B)$ as a crepant blowup of $(W/Z,\Delta)$ with the same blown-up divisors D as for the models X_i/Z. This model can also be constructed as a strictly log minimal model with multiplicities 1 for other exceptional divisors/W on an initial model. The pair $(X/Z,B)$ satisfies (1).

Assumption (2) is void because (X,B) and (X,B_i) are klt.

In (3) we take all boundary components of B and the exceptional divisors E with $b(E,X,B)\geq 0$. Then assumptions (4) and (5) hold for an appropriate subsequence as in Step 2 in the proof of Corollary 12. □

ACKNOWLEDGMENTS

This work was partially supported by the NSF grants DMS-0400832 and DMS-0701465.

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Translated by the author