3-FOLD LOG FLIPS

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3-FOLD LOG FLIPS*

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ABSTRACT. We prove that 3-fold log flips exist. We deduce the existence of log canonical and $\mathbb{Q}$-factorial log terminal models, as well as a positive answer to the inversion problem for log canonical and log terminal adjunction.

CONTENTS

§0.

Let X be a normal algebraic (or analytic) 3-fold with a marked $\mathbb{Q}$-divisor $B = B_X$ (the *boundary* of X); we write $K = K_X$, and consider the log canonical divisor $K+B$ as in Kawamata–Matsuda–Matsuki [8]. Suppose that $f\colon X \to Z$ is a birational contraction of X such that $K + B$ is numerically nonpositive relative to f. A *flip* of f is a birational (or bimeromorphic) modification

$$\begin{array}{ccc} X & \stackrel{\operatorname{tr} f}{\dashrightarrow} & X^+ \\ {\scriptstyle f}\searrow & & \swarrow {\scriptstyle f^+}, \\ & Z & \end{array}$$

where f^+ is a small birational contraction whose modified log canonical divisor $K_{X^+} + B^+$ is numerically positive relative to f^+; it is known (see [25] (2.13), and [8], 5-1-11) that the flip $\operatorname{tr} f$ is unique if it exists.

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**Editor's note*. The present translation includes the author's substantially corrected version of §8, a section §10 of translator's comments, and a number of other minor corrections and additions. A number of the translator's footnotes have been moved into the main text with the author's permission. All the remaining footnotes are by the translator.

Theorem. *The flip of f exists if $\lfloor B \rfloor = 0$ and $K + B$ is log terminal.*

Corollary. *Let $f\colon X \to Z$ be a projective morphism of algebraic (or analytic) 3-folds, and suppose that $\lfloor B \rfloor = 0$ and $K+B$ is log terminal. Then for every extremal face R of the Kleiman–Mori cone $\overline{\mathrm{NE}}(X/Z)$ (in the analytic case, of $\overline{\mathrm{NE}}(X/Z\,;W)$, where $W \subset Z$ is compact) contained in the halfspace $K+B<0$, the contraction morphism cont_R associated with R is either a fiber space of log Fanos over a base of dimension ≤ 2, or has a flip tr_R (respectively, the same statement over a neighborhood of W).*

The proof of the theorem, or more precisely of the equivalent Theorem 1.9, consists of a series of reductions. First of all, in §6, the construction of the flip reduces to the *special* case, which is classified according to its complementary index. This classification is similar, and in fact closely related to, Brieskorn's classification of log terminal surface singularities (see [2] and [4]). Index 1 special flips exist, and correspond to the flops or 0-flips of [7], (6.1), or [11], (6.6). Next, in §7, we construct exceptional index 2 flips, and carry out a reduction of the existence of the remaining exceptional flips of index 3, 4, and 6 to the case of special flips of index 1 or 2. The proof is completed by the reduction in §8 of special index 2 flips to exceptional index 2 flips. Here a significant role is played by a result of Kawamata on the minimal discrepancy of a terminal 3-fold singularity; this proof is given in the Appendix kindly provided by Professor Kawamata. Furthermore, as we will see in the proof, a flip can be decomposed as a composite of resolutions of singularities, birational contractions given by the eventual freedom theorem and the contraction theorem ([8], 3-1-2 and 3-2-1), and flips of types I–IV, defined in §2. We note also that the proof does not use Mori's flips ([16], (0.2.5)) in the case $B=0$ when K itself is terminal, and so gives a new approach to proving the existence of Mori's flips. In addition to the definitions and related general facts, the introductory §1 contains a statement of the main results proved in §§6–8; applications of these are given in §9. The main technique is contained in §§2–5.

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§1. Singularities and models

We generally use the terminology and notation of [8], [25] and [26]. The geometric objects we work with are either normal complex analytic spaces or normal algebraic varieties over a base field k of characteristic 0. The first is the *analytic* case, the second the *algebraic* case. For example, a *morphism* is either a holomorphic or a regular map; and a *modification* is assumed to be bimeromorphic in the analytic case, and birational in the algebraic case.

A *contraction* is a proper morphism $f\colon X \to Y$ with $f_*\mathcal{O}_X = \mathcal{O}_Y$, and is *projective* if f is. If $f\colon X \to Y$ is a contraction between varieties of the same dimension, that is, $\dim X = \dim Y$, then f is one-to-one at the generic point, and is a modification; such an f is a *birational contraction* or *blowdown* when Y is viewed as constructed from X, or an *extraction* or *blowup* when X is viewed as constructed from Y (see (10.8.4) for notes on terminology). An extraction or birational contraction whose exceptional set has codimension ≥ 2 is *small.*

We write $\rho(X/Y)$ (or $\rho(X/Y\,;W)$ in the analytic case) for the *relative Picard number* of f (respectively, of f over a compact analytic subset $W \subset Y$). As a rule, in the analytic case, we work infinitesimally over suitable neighborhoods of W, even when we do not say so explicitly; W is often *projective*, the fiber of a projective morphism, having tubular neighborhoods over Stein domains. (This condition is a particular case of *weakly* 1-*complete* in the sense of [18], (0.4); it means in particular that Serre vanishing can be used e.g. in the proof of Corollary 5.19 below.) An extraction or birational contraction f is *extremal* (over W) if $\rho(X/Y) = 1$ (respectively $\rho(X/Y; W) = 1$).

A *divisor* is usually understood as an $\mathbb{R}$-Weil divisor $D = \sum d_i D_i$, with D_i distinct prime Weil divisors on X and $d_i \in \mathbb{R}$, called the *multiplicity* of D_i in D. This terminology will be generalized later to include the multiplicity of D at prime divisors of an extraction $Y \to X$; see just before Lemma 8.7, and (10.8.5). We say that D is a $\mathbb{Q}$-*divisor* (respectively, an *integral* divisor) if $d_i \in \mathbb{Q}$ (or $d_i \in \mathbb{Z}$) for all D_i. Note that a $\mathbb{Q}$-divisor is $\mathbb{Q}$-Cartier if and only if it is $\mathbb{R}$-Cartier. More-or-less by definition, $\mathbb{R}$-Cartier divisors with support in a finite union $\bigcup D_i$ form a vector subspace defined over $\mathbb{Q}$ of the space $\bigoplus \mathbb{R}D_i$ of all divisors supported in $\bigcup D_i$ (in the analytic case, in a neighborhood of any compact subset of X).

1.1. **Negativity of a birational contraction.** *Let $f: X \to Z$ be a birational contraction and D an $\mathbb{R}$-Cartier divisor. Suppose that*

(i) *f contracts all components of D with negative multiplicities;*

(ii) *D is numerically nonpositive relative to f; and for each D_i, either D_i has multiplicity 0 in D, or D is not numerically 0 over the general point of $f(D_i)$.*

Then D is effective. Moreover, for each D_i either $D = 0$ in a neighborhood of the general fiber of $f: D_i \to f(D_i)$ or $d_i > 0$.

Proof (compare [Pagoda], (0.14)). First of all, passing to a general hyperplane section (that is, a general element of a very ample linear system) and using induction on the dimension of X reduces 1.1 to the assertion over a fixed point $P \in Z$; that is, after replacing Z by a suitable neighborhood of P if necessary, we can assume that all the components D with $d_i < 0$ are contracted to P. It is enough to prove the assertion on some blowup of X. Since the assertion is local over Z, we can also assume that X has an effective divisor E contracted by f that is numerically nonpositive and not numerically 0 over P; for example, we could take E to be the difference $f^*H - f^{-1}H$, where H is a general hyperplane section through P. (Throughout the paper, f or f^{-1} applied to a divisor denotes its *birational image* or *birational transform*, never the set-theoretic image, see below and (10.8.3).)

Using resolution of the base locus by Hironaka, we can assume that $|f^{-1}H|$ is a free linear system, which guarantees that $E = f^*H - f^{-1}H$ is numerically nonpositive. E is not numerically trivial over P since $|f^{-1}H|$ intersects $f^{-1}P$; it follows, of course, that $\operatorname{Supp} E \supset f^{-1}P$. If there exists a prime divisor D_i contracted to P and with negative multiplicity d_i in D, then there is a minimal value of $\varepsilon > 0$ such that $D + \varepsilon E$ is an effective divisor satisfying the assumptions (i) and (ii); if $D = \sum d_i D_i$ and $E = \sum e_i D_i$ where all $e_i > 0$, then $\varepsilon = \min\{-d_i/e_i \mid d_i < 0\}$, and some $d_i < 0$ by assumption. Then since ε is minimal, by (i) there is an exceptional divisor over P with multiplicity 0 in $D + \varepsilon E$. But $f^{-1}P$ is connected since Z is normal. Now $D + \varepsilon E$ is effective and numerically nonpositive over P, so this is only possible if $D + \varepsilon E = 0$ in a neighborhood of $f^{-1}P$, hence is numerically 0. By the choice of E and since $\varepsilon > 0$, this contradicts (ii). Hence D is effective. By the same argument the multiplicity $d_i > 0$ if D is nontrivial on D_i. Q.E.D.

We write B_X or simply B to denote a divisor $B = \sum b_i D_i$ with $0 \le b_i \le 1$, called the *boundary* of X. A *reduced* divisor B, with all $b_i = 0$ or 1, is of this form. The divisor $S = \lfloor B \rfloor$ is the *reduced part* of the boundary; the reduced divisor $\operatorname{Supp} B = \lceil B \rceil$ is identified with the support of the boundary. B is viewed as an extra structure added to X except where stated otherwise. The log canonical and log terminal conditions discussed below are restrictions not just on the singularities of X, but also on those of the boundary B. Boundaries are generalized to *subboundaries* in §3 by dropping the restriction $b_i \geq 0$.

Consider a correspondence between X and Y, that is, a partially defined, possibly multivalued, map $f: X \dashrightarrow Y$, and a prime divisor D_i on X. The image of D_i under f is the divisor $f(D_i) = \sum P_{ij}$, where P_{ij} are the divisorial components of the image under f of the generic point of D_i. This map extends to a homomorphism of divisors

$$D = \sum d_i D_i \mapsto f(D) = \sum d_i f(D_i);$$

$f(D)$ is the *image* of D. For a modification f, we usually call this the *birational transform* of D. A divisor D whose image is 0 is *contracted* or *blown down* by f, or is *exceptional* for f, but we sometimes also say that D is *extracted* or *blown up* by f. A *contracting modification* f is a modification such that f^{-1} does not contract any divisors; an extraction or birational contraction is of this kind. Similarly, a modification f is *small* if neither f nor f^{-1} contracts any divisors; small contractions and small extractions are of this nature.

The modified boundary B_Y of a boundary B of X under a modification $f: X \dashrightarrow Y$ can be defined in various ways even under the restrictions $0 \leq b_i \leq 1$. One usually takes B_Y to be the birational transform $f(B)$, although we could also take a divisor of the form

$$B_Y = f(B) + \sum e_i E_i,$$

where E_i are divisors on Y contracted by f^{-1} and all the multiplicities satisfy $0 \leq e_i \leq 1$. In what follows B_Y will denote this divisor with $e_i = 1$ for each i; this is the *log birational transform*, see (10.3.2).

We write K_X or simply K for a canonical divisor of X. A *log divisor* is a sum of the form $K + D$, where D is arbitrary. However, we are mainly interested in log divisors of the form $K + B$, where B is a boundary, assumed to be log canonical unless otherwise stated. This means in particular that $K + B$ is an $\mathbb{R}$-Cartier divisor, and hence its pullback $g^*(K + B)$ by any morphism $g: Y \to X$ is defined. If g is an extraction, and K_Y a suitable canonical divisor of Y, the pullback $g^*(K + B)$ on Y only differs from $K_Y + B_Y$ at exceptional components, that is,

$$K_Y + B_Y = g^*(K_X + B) + \sum a_i E_i.$$

Each a_i is real, and is independent of the model Y, as long as E_i appears as a divisor on it; we call it the *log discrepancy* of $K + B$ at E_i. It is independent of the choice of the canonical divisor K_X, since although the divisors K_X and K_Y are only defined up to linear equivalence, they can be intrinsically compared across a birational modification: there is an intrinsic identification of the sheaves $\mathscr{O}_X(mK_X) \cong \mathscr{O}_Y(mK_Y)$ outside the exceptional sets. The log discrepancy coefficient a_i of $K + B$ at E_i is $1 +$ its ordinary discrepancy. ([1]) For nonexceptional prime divisors D_i, it is natural to define the log discrepancy by the relation $a_i = 1 - b_i$, and the *discrepancy* to be $-b_i$. Thus for a log canonical divisor $K + B$, the log discrepancy of all blown up divisors is ≥ 0. Moreover, it is enough to verify this

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([1]) The intention here is: if $B_X = 0$, and every divisor in sight is $\mathbb{Q}$-Cartier, you can compare K_Y and K_X and get the (genuine) discrepancy, canonical etc. Even when $B_X = 0$ it still makes sense to work in the log category, define the *log birational transform* of $0 = B_X$ to be $B_Y = (g^{-1})^{1,\log}(0) = \sum E_i$, and define the log discrepancy by $K_Y + B_Y = g^*(K_X + B) + \sum a_i E_i$. Then the statement in the text is true; if $B \neq 0$ the birational transform of B has multiplicity 0 at each E_i, so one could also fix up a category of "birational pairs" in which it holds (Iitaka did this kind of thing around 1980).

inequality for exceptional divisors E_i of a resolution of singularities g on which the irreducible components of $\lceil B_Y \rceil$ are nonsingular and cross normally.

The divisor $K+B$ is *log terminal* if $a_i > 0$ for every exceptional divisor of one such resolution; $K+B$ is *strictly log terminal* if in addition X is $\mathbb{Q}$-factorial. The definition of log canonical and log terminal given here is somewhat wider than that of [8], 0-2-10, and in particular we do not assume that the b_i are rational, or the inequality $b_i < 1$ in the log terminal case (compare [8], 0-2-10, (i) and (ii)). This leads to an asymmetry between the given notions: for a log terminal $K+B$ we do not always have all $a_i > 0$, even for the exceptional divisors of the resolution indicated above (compare [8], 0-2-12). However, if all the exceptional divisors have $a_i > 0$ then $K+B$ is *purely* log terminal. This holds for log terminal $K+B$ precisely when on the normal crossing resolution the birational transforms of the irreducible components of the reduced part of the boundary do not intersect, which happens in particular if the reduced part of the boundary is irreducible or empty. Purely log terminal is Kawamata's notion of log terminal (compare [8], 0-2-10, (i)), except for the rationality of b_i; the Utah seminar ([Utah], (2.13)) uses the terminology

$$\lfloor B \rfloor = 0 \text{ and purely log terminal} = \text{Kawamata log terminal},$$
$$\text{strictly log terminal} = \mathbb{Q}\text{-factorial and log terminal}.$$

Example. To understand what's going on here, calculate the log discrepancy for the blowup of $P \in B \subset X$, where P is a node of a curve B on a nonsingular surface X. This X with B is (strictly) log terminal but not purely log terminal. This problem goes back to Iitaka around 1975: $B \subset X$ has infinitely many different log minimal models over P, and one needs to refine the definition to *minimal minimal* models.

Weakly log terminal is here understood as in [8], 0-2-10, (ii), although we do not assume that the boundary B is a $\mathbb{Q}$-divisor. Strictly log terminal is obviously stronger than weakly log terminal.

For adjunction, the following weakened version of weakly log terminal is important. The divisorial notion of *log terminal* is obtained when the exceptional set appearing in its definition is assumed to be divisorial. In this case, the reduced components of $g^{-1}(\lfloor B \rfloor)$ do not have any double or higher order crossing on the exceptional set of g. Say that $f\colon E \to X$ *maps at general points to normal crossing* of $\lfloor B \rfloor$ if $f(E) \subset X$ is defined at its generic point by components of $\lfloor B \rfloor$ crossing normally; the image in codimension k is given by intersection of k components. Thus log terminal is divisorial if and only if the exceptional divisors with log discrepancy 0 map at general points to normal crossing of reduced components of $\lfloor B \rfloor$, and the exceptional set does not lie over general points of normal crossings. In particular, if X is a 3-fold, for the divisorial log terminal property, exceptional divisors with log discrepancy 0 lie over double or triple normal crossings. Note that if X is not $\mathbb{Q}$-factorial, it can happen that two reduced components of $\lfloor B \rfloor$ can intersect in a point only, and then for $K+B$ to be divisorially log terminal, the log discrepancy over such a point must be > 0. Since strictly log terminal is stronger than divisorially log terminal, if it holds then exceptional divisors with log discrepancy 0 lie over normal crossings of $\lfloor B \rfloor$ and they only occur for $\lfloor B \rfloor$, as we will prove in Corollary 3.8.

1.2. **Example.** If X is nonsingular and if the irreducible components of $\lceil B \rceil$ are nonsingular and cross normally, then $K+B$ is strictly log terminal.

1.3. **Properties of log divisors.**

(1.3.1) *Convexity.* The set of boundaries B for which $K+B$ is log canonical (respectively nef, numerically positive or ample) is convex.

(1.3.2) *Rational polyhedral.* The set of boundaries B with support in a finite union $\bigcup D_i$ for which $K+B$ is log canonical is a rational convex polyhedron in $\bigoplus \mathbb{R}D_i$ (in the analytic case, in a neighborhood of any compact subset $W \subset X$).

(1.3.3) *Monotonicity.* If $B \geq B'$ are such that $K+B$ is log canonical (or log terminal) and $K+B'$ is $\mathbb{R}$-Cartier (automatic if X is $\mathbb{Q}$-factorial), then $K+B'$ is also log canonical (respectively log terminal). Moreover, the log discrepancies of $K+B$ and $K+B'$ at an exceptional component E_i satisfy $a_i' \geq a_i$, and $a_i' > a_i$ if E_i lies over the locus where $B > B'$, that is, if E_i is contracted into the support of $B - B'$.

(1.3.4) *Stability.* If B and B' are boundaries such that B' has support in $\operatorname{Supp} B$, $K+B$ is log terminal, $K+B'$ is $\mathbb{R}$-Cartier (automatic if X is $\mathbb{Q}$-factorial), and B' is close to B, then $K+B'$ is also log terminal (in the analytic case, in a neighborhood of a compact subset $W \subset X$). If in addition $K+B$ is purely log terminal and B' has support in a finite union $\bigcup D_i$ then $K+B'$ is also purely log terminal.

(1.3.5) *Rational approximation.* The set of rational boundaries B is dense among all boundaries for which $K+B$ is log canonical (in the analytic case, in a neighborhood of a compact subset $W \subset X$). If X is $\mathbb{Q}$-factorial, the set of rational boundaries B with $\lfloor B \rfloor = 0$ is dense among all boundaries for which $K+B$ is log terminal.

Here, except where stated otherwise, log terminal can be taken to be any of the notions introduced above, and distance between divisors is measured coefficient-by-coefficient.

Proof. (1.3.1) and (1.3.3–4) come directly from the definitions. By Example 1.2, (1.3.2) holds if the D_i are nonsingular and cross normally; the polyhedron will be the cube $0 \leq b_i \leq 1$. In general, consider a resolution $g: Y \to X$ on which E_i and $g^{-1}D_i$ are nonsingular and cross normally; the set of exceptional divisors E_i is finite (in the analytic case, in a neighborhood of $W \subset X$). The inclusion g^* of $\mathbb{R}$-Cartier divisors of X extends in the numerical sense to divisors D such that $g^{-1}D + \sum d_i E_i$ is numerically 0 relative to g for some real d_i. The d_i with this property are uniquely determined, as follows from negativity of a contraction, 1.1. Defining the numerical log canonical property of $K+B$ for g, it is not hard to check the rationality of the corresponding convex polyhedron in $\bigoplus \mathbb{R}D_i$. This polyhedron is the image under the rational projection $D \mapsto g(D)$ of the analogous polyhedron in $(\bigoplus \mathbb{R}g^{-1}D_i) \oplus (\bigoplus \mathbb{R}E_i)$, cut out in the polyhedron $\{D = \sum d_i D_i + \sum e_i E_i \mid 0 \leq d_i \leq 1$ and $e_i \leq 1\}$ by the relations $(K_Y + D) \cdot C = 0$ for all curves C contracted by g. But the divisors $D = \sum d_i D_i$ for which $K+D$ is $\mathbb{R}$-Cartier form an affine linear subspace of $\bigoplus \mathbb{R}D_i$ defined over $\mathbb{Q}$. Intersecting the polyhedron with this gives what we want.

Without the assumption $\lfloor B \rfloor = 0$, (1.3.5) follows from (1.3.2) and (1.3.4) in the log canonical (log terminal) case. When X is $\mathbb{Q}$-factorial and $\lfloor B \rfloor = 0$, it is obvious. Q.E.D.

A proper morphism $f: X \to Z$ is *log canonical*, or X is a *log canonical model* over Z, if $K+B$ is log canonical and numerically ample relative to f. The morphism $f: X \to Z$ is *log terminal*, or a *log minimal model* over Z, if $K+B$ is log terminal and $K+B$ is nef relative to f. And f is *strictly log terminal*, or a *strict log minimal model* over Z, if in addition $K+B$ is strictly log terminal and f is projective. Here an $\mathbb{R}$-Cartier divisor D is *numerically ample* if it is ample in the sense of Kleiman [10] (in the analytic case, in a neighborhood of any compact subset $Z \subset X$), and *nef* (respectively *numerically positive, numerically negative, numerically nonpositive, numerically* 0) if $D\cdot\Gamma \geq 0$ (respectively $D\cdot\Gamma > 0$, < 0, ≤ 0, $= 0$) for every curve Γ of X/Z. Here curves of X/Z means curves of X contracted to points in Z, that is, $\Gamma \subset X$ is contained in a fiber of f. The vector space $N_1(X/Z)$ is the $\mathbb{R}$-vector subspace of $N_1(X)$ spanned by such curves, see [8]. For $\mathbb{Q}$-Cartier divisors, numerical ampleness of f is equivalent to ampleness in the usual sense (in the analytic case, in a neighborhood of any compact subset $Z \subset X$) by [10]. If f is a small contraction and X a 3-fold, then the fibers of f are curves (possibly reducible), and f is numerically ample if and only if it is numerically positive.

There are two absolute cases. If f is a map to a point then X is *log canonical* (respectively *log terminal, strictly log terminal*). If f is the identity, then to say that f is log canonical (respectively log terminal, strictly log terminal) just means that $K+B$ has log canonical (respectively log terminal, strictly log terminal) singularities.

1.4. **Properties of morphisms.**

(1.4.1) *Convexity.* The set of boundaries B for which f is log canonical is convex.

(1.4.2) *Rational approximation.* The set of rational boundaries B is dense in all boundaries for which f is log canonical (in the analytic case, over a neighborhood of any compact subset $W \subset X$).

(1.4.3) *Projectivity.* If f is log canonical then it is projective (in the analytic case, over a neighborhood of any compact subset $W \subset X$).

Under our assumptions B is rational if and only if $K + B$ is $\mathbb{Q}$-Cartier. Thus in the study of log canonical morphisms f, we can manage with only $\mathbb{Q}$-divisors B and $K + B$ (compare [8], 0-3-10).

Proof. (1.4.1) follows from (1.3.1), (1.4.2) from (1.3.2), and (1.4.3) from (1.4.2) and [10]. Q.E.D.

A *modification* of a proper morphism $f\colon X \to Z$ to a proper morphism $g\colon Y \to Z$ is a commutative diagram

$$\begin{array}{ccc} X & \overset{t}{\dashrightarrow} & Y \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g}, & \\ & Z & \end{array}$$

with $t\colon X \dashrightarrow Y$ a modification; we say that g is a *model* of f. Obviously a model of a birational contraction is again a birational contraction. A *log canonical* (respectively *log terminal, strictly log terminal*) model of a proper morphism $f\colon X \to Z$ (with $K+B$ not necessarily log canonical, or indeed $\mathbb{R}$-Cartier) is a modification $g\colon Y \to Z$ of f such that g is log canonical (respectively log terminal, strictly log terminal), and the log discrepancy coefficients a_i of $K_Y + B_Y$ satisfy

$$a_i \geq 1 - b_i \qquad \text{(respectively } a_i > 1 - b_i\text{)}$$

for all divisors $D_i \subset X$ that are exceptional with respect to t.

Note once again the two absolute cases: if f is the identity morphism, then a log canonical (respectively log terminal, strictly log terminal) model $g\colon Y \to X$ of f is a relative log canonical (respectively log terminal, strictly log terminal) model of X for $K + B$ (compare [20], (6.3)); this is a partial resolution of the noncanonical singularities that leaves $K+B$ nef relative to g. If Z is a point, then a log canonical model (respectively strictly log terminal model) Y is projective.

1.5. **Properties of log models.**

(1.5.1) *Well defined.* A log canonical model is unique if it exists.

(1.5.2) *Log birational invariance.* If no component $D_i \subset X$ is contracted on a model $g\colon Y \to Z$, that is, the D_i also appear as divisors on Y, then the log canonical (respectively log terminal, strictly log terminal) model f for $K+B$ coincides with the corresponding model of the modification g for $K_Y + B_Y$.

(1.5.3) *Characterization as* Proj. If $K+B$ is log canonical, $\mathbb{Q}$-Cartier and g is the log canonical model of f, then $\mathscr{R}(f, K+B) = \bigoplus_{n\geq 0} f_*\mathscr{O}_X(n(K+B))$ is a finitely generated sheaf of graded $\mathscr{O}_Z$-algebras, and

$$Y = \operatorname{Proj} \mathscr{R}(f, K+B)$$

(in the analytic case, the same over any compact subset of Z).

(1.5.4) *Equivariance.* A birational selfmap of X that lies over a biregular automorphism of Z and maps each generic point of B to B induces a regular automorphism of the log canonical model g of $K+B$ (holomorphic automorphism in the analytic case).

(1.5.5) *Behavior in codimension* 1. If $K+B$ is log canonical, then the modification t to the log canonical model g is contracting; a modification to a log terminal model is not necessarily contracting, but the log discrepancy of $K+B$ does not exceed the corresponding log discrepancy for $K_Y + B_Y$.

(1.5.6) *Discrepancies decrease.* If f is a birational contraction, and $K+B$ is log canonical and numerically nonpositive relative to f, then the log canonical model g is a small contraction. Moreover, the log discrepancies of $K+B$ do not exceed the corresponding log discrepancies for $K_Y + B_Y$, and are strictly smaller for divisors lying over the union of the fibers of f where $K+B$ is not numerically 0. If in addition $K+B$ is (purely) log terminal and numerically negative relative to f, then the log canonical model g is in addition (purely) log terminal. (Compare [25], 2.13.3.)

(1.5.7) Let $f\colon Y \to X$ be a *weakly log canonical* model of X for $K+B$, that is, an extraction such that $K_Y + B_Y$ is log canonical on Y and nef relative to f. Then

$$f^*(K+B) = K_Y + B_Y + \sum d_i E_i$$

with $d_i \geq 0$ for all the exceptional divisors E_i of f. Moreover, all $d_i = 0$ if and only if $K+B$ is log canonical. f is the identity if and only if f and $K+B$ are both log canonical. f is small and purely log terminal if $K+B$ is purely log terminal. f is the identity if $K+B$ is purely log terminal and X is $\mathbb{Q}$-factorial, that is, $K+B$ is purely and strictly log terminal.

In view of the fact that the log canonical model of f is well defined, it can be constructed locally over neighborhoods of points of Z. Note that if in the formula (1.5.7) we take the sum of exceptional divisors E_i over to the left, we get a result that is the exact opposite of the definition of log canonical: all $a_i = -d_i \leq 0$. An extraction with all $d_i = 0$ is *log crepant* (compare [20], (2.12)).

Proof. (1.5.2) holds by definition, (1.5.3) is well known (putting together (1.5.2) and [8], 0-3-12), and (1.5.1) follows from (1.5.3) by taking a nonsingular model of f according to (1.5.2) and (1.4.2). (1.5.4) follows from (1.5.1). (1.5.5–6) are proved as in [25], (2.13), using a *Hironaka hut*

$$\begin{array}{ccccc} & & W & & \\ & {}^{f'}\swarrow & & \searrow^{g'} & \\ X & & \overset{t}{\dashrightarrow} & & Y \\ & {}_{f}\searrow & & \swarrow_{g} & \\ & & Z & & \end{array}$$

here W is nonsingular, so that $f'\colon W \to X$ and $g'\colon W \to Y$ are resolutions. (The Russian usage *domik Hironaka* is traditional, *domik* = little house, hut.) In the analytic case, the same over a neighborhood of a compact subset of Z. If g is log canonical, $g'^*(K_Y + B_Y)$ is nef relative to $g \circ g' = f \circ f'$ and f'. Thus the difference $\Delta = f'^*(K + B) - g'^*(K_Y + B_Y)$ is nonpositive relative to f' and its support is contained in divisors that are exceptional for both f' and g'. If E_i is exceptional for g' but not for f', its multiplicity in Δ is $b_i - 1 + a_i$, which is ≥ 0 by definition of the log canonical model, where a_i is the log discrepancy in E_i of $K_Y + B_Y$. By negativity of a contraction, 1.1, Δ is an effective divisor. Moreover, if E_i is exceptional for f' but not for g', then Δ or $-\Delta$ has maximal numerical Kodaira dimension on E_i, is not numerically 0 over the generic point of $f'E_i$, and its multiplicity $1 - a_i - 1 = -a_i$ at E_i is > 0 by negativity of a contraction, 1.1; here a_i is the log discrepancy of $K + B$ at E_i. This is impossible, and the modification t is contracting. The log discrepancy of $K_Y + B_Y$ at a prime divisor $D \subset W$ is greater than the corresponding log discrepancy of $K + B$ by the multiplicity of Δ at D, since $B_W \leq (B_Y)_W$.

In (1.5.6), $g'^*(K_Y + B_Y)$ is nef relative to $g \circ g' = f \circ f'$. Hence the difference $\Delta = f'^*(K + B) - g'^*(K_Y + B_Y)$ is nonpositive relative to f', and is supported in divisors that are exceptional for f' or g'. As above, it is effective on divisors E_i that are exceptional for g' but not for f'. Again by negativity of a contraction 1.1, the difference is effective and does not involve the E_i that are not exceptional for f' and g'. If the image $g'(E_i)$ is exceptional for g then also $f'(E_i)$ is an exceptional divisor for f. This is impossible by negativity of a contraction, 1.1, and since Δ is not numerically 0 over the general point of $g \circ g'E_i$. Therefore g is small.

The multiplicity of Δ at D is 0 only if Δ is trivial over the general point of $f \circ f'D$, that is, $f'D$ is not contained in the union V of fibers of f on which $K + B$ is not numerically 0. Thus by what we have said, the log discrepancies of $K + B$ are less than the corresponding log discrepancies of $K_Y + B_Y$ for divisors over V. If $K + B$ is log terminal and numerically negative relative to f then the points of indeterminacy of t land in V, which in this case is the union of positive dimensional fibers of f, and t is an isomorphism outside this set. Hence, outside V, we can take as resolution g' a suitable resolution of X as in the definition of log terminal and resolve its indeterminacy over V. Finally, the relation of (1.5.7) follows immediately from the definition of the extraction f and negativity of a contraction, 1.1, and the remaining assertions of (1.5.7) also follow easily from this. Q.E.D.

[8], 0-4-5, states the conjecture that log terminal models exist in the general case. To construct the log canonical model from a log terminal model $g\colon Y \to Z$ one must contract the curves $\Gamma \subset Y$ with $(K_Y + B_Y) \cdot \Gamma = 0$; when the boundary B_Y is a $\mathbb{Q}$-divisor, the existence of this model is equivalent to the conjectured abundance of $K_Y + B_Y$ (see [8], 6-1-14).

There is a *general philosophy*—Mori theory, or the theory of extremal rays—of how to go about constructing a log terminal model of a projective morphism $f\colon X \to Z$ over Z (in the analytic case, over a neighborhood of any compact projective subvariety of Z). First, ([2]) resolving the singularities of X and of the boundary B to a nonsingular variety and a divisor with normal crossings, we can assume that f is projective and $K+B$ strictly log terminal by (1.5.2). By the theorem on the cone ([8], (4-2-1), or [18], (4.12), in the analytic case), if there exists a curve Γ of X/Z (that is, $\Gamma \subset X$ is contained in a fiber of f) on which $K+B$ is negative, then the Kleiman-Mori cone $\overline{\mathrm{NE}}(X/Z)$ has a locally polyhedral extremal ray R with $(K+B)R<0$. By the contraction theorem ([8], (4-2-1), or [18], (4.12), in the analytic case), there exists a contraction $\mathrm{cont}_R\colon X \to Y$ of R over Z. (In applying these theorems, if it is not already so, we first perturb B slightly to make it a $\mathbb{Q}$-divisor.) If cont_R is not birational it is a nontrivial fiber space of log Fanos, and the minimal model program comes to a stop.

Otherwise cont_R is a birational contraction. Then one carries out a modification $t\colon X \dashrightarrow X^+$ from X to X^+ over Z; the modification will be simply $X^+ = Y$ if cont_R is a divisorial contraction. Otherwise, it will be a flip $X^+ \to Y$ over Z, if this exists (see Lemma 1.7). As is well known (see [25], (2.13)), the modification $t\colon X \dashrightarrow X^+$ does not decrease the log discrepancies of $K+B$, and for a divisorial contraction of E_i the log discrepancy increases: $a_i^+ > 1 - b_i$. Hence by (1.5.5), the log terminal (log canonical) models of X and X^+ over Z coincide. However, $K^+ + B^+$ is again strictly log terminal, f^+ is projective, and one conjectures that X^+ is simpler than X in some measurable respect, which means that any sequence of such modifications eventually terminates. Hence as a result of a sequence of modifications, f either becomes a nontrivial fiber space of log Fanos, or becomes a log terminal morphism. Therefore the problematic ingredients of this construction are the *existence* and *termination* of flips. On the other hand, starting from a terminal model f, by the theorem on eventual freedom of [8], 3-1-2, we get a log canonical model when B is a $\mathbb{Q}$-divisor with $\lfloor B \rfloor = 0$ and $K+B$ is big relative to f; *big* relative to f means that the restriction of $K+B$ to a general fiber of f has Kodaira dimension equal to the dimension.

LSEPD divisors. The trouble with the above general philosophy is that, in the procedure we have just described, even if we start with no reduced boundary components, these may appear at the time of the initial resolution, and may not be contracted by subsequent modifications. However, we now introduce a systematic method of decreasing B, while preserving the intersection number of $K+B$ with all curves of X/Z and preserving rationality, under an extra condition. Let $f\colon X \to Z$ be a contraction and $K+B$ a log divisor on X. We say that B is *the support of an effective principal divisor locally over* Z or is an *LSEPD divisor* if in a neighborhood of any connected component of B there exists an effective Cartier divisor D on X which is f^* of a principal divisor on Z, and such that

$$\lfloor B \rfloor \le \operatorname{Supp} D \le \lceil B \rceil = \operatorname{Supp} B.$$

The point is that locally over Z, which is sufficient for the construction of the log canonical model, D contains all the reduced components of B with $b_i = 1$, and all the components of D are contained in B, so that perturbing the boundary from B to $B - \varepsilon D$ with $0 < \varepsilon \ll 1$ leaves it effective, but pokes out the reduced components. ([3]) Moreover, by what we have said, the condition on D can be weakened to numerically

([2]) In Utahn dialect, this is called "running the MMP" (minimal model program, [Utah], 2.26).

([3]) I have slightly edited this section. See my commentary (10.4) for a brief explanation of the special role played in the log category by the reduced boundary components (with $b_i = 1$) and the condition $\lfloor B \rfloor = 0$ (that is, no $b_i = 1$ are allowed) in the Kawamata–Shokurov technique. As explained there, the LSEPD device Shokurov introduces in this section extends the Kawamata technique to the reduced case in important cases, e.g., the theorem on eventual freedom for nef and log big divisors. Shokurov

0 relative to f, even if f is weakly log canonical, that is, $K+B$ is nef relative to f, B is a $\mathbb{Q}$-divisor, and $K+B$ is big relative to f (compare (1.5.7)). In this case we say that B is numerically LSEPD.

Returning to the general philosophy, we note that the whole picture is somewhat simplified if we assume that f is finite over a general point of Z, for example, a birational contraction. First of all, this ensures that the final model of f will be a strictly log terminal birational contraction, and not a fiber space of log Fanos. Secondly, any $\mathbb{Q}$-Cartier divisor on X, and in particular $K+B$, will be big relative to f. Hence under the given assumption, in the process of constructing the log canonical model, it is natural to assume that the original f has boundary B that is an LSEPD $\mathbb{Q}$-divisor. Moreover, if f is strictly log terminal, it is enough to assume that B is numerically LSEPD; when f is extremal, this holds if and only if $\operatorname{Supp} B$ either has only components that are numerically 0 relative to f, or has both components that are negative and components that are positive relative to f. Thirdly, if X is nonsingular outside B, then LSEPD divisors are preserved both by the initial resolution of singularities and by subsequent modifications. Moreover, in (4.5), we perfect our general philosophy so that, in order to be able to construct a log canonical model from a strictly log terminal model, it will be sufficient to know that on the original model X (whose boundary B is not necessarily a $\mathbb{Q}$-divisor) there exists an LSEPD divisor B' with $\lfloor B \rfloor \le B' \le \lceil B \rceil$, such that $K+B$ is log terminal outside B'. Thus the task remaining is to achieve a strictly log terminal model of f, and by the above discussion, we must succeed in the construction and the termination of flips. For a surface X, flips and their termination are OK, so that we get the next result.

1.6. **Example.** If $f\colon X \to Z$ is a morphism of a surface X, finite over the general point of Z, then there exists a strictly log terminal model of f, even if $K+B$ is not log canonical; respectively, there exists a log canonical model of f provided that B passes through all points at which $K+B$ is not log terminal. In this case, the birational contraction to a log canonical model can be transformed to the numerically negative case using negativity of a contraction, 1.1, and then its existence can be proved using the contraction theorem ([8], 3-2-1). In particular, X has a strictly log terminal model; respectively, a log canonical model provided that B passes through all the points at which $K+B$ is not log terminal. From this and from (1.5.7), in the 2-dimensional case any notion of log terminal is always strict; since, quite generally, for a purely log terminal $K+B$ a strictly log terminal model is small, but for surfaces is the identity. By [8], 1-3-6, this follows anyway from the rationality of weakly log terminal singularities, which in the surface case is equivalent to log terminal. Note also that a strictly log terminal model, or a model as a fiber space of log Fanos exists for any projective morphism $f\colon X \to Z$ from a surface. (The material here is all elementary and well known. It's an exercise to understand all this in terms of collections of curves on surfaces and Zariski decomposition of a log divisor $K+B$ on the resolution; compare [Kawamata].)

According to Lemma 1.7 below, the previous considerations can also be used to construct the flips themselves, provided that termination is known, and that in constructing a flip we need only flips of a simpler type. We only note here that a *flip* of a birational contraction f *with respect to* D (where in general D is an $\mathbb{R}$-divisor of

calls this property *supports a fiber relative to* f, and later allows this to degenerate to *forms a fiber*; his equivalent definition is that there exists a boundary B' with $\lfloor B \rfloor \le B' \le \operatorname{Supp} B$ such that each connected component of B' equals the support of a fiber of a composite morphism (not necessarily proper) $X \to Z \to C$ of X to a curve. Compare [Utah], Definition 2.30.

X) is defined as a modification

$$\begin{array}{ccc} X & \xrightarrow{\operatorname{tr} f} & X^+ \\ & {\scriptstyle f}\searrow \quad \swarrow {\scriptstyle f^+}, & \\ & Z & \end{array}$$

where f^+ is a small contraction for which the modified divisor $D^+ = \operatorname{tr} f(D)$ is numerically ample. (*Modified divisor* D^+ means the birational transform; it moves under linear equivalence if D does.) A flip is obviously unique if it exists. For this reason it follows that it is equivariant: that is, automorphisms of X/Z preserving D act biregularly on X^+.

In applications, D is usually negative, and even antiample relative to f, and the contraction f itself is small and even extremal. In the case of an extremal and projective f, D is negative relative to f if and only it is antiample, and the flip does not depend on the choice of such D if Y has rational singularities, and is also extremal if X is $\mathbb{Q}$-factorial. If $f = \operatorname{cont}_R$ is a small contraction of an extremal ray, then f or R or the curves contracted by f are *flipping*, and otherwise *divisorial*. However, when X is not $\mathbb{Q}$-factorial, a flip of a divisorial contraction may differ from it, that is, it may not be regular (see [25], (2.11) and (2.9) and [8], 5-1-6): if the contracted divisor is not $\mathbb{Q}$-Cartier then K_Z is not $\mathbb{Q}$-Cartier, and a flip is needed to return to the inductive category. Thus the above extension to any birational contraction of the notion of flip is quite natural.

1.7. **Lemma.** *Let $f: X \to Z$ be a birational contraction such that $K+B$ is numerically nonpositive relative to f; then a log canonical model of f is a flip with respect to $K+B$, and conversely. A flip or log canonical model of f is also a log canonical model of Z for $K_Z + f(B)$.*

Proof. ([4]) If $g: Y \to X$ is a log canonical model of f then by (1.5.6) and by definition g is a small contraction, with $K+B$ numerically ample relative to g, that is, g is a flip of f. The converse follows from standard properties of flips (see [25], 2.13 and the properties of log flips in §1.12, below). The case of a log canonical model is treated similarly. Q.E.D.

1.8. **Example** (Tsunoda). Suppose that $f: X \to Z$ is a projective birational contraction of an algebraic (or analytic) 3-fold X, and that the boundary B is *semistable* relative to f. This means that B is linearly 0 relative to f and has a projective resolution of singularities $g: Y \to X$ such that $g^*B = g^{-1}B + \sum E_i$, which is the sum of nonsingular prime divisors with normal crossings, where $\sum E_i$ is the exceptional set of f. The minimal number d of exceptional divisors of such a resolution is the *depth* of B. I assert the existence of a flip of f with respect to a numerically nonpositive $K+B$ when the exceptional set of f is contained in B. By Lemma 1.7, this coincides with the log canonical model of f. Composing with the above resolution, we can assume that X is nonsingular, and the boundary B is linearly 0 relative to f, reduced, and consists of nonsingular prime divisors with normal crossings. It is enough to construct a strictly log terminal model of f. The existence of flips of extremal rays with $K+B$ negative is known in this case by [23], Theorem 1, or [7],

([4]) Explanation: The alternative to f being a divisorial contraction is that $\operatorname{cont}_R: X \to Y$ is a small or *flipping* contraction over Z; then R is a *flipping ray*. The required modification is a relative log canonical model $X^+ \to Y$ of Y; if this exists, the results of Proposition 1.5 are applicable to it, so that by (1.5.6), $X^+ \to Y$, if it exists, is a small extraction. Then $t: X \dashrightarrow X^+$ is a *flip* over Z; in the 3-fold case, $\operatorname{cont}_R: X \to Y$ contracts a curve Γ (possibly reducible), the *flipping curve* of R, and $X^+ \to Y$ extracts a curve Γ^+, the *flipped curve*, so that the modification $t: X \to X^+$ just replaces the neighborhood of a flipping curve $\Gamma \subset X$ by that of a flipped curve $\Gamma^+ \subset X^+$.

(10.1). Moreover, the semistability of B is preserved by such flips, and its depth increases by at most 1 under a divisorial contraction, and decreases under a flip (see [26], (9.1), and [28]). Hence termination holds and the log terminal model of f exists. If we discard B, termination also follows by termination in the terminal case for K (see [25], (2.17)).

The main theorem of §0 is equivalent to the following result.

1.9. **Main Theorem.** *Let $f\colon X \to Z$ be a birational contraction of an algebraic (or analytic) 3-fold X; suppose that $K+B$ is log terminal outside B and nonpositive relative to f, and that the boundary of B is LSEPD. Then the flip of f exists.*

Proof of equivalence. By the uniqueness of a flip, we can restrict to the local situation. If f is the birational contraction of the theorem of §0 and H a hyperplane section containing $f(\operatorname{Supp} B)$, then by stability (1.3.4), for small rational $\varepsilon > 0$ the log divisor $K + B + \varepsilon f^*H$ satisfies all the requirements of Theorem 1.9. For the converse, subtracting off a multiple of the principal divisor provided by the LSEPD assumption ensures that $\lfloor B \rfloor = 0$, and by monotonicity (1.3.3) makes $K+B$ log terminal. Q.E.D.

The proof of Theorem 1.9 and Theorem 1.10 below makes up the bulk of this paper; this is obtained first in a jumbled form in §6, and then, in an organised form, in §§6–8 after reducing to special flips.

1.10. **Theorem.** *A small proper morphism $f\colon X \to Z$ of an algebraic (or analytic) 3-fold X that is finite over the general point of Z has a strictly log terminal model for $K+B$ (in the analytic case, over a neighborhood of a projective subset $W \subset Z$), even if X is not $\mathbb{Q}$-factorial and $K+B$ not log canonical.*

In the analytic case, *over a neighborhood of W* also means that the strictly log terminal property of the model is preserved on shrinking the neighborhood of W.

1.11. **Corollary.** *A small proper morphism $f\colon X \to Z$ of an algebraic (or analytic) 3-fold X that is finite over a general point of Z has a log canonical model for $K+B$, even if X is not $\mathbb{Q}$-factorial and $K+B$ is not log canonical, provided that B is LSEPD, and $K+B$ is log terminal outside the principal divisor provided by the LSEPD assumption.*

The final condition is satisfied, for example, if the boundary B is LSEPD and $K+B$ is log terminal outside B. If we apply Theorem 1.10 and Corollary 1.11 to the identity morphism id_X then we get respectively the log canonical and log terminal model of X for $K+B$ (see §9). Moreover, the local divisor D supports a fiber relative to id_X, or, as we will also say, simply *is a fiber*, when locally there is an effective Cartier divisor D' having the same support, $\operatorname{Supp} D' = \operatorname{Supp} D$. This holds, for example, if D is effective and $\mathbb{Q}$-Cartier, and by rational approximation, even when it is $\mathbb{R}$-Cartier. Note that the corollary holds for the boundary B, as soon as Theorem 1.10 is established for it (see the proof in §4). From this we get the following proof.

Proof of the corollary of §0. The cone $\overline{\mathrm{NE}}(X/Z)$ (respectively $\overline{\mathrm{NE}}(X/Z\,;W)$) is locally polyhedral in the region $K+B<0$, by [13], (5.4). The contraction cont_R then exists by [8], 3-1-2 (in the analytic case we can use the arguments of [18], (5.8)), rational approximation (1.3.5), and stability (1.3.4) of purely log terminal divisors. It is known that cont_R is either a nontrivial fiber space of log Fanos or is birational, that is, a birational contraction. In the final case, the flip exists by the theorem of §0. Q.E.D.

To conclude §1 we give some facts that will be needed later.

1.12. **Properties of log flips.** Let $g^+\colon X^+ \to Y$ be a flip over Z of a birational contraction $g\colon X \to Y$ with respect to the divisor $K+B$. Then $K^+ + B^+$ is log canonical if $K+B$ is numerically nonpositive relative to g. Respectively, $K^+ + B^+$ is divisorially log terminal if $K+B$ is divisorially log terminal, and the exceptional locus of g does not contain any generic points of normal crossing of $\lfloor B \rfloor$ (compare the paragraph before Example 1.2). Moreover, if in addition $K+B$ is log terminal, respectively purely or weakly log terminal, and negative relative to g then

(1.12.1) $D^+ = K^+ + B^+$ is log terminal, respectively purely or weakly log terminal.

(1.12.2) If a $\mathbb{Q}$-Cartier divisor D is numerically 0 relative to g, then $D^+ = \operatorname{tr} f(D)$ is a $\mathbb{Q}$-Cartier divisor, numerically 0 relative to g^+ and having the same index as D (in the analytic case, over a neighborhood of any compact subset of Z).

(1.12.3) X^+ is projective over Z (in the analytic case, over a neighborhood of any compact subset of Z) if X is projective over Z and g is extremal.

(1.12.4) If X is $\mathbb{Q}$-factorial and g is extremal then X^+ is $\mathbb{Q}$-factorial; if in addition g is small then g^+ is extremal. Moreover,

$$\rho(X^+/Z) = \begin{cases} \rho(X/Z) - 1 & \text{if } g \text{ is a divisorial contraction;} \\ \rho(X/Z) & \text{if } g \text{ is a small contraction} \end{cases}$$

(in the analytic case, $\rho(X^+/Z\,;W) = \rho(X/Z\,;W) - 1$ or $\rho(X^+/Z\,;W) = \rho(X/Z\,;W)$).

Proof. This is either known, or can be obtained by modifying [25] and [8] (see also the properties of log models in §1.5 together with Lemma 1.7). Note that in the 3-fold case, the condition that the exceptional locus of g does not contain any generic points of normal crossing of $\lfloor B \rfloor$ is satisfied if it does not contain any triple points, double curves or prime components of $\lfloor B \rfloor$. Q.E.D.

§2. The covering trick

Let $f\colon \widetilde{X} \to X$ be a quasifinite morphism between normal varieties. A prime Weil divisor D on X defines an integral effective divisor $f^*D = \sum m_i E_i$ with E_i prime. f^*D is meaningful, since D is Cartier in codimension 1, where f is a finite extension of DVRs. This a "birational transform" construction; in terms of divisorial sheaves, $\mathcal{O}_{\widetilde{X}}(f^*D)$ is the double dual $\mathcal{O}_{\widetilde{X}}(f^*D) = (f^*(\mathcal{O}_X(D)))^{**}$.

The coefficient $m_i = \operatorname{mult}_{E_i}(f)$ is the *multiplicity* of f at (or along) E_i. The number $r_i = r_{E_i}(f) = m_i - 1$ is the *ramification index* of f along E_i. Divisors with positive ramification indices are called *ramification divisors* of f; these form the support of the *ramification divisor* $R = \sum r_E(f)E$, and the reduced divisor

$$R_{\text{red}} = \sum_{r_E(f)>0} E$$

is the *ramification locus* of f.

2.1. **The pullback formula.**

$$K_{\widetilde{X}} + f^{-1}D = f^*(K_X + D) + \sum_i \sum_{f(E_j)=D_i} (1-d_i) r_j E_j$$

for any Weil divisor $D = \sum d_i D_i$ on X, where $f^{-1}D = \sum d_i f^{-1} D_i$ and $f^{-1}D_i$ are the reduced divisors $\sum E_j$ obtained as the set-theoretic inverse image of $\operatorname{Supp} D_i$.

Proof. The relation

$$f^*D = \sum_i \sum_{f(E_j)=D_i} d_i m_j E_j$$

reduces the verification to the particular case when $D = 0$. Then the pullback formula becomes a higher-dimensional analog of the Hurwitz formula for canonical divisors [27], $K_{\widetilde{X}} = f^*K_X + R$, where R is the ramification divisor of f. By the same argument, it is also equivalent to the log analog of the Hurwitz formula $K_{\widetilde{X}} + f^{-1}B = f^*(K_X + B)$ when B is a reduced divisor containing the ramification locus $f(R_{\text{red}}) \subset X$. Q.E.D.

Remark. Grothendieck duality provides a less pedestrian definition of R and proof of (2.1): since $f_*\omega_{\widetilde{X}} = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_{\widetilde{X}}, \omega_X)$, there is an intrinsic evaluation homomorphism $f_*\omega_{\widetilde{X}} \to \omega_X$, hence an intrinsic homomorphism $J\colon \omega_{\widetilde{X}} \to f^*\omega_X$, extending the map $\Omega^n_{\widetilde{X}} \to f^*\Omega^n_X$ defined on the nonsingular locus by the determinant of the Jacobian matrix; $\mathrm{div}(J)$ is the ramification divisor.

The pullback formula can also be conveniently expressed in the form $K_{\widetilde{X}} + B_{\widetilde{X}} = f^*(K + B)$, where

$$\begin{aligned} B_{\widetilde{X}} &= f^{-1}B - \sum_i \sum_{f(E_j)=D_i} (1 - b_i) r_j E_j \\ &= \sum_i \sum_{f(E_j)=D_i} (b_i m_j - r_j) E_j. \end{aligned}$$

Here $B = D = \sum b_i D_i$ is a boundary of X. The inequality $b_i m_j - r_j \le 1$ follows from $b_i \le 1$. Hence to ensure that $B_{\widetilde{X}}$ is a boundary of $\widetilde{X}$, we need only the inequality $b_i m_j - r_j \ge 0$, or equivalently

$$b_i \ge \frac{r_j}{m_j} = \frac{m_j - 1}{m_j}. \tag{2.1.1}$$

2.2. **Corollary.** *For a finite morphism f, the divisor $K_{\widetilde{X}} + B_{\widetilde{X}}$ is log canonical (respectively purely log terminal) if and only if $K + B$ is.*

Proof (compare [Pagoda], (1.9)). First of all, I claim that $K + B$ is an $\mathbb{R}$-Cartier divisor if and only if its pullback $K_{\widetilde{X}} + B_{\widetilde{X}} = f^*(K_X + B)$ is. This follows from the fact that the pullback f^* and pushforward f_* by a finite morphism f preserve $\mathbb{R}$-Cartier divisors, since the composite $f_* \circ f^*$ multiplies divisors by $\deg f$.

The pullback of a finite morphism f by a birational contraction $g\colon Y \to X$ fits in a commutative diagram

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\widetilde{f}} & Y \\ \widetilde{g}\downarrow & & \downarrow g \\ \widetilde{X} & \xrightarrow{f} & X \end{array}$$

where $\widetilde{f}$ is again a finite morphism, and $\widetilde{g}$ a birational contraction. I can assume by the above that $K + B$ and $K_{\widetilde{X}} + B_{\widetilde{X}}$ are $\mathbb{R}$-Cartier divisors. By the final version of the pullback formula and the definition of the log discrepancies a_i, we have

$$\begin{aligned} K_{\widetilde{Y}} + B_{\widetilde{Y}} &= \widetilde{f}^*(K_Y + B_Y) = \widetilde{f}^*\left(g^*(K + B) + \sum a_i E_i\right) \\ &= \widetilde{g}^*(K_{\widetilde{X}} + B_{\widetilde{X}}) + \sum a_i \widetilde{f}^*(E_i). \end{aligned}$$

Thus the log discrepancy $\widetilde{a}_j$ of a prime divisor E_j contracted by $\widetilde{g}$ can be computed by the log discrepancy a_i of $D_i = \widetilde{f}(E_j)$:

$$\widetilde{a}_j = a_i m_j,$$

where m_j is the multiplicity of $\widetilde{f}$ at E_j. Hence the inequalities $\widetilde{a}_j \geq 0$ and $a_i \geq 0$ (or $\widetilde{a}_j > 0$ and $a_i > 0$) are equivalent. It only remains to check that each divisor contracted by $\widetilde{X}$ appears as an exceptional divisor of $\widetilde{g}$ for a suitable g. Q.E.D.

2.3. **Construction of cyclic cover.** Let D be a primitive principal divisor on X. Then for any natural n there is a finite cyclic cover $f\colon \widetilde{X} \to X$ of degree n (a Galois cover with Galois group $\mathbb{Z}/(n)$) such that the pullback f^*D is divisible by n as a principal divisor. Indeed, $D = \operatorname{div}(\varphi)$ is the divisor of a function φ. Here primitive means that D is not divisible by $m \geq 2$ as a principal divisor, which implies that the polynomial $x^n - \varphi$ is irreducible over the function field $\mathscr{R}(X)$ (respectively, the meromorphic function field $\mathscr{M}(X)$ in the analytic case). For f we can take the normalization of X in the finite extension $\mathscr{R}(\sqrt[n]{\varphi})$. (See [14]; in the analytic case, the normalization of X in a finite extension $\mathscr{L}$ of $\mathscr{M}(X)$ is a finite holomorphic map $f\colon \widetilde{X} \to X$ such that for every Stein open U of X the ring $\Gamma(f^{-1}U, \mathscr{O}_{\widetilde{X}})$ is the integral closure of $\Gamma(U, \mathscr{O}_X)$ in $\mathscr{L} \otimes_{\mathscr{M}(X)} \mathscr{M}(U)$, where $\mathscr{M}(U)$ is the field of meromorphic functions on U.) The name comes from the fact that $\widetilde{X}$ is the normalization of the graph $\Gamma_\psi \subset X \times \mathbb{P}^1$ of the function $\psi = \sqrt[n]{\varphi}$, a many-valued function on X that becomes single-valued on $\widetilde{X}$. Locally, the normalization is uniquely defined by D. By definition $f^*D = \operatorname{div}(f^*\varphi) = \operatorname{div}(\psi^n) = n\operatorname{div}(\psi)$. The irreducibility of $\widetilde{X}$ comes from the irreducibility of $x^n - \varphi$ over $\mathscr{R}$. By construction f is ramified only over components D_i of $D = \sum d_i D_i$ and has ramification multiplicity

$$m_i = \operatorname{mult}_{D_i}(f) = \frac{n}{\operatorname{hcf}(d_i, n)}$$

along D_i. The cover f is called *taking the nth root* of D. Of especial interest is the case when $n = \operatorname{lcm}(d_i)$ (in order for the l.c.m. to exist in the analytic case, we must assume that D is finite, which holds in a neighborhood of any compact subset of X). In this case $m_i = n/|d_i|$ for $d_i \neq 0$, hence

$$\frac{1}{n} f^*D = \sum \operatorname{Sgn}(d_i) f^{-1} D_i.$$

In particular, if D is effective then $(1/n)f^*D$ is a principal reduced divisor.

2.4. **Examples of cyclic covers.**

(2.4.1) Suppose that $K + B$ is a log divisor, not necessarily log canonical, with boundary B of index n such that $n(K+B) \sim 0$ is linearly 0 on X. The corresponding finite cyclic cover $f\colon \widetilde{X} \to X$ of degree n is ramified only over the components D_i of B, and $b_i = k_i/m_i$ is a proper fraction, with $k_i \leq m_i$ and m_i the multiplicity of f at D_i. Indeed,

$$\operatorname{mult}_{D_i}(f) = \frac{n}{\operatorname{hcf}(nk_i/m_i, n)} = \frac{nm_i}{\operatorname{hcf}(nk_i, nm_i)} = \frac{m_i}{\operatorname{hcf}(k_i, m_i)} = m_i.$$

By (2.1.1), $B_{\widetilde{X}}$ is a boundary if and only if $b_i = 1$ or $(m_i - 1)/m_i$ for some natural number m_i, and the boundary $B_{\widetilde{X}} = f^{-1}\lfloor B \rfloor$ is reduced in this case. This f is the *index* 1 *cover* of the log divisor $K + B$. By Corollary 2.2, for such covers the divisor $K_{\widetilde{X}} + B_{\widetilde{X}}$ is log canonical (or purely log terminal) if and only if the same holds for $K + B$. However, by construction $K_{\widetilde{X}} + B_{\widetilde{X}}$ has index 1. The divisor $K + B$ is purely log terminal and $b_i = (m_i - 1)/m_i$ if and only if $B_{\widetilde{X}} = 0$ and $K_{\widetilde{X}}$ is canonical. The following two particular cases are of special interest.

(2.4.2) Suppose that $K+B$ is a log canonical divisor of index 2 for which $2(K+B)$ is linearly 0. Then $B_{\widetilde{X}} = f^{-1}\lfloor B \rfloor$ is a boundary and $K_{\widetilde{X}} + B_{\widetilde{X}}$ is log canonical of index 1. If moreover $K+B$ is log terminal and $\lfloor B \rfloor = 0$ then $B_{\widetilde{X}} = 0$ and $K_{\widetilde{X}}$ is canonical, that is, in this case $\widetilde{X}$ is Gorenstein with canonical singularities.

(2.4.3) Suppose that X is $\mathbb{Q}$-Gorenstein, $B = 0$, and K has index n. Then a cyclic cover $f\colon \widetilde{U} \to U$ is defined over a suitable neighborhood U of any point of $\widetilde{X}$, known as the *index* 1 *cover* (see [20], (1.9)). Here f is etale in codimension 1, and K_U is log canonical or log terminal if and only if $K_{\widetilde{U}}$ is. But by construction $K_{\widetilde{U}}$ has index 1 and $K_{\widetilde{U}}$ is log terminal if and only if it is canonical, or by [8], 0-2-16, if and only if it is rational and Gorenstein. In particular this allows us to reduce the study of log terminal singularities to the case of canonical singularities of index 1. This approach to the study of singularities, introduced by M. Reid ([20], (1.9)) and independently by J. Wahl, is called the *covering trick*.

(2.4.4) Suppose now that $f\colon X \to Z$ is a proper morphism, and the boundary B is a reduced LSEPD divisor, that is, locally over Z, there is an effective primitive principal divisor $D = \sum d_i D_i$ with support B. Then according to the construction, locally over Z there is a finite cyclic cover $\pi\colon \widetilde{X} \to X$ of degree $n = \operatorname{lcm}\{d_i\}$, ramified only over components of B, and such that, locally over Z, $\pi^{-1}B$ is a reduced principal divisor. By Corollary 2.2, under such a cover the log canonical (or purely log terminal) property of $K+B$ is preserved and the index of $K_{\widetilde{X}} + \pi^{-1}B = \pi^*(K+B)$ divides that of $K+B$.

The covering trick is also used in the construction of flips (see [7], (8.5), and [16], (0.4.4)). Its use is based essentially on the equivariance of flips. Let $\widetilde{f}\colon \widetilde{X} \to X$ be a finite cover and $g\colon X \to Z$ a birational contraction. Then in the category of normal varieties or complex spaces, $\widetilde{f}$ is obtained as the pullback

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & X \\ {\scriptstyle \widetilde{g}}\downarrow & & \downarrow{\scriptstyle g} \\ \widetilde{Z} & \xrightarrow{f} & Z \end{array}$$

of a finite cover $f\colon \widetilde{Z} \to Z$, the normalization of Z in a finite extension $\widetilde{\mathscr{R}}/\mathscr{R}$, where $\mathscr{R}$ and $\widetilde{\mathscr{R}}$ are the rational (meromorphic) function fields of X and $\widetilde{X}$ respectively. Note that the rational (meromorphic) function field of Z or any other modification of X is just $\mathscr{R}$. The pullback $\widetilde{g}$ is also a birational contraction, and is small if g is.

2.5. **Lemma.** *The flip of g with respect to a divisor D exists if and only if the flip of the birational contraction $\widetilde{g}$ with respect to $\widetilde{f}^*D$ exists.*

Proof. Suppose first that a flip $t\colon X \dashrightarrow X^+$ exists. Then it has a pullback

(2.5.1)
$$\begin{array}{ccc} \widetilde{X} & \overset{\widetilde{t}}{\dashrightarrow} & \widetilde{X}^+ \\ {\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle \widetilde{f}^+} \\ X & \overset{t}{\dashrightarrow} & X^+ \end{array}$$

where $\widetilde{f}^+\colon \widetilde{X}^+ \to X^+$ is the normalization of X^+ in the function field $\widetilde{\mathscr{R}}$. Let $\widetilde{g}^+\colon \widetilde{X}^+ \to \widetilde{Z}$ be the pullback of the birational contraction $g^+\colon X^+ \to Z$. Since $\widetilde{Z}$ is finite over Z and $\widetilde{X}^+$ finite over X^+, it follows that $\widetilde{g}^+$ is a small contraction, and $(\widetilde{f}^*D)^+ = (\widetilde{f}^+)^*D^+$ is ample relative to $\widetilde{g}^+$. Hence $\widetilde{t}$ is a flip of $\widetilde{g}$ with respect to $\widetilde{f}^*D$.

Conversely, suppose that a flip $\widetilde{t}$ is given. By the above, after normalizing $\widetilde{f}$ in the sense of Galois theory, which corresponds to the normal closure of $\widetilde{\mathscr{R}}/\mathscr{R}$, we can assume that $\widetilde{f}$ is a Galois cover with Galois group G. Then the divisor $\widetilde{f}^*D$ is invariant under G. Hence by equivariance of the flip $\widetilde{t}$, G acts biregularly (in the analytic case, biholomorphically) on $\widetilde{X}^+$, and the diagram

$$\begin{array}{ccc} \widetilde{X} & \overset{\widetilde{t}}{\dashrightarrow} & \widetilde{X}^+ \\ {\scriptstyle \widetilde{f}}\downarrow & & \\ X & & \end{array}$$

can be completed to a pullback (2.5.1), where $\widetilde{f}^+ \colon \widetilde{X}^+ \to X^+ = \widetilde{X}^+/G$ is obtained by taking the quotient by the action of G. It is not hard to check that the modification X^+ obtained in this way is the flip with respect to D. Q.E.D.

2.6. **Proposition. Flip of Type I.** *Suppose that X is a 3-fold, $f\colon X \to Z$ a birational contraction, and $K + B$ a log divisor such that*

(i) *$K + B$ is nonpositive relative to f;*

(ii) *the boundary B is a reduced LSEPD divisor and contains the exceptional of f;*

(iii) *X is nonsingular outside B.*

Then the flip of f relative to $K + B$ exists.

By the eventual freedom theorem ([8], 3-1-2), in order to ensure that the boundary is LSEPD, both here and in Propositions 2.7–8 below, it is enough to assume it is numerically LSEPD.

Proof. Since the construction of a flip is local in nature, we can restrict to a neighborhood of a fiber of f, and by the above, after contracting curves on which $K + B$ is numerically 0 if necessary, we can assume that the birational contraction f is projective with $K + B$ antiample relative to f (compare [25], Proposition 2.3). By the semistable reduction theorem (see [9], and [Shokurov]), after possibly shrinking X to a suitable neighborhood of the contracted fiber, there exists a finite cover $g\colon \widetilde{X} \to X$, ramified only over B, on which the boundary $g^{-1}B$ is semistable (as in Example 1.8). Hence by Corollary 2.2, $K_{\widetilde{X}} + g^{-1}B = g^*(K + B)$ is log canonical, and by (i) is nonpositive relative to the pulled-back contraction $\widetilde{f}\colon \widetilde{X} \to \widetilde{Z}$. Then by Example 1.8 there exists a flip of $\widetilde{f}$ with respect to $\widetilde{g}^*(K + B)$, so that descending by Lemma 2.5 we get the required flip of f. Q.E.D.

2.7. **Proposition. Flip of Type II.** *Suppose that X is a 3-fold, $f\colon X \to Z$ a birational contraction, and $K + B$ a log divisor such that*

(i) *$K + B$ is numerically 0 relative to f;*

(ii) *the boundary B is a reduced LSEPD divisor;*

(iii) *X has log terminal singularities outside B.*

Then the flip of f relative to any $\mathbb{Q}$-divisor exists.

Proof. Again we can restrict to a neighborhood of a fiber of f. By the theorem on eventual freedom and (i)–(iii) it follows that $K + B$ is linearly 0 relative to f, that is, $K+B$ descends relative to f as a $\mathbb{Q}$-Cartier divisor. Thus it is enough to consider the case that f is the identity. By (2.4.1), (2.4.3) outside B, and Lemma 2.5, the assertion reduces to the case that $K + B$ has index 1; then in (iii), log terminal is replaced by canonical singularities outside B. In the same way, by (2.4.4) we can also assume that B is a reduced Cartier divisor. Then K is Gorenstein and has canonical singularities by monotonicity (1.3.3). Hence the required flip exists by Kawamata ([7], (6.1)) or Kollár's version ([11], (6.6)). Q.E.D.

2.8. **Proposition. Flip of Type III.** *Suppose that X is a 3-fold, $f\colon X \to Z$ a birational contraction, and $K+B$ a log divisor such that*

(i) *$K+B$ has index 2;*
(ii) *the reduced part of the boundary $S = \lfloor B \rfloor$ is LSEPD;*
(iii) *$K+B$ is numerically 0 relative to f;*
(iv) *$K+B$ has log terminal singularities outside $S = \lfloor B \rfloor$.*

Then the flip of f relative to any $\mathbb{Q}$-divisor exists.

Proof. As in the preceding proof we can restrict to the local assertion and assume that f is the identity and $K+B$ has index 1 or 2. By Proposition 2.7, I need only consider the case of index 2. Then by (2.4.2) and Lemma 2.5, after making a double cover, the existence of the required flip again reduces to a flip of Type II. Here under the conditions (ii) and (iv) the reduced part of the boundary $S = \lfloor B \rfloor$ is replaced by the full boundary B, and log terminal in (iv) by canonical outside B. Q.E.D.

2.9. **Proposition. Flip of Type IV.** *Suppose that X is a 3-fold, $f\colon X \to Z$ a birational contraction, and $K+B$ a log divisor such that*

(i) *$K+B$ is purely log terminal and of index 2;*
(ii) *$K+B$ is numerically 0 relative to f;*
(iii) *the components of $\lfloor B \rfloor$ are not exceptional relative to f.*

Then the flip of f relative to any $\mathbb{Q}$-divisor exists.

Proof. As above, we can restrict to the local assertion and assume that f is the identity. To prove that $K+B$ descends as a $\mathbb{Q}$-Cartier divisor we must apply the eventual freedom theorem of [8], 3-1-2, having first, as a preliminary step, made a extension of the boundary B by adding to it εf^*H for some small $\varepsilon > 0$, where H is a hyperplane section such that $B + f^*H$ is an LSEPD divisor. To construct H, we take a general hyperplane section H' of Z through $f(\lfloor B \rfloor)$. Then it is enough to take H to be a general hyperplane section through $H' - f(\lfloor B \rfloor)$.

If $S = \lfloor B \rfloor$ has a reduced component, it can be replaced in the boundary B by the divisor $\frac{1}{2}D$, where D is a general element of the linear system $|2S|$. Indeed, for any fixed resolution $g\colon Y \to X$,

$$g^*(\tfrac{1}{2}D - S) = g^{-1}(\tfrac{1}{2}D - S) - \sum e_i E_i,$$

where all $e_i \geq 0$, since every function in $\mathscr{O}(2S)$, possibly after adding a constant, does not have a 0 along the exceptional divisors E_i. Hence the log discrepancy of $K+B$ with the new boundary differs by a contribution e_i at E_i, and is again > 0. Now D is reduced by Bertini's theorem, and hence the new log divisor $K+B$ is purely log terminal and $\lfloor B \rfloor = 0$. Since $2(K+B)$ is linearly 0, the existence of the flip again reduces by Example 2.4.2 to [7], (6.1), or to [11], (6.6). Q.E.D.

The construction of flips of type IV may at first sight seem to be a inessential generalization of Kawamata's sufficient condition ([7], §8) for the existence of flips. However, these flips plays an important role in what follows.

§3. Adjunction of log divisors

Consider a prime Weil divisor $S \subset X$ and its normalization $\nu\colon S^\nu \to S \subset X$. For an arbitrary divisor D on X whose support does not contain S, the *restriction*

$D_{|S^\nu} = \nu^* D$ can be defined as a divisor on S^ν as follows. Taking general hypersurface sections reduces the definition to the case when X is a surface; let $f\colon Y \to X$ be a resolution of singularities of X, which also resolves the singularities of S, and hence normalizes S. Then the *numerical pullback* of D is defined (following Mumford) by $f^*D = f^{-1}D + E$, where E is the exceptional $\mathbb{Q}$-divisor for f determined by $f^*D \cdot E_i = (f^{-1}D + E) \cdot E_i = 0$ for all exceptional curves E_i ($f^{-1}D$ is the birational transform). The existence and uniqueness of E follows from the fact that the intersection matrix $\{E_iE_j\}$ is negative definite; this also implies that E is effective if D is (by negativity of a contraction, 1.1).

By construction, the normalization ν can be identified with f on $S^\nu = f^{-1}S$. Hence it is natural to set $D_{|S^\nu} = f^*D_{|S^\nu}$. It is easy to check that the map $_{|S^\nu} = \nu^*$ is well defined, and is a partially defined homomorphism of Weil divisors of X to those of S^ν, and on $\mathbb{R}$-Cartier divisor is the $\mathbb{R}$-linear extension of the pullback of Cartier divisors under ν. In the case of log divisors, we have the pullback $f^*(K+S+D) = K_Y + S^\nu + D'$, where $D' = f^{-1}D + E$ is defined by the equations $(K_Y + S^\nu + D') \cdot E_i = 0$ on the exceptional components E_i. We call the restriction $D'_{|S^\nu}$ the *different* of D on S^ν, and denote it by D_{S^ν}. One can check that the different does not depend on the choice of the canonical divisor K or on the choice of the resolution f, and that its definition extends naturally to higher dimensions. ([5])

3.1. **Adjunction formula.** *If S is not contained in* Supp D, *then*

$$K_{S^\nu} + D_{S^\nu} = (K + S + D)_{|S^\nu}$$

for a suitable canonical divisor K_{S^ν}. In the general case, this should be understood as equality of linear equivalence classes. (*See* (10.6).)

Proof. Again it is enough to prove the formula for surfaces. But then, by the adjunction formula in the nonsingular case and the construction just discussed,

$$\begin{aligned} K_{S^\nu} + D_{S^\nu} &= (K_Y + S^\nu + D')_{|S^\nu} \\ &= f^*(K + S + D)_{|S^\nu} = (K + S + D)_{|S^\nu}. \quad \text{Q.E.D.} \end{aligned}$$

Differents have certain remarkable properties when $D = B$ is a boundary, or a divisor of the following more general type. A *subboundary* is a divisor $D = \sum d_i D_i$ with $d_i \le 1$; thus effective subboundaries are boundaries. If we write the definition of log discrepancy for a birational contraction $f\colon Y \to X$ in the form

$$f^*(K + B) = K_Y + B^Y,$$

where $B^Y = B_Y - \sum a_i E_i$, then the condition $a_i \ge 0$ for $K + B$ to be log canonical means that B^Y is a subboundary. Similarly, a log divisor $K + D$ is *log canonical* if it is $\mathbb{R}$-Cartier and for any extraction f the divisor D^Y defined in the same way is a subboundary. In particular, D itself is a subboundary. One sees easily that it is sufficient that this condition should hold for a resolution of singularities of f on which all the exceptional divisors E_i and irreducible components of $f^{-1}D$ are nonsingular and cross normally. If for one such resolution D^Y is a subboundary, and has multiplicities < 1 for all exceptional divisors of f, then $K + D$ is *log terminal*. In a similar way, we transfer to the case of subboundary the definitions of *strictly, purely, weakly* and *divisorially* log terminal. By what we have said, for genuine boundaries the given definitions are equivalent to the usual notions of log canonical, log terminal and the various flavors of log terminal. Note also the following obvious

([5]) See 10.6. and [Utah], Chapter 19.

fact: if $K+D$ is $\mathbb{R}$-Cartier, then for any extraction f, $K+D$ is log canonical if and only if $K_Y + D^Y$ is.

3.2. **Properties of the different.**

(3.2.1) *Semiadditivity.* $D_{S^\nu} = 0_{S^\nu} + D_{|S^\nu}$.

(3.2.2) *Effectivity.* $D_{S^\nu} \geq 0$ if $D \geq 0$.

(3.2.3) *Log canonical* (*divisorially log terminal*). If $K+S+D$ is log canonical (divisorially log terminal) then the divisor $K_{S^\nu} + D_{S^\nu}$ is also log canonical (respectively divisorially log terminal). If moreover D is a boundary, then D_{S^ν} is also a boundary. If moreover D is a boundary and $K+S+D$ is purely log terminal, then D_{S^ν} is a boundary and $K_{S^\nu} + D_{S^\nu}$ is purely log terminal with $\lfloor D_{S^\nu} \rfloor = 0$.

Note that when X is a 3-fold (3.2.3) holds for weakly and strictly log terminal, since by Example 1.6 all the flavors of log terminal coincide on a surface. For purely log terminal $K+S+D$ (3.2.2–3) together with the necessary definitions generalize to the case that S is a reduced, but possibly reducible divisor (compare Lemma 3.6).

Proof. All the assertions are local, and the first two reduce to the surface case. By the additivity of f^* we get

$$f^*(K+S+D) = f^*(K+S) + f^*D = K_Y + S^\nu + 0' + f^*D$$

(where $0'$ is defined as in the paragraph before adjunction formula 3.1 by $f^*(K+S) = K_Y + S^\nu + 0'$), hence

$$D_{S^\nu} = 0'_{|S^\nu} + f^*D_{|S^\nu} = 0_{S^\nu} + D_{|S^\nu}.$$

Hence it is enough to prove effectivity (3.2.2) for $D=0$. If we take f to be a minimal resolution of singularities of X and S, then $0' \geq 0$ by negativity of a contraction, 1.1, since $K_Y + S^\nu$ is nef on exceptional curves of this resolution. Indeed, for (-1)-curves E_i we have $K_Y E_i = -1$ and $S^\nu E_i \geq 2$, and for the other exceptional curves $K_Y E_i \geq 0$.

We have to check the final property in complete generality. By the adjunction formula $K_{S^\nu} + D_{S^\nu}$ is an $\mathbb{R}$-Cartier divisor on S^ν. To compute the log discrepancies, consider the resolution of singularities $f\colon Y \to X$ from the definition of $K+S+D$ log canonical (respectively divisorially log terminal). In particular, $f^{-1}S$ is then nonsingular, so that we have the commutative diagram

$$\begin{array}{ccc} & & f^{-1}S \subset Y \\ & \overset{g}{\swarrow} & \downarrow {\scriptstyle f|_{f^{-1}S}} \\ S^\nu & \overset{\nu}{\searrow} & \\ & & S \subset X \end{array}$$

where $g\colon f^{-1}S \to S^\nu$ is a resolution of singularities of S^ν. By definition of a log canonical divisor,

$$K_Y + (S+D)^Y = f^*(K+S+D),$$

where $(S+D)^Y$ is a subboundary (and in the log terminal case its multiplicities for the exceptional components of f are < 1). By the adjunction formula

$$\begin{aligned} K_{f^{-1}S} + \left((S+D)^Y - f^{-1}S\right)_{|f^{-1}S} &= f^*(K+S+D)_{|f^{-1}S} \\ &= g^*(K+S+D)_{|S^\nu} = g^*(K_{S^\nu} + D_{S^\nu}). \end{aligned}$$

Thus

$$D_{S^\nu} = g\big(((S+D)^Y - f^{-1}S)_{|f^{-1}S}\big) \quad \text{and} \quad (D_{S^\nu})^{f^{-1}S} = ((S+D)^Y - f^{-1}S)_{|f^{-1}S}.$$

But $f^{-1}S$ appears in $(S+D)^Y$ with multiplicity 1. Hence by the preceding relation we have

$$K_{f^{-1}S} + (D_{S^\nu})^{f^{-1}S} = g^*(K_{S^\nu} + D_{S^\nu}),$$

where $(D_{S^\nu})^{f^{-1}S}$ is a subboundary: all its multiplicities are ≤ 1 (and, in the log terminal case, < 1 for the multiplicities of the intersections with exceptional divisors for f, and moreover these intersections contain all the exceptional set of $f^{-1}S$ over S^ν and after an additional blowup the intersection of the exceptional set of $f^{-1}S$ becomes divisorial over S^ν), by normal crossings of $f^{-1}S$ with the components of the divisor $(S+D)^Y - f^{-1}S$. But this is the definition of $K_{S^\nu}+D_{S^\nu}$ being log canonical (respectively, divisorially log terminal). It remains only to note that the fact that the components of $(D_{S^\nu})^{f^{-1}S}$ and of the exceptional divisors of the form $E_{i|f^{-1}S}$ are nonsingular, and that they cross normally, follows from the same requirements for the resolution f. The remaining assertions now follow from (3.2.2). Q.E.D.

The following stands out among the standard problems concerning log models:

3.3. **Inversion of adjunction** (inversion of the log canonical and log terminal properties). Does the implication (3.2.3) between the log canonical (respectively log terminal) conditions for $K+S+D$ and $K_{S^\nu}+D_{S^\nu}$ have a converse?

Let $D = \sum d_i D_i$ be an effective divisor with $S \not\subset \operatorname{Supp} D$ such that

(i) $K+S+D$ is $\mathbb{R}$-Cartier;

(ii) $K_{S^\nu}+D_{S^\nu}$ is log canonical (respectively log terminal and $\lfloor D_{S^\nu} \rfloor = 0$, or divisorially log terminal and normal crossings of reduced components of D_{S^ν} extend generically in a neighborhood of S).

The problem of inversion of adjunction asks whether (i) and (ii) imply that $K+S+D$ is log canonical (respectively purely log terminal and $\lfloor D \rfloor = 0$, or divisorially log terminal) in a neighborhood of S, and in particular $S+D$ is a boundary.

Here the condition on extending normal crossings of D_{S^ν} in the third case of (ii) means the following: whenever $\bigcap D_i' \subset S^\nu$ is a generically normal intersection of k reduced components $D_i' \leq D_{S^\nu}$, then we require that in a neighborhood of S, there exist k reduced divisors $D_i \leq D$ extending the D_i' and having normal crossing with S:

$$D_i' = D_{i|S^\nu} \quad \text{and} \quad \bigcap D_i' = S \cap \bigcap D_i$$

with generic normal crossings at general points, which includes the normality of S at the given general points. In particular, this includes the requirement that each component of D_{S^ν} with multiplicity 1 is generically the normal intersection of S and a component of $\lfloor D \rfloor$. Note that the inversion problem is certainly false in the log terminal version if we do not assume the above restriction (see Corollary 3.16, and Examples 3.5 and 3.17).

Our results on this problem for 3-folds are contained in Proposition 5.13 and Corollary 9.5.

3.4. **Conditional inversion of adjunction.** Suppose that, in addition to the assumptions of 3.3, X locally has a weakly log canonical model for $K+S+B$ in a neighborhood of any point of S, where $B = \sum \min\{1, d_i\} D_i$ (for any boundary B). Then the inversion problem 3.3 of the log canonical (respectively log terminal) conditions has a positive answer for $K+S+D$ (for any D).

Proof. The required assertion is local. Hence in the arguments that follow we can restrict to a neighborhood of a point $P \in S$. Take the boundary $B = \sum \min\{1, d_i\} D_i$. Then by the assumptions, there exists a weakly log canonical model $f: Y \to X$ for $K + S + B$. By (1.5.7), negativity of a contraction, 1.1, and the numerical nonpositivity relative to f in a neighborhood of $f^{-1}P$, we get that

$$E = (S+D)^Y - (S+B)_Y = f^*(K+S+D) - (K_Y + f^{-1}S + B_Y)$$

is effective. Moreover, either it is zero, or some exceptional divisor E meeting $f^{-1}P$ appears in it with positive multiplicity. In the first case, $K+S+D$ log canonical in a neighborhood of P follows from $K_Y + f^{-1}S + B_Y$ log canonical.

The second case is impossible. We prove this below using an almost obvious property of the different (see Corollary 3.11 and the end of the proof after it). On the way, we establish other properties of the different that we need in our subsequent treatment.

3.5. **Example.** Let $P \in X$ be a surface singularity, $P \in S \subset X$ a curve through P, and suppose that $f: Y \to X$ is a resolution of $P \in X$ having a unique exceptional curve $E \cong \mathbb{P}^1$ with $m = -E^2$ that intersects $f^{-1}S$ normally in one point. Then $P \in S$ is a nonsingular point, the divisor $K_X + S$ is log terminal at P, and the different 0_S at P has multiplicity $(m-1)/m$. If moreover there is another nonsingular curve S' through P whose birational transform $f^{-1}S'$ meets E transversally in one point and is disjoint from $f^{-1}S$, then $K+S+S'$ is log canonical at P, $S'_S = 1$, and by (3.16) below $K+S+S'$ is log terminal only if $m = 1$. In the example just described,

$$(P \in S + S' \subset X) \cong \left(0 \in (xy = 0) \subset \mathbb{A}^2\right)/(\mathbb{Z}/m),$$

where $\mathbb{Z}/m$ acts by $x, y \mapsto \varepsilon x, \varepsilon y$, with ε a primitive mth root of 1.

In the general case we have a similar result.

3.6. **Lemma.** *If $K + B$ is purely log terminal, then the reduced part of the boundary $\lfloor B \rfloor$ is normal. In particular, there are no selfintersections and the connected components of $\lfloor B \rfloor$ are irreducible.*

Proof. Let $f: Y \to X$ be the resolution of singularities of X from the definition of $K+B$ log terminal. Then f is proper,

$$-f^*(K+B) = -K_Y - B_Y + \sum a_i E_i$$

is big relative to f, and all the components of the fractional divisors $\{-f^*(K+B)\} = \{-B_Y + \sum a_i E_i\}$ cross normally. Hence by the Kawamata-Viehweg vanishing theorem ([8], 1-2-3),

$$\begin{aligned} R^1 f_* \mathcal{O}_Y(-f^{-1}(\lfloor B \rfloor) + E) &= R^1 f_* \mathcal{O}_Y\left(\left\lceil -B_Y + \sum a_i E_i \right\rceil\right) \\ &= R^1 f_* \mathcal{O}_Y(K_Y + \lceil -f^*(K+B) \rceil) = 0, \end{aligned}$$

where E is an effective divisor, and the first equality holds by the log terminal assumption: all $a_i > 0$. Note that by the pure log terminal assumption the irreducible components of $f^{-1}(\lfloor B \rfloor)$ are disjoint. Now applying the direct image functor f_* to the short exact sequence

$$0 \to \mathcal{O}_Y(-f^{-1}(\lfloor B \rfloor) + E) \to \mathcal{O}_Y(E) \to \mathcal{O}_{f^{-1}(\lfloor B \rfloor)}(E_{|f^{-1}(\lfloor B \rfloor)}) \to 0$$

gives

$$0 \to f_* \mathcal{O}_Y(-f^{-1}(\lfloor B \rfloor) + E) \to f_* \mathcal{O}_Y(E) \to f_* \mathcal{O}_{f^{-1}(\lfloor B \rfloor)}(E_{|f^{-1}(\lfloor B \rfloor)}) \to 0.$$

By normal crossings and (3.2.3) the multiplicities of $\{B\}_{\lfloor B\rfloor^\nu}$ are equal to appropriate multiplicities of the subboundary $B^Y = B_Y - \sum a_i E_i$, and by (3.2.2) are ≥ 0. In other words, $a_i \leq 1$ when $E_{i|f^{-1}(\lfloor B\rfloor)}$ is not exceptional on $f^{-1}(\lfloor B\rfloor)$. Hence the support of the divisor E and its restriction $E_{|f^{-1}(\lfloor B\rfloor)}$ are exceptional for f. From this by negativity of a contraction, 1.1, it follows that for any open subset $U \subset X$ we have

$$\Gamma(f^{-1}U, \mathscr{O}_Y(-f^{-1}(\lfloor B\rfloor) + E)) = \Gamma(f^{-1}U, \mathscr{O}_Y(-f^{-1}(\lfloor B\rfloor));$$
$$\Gamma(f^{-1}U, \mathscr{O}_Y(E) = \Gamma(f^{-1}U, \mathscr{O}_Y);$$

and

$$\Gamma(f^{-1}(\lfloor B\rfloor) \cap f^{-1}U, \mathscr{O}_{f^{-1}(\lfloor B\rfloor)}(E_{|f^{-1}(\lfloor B\rfloor)})) = \Gamma(f^{-1}(\lfloor B\rfloor) \cap f^{-1}U, \mathscr{O}_{f^{-1}(\lfloor B\rfloor)})).$$

Thus we can omit E in the final exact sequence, to get the exact sequence

$$0 \to \mathscr{O}_X(-S) \to \mathscr{O}_X \to f_*\mathscr{O}_{f^{-1}(\lfloor B\rfloor)} \to 0.$$

Hence $\mathscr{O}_S = f_*\mathscr{O}_{f^{-1}S}$, that is, the irreducible components of S are normal and disjoint. In the analytic case by [18], (3.6), the arguments we have given work over a neighborhood of any point X, which is enough to verify what we want. Q.E.D.

3.7. **Corollary.** *If $K+S+B$ is strictly log terminal, and S in a neighborhood of P is nonsingular and does not pass through codimension 2 singular points of X, then the index of S at P is 1 and $P \in X$ is nonsingular.*

Proof. By monotonicity (1.3.3) we can assume that $K+S$ is purely log terminal, and it is enough to check that the index of S at P is 1. In the opposite case, by (2.4.4) there is a finite cover $\pi\colon Y \to X$ ramified over P and unramified in codimension 1 near P on S. Moreover, $K_Y + \pi^{-1}S$ is purely log terminal, so that by Lemma 3.6, $\pi^{-1}S$ is normal. Then since $P \in S$ nonsingular, the cover π is locally trivial in the analytic sense over P on S, and is hence unramified over P on X. Q.E.D.

3.8. **Corollary.** *If $K+B$ is divisorially log terminal and all the irreducible components of B are $\mathbb{Q}$-Cartier, then these components are normal and cross normally. In particular this holds if $K+B$ is strictly log terminal.*

The condition that a divisor has normal crossing includes that its components are normal varieties (or analytic spaces) and cross normally at generic points of intersection of k components. Hence by the corollary, exceptional divisors with log discrepancy 0 lie over the intersections of the reduced part of the boundary. For example, in the 3-fold case components with 0 log discrepancy coefficients for $K+B$ lie over triple points and double curves of $\lfloor B\rfloor$; in the final case it is assumed that the image of the exceptional divisor is a component of such a double curve. From this it follows that the restriction of $K+B$ to a reduced component is purely log terminal outside triple points (in the higher-dimensional case, outside triple and higher order crossings).

Proof. By monotonicity (1.3.3) and Lemma 3.6 we get that the irreducible components S of the reduced part of the boundary B are normal. By (3.2.3) the restriction $(K+B)_{|S}$ is divisorially log terminal, and by the proof of (3.2.3) generic normal crossings extend in a neighborhood of S. Hence to do an induction on the dimension, it is enough to check that $S'_{|S}$ is normal for every irreducible component $S' \neq S$ of the reduced part of the boundary B; we can restrict ourselves to a boundary with just two such components $\lfloor B\rfloor = S + S'$. In this case, since generic normal crossings extend in a neighborhood of S, the log divisor $(K+B)_{|S}$ is purely log terminal and the reduced part of its boundary coincides with $S \cap S'$. Hence by Lemma 3.6 this intersection is normal and its connected components are disjoint. Q.E.D.

We say that $K+B$ is *log terminal in codimension* 2 if it is log terminal along any codimension 2 subvariety $W \subset X$ (in the analytic case, analytic subspace). Here and in what follows, *along* means *at the generic point.*

3.9. **Proposition** (Properties of the different). *Suppose that $K+S+B$ is log terminal in codimension 2. Then*

(3.9.1) *for a prime divisor $P \subset S$ the multiplicity of P in the different 0_S is of the form $(m-1)/m$, where m is a natural number, the index of $K+S$ along P;*

(3.9.2) *X is nonsingular along P if and only if $m=1$;*

(3.9.3) *the index of any integral divisor along P divides m.*

By Example 1.6 and Corollary 3.8, S is normal at a codimension 1 point P. Thus $S = S^\nu$ along P.

Remark. The proposition is elementary, since at a general point of P, by the classification of surface log canonical singularities [Kawamata], $S \subset X$ is transversally $(x = 0) \subset \mathbb{A}^2$ divided by the action of $\mathbb{Z}/m$ by $(x, y) \mapsto (\varepsilon x, \varepsilon^a y)$ with $\operatorname{hcf}(m, a) = 1$. What follows is contained in a more explicit form in Hirzebruch's continued fractions treatment of these quotient singularities, see for example [Oda], (1.6), and [Utah], §3. Compare (5.2.3) below.

Proof. All the assertions are local, and taking general hypersurface sections reduces them to the case of X a surface. Let $f\colon Y \to X$ be a minimal resolution of singularities in a neighborhood of $P \in S \subset X$. By Example 1.6, and monotonicity (1.3.3), since S is irreducible, $K+S$ is purely log terminal, so that

$$K_Y + f^{-1}S + \sum E_i = f^*(K+S) + \sum a_i E_i,$$

where E_i are the exceptional curves and all $a_i > 0$. It is not hard to check that in this case, the curves $f^{-1}S$ and E_i are nonsingular, cross normally, and form a chain $f^{-1}S, E_1, \dots, E_n$, and $E_i \cong \mathbb{P}^1$ (see [7], (9.8)). That is,

$$f^{-1}S \cdot E_1 = E_1 \cdot E_2 = \cdots = E_{n-1} \cdot E_n = 1,$$

and all other intersection numbers of $f^{-1}S$ and E_i are 0. Thus the multiplicity of P in 0_S equals $1 - a_i$, and we must check that $a_1 = 1/m$. Note now that

$$\left(K_Y + f^{-1}S + \sum_{i=1}^{n} E_i\right) \cdot E_j = \begin{cases} 0 & \text{if } 1 \le j \le n-1; \\ -1 & \text{if } j = n. \end{cases}$$

Thus the a_i can be found by solving the system of linear equations

$$\left.\begin{aligned} a_1 {E_1}^2 + a_2 &= 0 \\ a_1 + a_2 {E_2}^2 + a_3 &= 0 \\ &\ \vdots \\ a_{n-2} + a_{n-1} {E_{n-1}}^2 + a_n &= 0 \\ a_{n-1} + a_n {E_n}^2 &= -1 \end{aligned}\right\}$$

Successively expressing $a_2, a_3, \dots, a_n$ in terms of a_1 from the first $n-1$ equations, we get that $a_i = k_i a_1$ with integers k_i. Then from the final equation $k_{n-1}a_1 +$

$k_n a_1 E_n^2 = -1$ we get that $a_1 = 1/m$ with m the integer $m = -k_{n-1} - k_n E_n^2$, hence $a_i = k_i/m$ (taking $k_1 = 1$). All $a_i > 0$, and m and k_i are natural numbers. Thus m is the smallest natural number for which all the products ma_i are integers. It follows from this by [8], 3-2-1, that m is the index of $K+S$ at P.

Since the resolution f is minimal, K_Y is nef relative to f, from which by negativity of a contraction 1.1 it follows that all $a_i < 1$ (see [15], Part 2, and compare Lemma 3.18). Hence if $m = 1$ then all the a_i are natural, and this is only possible for $n = 0$, that is if $P \in S \subset X$ is a nonsingular point of S and of X. The converse follows from the adjunction formula in the nonsingular case, and this proves (3.9.2). The proof of the final assertion can be reduced by the covering trick (2.4.1) to $m = 1$, when all integral divisors in a neighborhood of P are Cartier. Q.E.D.

The next result follows at once from (3.9.1) and (3.9.3).

3.10. **Corollary.** *If $K+S$ is log terminal in codimension 2 and $D = \sum d_i D_i$, then the multiplicity of the different D_S at a prime divisor P is given by*

$$p = \frac{m-1}{m} + \sum \frac{k_i}{m} d_i,$$

where the sum runs over irreducible components D_i containing P, and k_i are natural numbers such that $D_{i|S}$ has multiplicity k_i/m at P.

If $K+S+B$ is purely log terminal, then by (3.2.3) we have the inequality

$$0 \le \frac{m-1}{m} + \sum \frac{k_i}{m} d_i < 1.$$

3.11. **Corollary.** *If $K+S+B$ is log canonical in codimension 2 and $B = \sum b_i D_i$, then the multiplicity p of the different of B_S at a prime divisor P increases compared to the multiplicity b_i of an irreducible boundary component D_i through P; that is, $b_i \le p$. In particular, if the reduced part of the boundary B passes through P then P is a reduced component of the boundary of B_S, and there is a unique prime component of the support of B passing through P.*

Proof. We can obviously restrict to the case $b_i > 0$. Then by monotonicity (1.3.3), property (1.5.7) of an extraction and the existence of a strictly log terminal model along P for $K+S$ (Example 1.6), we get a purely log terminal model of $K+S$ along P. Hence by Corollary 3.10,

$$p \ge \frac{m-1}{m} + b_i \frac{k_i}{m} \ge b_i \frac{m-1}{m} + b_i \frac{1}{m} = b_i. \quad \text{Q.E.D.}$$

Conclusion of proof of 3.4. In the second case, the support of E is nonempty and by construction is contained in the reduced part of B_Y. Moreover $f^{-1}S$ intersects the connected fiber $f^{-1}P$, and it intersects E because E is numerically nonpositive relative to f. The intersection $f^{-1}S \cap \operatorname{Supp} E$ contains a divisor Q on $f^{-1}S$; this is obvious if E is $\mathbb{Q}$-Cartier, and in the general case one can use rational approximation to arrange that E is $\mathbb{Q}$-Cartier without changing the support (see just before (1.1)). By Corollary 3.11, the multiplicity of the boundary $(B_Y)_{f^{-1}S}$ along Q is 1. But then the restriction

$$\begin{aligned}(K_Y + f^{-1}S + B_Y + E)_{|f^{-1}S^\nu} &= (K_Y + (S+D)^Y)_{|f^{-1}S^\nu} \\ &= f^*(K+S+D)_{|f^{-1}S^\nu} = f_S^*(K_{S^\nu} + D_{S^\nu}) = K_{f^{-1}S^\nu} + (D_{S^\nu})^{f^{-1}S^\nu}\end{aligned}$$

is not log canonical, at least along Q, where $f_S \colon f^{-1}S^\nu \to S^\nu$ is the map of the normalizations induced by the restriction $f_{|f^{-1}S}$. More precisely, $(D_{S^\nu})^{f^{-1}S^\nu}$ is not a boundary, since its multiplicity at Q is > 1. Thus we get a contradiction to $K_{S^\nu} + D_{S^\nu}$ log canonical. In the case when $K_{S^\nu} + D_{S^\nu}$ is divisorially log terminal, the divisor $K + S + D$ is at least log canonical.

If the image M of an exceptional divisor with log discrepancy 0 passes through P and the log discrepancy over its intersection with S (more precisely, over $\nu^{-1}(M \cap S)$ for $K_{S^\nu} + D_{S^\nu}$ in a neighborhood of P) is > 0, then by the above $K + S + D + \varepsilon H$ is log canonical for a general hypersurface through M in a neighborhood of P, because $K_{S^\nu} + (D + \varepsilon H)_{S^\nu}$ is log terminal. This contradicts the choice of M and H. Hence there is an exceptional divisor for S^ν with image M' in $\nu^{-1}(M \cap S)$ that passes through P and has log discrepancy 0 for $K_{S^\nu} + D_{S^\nu}$. Then since the final divisor is divisorially log terminal, since normal crossings extend, and $K + S + D$ is log canonical, it follows that the boundary $S + D$ is normal at general points of M' and the exceptional 0 log discrepancies of $K + S + D$ near these general points lie only over normal crossings, that is, M lands in one of them locally. This proves that $K + S + D$ is divisorially log terminal. The same arguments work assuming that $K_{S^\nu} + D_{S^\nu}$ is log terminal and $\lfloor D_{S^\nu} \rfloor = 0$. Moreover, in the final case there are no normal crossings, apart from the trivial S, so that $\lfloor D \rfloor = 0$ near S. Q.E.D.

We get the next result from conditional inversion 3.4, Example 1.6 and Corollary 3.11:

3.12. **Corollary.** *The inversion problem* 3.3 *holds for surfaces.*

3.13. **Weak inversion of adjunction.** In addition to the assumptions of 3.3, suppose that X is $\mathbb{Q}$-factorial. Then $S + D$ is a boundary in a neighborhood of S; moreover, in the case when $K_{S^\nu} + D_{S^\nu}$ is log terminal and $\lfloor D_{S^\nu} \rfloor = 0$, we have $\lfloor D \rfloor = 0$, as follows obviously from Corollary 3.12.

3.14. **Definition.** In the study of log divisors $K + D$, an important role is played by the *locus of log canonical singularities* $\mathrm{LCS}(K + D)$, the union of the images of all divisors with log discrepancy ≤ 0, that is, the union of components of D with multiplicities ≥ 1 and the images of exceptional divisors with log discrepancy ≤ 0. If X is nonsingular and the support of D consists of nonsingular divisors crossing normally, then $\mathrm{LCS}(K + D)$ is the union of components of D with multiplicities ≥ 1. It is easy to deduce from this that $\mathrm{LCS}(K + D)$ is always a closed subvariety of X (or analytic subspace in the analytic case). $\mathrm{LCS}(K + D) = \varnothing$ for an effective divisor D if and only if $\lfloor D \rfloor = 0$ and $K + D$ is purely log terminal.

3.15. **Corollary.** *If D is effective, but $K + S + D$ possibly not log canonical, then the locus of log canonical singularities of $K_{S^\nu} + D_{S^\nu}$ contains the prime divisors P whose image $\nu(P)$ is contained in the support of $\lfloor D \rfloor \cap S$. If moreover X is $\mathbb{Q}$-factorial then* $\mathrm{LCS}(K_{S^\nu} + D_{S^\nu})$ *contains* $\nu^{-1}(\lfloor D \rfloor \cap S)$.

Proof. If the multiplicity of D_{S^ν} at P is ≤ 1 then the assertion follows along P from Corollaries 3.12 and 3.11. In the opposite case the statement is obvious. Q.E.D.

3.16. **Corollary.** *If D is an effective divisor, then $K + S + D$ is log canonical in codimension 2 in a neighborhood of S if and only if the different D_{S^ν} is a boundary; $K + S + D$ is log terminal in codimension 2 in a neighborhood of S if and only in addition every reduced irreducible component P of D_{S^ν} lies on a unique prime component D_i of D, and the intersection $S \cap D_i$ is normal along P.*

Proof. This follows from (3.2.3) and from Corollaries 3.12 and 3.11. To verify that S and D_i cross normally along P if $K + S + D$ is log terminal we have to use the fact that it is strictly log terminal in codimension 2, together with Corollary 3.8. Q.E.D.

From now until the end of §3, X is a surface.

3.17. **Example.** The inversion of adjunction 3.3, proved in Corollary 3.12 in dimension 2, is useful for checking log canonical and log terminal. If the surface X and a curve $S \subset X$ are both nonsingular, then $K + S + B$ is log canonical (purely log terminal) in a neighborhood of a point $P \in S$ if and only if $(S \cdot B)_P \leq 1$ (respectively $(S \cdot B)_P < 1$), where $(\cdot)_P$ is the local intersection number at P. Indeed, by Proposition 3.9 and (3.2.1), in a neighborhood of P

$$K_S + B_S = K_S + 0_S + B_{|S} = K_S + (S \cdot B)_P.$$

For example if $B = bD$ with $b \geq 0$, and D is a nonsingular curve having simple tangency to S at P, then $K + B$ is log canonical (log terminal) in a neighborhood of P if and only if $b \leq 1/2$ (respectively $< 1/2$); but by Corollary 3.16 it is not log terminal if $b = 1/2$.

3.18. **Lemma.** *Suppose that S and S' are curves through a point $P \in X$, and that $K + S + S'$ is log canonical at P. Then in a neighborhood of P:*

(3.18.1) *S and S' are irreducible and nonsingular.*
On a minimal resolution of singularities $f\colon Y \to X$, the following hold:

(3.18.2) *$f^{-1}S, E_1, \dots, E_n, f^{-1}S'$ is a chain of nonsingular curves, with $E_i \cong \mathbb{P}^1$ exceptional curves of f.*

(3.18.3) *The log discrepancy coefficients of E_i for $K+S+S'$ are all equal to 0; for $K+S$ they are all contained in $(0, 1) \cap \mathbb{Q}$, and for K either they are all contained in $(0, 1) \cap \mathbb{Q}$, or $P \in X$ is a Du Val singularity of type A_n for some $n \geq 0$, and all are equal to 1 (that is, the (genuine) discrepancy is 0, the log discrepancy 1).*

(3.18.4) *If $f^*S = f^{-1}S + \sum e_i E_i$ then $e_i \in (0, 1) \cap \mathbb{Q}$.*

(3.18.5) *If the index of $K + S$ equals m and $e_1 = (m-1)/m$ then $m = n + 1$ and all the E_i are (-2)-curves, that is, $P \in X$ is a Du Val singularity of type A_n.*

(3.18.6) *If $f^*(S + S') = f^{-1}S + f^{-1}S' + \sum e_i' E_i$ then either all $e_i' \in (0, 1)$, or they are all equal to 1 and $P \in X$ is a Du Val singularity of type A_n.*

(3.18.7) *Suppose that we set*

$$f^*(S + cS') = f^{-1}S + cf^{-1}S' + \sum e_i'' E_i$$

for some $0 < c \leq 1$; then $e_1'' > 1/2$ and $E_1{}^2 \leq -3$ is only possible in the case when $E_1{}^2 = -3$, $E_2{}^2 = \cdots = E_n{}^2 = -2$ and $c > 1/2$.

Note that $K + S + S'$ log canonical implies that

$$(S + S' \subset X) \cong ((xy = 0) \subset \mathbb{A}^2)/(\mathbb{Z}/m),$$

where $\mathbb{Z}/m$ acts diagonally (see [Kawamata]).

Proof. All the assertions except for (3.18.4–7) are well known ([7], (9.6)), and can also be deduced easily using the general technique of §1 (Example 1.6). Note that f is log terminal and is a minimal model of $K + S + S'$. Here minimal means that any other log terminal extraction factors through it. (3.18.4–6) follow from (3.18.3) using the log crepant components of (1.5.7), after possibly interchanging S and S'. It follows from the assertion (3.18.7) that the log discrepancy of K on contracting E_1 is less than $1/2$, and hence $E_1{}^2 = -3$. If $E_2{}^2 = \cdots = E_i{}^2 = -2$ and $E_{i+1}{}^2 \leq -3$ then $K_Y + (1/2)(E_1 + \cdots + E_{i+1})$ is nef on $E_1, \dots, E_n$, which is impossible for

$e_1'' > 1/2$. Hence $E_1{}^2 = -3$, $E_2{}^2 = \cdots = E_n{}^2 = -2$ and by the same argument $c > 1/2$. Q.E.D.

§4. Two terminations

Except where otherwise stated, X is a 3-fold throughout the remainder of the paper.

4.1. **Theorem on special termination.** *Let $f\colon X \to Z$ be a projective morphism; we consider a chain of successive modifications of $f\colon X \to Z$ in extremal rays with $(K+B)R < 0$. Assume that X is $\mathbb{Q}$-factorial and the support of every flipping ray R of the chain lies in the reduced part of the boundary $\lfloor B \rfloor$. Then the chain terminates (in the analytic case, over a neighborhood of any compact subspace of Z).*

Here the *support* $\operatorname{Supp} R$ of an extremal ray R is the exceptional subvariety (or analytic subspace) of cont_R, that is, the union of curves C with $\operatorname{cont}_R C = pt$.

4.2. **Lemma.** *Suppose that*

$$b_i = \frac{n_i - 1}{n_i} + \sum_j \frac{k_{ij}}{n_i} d_j < 1,$$

where n_i and k_{ij} are natural numbers, d_j is a finite (ordered) set of numbers in the interval $(0, 1)$, and

$$p = \frac{m-1}{m} + \sum_i \frac{l_i}{m} b_i < 1,$$

with natural numbers m and l_i. Then substituting the b_i gives an expression of the same type for p in terms of the d_j.

Proof. If $n_i = 1$ for all i with $l_i \geq 1$ then this is obvious:

$$p = \frac{m-1}{m} + \sum_j \frac{\sum_i l_i k_{ij}}{m} d_j.$$

Otherwise, there exists a unique i_0 such that $n_{i_0} \geq 2$ and $l_{i_0} \geq 1$, for if there were 2 or more then

$$p \geq \frac{m-1}{m} + \frac{1}{m}\left(\frac{1}{2} + \frac{1}{2}\right) = 1.$$

Now $l_{i_0} = 1$ for the same reason, since if $l_{i_0} \geq 2$ then

$$p \geq \frac{m-1}{m} + \frac{2}{m} \times \frac{1}{2} = 1.$$

Hence

$$p = \frac{m-1}{m} + \frac{1}{m}\left(\frac{n_{i_0} - 1}{n_{i_0}} + \sum_j \frac{k_{i_0 j}}{n_{i_0}} d_j\right) + \sum_{i \neq i_0} \frac{l_i}{m}\left(\sum_j k_{ij} d_j\right)$$

$$= \frac{n_{i_0} m - 1}{n_{i_0} m} + \sum_j \frac{k_{i_0 j} + \sum_{i \neq i_0} n_{i_0} l_i k_{ij}}{n_{i_0} m} d_j. \quad \text{Q.E.D.}$$

Proof of Theorem 4.1. In the analytic case, since f is projective, we can shrink Z to a neighborhood of a compact set in such a way that the relative Picard number $\rho(f) = \rho(X/Z)$ is finite. For the proof below, we also need to know that there are only finitely many components of the boundary and their intersections relative to f.

As usual (see [8], 5-1), since a divisorial extremal contraction reduces the relative Picard number $\rho(f)$ by 1, we can restrict ourselves to a chain consisting of flips only. Note that by the projectivity of f (and by choice of Z in the analytic case), we can also assume that the reduced boundary $\lfloor B \rfloor$ has only finitely many irreducible components S, and restricts to only finitely many irreducible curves on each S, or, more precisely, on the normalization S^ν in the locus of log canonical singularities of the restriction $(K+B)_{|S^\nu}$. After a flip in a ray R with $(K+B)R < 0$, the assumption that $K+B$ is log canonical implies that the modified $K+B$ is purely log terminal along the flipped curves (since discrepancies decrease, see (1.5.6) and Lemma 1.7, [25], (2.13.3), or [8], 5-1-11 (3)).

Hence if as a result of a flip one such curve again lands on a component S of the reduced part of the boundary $\lfloor B \rfloor$, then by Corollary 3.11 and (3.2.3), it does not lie in the locus of log canonical singularities of the modified log divisor $(K+B)_{|S^\nu}$. Therefore the number of irreducible components of $\mathrm{LCS}(K+B)_{|S^\nu}$ does not increase under modifications, and decreases if one of the flipping curves happens to be contained in it. Hence we can restrict to the case that the supports of flipping rays are not contained in $\mathrm{LCS}(K+B)_{|S^\nu}$. By the same arguments, $(K+B)_{|S^\nu}$ is purely log terminal after a modification in a neighborhood of a flipped curve. But there are only a finite number of points at which $(K+B)_{|S^\nu}$ is not log terminal, and we can suppose that $(K+B)_{|S^\nu}$ is log terminal at the points P of intersection of $\mathrm{LCS}(K+B)_{|S^\nu}$ with the support of flipping rays. Moreover, in a neighborhood of such a point P, the set $\mathrm{LCS}(K+B)_{|S^\nu}$ will be a nonsingular curve C, and from two applications of Corollary 3.10 on the coefficients of the different, by Lemma 4.2, the multiplicity of P in the boundary appearing in the adjunction formula for $(K+B)_{|S^\nu|C}$ is of the form

$$p = \frac{m-1}{m} + \sum_i \frac{l_i}{m} b_i,$$

with m and l_i natural numbers; recall that b_i are the multiplicities of the boundary B. We can again assume that the set of points P with $p > 0$ is finite, and it's exactly through these points that the flipping curves are allowed to pass. In fact, p decreases under flips, since on the original model it decreases on contracting a curve Γ with $(K+B)_{|S^\nu} \cdot \Gamma < 0$; and on the flipped model, p does not increase on extracting a curve Γ^+ with $(K+B)_{|S^\nu} \cdot \Gamma^+ \geq 0$. By the purely log terminal condition $p < 1$, and obviously, there are only a finite number of possible decreases of p under flips in curves through P. Thus we can suppose that the flipped curves are disjoint from $\mathrm{LCS}(K+B)_{|S^\nu}$. It is not hard to verify that the number *Hell* of curves (including nonexceptional curves) (6) not lying in the locus of canonical singularities, and not contracted to it, and having log discrepancy < 1 (compare Lemma 8.7 below) is finite. By the proof of (3.2.3) this number *Hell*(X with B), rather like the *difficulty* of the log terminal case ([25], Definition 2.15), does not increase under flips, since the log discrepancy coefficients of $(K+B)_{|S^\nu}$ do not decrease, by standard assertions (see (1.5.6) and Lemma 1.7, [25], (2.13.3), or [8], 5-1-11 (3)), and similarly the multiplicities of the boundary of $(K+B)_{|S^\nu}$ do not

(6) Exercise: Find a better name for the number *Hell*(X with B). I wrote *Hello* referring to the famous computer program of that name, and Shokurov modified it referring to the works of Dante Alighieri (1265–1321); neither is particularly logical.

increase. Thus the multiplicities of the boundary b_i of $(K+B)_{|S^\nu}$ can take only a finite number of standard values

$$b = \frac{m-1}{m} + \sum_i \frac{l_i}{m} b_i.$$

New multiplicities of this kind can only arise when some flipped curves again land on S; then their log discrepancy for $(K+B)_{|S^\nu}$ increases. Hence in proving termination we can assume that after each flip the flipped curves lie outside the modified $\lfloor B \rfloor$. Hence the termination follows from the boundedness of the relative Picard number $\rho(f_{|S^\nu})$. Q.E.D.

From the termination just proved we get the following result.

4.3. **Proposition.** *Let X be a normal algebraic (or analytic) 3-fold, and suppose that the log divisor $K+B$ and the boundary B are such that*

(i) *$K+B$ is possibly not log canonical*;

(ii) *the boundary B is a reduced LSEPD divisor (in the analytic case in a neighborhood of a projective subspace $W \subset X$)*;

(iii) *X has canonical singularities (respectively is nonsingular) outside B.*

Then in a neighborhood of B (in the analytic case, over a neighborhood of W) there exists a log canonical (respectively strictly log terminal) model $f\colon Y \to X$ of X for $K+B$.

Proof. Recall that a log canonical model of X for $K+B$ is a log canonical model $f\colon Y \to X$ of the identity morphism of X for $K+B$ (see (1.4–5) above for definitions and properties). As the first approximation to such a model, take a projective resolution of singularities $f\colon Y \to X$ (in the analytic case, in a neighborhood of W), such that all the exceptional divisors E_i that contract to B, and all the irreducible components of $f^{-1}B$, are nonsingular and cross normally. Shrinking X if necessary to a neighborhood of B, we can assume by the classification of canonical surface singularities that the exceptional divisors not contracted by f to B are contracted by f along a ruling to curves, outside $f^{-1}B$.

First, we establish the existence of a strictly log terminal model of f for $K_Y + f^{-1}B + \sum E_i$, where the sum runs only over the exceptional divisors over B; we can omit the exceptional divisors contracted to curves outside B from the boundary by assumption (iii). The existence of this model is proved according to the general ideology explained in §1 after 1.5. Since $K_Y + f^{-1}B + \sum E_i$ is strictly log terminal, if $K_Y + f^{-1}B + \sum E_i$ is nef relative to f then f will itself be the required model. Otherwise, by the theorem on the cone and the contraction theorem ([8], 4-2-1 and 3-2-1) there is an extremal contraction $g\colon Y \to Z$ over X, with $K_Y + f^{-1}B + \sum E_i$ negative relative to g; it is birational since f is. If g is a divisorial contraction then $K_Z + g(f^{-1}B + \sum E_i)$ is again strictly log terminal, and the modification f is projective by the Properties of log flips, 1.12.

Now write Y for Z and $f\colon Y \to X$ for its contraction to X. For flipping contractions g we do the same thing. Thus using termination 4.1, we eventually get to the model we want provided that flips exist and their curves lie in the boundary. Now note that by induction on the number of transformations one can check that the exceptional divisors for f outside $f^{-1}B$ are nonsingular ruled surfaces, and that the flipping curves of extremal rays lie over B. In the same way, our modifications leave Y nonsingular outside the boundary. Indeed, the boundary $f^{-1}B + \sum E_i$ lies over B and by (ii) is an LSEPD divisor relative to f, hence also relative to g. Thus flipping curves always lie on the boundary, both before and after flipping, and it is enough to check that nonsingularity is preserved by a contraction g of a surface to a curve outside the boundary. But by the classification of extremal contractions of surfaces in the terminal case, and by induction, such a contracted surface is nonsingular outside $f^{-1}B$, ruled with fibers negative with respect to K_Y and not intersecting the boundary, and contracts to a set that is nonsingular outside the boundary.

Suppose now that g is a small contraction over X, on which $K_Y + f^{-1}B + \sum E_i$ is negative. Since the boundary $f^{-1}B + \sum E_i$ is a reduced LSEPD divisor over g, and X is nonsingular outside it, the flip exists by Proposition 2.6.

By the theorem on eventual freedom ([8], 3-1-2) and by (ii), since the model f is strictly log terminal, we get the required log canonical model by contracting curves Γ over X with $(K_Y + f^{-1}B + \sum E_i)\cdot\Gamma = 0$. Indeed, by (iii) the boundary of this model contains all the exceptional divisors. In particular, the exceptional divisors with multiplicity 0 in the boundary contract outside the boundary to curves of canonical singularities. Note that if X has no singularities outside B then we can choose f such that all its exceptional divisors lie over B, which allows us to construct a strictly log terminal model of X for $K+B$. Q.E.D.

4.4. **Corollary.** *Suppose that $K+S+B$ is such that*

(i) $K+S+B$ *is $\mathbb{R}$-Cartier but a priori not log canonical*;

(ii) *the boundary $S+B$ is a reduced LSEPD divisor, for example is $\mathbb{Q}$-Cartier*;

(iii) S *is an irreducible surface on which $K_{S^\nu} + B_{S^\nu}$ is log canonical*;

(iv) X *has canonical singularities outside $S+B$.*

Then $K+S+B$ is log canonical in a neighborhood of S.

This is a first particular case of Problem 3.3 on the inversion of the log canonical condition, which we prove for 3-folds (see Proposition 5.13 and Corollary 9.5 below).

Proof. The result is local, so we can restrict to a neighborhood of some point S. But then by Proposition 4.3 there exists a log canonical model of X for $K+S+B$. Hence what we want follows from conditional inversion, 3.4. Q.E.D.

We now upgrade somewhat the philosophy of §1. As a preliminary step, we add something to the boundary to make it an LSEPD divisor relative to f. Next we construct a log terminal model for the increased boundary. Then we modify this model by reducing the additions to the boundary (see Example 4.7 and the proof of reductions 6.4–5). For this termination 4.1 is sufficient. Hence the main difficulties here are concerned with the construction of flips, and in particular with flips of 0-contractions arising when the additions to the boundary are reduced.

4.5. **0-contractions** (flops). Let $f\colon X \to Z$ be a projective morphism that is finite over the general point of Z (in the analytic case we require that there exists a big Cartier divisor on Z in a neighborhood of a compact subset $W \subset Z$; this holds, for example, if Z is a Stein space or W is projective). Suppose that f and the boundary $B+H$ satisfy

(4.5.1) H is an effective divisor possibly having components in common with B.

(4.5.2) There exists an LSEPD divisor D relative to f with $\lfloor B+H \rfloor \le D \le \lceil B+H \rceil$.

(4.5.3) $K+B$ is strictly log terminal (in the analytic case, over a neighborhood of W).

(4.5.4) For some ε with $0 < \varepsilon \le 1$ the divisor $K+B+\varepsilon H$ is log canonical.

(4.5.5) $K + B + \varepsilon H$ is nef relative to f.

Here we do not have to assume that X is 3-dimensional. An *extremal* 0-*contraction* over Z is an extremal contraction $g: X \to Y$ over Z such that

(4.5.6) there exists ε' with $0 < \varepsilon' \leq \varepsilon$ for which $K + B + \varepsilon' H$ is nef relative to f and numerically 0 relative to g.

(4.5.7) $K + B$ is numerically negative relative to g.

Under assumptions (4.5.1–5) I claim that either $K + B$ is nef relative to f, or there exists a 0-contraction (either way, in the analytic case, over a neighborhood of W). Moreover, a modification of 0-contractions relative to $K + B$ (if it exists) preserves all the assumptions (4.5.1–5). Hence if for a fixed ε_0 with $\varepsilon \geq \varepsilon_0 \geq 0$ we know that small 0-contractions with $\varepsilon \geq \varepsilon' > \varepsilon_0$ can be flipped relative to $K + B$, and that chains of such flips terminate, then eventually we get a strictly log terminal model f for $K + B + \varepsilon_0 H$ when $\varepsilon > \varepsilon_0$ or the initial divisor $K + B + \varepsilon_0 H$ is log terminal, and a log canonical model for $K + B + \varepsilon_0 H$ when $\varepsilon_0 > 0$, or for $\varepsilon_0 = 0$ when (4.5.2) holds with $H = 0$.

Proof. Every Cartier divisor on X is big after adding divisors pulled back from Z if necessary, and has effective multiples. Because f is finite over the general point of Z, this statement is equivalent to the existence of a big Cartier divisor on Z. This always holds in the algebraic case (and by assumption in the analytic case).

Suppose that $K + B$ is numerically negative on some curve over Z. Then we can choose a minimal $\varepsilon > 0$ for which all the assumptions (4.5.1–5) hold. Then by the Kleiman ampleness criterion [10], $(K + B + \varepsilon H)^{\perp}$ is a supporting hyperplane of the Kleiman-Mori cone $\overline{\mathrm{NE}}(X/Z)$ (in the analytic case, $\overline{\mathrm{NE}}(X/Z\,;W)$, and similarly below). I claim that it is rational polyhedral in a neighborhood of the face $M = (K + B + \varepsilon H)^{\perp} \cap \overline{\mathrm{NE}}(X/Z)$. This means that M is spanned by a finite set $\{R_i\}$ of extremal rays, the whole cone $\overline{\mathrm{NE}}(X/Z)$ is spanned by M together with the complement of some neighborhood of M, and there exists a Cartier divisor D which is negative on $M \setminus 0$ and such that

$$\overline{\mathrm{NE}}^{D}(X/Z) = \{v \in \overline{\mathrm{NE}}(X/Z) \mid Dv < 0\}$$

is a neighborhood of M. Indeed, we can assume by the previous argument that D is effective. The sets $\overline{\mathrm{NE}}^{K+B+\varepsilon H+\delta D}(X/Z)$ for any $\delta > 0$ are also neighborhoods of M. If $K + B + \varepsilon H + \delta D$ were strictly log terminal, then the theorem on the cone ([8], 4-2-1) would imply our assertion that $\overline{\mathrm{NE}}(X/Z)$ is polyhedral near M. But by (4.5.2–4) we can reduce the multiplicities of the boundary to get $B' < B + \varepsilon H$ such that $K + B'$ is strictly log terminal, $\lfloor B' \rfloor = 0$, and the intersection number of $K + B + \varepsilon H + \delta D$ with curves over Z is preserved, that is,

$$(K + B + \varepsilon H + \delta D) \cdot v = (K + B' + \delta D) \cdot v \quad \text{for all } v \in N_1(X/Z).$$

Hence if we replace $K + B + \varepsilon H + \delta D$ by $K + B' + \delta D$ for small δ, by the fact that D is effective and by stability (1.3.4), we get what we want. Note that $K + B + \varepsilon' H$ strictly log terminal for all $0 < \varepsilon' < \varepsilon$ is equivalent to conditions (4.5.3–4). By the choice of ε and by the polyhedral property of $M \neq 0$ just proved, there exists an extremal ray R in a neighborhood of M, and therefore in M, on which $K + B + \varepsilon' H$ is negative, but $(K + B + \varepsilon H)R = 0$, since R is in M. It follows that $HR > 0$ and $(K + B)R < 0$. This ray defines the required 0-contraction because f is finite over the general point of Z.

A modification in a 0-contraction relative to $K + B$ preserves assumptions (4.5.1–2) obviously, and preserves (4.5.3–5) and the projectivity of f by the standard properties of flips 1.12; as the new ε we take ε'. Note also that if the divisor $H \neq 0$ then it is not contracted by a 0-contraction since it is nef with respect to it, and hence it remains $\neq 0$ under our modifications. Hence if these modifications with $\varepsilon' > \varepsilon_0$ terminate, then decreasing ε down to ε_0, by (4.5.3) we eventually get a strictly log terminal model of f for $K+B+\varepsilon_0 H$. Then for $\varepsilon_0 > 0$, by the polyhedral result just proved and the contraction theorem ([8], 3-2-1) we get a log canonical model of f for $K+B+\varepsilon_0 H$. It is obtained by contracting the face M. By the same arguments all $\mathbb{R}$-Cartier divisors numerically 0 on M, and in particular $K+B+\varepsilon_0 H$, descend to this model. When $\varepsilon_0 = 0$, we need to use the additional condition and again the same arguments with $H = 0$. Q.E.D.

From now on X is again a 3-fold.

Proof of Corollary 1.11. First of all, a log canonical model can be constructed locally. Hence we can assume that B is an LSEPD divisor such that the principal Cartier divisor D with $\lfloor B \rfloor \leq D \leq \lceil B \rceil$ contains the locus of log canonical singularities of $K+B$. Consider a strictly log terminal model $g\colon Y \to X$. By (1.5.5), and since $K+B$ is obviously purely log terminal outside D, the contraction g fails to be small only over D. Thus D_Y is an LSEPD divisor for g and $\lfloor B_Y \rfloor \leq \operatorname{Supp} D_Y \leq \lceil B_Y \rceil$. By the arguments at the end of the preceding proof and the finiteness of g over the general point of Z we get the contraction to the log canonical model of f. Q.E.D.

4.6. **Corollary.** *Under the assumptions* (4.5.1–5), *for fixed* $\varepsilon \geq \varepsilon_0 \geq 0$, *the existence of flips of small* 0*-contractions with* $\varepsilon \geq \varepsilon' > \varepsilon_0$ *and the fact that the curves contracted by these lie on the reduced part of the boundary of* B *imply that there exists* (1) *a strictly log terminal model of* f *for* $K+B+\varepsilon_0 H$ *when* $\varepsilon > \varepsilon_0$ *or the original divisor* $K+B+\varepsilon H$ *is log terminal, and* (2) *a log canonical model for* $K+B+\varepsilon_0 H$ *when* $\varepsilon_0 > 0$ *or when* $\varepsilon_0 = 0$ *and* (4.5.2) *holds with* $H = 0$.

Proof. Direct from 4.5 and termination 4.1. Q.E.D.

4.7. **Example.** Suppose that the boundary B is a reduced LSEPD divisor, but $K+B$ is not necessarily log canonical. Then to construct the log canonical model of X for $K+B$ in a neighborhood of B we complement the boundary to $B+H$ in a neighborhood of a fixed point $P \in B$ by adding a reduced Cartier divisor H such that X is nonsingular outside $B+H$ and $B+H$ is LSEPD in a neighborhood of P. For H we can take a general hyperplane through P and through the curves of singularities of X near P. By Proposition 4.3 there is a strictly log terminal model $f\colon Y \to X$ for $K+B+H$. We can apply Corollary 4.6 to the birational contraction f and to the log divisor $K_Y + f^{-1}B + f^{-1}H + \sum E_i$, where E_i are exceptional for f. Here for H we take $f^{-1}H$, and for B we take the remainder of the boundary $f^{-1}B + \sum E_i$. (Of course, there are big Cartier divisors on X in a neighborhood of P.) Since X is nonsingular outside $B+H$, all exceptional divisors of f lie over $B+H$. Hence the boundary $f^{-1}B + f^{-1}H + \sum E_i$ is LSEPD for f, since the same hold for $B+H$. The remaining assumptions of 4.5 are satisfied by the fact that f is strictly log terminal for $K_Y + f^{-1}B + f^{-1}H + \sum E_i$, with $\varepsilon = 1$.

Take $\varepsilon_0 = 0$. Shrinking the neighborhood of P we can assume that all the exceptional divisors E_i that contract to curves have irreducible fibers (in fact, by Corollary 3.8, nonsingular fibers) outside $f^{-1}P$, relative to the contraction induced by f. Hence the curves of small 0-contractions lie on the reduced part of the boundary $f^{-1}B + \sum E_i$ over P, which is LSEPD for f. Moreover, since the intersection of such curves with $f^{-1}H$ is positive and H is Cartier, some exceptional surface E_{i_0} has negative intersection with them. But $f^{-1}B + \sum E_i$ is also LSEPD for f, and hence $f^{-1}B$ or one of the exceptional surfaces over B intersects them nonnegatively (positively if $f(E_{i_0}) \subset B$). (Moreover, positively by Corollary 3.8, since $f^{-1}B + f^{-1}H + \sum E_i$ is strictly log terminal.)

Thus *we need to establish the existence of flips in the case that $K + B$ is strictly log terminal and negative relative to a small extremal contraction, and the boundary B is reduced and has two irreducible components, intersecting the contracted curves, one numerically negative and the other nonnegative relative to the contraction.* In what follows, in Corollary 5.15 we will establish the existence of such flips. Thus according to the previous corollary we get a strict log terminal model $f: Y \to X$ for $K + B$ in a neighborhood of P. If in addition $K + B$ has log terminal singularities outside B, then it is purely log terminal outside B, and by (1.5.7) all exceptional surfaces of f lie over B. Thus the reduced boundary of $K_Y + f^{-1}B + \sum E_i$ is an LSEPD divisor, hence by Corollary 4.6, X has a log canonical model for $K + B$ in a neighborhood of P.

4.8. **Definition.** A *limiting chain* of length n is an ordered set of real numbers $0 < d_1, d_2, \dots, d_n < 1$, for which there exist natural numbers $n_i \neq 0$ and k_{ij} such that

(4.8.1) $\sum p_i = 1$ or 2, where

(4.8.2) $p_i = (n_i - 1)/n_i + \sum_j k_{ij} d_j / n_j < 1$, and

(4.8.3) for each $j = 1, \dots, n$ at least one $k_{ij} \neq 0$.

See Definition 6.1 and Proposition 6.2 for an explanation of the term *limiting chain.* We can introduce a partial order on such limiting chains:

$$\{d_j\}_{j=1,\dots,n} \geq \{d'_j\}_{j=1,\dots,m} \quad \Longleftrightarrow \quad \begin{cases} n < m \text{ or } n = m \text{ and} \\ d_j \geq d'_j \text{ for } j = 1, \dots, n. \end{cases}$$

4.9. **Second termination** (Chicago Lemma). *Limiting chains satisfy the a.c.c. with respect to this partial order.*

Proof. Consider an increasing sequence of limiting chains d^l_j for $l = 1, \dots$ of length n_l. Note that when n_l decreases the sequence increases. Hence, restricting to a subsequence if necessary, we can assume that all chains have the same length $n_l = n$. But then all the d^l_j are bounded below by a positive constant $\min d^1_j$, which implies that nonzero numbers of the form

$$p^l_i = \frac{n^l_i - 1}{n^l_i} + \sum_j \frac{k^l_{ij}}{n^l_i} d^l_j$$

are bounded from below by a positive constant, and hence by (4.8.1) are finite in number.

This finite number is universal: the number of nonzero p^l_i is bounded by a universal constant independent of l. Hence, once more restricting to a subsequence if necessary, and renumbering as appropriate, we can assume that there are always exactly m nonzero p^l_i:

$$1 > p^l_1, p^l_2, \dots, p^l_m > 0.$$

I claim that each sequence p^1_i, p^2_i, ... does not have a decreasing subsequence. Indeed, if there is such a subsequence, then it is bounded by a constant < 1. Hence by (4.8.2) in it the n^l_i are bounded, and hence restricting to a subsequence we can assume that they are constant: $n^l_i = n_i$. Thus the fact that the d^l_j are bounded from below by a positive constant implies that the natural numbers k^l_{ij} are bounded and finite independently of l. Again restricting to a subsequence we can assume the k_{ij}^l are constant: $k_{ij}^l = k_{ij}$. But then the p_i^l decrease, which contradicts the fact that $d_j^1 \le d_j^2 \le \cdots$ is nondecreasing. Moreover, it follows by the same argument that p_i^l has a constant subsequence if and only if, possibly after passing to a suitable subsequence d_j^l, we get $n_i^l = n_i$, $k_{ij}^l = k_{ij}$ and $d_j^l = d_j$ for $k_{ij} \ne 0$. Thus restricting to a subsequence d_j^l we can assume that all sequences p_i^l are monotonically nondecreasing, and by (4.8.3), one of them is increasing. But this contradicts the finiteness of values of the sum $\sum p_i^l = 1$ or 2 from (4.8.1). Q.E.D.

§5. Complementary log divisors

In this section X is first arbitrary, then a surface, and towards the end a $\mathbb{Q}$-factorial 3-fold. Except where stated otherwise, we write a subboundary in the form $S+D$ to distinguish its reduced part:

$$S = \sum_{d_i=1} D_i \quad \text{and} \quad D = \sum_{d_i<1} d_i D_i.$$

5.1. **Definition** (n-complement). A log divisor $K+S+D$, not necessarily log canonical, is *n-complementary* for a natural number n if there exists a Weil divisor D^+ such that

(5.1.1) $\quad D^+ \ge S + (1/n)\lfloor (n+1)D \rfloor$;

(5.1.2) $\quad K + D^+$ is log canonical;

(5.1.3) $\quad n(K + D^+)$ is linearly 0.

In particular, D^+ is then a subboundary, $K + D^+$ has index n and nD^+ is an integral divisor. The condition (5.1.1) says that $\overline{D} = nD^+ - nS - \lfloor (n+1)D \rfloor$ is an effective divisor; this is an *n-complement* or simply *complement* of $K + S + D$. By (5.1.3), $\overline{D} \in |-nK - nS - \lfloor (n+1)D \rfloor|$, so that $K + S + D$ is n-complementary if and only if there exists $\overline{D} \in |-nK - nS - \lfloor (n+1)D \rfloor|$ such that

$$K + D^+ = K + S + (1/n)\left(\lfloor (n+1)D \rfloor + \overline{D}\right)$$

is log canonical, where $D^+ = S + (1/n)\left(\lfloor (n+1)D \rfloor + \overline{D}\right)$. The case we are ultimately interested in is when the subboundary $S+B$ is a boundary. Then by (5.1.1), B^+ is also a boundary.

5.2. **Examples.**

(5.2.1) $\quad$ On $\mathbb{P}^1$, consider a nonpositive divisor $K + B$ with $\lfloor B \rfloor = 0$, that is,

$$B = \sum_{i \ge 1} b_i P_i, \quad \text{with } 0 \le b_i < 1 \text{ and } \sum b_i \le 2,$$

where P_i are distinct points of $\mathbb{P}^1$. Then $K+B$ is 1-, 2-, 3-, 4- or 6-complementary.

Moreover, if we assume in addition that $b_1 \geq b_2 \geq \cdots$, then

$K+B$ is not 1-complementary	$\Longleftrightarrow$	$b_1, b_2, b_3 \geq \frac{1}{2}$;
$K+B$ is not 1- or 2-complementary	$\Longleftrightarrow$	$b_1, b_2 \geq \frac{2}{3}$ and $b_3 \geq \frac{1}{2}$, or $b_1 = \frac{2}{3}$, $b_2 = b_3 = \frac{1}{2}$ and $b_4 = \frac{1}{3}$;
$K+B$ is not 1-, 2- or 3-complementary	$\Longleftrightarrow$	$b_1 \geq \frac{3}{4}$, $b_2 \geq \frac{2}{3}$ and $b_3 \geq \frac{1}{2}$, or $b_1 = \frac{2}{3}$, $b_2 = b_3 = \frac{1}{2}$ and $b_4 = \frac{1}{3}$;
$K+B$ is not 1-, 2-, 3- or 4-complementary	$\Longleftrightarrow$	$b_1 \geq \frac{4}{5}$, $b_2 \geq \frac{2}{3}$ and $b_3 \geq \frac{1}{2}$.

Of course, a general element $\overline{B} \in |-nK - \lfloor (n+1)B \rfloor|$ will always do as an n-complement, provided that this linear system is free and $\lfloor B \rfloor = 0$. But on $\mathbb{P}^1$, the linear system $|D|$ of an integral divisor D is free if and only if it has $\deg D \geq 0$. Hence in our example, a divisor $K+B$ is n-complementary if and only if $\deg\big((K + (1/n)\lfloor (n+1)B \rfloor\big) \leq 0$, or equivalently,

$$\deg \lfloor (n+1)B \rfloor = \sum_i \lfloor (n+1)b_i \rfloor \leq 2n.$$

For example, $K+B$ fails to be 1-complementary if and only if $\sum \lfloor 2b_i \rfloor > 2$, that is, b_1, b_2, and $b_3 \geq 1/2$. The subsequent more precise statements are proved similarly case-by-case, and finally we prove that $K+B$ is 6-complementary if $b_1 \geq 4/5$, $b_2 \geq 2/3$ and $b_3 \geq 1/2$. (Recall that $\sum b_i \leq 2$.) It is not hard to deduce from what we have said that if $K+B$ is n-complementary but not 1- or 2-complementary then the boundary of the n-complement of $K+B$ does not have reduced components for $n = 3$, 4 or 6, that is, $\lfloor B^+ \rfloor = 0$.

Note that the special case $(2/3, 1/2, 1/2, 1/3)$ does not occur if $K+B$ is numerically negative, or equivalently $\sum b_i < 2$.

(5.2.2) Although we usually assume that X is irreducible, we relax this requirement in the following example, which will occur in the proof of Theorem 5.6 below. Suppose that X is a connected, possibly reducible or incomplete curve with nodes (ordinary double points). The log canonical divisor $K+B$ is then taken to be the nonsingular canonical divisor K (that is, the Cartier divisor corresponding to the dualizing sheaf) plus a nonsingular boundary $B = \sum b_i P_i$. The nonsingularity of the divisor means that its support is contained in the nonsingular part of X, and log canonical is defined by the inequalities $0 \leq b_i \leq 1$, which is the same as the definition of boundary. However, we assume more, namely that

$$\lfloor B \rfloor = 0, \quad \text{that is,} \quad 0 \leq b_i < 1,$$

and that $K+B$ is numerically nonpositive on each irreducible component of a connected complete algebraic subvariety (or analytic subspace) $S \subset X$. Then $K+B$ is 1-, 2-, 3-, 4- or 6-complementary in a neighborhood of S, and is 1- or 2-complementary if X is singular, in particular if it has more than one irreducible component. More precisely, if $S = X$ is a curve of arithmetic genus 1, an "elliptic curve", then $K+B$ numerically nonpositive implies that X is a "wheel", and $B = 0$ so that K is 1-complementary. In the opposite case, S is a chain of $\mathbb{P}^1$ s. If $S = pt.$ then the chain is empty and $K+B$ is 1-complementary.

The case when $S = X$ is a chain of a single element $\mathbb{P}^1$ in a neighborhood of S was treated in (5.2.1). If the chain has two or more curves, or one curve and

$S \neq X$ in a neighborhood of S, then again by numerical nonpositivity of $K+B$ the boundary B is concentrated on the two end curves S of the chain when $S = X$, and possibly on one of the two end curves when $S \neq X$; corresponding to this the boundary B breaks up into a sum $B = B' + B'' = \sum_{i\geq 1} b_i' P_i' + \sum_{i \geq 1} b_i'' P_i''$ or $B = B' = \sum_{i\geq 1} b_i' P_i'$, where P_i' and P_i'' are points of the end curves. In this case $K+B$ is 1- or 2-complementary, and if we suppose that $b_1' \geq b_2' \geq \cdots$ and $b_1'' \geq b_2'' \geq \cdots$ then $K+B$ fails to be 1-complementary if and only if $b_1' = b_2' = 1/2$ and $b_1'' = b_2'' = 1/2$. In particular, $K+B$ is 1-complementary if one of the extreme components is missing, or if $K+B$ is negative on one of them.

(5.2.3) *Surface quotient singularities.* It is well known (see for example [6], (1.9)) that an isolated surface singularity $P \in X$ is log terminal if and only if it is a quotient singularity $\mathbb{C}^2/G$ with $G \subset \mathrm{GL}(2, \mathbb{C})$. Then K is 1-, 2-, 3-, 4- or 6-complementary. Moreover, except for the canonical surface singularities which have 0 as complement,

K is not 1-complementary	$\Longleftrightarrow$	$P \in X$ has graph D_n, E_6, E_7 or E_8
K is not 1- or 2-complementary	$\Longleftrightarrow$	$P \in X$ has graph E_6, E_7 or E_8
K is not 1-, 2- or 3-complementary	$\Longleftrightarrow$	$P \in X$ has graph E_7 or E_8
K is not 1-, 2-, 3- or 4-complementary	$\Longleftrightarrow$	$P \in X$ has graph E_8

The singularities having the exceptional graphs E_6, E_7 and E_8 have been extensively treated [15]. According to Brieskorn's classification (see [2], [4], and [Utah], Chapter 3) the minimal resolution of a surface quotient singularity has exceptional curves $E_i \cong \mathbb{P}^1$ crossing normally, and the graph of the resolution is one of A_n, D_n, E_6, E_7 or E_8; it is marked with the selfintersection numbers $E_i{}^2 \leq -2$. In the analytic case, the required assertions comes from this and a case-by-case analysis.

For example, in type A_n the exceptional set of the minimal resolution $f\colon Y \to X$ is a chain $E_1, \ldots, E_n$ of $\mathbb{P}^1$ s with normal crossings, and it can be complemented by two curves S and S' that cross the two extreme curves E_1 and E_n normally (see Proposition 3.9 above and the remark following it). By the adjunction formula the Cartier divisor $K_Y + S + S' + \sum E_i$ is numerically 0 relative to f, and also linearly 0 since log terminal singularities are rational. Hence the pushdown $f(S+S')$ is a 1-complement in a neighborhood of a singularity of type A_n. The remaining cases can be analyzed in a similar way, and the algebraic case reduces to the analytic case by standard cohomological arguments. By Corollary 5.9 below, the classification of log canonical surface singularities with boundaries in terms of complements does not become more complicated.

(5.2.4) (*Alekseev, Shokurov, Reid* [1], [21], [24]). The canonical divisor K of a $\mathbb{Q}$-Fano variety with log terminal singularities of index n is n-complementary if X has Fano index $m/n > \dim X - 2$ (or $= 1$ for 3-folds). ([7]) Let H be the ample generator of $\operatorname{Pic} X$, $H' \in |mH|$ a general element, and set $0^+ = (1/n)H'$. (In the case of a 3-folds with rational Gorenstein singularities, choose a general element $0^+ \in |-K_X|$.) From [1] one deduces that for $m \geq 2$ the linear system $|mH|$ is

([7]) Here $n =$ index of singularities, so $-nK_X$ is Cartier, and $-nK_X = mH$; the *Fano index* is m/n.

free, and for $m = 1$ the Fano index is $> \dim X - 2$ only for surfaces. Furthermore, choosing an n-complement in this way gives a log terminal $K + 0^+$ (canonical for the 1-complement in the 3-fold case). In this case we say that K is *strongly n-complementary.*

(5.2.5) (*Reid* [22], (6.4)). K is strongly 1-complementary in a neighborhood of any 3-dimensional terminal singularity.

(5.2.6) (*Mori*). [12], (7.1), says that for an analytic 3-fold with terminal singularities, K is strongly 1-complementary ([8]) in a neighborhood of an irreducible curve of a flipping extremal ray R with $KR < 0$.

(5.2.7) (*Mori, Morrison and Morrison* [17]). K is not always strongly 1- or 2-complementary in a neighborhood of a 4-fold terminal singularity (or even an isolated quotient singularity by a cyclic group of prime order p). Is K 1- or 2-complementary? If not, for what n does an n-complement exist?

We start with some general facts coming more-or-less from the definitions.

5.3. **Lemma.** *If D' is a subboundary and $K + D'$ is n-complementary, then $D' \geq D$ implies that $K + D$ is n-complementary.*

Proof. Set $D^+ = D'^+$. Q.E.D.

5.4. **Lemma.** *For a birational contraction $f\colon X \to Y$*

$$K + D \ n\text{-complementary} \quad \Longrightarrow \quad K_Y + f(D) \ n\text{-complementary.}$$

Proof. Set $f(D)^+ = f(D^+)$. Q.E.D.

5.5. **Proposition.** *Suppose*

(i) *$K + S + D$ is log canonical and a $\mathbb{Q}$-divisor (or log terminal and X is $\mathbb{Q}$-factorial);*

(ii) *$f\colon X \to Z$ is a birational contraction such that $K + S + D$ is numerically antiample relative to f.*

Then $K + S + D$ is n-complementary in a neighborhood of any fiber of f.

Proof. Since D is a $\mathbb{Q}$-divisor, by the Kleiman ampleness criterion [10] there exists a natural number n such that $-n(K + S + D)$ is very ample relative to f. In this case as n-complement we can take a general hyperplane section

$$\overline{D} \in |-nK - nS - \lfloor (n+1)D \rfloor| = |-n(K + S + D)|.$$

Adding a general $\overline{D}$ preserves the condition that $K+S+(1/n)\lfloor (n+1)D \rfloor = K+S+D$ is log canonical, and a fortiori, so does adding $(1/n)\overline{D}$. The case when $K + S + D$ is log terminal and X is $\mathbb{Q}$-factorial can be reduced using stability (1.3.4) and Lemma 5.3 to the preceding case after increasing slightly the multiplicities of the subboundary D to make them rational numbers. Q.E.D.

For surfaces we have a more exhaustive and precise result.

5.6. **Theorem.** *Let $f\colon X \to Z$ be a birational contraction of a surface X, and D a subboundary such that*

(i) *f contracts all the curves with negative multiplicities in D;*

(ii) *$K + D$ is numerically nonpositive relative to f;*

(iii) *$K + D$ is log canonical.*

([8]) Either 1- or 2-complementary in [16]. Mori and Kollár have proved 1-complementary ([Mori–Kollár], Theorem 1.7), but the proof is very indirect.

Then $K + D$ *is* 1-, 2-, 3-, 4- *or* 6-*complementary in a neighborhood of any fiber of* f. *More precisely, if* $K + D$ *is not* 1- *or* 2-*complementary, then* $K + D$ *is either* 3-, 4- *or* 6-*complementary in such a way that for any log terminal extraction* $Y \to X$ *there is a unique irreducible component of* D^{+Y} *which is reduced, and it is exceptional on* Z. (*The notation* D^{+Y} *was introduced after adjunction formula* 3.1.)

A complement, or more generally a log divisor $K + D$, is *exceptional* if for any extraction $Y \to X$, there is at most one irreducible component of D_Y which is reduced. Thus the 3-, 4- or 6-complements of the theorem are exceptional. In the exceptional types E_6, E_7 and E_8 of Example (5.2.3), K has an exceptional complement.

Proof. Fix a fiber $f^{-1}P$ for $P \in Z$. Adding to $K + D$ an effective and numerically nonpositive divisor, for example f^*H for a general hyperplane section H through P, we can arrange that $K + D$ is *actually log canonical*. This means that there exists an extraction $Y \to X$ for which D^Y has a reduced component that intersects the inverse image of the fiber. Then by Lemma 5.3, the required complement reduces to the same type of complement for the new subboundary. On the other hand, by Lemma 5.4, the theorem reduces to the case that the surface X is nonsingular, the support of the subboundary D consists of nonsingular irreducible curves crossing normally. By the above, we can also suppose that D has a reduced component intersecting the fiber. We now write the subboundary $S + D$ according to our convention, where S is reduced and intersects the fiber, and $\lfloor D \rfloor \le 0$. Then assumptions (ii) and (iii) of the theorem take the form

(ii′) $K + S + D$ is numerically nonpositive relative to f;

(iii′) $K + S + D$ is log canonical.

Using the following assertion we can suppose that S is connected in a neighborhood of the fibers.

5.7. **Connectedness Lemma.** *Let* D *be a divisor on a surface* X *and* $f\colon X \to Z$ *a birational contraction such that*

(i) f *contracts the components of* D *with negative multiplicities*;

(ii) $K + D$ *is numerically nonpositive relative to* f.

Then the locus of log canonical singularities of $K+D$ *is connected in a neighborhood of any fiber of* f.

To prove the lemma and Theorem 5.6, we need a further result.

5.8. **Nonnegativity Lemma.** *Let* X *be a nonsingular surface,* $S \subset X$ *a nonsingular curve, and* D *a numerically contractible divisor such that* $K + S + D$ *is numerically nonpositive on its support. Then* S *intersects only components of* D *with nonnegative multiplicities.*

Proof. A curve D is *numerically contractible* if it is complete (compact in the analytic case) and its components have negative definite intersection matrix; a divisor is numerically contractible if its support is contained in a numerically contractible curve. Discarding the effective part of the divisor D, we can assume that the multiplicities of D are all negative. If now $D \neq 0$ then by negative definiteness there exist an irreducible curve $E \subset \operatorname{Supp} D$ with $DE > 0$. It follows from this that $(K+S)E < 0$. Hence E is a (-1)-curve, and S is disjoint from E. Let $g\colon X \to Y$ be the contraction of E. Then after substituting $S \mapsto g(S)$ and $D \mapsto g(D)$, the log divisor $K_Y + g(S) + g(D)$ satisfies the previous conditions. However, the number of components of D has decreased. After a number of such contractions, $D = 0$, when the conclusion of the lemma is obvious. Q.E.D.

Proof of Lemma 5.7. Suppose that $Y \to X$ is a resolution of singularities on which the support of D^Y consists of nonsingular curves crossing normally. Then the locus of log canonical singularities of $K + D^Y$ is the union of components with multiplicities ≥ 1, and its image is the locus of log canonical singularities of $K + D$. Hence verifying the lemma reduces to the case that X is a nonsingular surface and $\operatorname{Supp} D$ consists of nonsingular curves crossing normally. We can also suppose that this set of curves contains exceptional curves. We can combine the components of a fiber into an effective divisor F which is numerically negative on the fiber. Suppose that there is a minimal $\varepsilon > 0$ for which the locus of log canonical singularities of $K+D+\varepsilon F$ has fewer connected components than that of $K+D$. Then $D+\varepsilon F$ has a reduced chain $\sum_{i=1}^n E_i$ of irreducible curves contained in the fiber with multiplicities 1, whose ends intersect the curves E_0 and E_{n+1} with multiplicities ≥ 1. By construction $K + D + \varepsilon F$ is numerically negative on the curves of the chain. Using the preceding lemma it is easy to check that the components of $K + D + \varepsilon F$ with negative multiplicities do not intersect the chain. Note for this that a curve made up of such components is numerically contractible, since by (i) it is contained in a fiber. Consequently, the log divisor $K + E_0 + E_1 + \cdots + E_n + E_{n+1}$ is negative on the curves of the chain, and in particular on E_1, hence $\deg K_{E_1} = (K + E_1) \cdot E_1 < -2$. This is of course impossible. Hence no such ε can exist, and by connectedness of the fiber this is only possible if the locus of log canonical singularities of $K + D$ is connected in a neighborhood of the fiber. Q.E.D.

We return to the proof of Theorem 5.6. Note that according to the assumptions on the support of $S + D$ the components of S are nonsingular and cross normally, and hence S is a curve with nodes (ordinary double points). By Lemmas 5.7–8 S is connected in a neighborhood of a fiber, and does not intersect components of D with negative multiplicities. To prepare to apply the Kawamata-Viehweg vanishing theorem, we make D into a $\mathbb{Q}$-divisor, of course without losing our assumptions. Indeed, if the multiplicity d_i of D_i is irrational and D_i is not contracted by f, then it can be reduced to d_i' in such a way that $\lfloor (n+1)d_i' \rfloor = \lfloor (n+1)d_i \rfloor$ for each of $n = 1, 2, 3, 4$ or 6. If D_i is contracted by f and $K + S + D$ is numerically negative relative to f then we do the same. The remaining D_i are numerically contractible and $K + S + D$ is numerically 0 on them, so that we deduce that their multiplicities are rational. Hence the Kawamata-Viehweg vanishing theorem applies to $-(n+1)(K + S + D)$, giving

$$R^1 f_* \mathscr{O}_X(-nK - (n+1)S - \lfloor (n+1)D \rfloor)$$
$$= R^1 f_* \mathscr{O}_X(K + \lceil -(n+1)(K + S + D) \rceil) = 0.$$

Thus in a suitable neighborhood of the fiber under study, the linear system

$$| -nK - nS - \lfloor (n+1)D \rfloor |$$

cuts out a complete linear system on S. By the assumption that the support has normal crossings, D does not pass through singularities of S. Therefore by nonsingularity of X, and possibly after a suitable choice of canonical divisor K, we can assume that the support of $K+S+D$, and hence also that of $-nK-nS-\lfloor (n+1)D \rfloor$, does not contain the curve S, and meets it in nonsingular points. Hence their restrictions

$$K_S + D_{|S} \quad \text{and} \quad -nK_S - \lfloor (n+1)D \rfloor_{|S}$$

to S are nonsingular. Since only the effective part of D meets S, $D_{|S}$ is effective. Again by normal crossings $\lfloor D_{|S} \rfloor = \lfloor D \rfloor_{|S} = 0$, and by (ii), $K_S + D_{|S}$ is numerically nonpositive on the intersection of the fiber $f^{-1}P$ with S. This set is connected, since S is connected in a neighborhood of $f^{-1}P$. Thus by Example (5.2.2) we see that for $n = 1, 2, 3, 4$ or 6 there exists an n-complement

$$\overline{D_{|S}} \in |-nK_S - \lfloor (n+1)D_{|S} \rfloor| = |-nK_S - \lfloor (n+1)D \rfloor_{|S}|.$$

Since $|-nK - nS - \lfloor (n+1)D \rfloor|$ restricts surjectively to S, it contains an effective divisor $\overline{D}$ which restricts to this: $(\overline{D})_{|S} = \overline{D_{|S}}$, and hence the restriction

$$(K + S + (1/n)(\lfloor (n+1)D \rfloor + \overline{D}))_{|S} = K_S + (1/n)\left(\lfloor (n+1)D_{|S} \rfloor + \overline{D_{|S}}\right)$$

is log canonical. Hence by inversion of adjunction 3.12 (compare Example 3.17), $K + D^+$ is log canonical in a neighborhood of S for the effective divisor $D^+ = S + (1/n)(\lfloor (n+1)D \rfloor + \overline{D})$. I claim that this gives the necessary n-complement. It remains to check that $K + D^+$ is log canonical. If not, by the preceding lemma on connectedness and the fact that $K + D^+$ is log canonical in a neighborhood of S, but not log canonical in a neighborhood of the fiber, it follows that there exists an irreducible curve C that is contained in the fiber, passes through a point that is not log canonical for $K + D^+$, is contained in D^+ with multiplicity 1, and intersects some component D^+ with multiplicity 1 at another point. But this is not possible, since $K + D^+$ numerically 0 on C implies that $C = \mathbb{P}^1$ and $(K + D^+)_{|C} = K_{\mathbb{P}^1} + D'$, where by the nonnegativity lemma $D' = (D^+ - C)_{|C}$ is an effective divisor having two points with multiplicities respectively > 1 by Corollary 3.12 and ≥ 1 by Corollary 3.15. This contradicts $K_{\mathbb{P}^1} + D'$ numerically 0, since $\deg K_{\mathbb{P}^1} = -2$. (Compare the end of the proof of the connectedness lemma.)

In conclusion we note that by (5.2.2), the 3-, 4- or 6-complements obtained in our construction occur only when the curve $S = \mathbb{P}^1$ is exceptional relative to f, and the boundary $(K + D^+)_{|S} = K_S + (1/n)(\lfloor (n+1)D_{|S} \rfloor + \overline{D}_{|S})$ has no reduced components. Hence by Corollary 3.12, in a neighborhood of S the log divisor $K + D^+$ is purely log terminal and has a unique irreducible components S, hence by Lemma 5.7 it follows that the complements obtained are exceptional. Q.E.D.

5.9. **Corollary.** *On a surface, $K + B$ is 1-, 2-, 3-, 4- or 6-complementary in a neighborhood of any point. In particular the same holds for K in a neighborhood of any log canonical surface singularity.*

5.10. **Corollary.** *At a log canonical, but not log terminal, surface singularity, the canonical divisor K has index 1, 2, 3, 4 or 6.*

5.11. **Corollary.** *Add to the assumptions of Theorem 5.6 the following: the existence of a reduced component of D meeting the fiber but not contained in it; to (ii) we add the condition that $K + D$ is numerically negative on the fiber; and to (iii) the condition that $K + D$ is log terminal. Then $K_Y + D^Y$ is 1-complementary in a neighborhood of the inverse image of the given fiber for any resolution $f\colon Y \to X$.*

Proof. As in the proof of the theorem we can assume that X is nonsingular and that the components of D are nonsingular and cross normally; moreover, we can preserve the preceding assumptions by increasing D slightly on the blown up curves. Then S, the reduced part of the divisor D, is a curve with nodes, is connected in a neighborhood of the fiber by Lemma 5.7, and by assumption is not contained in the fiber. By Lemma 5.8 and assumption (i) of Theorem 5.6, D is a boundary in a neighborhood of S. As we see from the end of the proof of the theorem, $K + D$ is

n-complementary if the restriction $(K+B)_{|S}$ is. But $(K+B)_{|S}$ is 1-complementary by (5.2.2), if we slightly increase the components of $D-S$. Q.E.D.

From now on in this section, and in the remainder of the paper, X is a $\mathbb{Q}$-factorial 3-fold.

5.12. **Theorem.** *Let $f\colon X \to Z$ be a small contraction of X with boundary $S+B$, and suppose that*

(i) *$K+S+B$ is log terminal;*
(ii) *$K+S+B$ is numerically nonpositive relative to f;*
(iii) *S is reduced and irreducible;*
(iv) *$\lfloor B \rfloor = 0$.*

Then $K+S+B$ is 1-, 2-, 3-, 4- or 6-complementary in a neighborhood of any fiber of f lying on S. More precisely, if $K+S+B$ is not 1- or 2-complementary then $K+S+B$ has an exceptional 3-, 4- or 6-complement.

Here $K+B^+$ is *exceptional* if $(K+B^+)_{|S}$ is exceptional (see the paragraph following Theorem 5.6).

Proof. Since X is $\mathbb{Q}$-factorial, by stability (1.3.4) we can perturb the irrational multiplicities of B to arrange that B is a $\mathbb{Q}$-divisor, while preserving all the conditions (i)–(iv) and $\lfloor (n+1)B \rfloor$ for $n = 1, 2, 3, 4$ or 6. Fix the fiber $f^{-1}P$ over $P \in Z$, and consider a resolution of singularities $g\colon Y \to Z$ such that the exceptional divisors and all irreducible components of $g^{-1}(K+S+B)$ are nonsingular and cross normally. By assumption and because Z is normal, the fiber $f^{-1}P$ under study is a connected curve (or a point) lying over S. This curve on S is also contracted by f. Moreover, since S is normal (see Corollary 3.8), there is a commutative square

$$\begin{array}{ccc} S & \subset & X \\ {\scriptstyle f_T}\downarrow & & \downarrow{\scriptstyle f} \\ T & \to & Z \end{array}$$

where T is a normal surface and f_T the contraction induced by f. The resolution g defines a similar square

$$\begin{array}{ccc} g^{-1}S & \subset & Y \\ {\scriptstyle g_T}\downarrow & & \downarrow{\scriptstyle f\circ g} \\ T & \to & Z \end{array}$$

where $g_T = f_T \circ (g_{|g^{-1}S})$ is a resolution of singularities of the normal surface T. Hence $(f\circ g)^{-1}P \cap g^{-1}S = (g_{|g^{-1}S})^{-1} f^{-1}P$ is a connected fiber contracted by g_T. Consider on Y the log divisor $K_Y + g^{-1}S + D$ with the subboundary $g^{-1}S + D = (S+B)^Y$, or in other words, with $f^*(K+S+B) = K_Y + g^{-1}S + D$. Since $K+S+B$ is purely log terminal, the divisor D is not only a subboundary, but its multiplicities are all < 1. By normal crossing $D_{g^{-1}S} = D_{|g^{-1}S}$ is also a subboundary with multiplicities < 1. Furthermore, the log divisor

$$\begin{aligned} K_{g^{-1}S} + D_{g^{-1}S} &= (K_Y + g^{-1}S + D)_{|g^{-1}S} = f^*(K+S+B)_{|g^{-1}S} \\ &= (g_{|g^{-1}S})^*(K+S+B)_{|S} = (g_{|g^{-1}S})^*(K_S + B_S) \end{aligned}$$

is log terminal, has no reduced boundary components and is numerically nonpositive relative to g_T. By (3.2.2) and since B is effective, $g_{|g^{-1}S}$, hence also g_T, contracts the components of $D_{g^{-1}S}$ with negative multiplicities. Thus the birational contraction g_T and the subboundary $D_{g^{-1}S}$ satisfy the conditions of Theorem 5.6. Hence the log divisor $K_{g^{-1}S}+D_{g^{-1}S}$ is 1-, 2-, 3-, 4- or 6-complementary; its complement $\overline{D_{g^{-1}S}}$ is an element of the linear system $|-nK_{g^{-1}S}-\lfloor(n+1)D_{g^{-1}S}\rfloor|$. Now note that $-(K_Y+g^{-1}S+D)$ is a $\mathbb{Q}$-divisor that is nef and big relative to the contraction $f\circ g$. Hence by the Kawamata-Viehweg vanishing theorem

$$\begin{aligned} &R^1(f\circ g)_*\mathscr{O}_Y(-nK_Y-(n+1)g^{-1}S-\lfloor(n+1)D\rfloor)\\ &=R^1(f\circ g)_*\mathscr{O}_Y(K_Y+\lceil-(n+1)(K_Y+g^{-1}S+D)\rceil)=0.\end{aligned}$$

Hence $|-nK_Y-ng^{-1}S-\lfloor(n+1)D\rfloor|$ on Y cuts out the complete linear system $|-nK_{g^{-1}S}-\lfloor(n+1)D_{g^{-1}S}\rfloor|$ on the surface $g^{-1}S$ in a suitable neighborhood of the fiber $(f\circ g)^{-1}P$. Here $\lfloor(n+1)D_{g^{-1}S}\rfloor=\lfloor(n+1)D\rfloor_{|g^{-1}S}$ holds by normal crossings. Thus there is a divisor $\overline{D}\in|-nK_Y-ng^{-1}S-\lfloor(n+1)D\rfloor|$ with $(\overline{D})_{|g^{-1}S}=\overline{D_{g^{-1}S}}$. I claim that $g(\overline{D})$ is an n-complement of $K+S+B$. To check this, introduce the divisor $D^+=g^{-1}S+(1/n)(\lfloor(n+1)D\rfloor+\overline{D})$. Then by construction, D^+ satisfies (5.1.1) and (5.1.3), but instead of (5.1.2) we only know that the restriction

$$(K_Y+D^+)_{|g^{-1}S}=K_{g^{-1}S}+\frac{1}{n}(\lfloor(n+1)D_{g^{-1}S}\rfloor+\overline{D_{g^{-1}S}})$$

is log canonical. Since being linearly 0 is preserved under birational contractions, $B^+=g(D^+)=S+(1/n)(\lfloor(n+1)B\rfloor+g(\overline{D}))$ also satisfies (5.1.1) and (5.1.3), but instead of (5.1.2) we only know that the restriction $(K+B^+)_{|S}=K_S+B_S^+$ is log canonical. Indeed,

$$(g_{|g^{-1}S})^*(K_S+B_S^+)=g^*(K+B^+)_{|g^{-1}S}=(K_Y+D^+)_{|g^{-1}S}$$

is log canonical. Hence it remains to check that $K+B^+$ is log canonical. For this, in addition to the restriction $(K+B^+)_{|S}=K_S+B_S^+$ being log canonical, we need the effectiveness

$$B^+-S=g(D^+)-S=\frac{1}{n}(\lfloor(n+1)B\rfloor+g(\overline{D}))\geq\frac{1}{n}(\lfloor(n+1)B\rfloor)\geq 0.$$

For $n=1$, $K+B^+$ log canonical follows from Corollary 4.4. Indeed, by (i) the log divisor $K+B^+$ has only log terminal singularities outside B^+, and in fact canonical singularities, since it has index 1. For $n\geq 2$ we need the following more general case of inversion of adjunction 3.3.

5.13. **Proposition.** *The inversion problem* 3.3 *is true for* 3-*folds under the present assumption that* X *is* $\mathbb{Q}$-*factorial.*

One can deduce from Proposition 5.13 that if S_t is a deformation of surface singularities $P_t\in S_t$ and P_0 is a log canonical (log terminal) singularity of S_0 and the total space of the deformation is $\mathbb{Q}$-Gorenstein, then $P_t\in S_t$ are log canonical singularities (respectively log terminal singularities) for t close to 0 (compare [3]). By inversion of ajunction 3.3, this holds (conjecturally) in any dimension and with boundaries. Moreover log canonical singularities (respectively log terminal singularities) can be treated as singularities of Kodaira dimension 0 (respectively $-\infty$), which leads to the conjecture on the upper semicontinuity of the Kodaira dimension of singularities under deformations (as in Ishii [5], for example).

For the proof of Proposition 5.13 we use the following result, which has essentially already been proved.

5.14. **Lemma.** *Suppose given a log divisor $K+S+B$ and a contraction $f\colon X\to Z$ that is small in a neighborhood of some fiber $f^{-1}P$, such that*

(i) $K+S+B$ *is log terminal*;

(ii) $K+S+B$ *is numerically negative relative to* f;

(iii) $\lfloor B\rfloor=0$;

(iv) S *is reduced*;

(v) S *has two irreducible components meeting $f^{-1}P$, one nef and one numerically nonpositive relative to f.*

Then $K+S+B$ is 1*-complementary in a neighborhood of $f^{-1}P$.*

Note that in (v) a component of S having nonzero intersection number with the fiber $f^{-1}P$ simply intersects it, and a component that is numerically 0 on $f^{-1}P$ and intersects it must contain it.

Proof. By (v), S has an irreducible component S^- which is nonpositive relative to f. Since X is $\mathbb{Q}$-factorial, by (i) S^- is a normal surface, and by connectedness the whole contracted fiber of f is contained in it. If we slightly reduce the multiplicity of the components $S'=S-S^-$ then the new log divisor $K+S^-+B'$ satisfies all the conditions of Theorem 5.12 and $S'+B\geq B'$. By (3.2.3) and (i), $K_{S^-}+(S'+B)_{S^-}$ is log canonical (even log terminal), and by (ii) it is numerically negative relative to the induced birational contraction of the fiber. From (v) and Corollary 3.11, it follows that there exists a reduced component $(S'+B)_{S^-}$ intersecting the fiber, but not contained in it. The log divisor $K_{S^-}+(S'+B)_{S^-}$ satisfies the assumptions of Corollary 5.11. Thus $K_T+(S'+B)^T_{S^-}$ is 1-complementary for an arbitrary extraction $g\colon T\to S^-$. In particular, this holds for the resolution of singularities $g_{|g^{-1}S^-}$ of the proof of Theorem 5.12. But the inequality $S'+B\geq B'$ implies that

$$(S'+B)_{S^-}^{g^{-1}S^-}\geq (B')_{S^-}^{g^{-1}S^-}=D_{|g^{-1}S^-},$$

where D is the divisor defined by $g^*(K+S^-+B')=K_Y+g^{-1}S^-+D$. Thus by Lemma 5.3,

$$K_{g^{-1}S^-}+D_{g^{-1}S^-}=(K_Y+g^{-1}S^-+D)_{|g^{-1}S^-}$$

has a 1-complement, hence by the part of the proof of Theorem 5.12 already established $K+S^-+B'$ is 1-complementary. But slightly decreasing the multiplicities of the components of S' gives $\lfloor 2B'\rfloor=S'+\lfloor 2B\rfloor$, and hence by (5.1.1) $K+S+B$ is also 1-complementary. Q.E.D.

5.15. **Corollary.** *If in addition to the assumptions of Lemma* 5.14 *f is an extremal birational contraction, $B=0$ and S has an irreducible component that is numerically negative relative to f, then a flip of f exists in a neighborhood of the indicated fiber.*

Proof. By the preceding lemma, in a neighborhood of the fiber there is a reduced boundary B^+ such that $K+B^+$ is a log canonical divisor of index 1, and $K+B^+$ is numerically 0 relative to f. Hence $B^+>S$, and B^+ is reduced and has a component intersecting the fiber positively. On the other hand, by assumption there exists a component of S intersecting the fiber negatively. Hence B^+ is an LSEPD divisor for f is a neighborhood of the exceptional fiber. It follows from (i) that $K+B^+$ has log terminal singularities outside B^+. Hence by Proposition 2.7 there exists a flip of type II. Q.E.D.

5.16. **Corollary.** *If the boundary B is reduced and $K+B$ is not necessarily log canonical (nor log terminal outside B), then in a neighborhood of any point $P\in B$*

there exists a strictly log terminal model (respectively a log canonical model) X of $K+B$, even in the case that X is not $\mathbb{Q}$-factorial, but B is LSEPD.

Proof. By Example 4.7, it is enough to have flips of small extremal contractions under the assumption that X is $\mathbb{Q}$-factorial, $K+B$ is strictly log terminal and negative relative to f, B is reduced and has both components that are numerically negative and positive relative to f. But such flips exist by Corollary 5.15. Q.E.D.

Now we can strengthen the flip of type I.

5.17. **Corollary.** *Suppose that X is a 3-fold, not necessarily $\mathbb{Q}$-factorial, and the log divisor $K+B$ and the birational contraction $f\colon X \to Z$ are such that*

(i) *$K+B$ is nonpositive relative to f;*

(ii) *the boundary B is a reduced LSEPD divisor for f containing the exceptional set of f;*

(iii) *$K+B$ is log terminal outside B.*

Then there exists a flip of f relative to $K+B$.

Proof. Decreasing B and applying the theorem on eventual freedom ([8], 3-1-2), we can check that $f(B)$ is a reduced LSEPD. By (ii) $K_Z+f(B)$ is log terminal outside $f(B)$. The existence of a flip is a local fact. Hence we can restrict to a neighborhood of a nontrivial connected fiber $f^{-1}P$ with $P \in Z$. By Corollary 5.16, in a neighborhood of P there is a log canonical model of Z for $K_Z+f(B)$. According to Lemma 1.7 this gives us the required flip of f. Q.E.D.

5.18. **Corollary.** *If, in addition to the assumptions of Lemma 5.14, the birational contraction f is extremal, then f has a flip in a neighborhood of the indicated fiber.*

Proof. We start from the fact that if S has an irreducible component numerically negative for f then it is normal by Corollary 3.8, since $K+S+B$ is strictly log terminal. Then S cannot have a component that is numerically 0 relative to f and which intersects the fiber. Indeed, otherwise this component would intersect the first curve outside the birational contraction, which by Corollary 3.16 contradicts $K+S+B$ log terminal. Hence by (v) S has a component that is positive relative to f. Therefore since f is extremal, in such a situation B^+ is numerically LSEPD for f, and hence LSEPD for f, since $K+S+B$ is strictly log terminal. But $K+B^+$ is log terminal outside B^+. Thus the flip of type II exists by Proposition 2.7.

Thus we can suppose that all the components of S are nef relative to f. Moreover, by (v) S has a component containing a fiber and numerically 0 on it. Again, since $K+S+B$ is log terminal and by Corollary 3.16, this is only possible if the remaining components of S in a neighborhood of the fiber intersect it positively. In particular, by (v) there is a component of S that is positive relative to f. If B^+ has a component which is negative relative to f then once more B^+ is an LSEPD divisor, and there exists a flip of type II by Proposition 2.7. In the opposite case all components of B^+ are nef relative to f. Then discarding components of B^+ intersecting the fiber of f positively, we arrive at the flip of Corollary 5.17. Q.E.D.

5.19. **Corollary.** *If $K+B$ is a log divisor, not necessarily log canonical, then in a neighborhood of $\lfloor B \rfloor$, there exists a strictly log terminal model of X for $K+B$, even if X is not $\mathbb{Q}$-factorial, but B is $\mathbb{R}$-Cartier and $\lfloor B \rfloor$ is an LSEPD divisor. Under the same assumptions, if $K+B$ is log terminal outside $\lceil B \rceil$ then a log canonical model exists. In the analytic case, the same conclusions hold in a neighborhood of a projective subspace $W \subset \lfloor B \rfloor$.*

Proof. The boundary B can be increased by adding an effective divisor H (in the analytic case, in a neighborhood of W) in such a way that the new boundary $B+H$

is reduced, Cartier, and X is nonsingular outside $B+H$. As $B+H$ we can take a sufficiently general hyperplane section passing through the singularities outside the support of the boundary B and through the support of B. The reducibility of such a divisor follows from Bertini's theorem, since for $m \gg 0$, the linear system $|D+mH-\sum C_i|$ on the projective closure of X has base locus $\bigcup C_i$ union the singularities of X (respectively, in a neighborhood of W), where D is an integral Weil divisor, H the hyperplane section and C_i irreducible closed subvarieties of codimension ≥ 2. This follows easily from the sheaf-theoretic description of linear systems and the Serre vanishing theorem (in the analytic situation, by [18], 0.4, in a suitable neighborhood of W).

From this point on, one can work as in Example 4.7. Applying Proposition 4.3 to $K+B+H$, we get a strictly log terminal extraction of f in a neighborhood of $\lfloor B \rfloor$ (in the analytic case, in a neighborhood of W), such that the whole boundary $f^{-1}B+f^{-1}H+\sum E_i$ is an LSEPD divisor for f, since it is contracted to the Cartier boundary $B+H$. Note also that H is $\mathbb{R}$-Cartier, since by assumption B is $\mathbb{R}$-Cartier. In the construction of the log terminal model of Example 4.7 we need to replace B by its reduced part $\lfloor B \rfloor$. The existence of flips can be checked locally over Z, and it follows by Corollary 5.18. However, to construct the log canonical model, the arguments of Example 4.7 are also good enough. Q.E.D.

Proof of Proposition 5.13. By inversion of adjunction 3.4, and the fact that X is $\mathbb{Q}$-factorial, this follows from Corollary 5.19. Q.E.D.

Conclusion of the proof of Theorem 5.12. We apply Proposition 5.13 with $D=B^+-S$. Note that in the case we consider S is normal, that is, $S^\nu = S$. The final assertion of the theorem follows from the fact that if $K+S+B$ does not have a 1- or 2-complement, then neither does $K_{g^{-1}S}+D_{g^{-1}S}$. But then by Theorem 5.6, $K_{g^{-1}S}+D_{g^{-1}S}$ has an exceptional 3-, 4- or 6-complement. According to the proof and definition, the same holds for $K+S+B$. Q.E.D.

We have the following variant of flips of types I and II.

5.20. **Corollary.** *Let $f\colon X \to Z$ be an extremal contraction and $K+B$ a log divisor such that*

(i) $K+B$ *is log terminal outside the support of the boundary* $\lceil B \rceil$;

(ii) $K+B$ *is numerically* 0 *relative to* f;

(iii) *the reduced part of the boundary* $\lfloor B \rfloor$ *has two irreducible components that are numerically negative and positive relative to* f.

Then there exists a flip of f with respect to any divisor.

Proof. Let S^+ be the component that is positive on the contracted curve. Since f is extremal, there is a unique nontrivial flip, and by (ii) and Lemma 1.7 it can be constructed as a log canonical model for $K+B-S^+$ over Z. Thus we can also restrict to the case of a connected curve contracted by f to a point P. Hence by Corollary 5.19 there exists a strictly log terminal extraction $g\colon Y \to X$. By (1.5.7) $g^*(K+B) = K_Y + B_Y$, and by construction this is strictly log terminal. Hence by (ii) $h = f \circ g$ is a strictly log terminal model of Z for $K_Z + f(B)$. By (iii) $f(\lfloor B \rfloor) = \lfloor f(B) \rfloor$ is LSEPD. But the reduced part of the boundary B_Y lies over $\lfloor f(B) \rfloor$ by (i), and hence is an LSEPD divisor for h. Moreover, we can apply Corollary 4.6 to $f=h$, $H=h^{-1}f(S^+)$ for $\varepsilon = 1$ and $\varepsilon_0 < 1$ close to 1. Again by (iii) it remains to check the existence of flips of small 0-contractions h over Z such that the reduced part of the boundary B_Y has irreducible components H and S that are respectively positive and negative relative to f. Note that, decreasing

H slightly, $K_Y + B_Y$ becomes negative relative to h and log terminal. Hence by Theorem 5.12 we deduce that for $\beta < 1$ and close to 1, the log divisor $K_Y + \beta B_Y$ is 1- or 2-complementary. Exceptional complements drop out by construction in the proof of Theorem 5.6, since S and H are preserved in a complement. In the case of a 1-complement, there exists a flip of type II. The case of a 2-complement reduces our corollary to the situation when the following additional condition holds:

(iv) $2(K+B)$ is linearly 0 in a neighborhood of the contracted curve.

Then $2(K_Y + B_Y)$ is also linearly 0 relative to g, which by Proposition 1.12 is preserved under modifications in 0-contractions. Then the required flips are of type III. The condition (iv) holds because, by construction, $K_Y + B_Y$ becomes log terminal on decreasing the multiplicity of H in the boundary B_Y. Q.E.D.

§6. Special flips

The convention of §5 is no longer in force here: when writing $K + S + B$, we do not necessarily assume $\lfloor B \rfloor \leq 0$.

6.1. **Definition.** A small projective birational contraction f of a connected curve is *limiting* for a log canonical divisor $K + S + B$ if the following conditions (6.1.1–5) hold:

(6.1.1) $K + S$ is strictly log terminal;

(6.1.2) S is an irreducible surface that intersects the contracted curve and is nonpositive relative to f;

(6.1.3) every irreducible component of the fractional part $\{B\}$ is negative relative to f;

(6.1.4) the log divisor $K + S + B$ is negative relative to f;

(6.1.5) in a neighborhood of the contracted curve, $K+S+B'$ is not log canonical for any $B' > B$ with the same support as B.

f is *special* if in addition f is extremal (in the analytic case, both (6.1.1) and the extremal property are preserved on shrinking to a neighborhood of the contracted curve, that is, over $W = pt.$, the image of the exceptional curve for f, so in particular the fiber is irreducible), and

(6.1.6) B is integral, that is, $\{B\} = 0$;

(6.1.7) $K + S + B$ is strictly log terminal.

The significance of this assortment of conditions should become clear from Proposition 6.2 and the proof of Reductions 6.4–5. The corresponding flips will be called *limiting* and *special*. Note that (6.1.5) follows automatically from (6.1.6), since if B is integral, increasing its multiplicities must either change the support or violate the boundary condition $b_i \leq 1$.

6.2. **Proposition.** *If f is a limiting contraction then the multiplicities $b_i \in (0, 1)$ of B form a limiting chain* (*see Definition* 4.8) *of length equal to the number of irreducible components of B with fractional multiplicities.*

Note that if there are coincidences between the multiplicities b_i of different prime divisors D_i, these can be viewed as a single multiplicity for their sum, which is a reduced but reducible divisor. Then Proposition 6.2 remains valid. In particular, we can take the *distinct* multiplicities b_i as a limiting chain, and we do this in our reduction below.

6.3. **Lemma.** *Let $P \in X$ be a point of a surface X at which $K + B$ is not log terminal, and suppose that in a neighborhood of P, the multiplicities b_i of irreducible and nonreduced components of B can be written*

$$b_i = \frac{n_i - 1}{n_i} + \sum_j \frac{k_{ij}}{n_i} d_j ,$$

where n_i, k_{ij} are natural numbers, and d_j is a finite ordered set of numbers from the interval $(0, 1)$. Then for some i, the d_j with $k_{ij} \neq 0$ form a limiting chain.

Proof. Let $f: Y \to X$ be a log terminal model of the surface singularity $P \in X$ (see Example 1.6). By assumption its fiber is nonempty, and by Lemma 5.7 it is a connected curve of S. Since P is log canonical, by (1.5.7) the log terminal divisor

$$K_Y + S + f^{-1}B = f^*(K + B)$$

is numerically 0 relative to f. If the chain d_i is empty then it is limiting by definition. Otherwise B has an irreducible component through P with multiplicity $b_i \in (0, 1)$. Its birational transform meets the fiber over P. From this and from Corollary 3.16 one deduces easily that the reduced part of the boundary $S + f^{-1}B$ is a chain of curves in a neighborhood of the fiber, and its contracted irreducible components are copies of $\mathbb{P}^1$. Here the birational transforms of irreducible components of B with $b_i \in (0, 1)$ can only intersect its ends, that are contracted to P. (Under our assumptions there is at most one end not contracted by f, and this can only be the birational transform of the unique reduced irreducible component of the boundary B through P.) For each of the contracted ends $\mathbb{P}^1$, we deduce by adjunction, the log terminal assumption on $K_Y + S + f^{-1}B$, and Corollaries 3.16 and 3.10 that

$$K_{\mathbb{P}^1} + (S + f^{-1}B)_{\mathbb{P}^1} = \delta I + \sum p_h P_h ,$$

where $\delta = 0$ or 1, and is 1 only at a point I of intersection of the component $\mathbb{P}^1$ with another component S, and

$$p_h = \frac{m_h - 1}{m_h} + \sum_i \frac{l_{hi}}{m_h} b_i < 1$$

with m_h, l_{hi} natural numbers. By Lemma 4.2, substituting for the b_i in terms of d_j gives a similar expression for the p_h as a sum of the d_j. Note that $\sum p_h = 1$ or 2, where the sum runs over all contracted ends, and each d_j with $k_{ij} \neq 0$ for some i necessarily appears with nonzero coefficient in some p_h. Q.E.D.

Proof of Proposition 6.2. First of all, (6.1.2) and the connectedness of the contracted curve imply that the contracted curve is contained in S. The surface S itself is normal by (6.1.1), and by (3.2.3) the restriction $K_S + B_S$ is log canonical. If $K_S + B_S$ is log terminal and $\lfloor B_S \rfloor = 0$ in a neighborhood of the contracted curve, then these properties are preserved on slightly increasing B, and by Proposition 5.13 the log canonical property of $K + S + B$ is preserved, which is impossible by assumption (6.1.5). Thus the locus of log canonical singularities $M = \mathrm{LCS}(K_S + B_S)$ is nonempty in a neighborhood of the curve, and by Lemma 5.7 is connected.

Suppose first that M contains one of the irreducible contracted curves C. Then by (6.1.3), every irreducible component of B with multiplicity $b_i \in (0, 1)$ passes through C. For a general hyperplane section H the divisor $K + H + S + B$ is log canonical in a neighborhood of the contracted curve, and $K + H + S + \lfloor B \rfloor$ is strictly log terminal. In particular, H is a normal surface. According to Corollary 3.10, the multiplicities of the boundary $(S + B)_H$ of the log canonical divisor $K_H + (S + B)_H$

have only multiplicities of the form $d_i = b_j$ in a neighborhood of the intersection $C \cap H$. By Corollary 3.16, when $\{B\} \neq 0$, $K_H + (S+B)_H$ is not log terminal at the points of $C \cap H$, since in this case $K+S+B$ and $K+H+S+B$ are not log terminal along C. Moreover, b_j coincides with some $d_i < 1$ in a neighborhood of $C \cap H$. Thus the chain b_j is limiting by Lemma 6.3. In the case $\{B\} = 0$ the chain b_j is empty and limiting by definition. Hence we also assume below that $\{B\} \neq 0$.

Thus it remains to consider the case that M intersects the contracted curve in a unique point P. By (6.1.3) every irreducible surface in B with fractional multiplicity passes through P. By Corollary 3.10, in a neighborhood of P the multiplicities of the boundary B_S have the form

$$d_i = \frac{n_i - 1}{n_i} + \sum_j \frac{k_{ij}}{n_i} b_j.$$

Now the multiplicities of the contracted curves of the boundary are < 1 and every b_j occurs in each of them. If at P the restriction $K_S + B_S$ is not log terminal, then again by Lemma 6.3 the chain b_j is limiting. On the other hand, if at P the restriction $K_S + B_S$ is log terminal, then in a neighborhood of P, the set M is a nonsingular irreducible curve through P. Moreover, if an irreducible component of B with fractional multiplicity does not contain M then one can increase its multiplicity while preserving the log canonical property of $K_S + B_S$, and hence also of $K+S+B$. This is impossible by (6.1.5). Hence in the case that $K_S + B_S$ is log terminal, in a neighborhood of P, every irreducible component of B with fractional multiplicity must pass through M. Hence just as in the first case considered it follows that the chain b_i is limiting. Q.E.D.

6.4. **Reduction.** *Theorem* 1.9 *is implied by the existence of special flips, and even by the existence of special flips of the types* (6.6.1–2) *below.*

6.5. **Reduction.** *Theorem* 1.10 *and Corollary* 1.11 *are implied by the existence of special flips, and even by the existence of special flips of the types* (6.6.1–2) *below.*

Proof of Reductions 6.4–5. By Lemma 1.7, to construct the flip of f with respect to $K+B$ it is enough to construct a log canonical model of Z for $K_Z + f(B)$; then the conclusions of Corollary 1.11 will be satisfied. Moreover, according to the proof of this corollary in §4, it is sufficient to construct a strictly log terminal model of f. Thus Reduction 6.4 reduces to Reduction 6.5,]and more precisely, to the reduction concerning Theorem 1.10.

We now turn to constructing a strictly log terminal model of f in Reduction 6.5; we add a reduced divisor H to the boundary B (in the analytic case, in a neighborhood of W) to get a boundary $B+H$ with the properties that X and the components of B are nonsingular and cross normally, $\lfloor B \rfloor + H$ is a Cartier divisor and is principal locally over Z (that is, f^* of a Cartier divisor on Z), and $K+B$ is log terminal outside H. For H we can take a general element of the linear system $|mf^*A - \lfloor B \rfloor - f^{-1}C|$ with $m \gg 0$, where A is an ample divisor on Z, and C is the image in Z of the curves contracted by f, the singularities of X and the irreducible components of the support of B, their nonnormal intersections and the points at which $K+B$ is not log terminal. Since f is small and finite over the general point of Z, both C and $f^{-1}C$ are at most 1-dimensional algebraic subsets (in the analytic case, analytic subsets) and the base locus of the linear system is exactly $f^{-1}C$ (compare the proof of Corollary 5.19). To prove this, using Zariski's main theorem (that is, the Stein factorization), we can contract the exceptional curves of f and suppose that f is finite. Then X is quasiprojective (in the analytic case, is 1-complete in a neighborhood of $f^{-1}W$, see [18], (0.1)).

Therefore by Bertini's theorem, H is reduced and nonsingular and the components of $H+B$ cross normally outside $f^{-1}C$, $\lfloor B\rfloor + H$ is principal locally over Z, and $K+B+H$ is log terminal. Now take a resolution of singularities $g\colon Y \to X$ with exceptional set only over $f^{-1}C$, and such that the composite morphism $f\circ g\colon Y \to X \to Z$ is projective and finite over the general point, and all the components of $B_Y + g^{-1}H$ are nonsingular and cross normally, in particular $K_Y + B_Y + g^{-1}H$ is strictly log terminal. First of all, by the philosophy of §1 over Z (in the analytic case, over a neighborhood of W) we construct a strictly log terminal model of $f\circ g$ for $K_Y + B_Y + g^{-1}H$. Next we apply Corollary 4.6 to this model, with the modified $g^{-1}H$ for H, $\varepsilon = 1$ and $\varepsilon_0 = 0$. For this we note that $\lfloor B_Y\rfloor + g^{-1}H$ is LSEPD for $f\circ g$, since by construction it has the same support as the effective Cartier divisor $g^*(\lfloor B\rfloor + H) \sim (f\circ g)^* mA$. By Proposition 1.12, this property is preserved under modifications of $g^*(\lfloor B\rfloor + H)$ over Z.

Therefore in both constructions flipping curves are contained in the reduced part of the boundary—in the second, since a modification of $g^{-1}H$ is a reduced component of a modification of $g^*(\lfloor B\rfloor + H)$ and is positive relative to a 0-contraction. Thus the construction of a strictly log terminal model of f for $K+B$ over Z reduces to the construction of flips of small extremal contractions, for which, after discarding fractional components of the modified boundary B_Y not negative relative to such birational contractions, conditions (6.1.1–4) are satisfied modulo the connectedness of the flipping curve. On the contrary, (6.1.5) does not hold since $K_Y + B_Y$ is strictly log terminal and, by stability (1.3.4), when there are fractional components of the modified boundary B_Y that are negative relative to such birational contractions.

Increasing equal multiplicities equally we get a limiting contraction, the boundary of which has a chain of distinct fractional multiplicities that is limiting by Proposition 6.2 and is $>$ the same chain for B. Here, in the algebraic case, we need to take a neighborhood of a connected component of the birational contraction, to ensure that its exceptional curve is connected and that the log divisor with the new boundary will be log canonical; in the analytic case, we can localize to replace the given birational contraction by the contraction of only one connected component; under this we may lose projectivity of $f\circ g$, $(f\circ g)^{-1}W$ is as a rule not a flipping curve and Y is $\mathbb{Q}$-factorial only after shrinking neighborhoods of $(f\circ g)^{-1}W$.

The conclusion is that *constructing a log terminal model of f for $K+B$ reduces either to finding special flips, or limiting flips with chains of distinct fractional multiplicities of the new boundary $>$ the same chain for B*. In particular, this localizes the problem of finding log terminal models.

Thus Reductions 6.4–5 are reduced to the existence of limiting extremal flips. It now remains to reduce the existence of limiting extremal flips to the existence of special flips. For this, consider an extremal limiting contraction f that is not special. Then immediately from the definition, we get that B either contains a component that is numerically negative relative to f, or in the case of reduced B the contraction f becomes special after discarding certain components that are nef relative to f. From this and from the extremal property of f it follows that in a neighborhood of the contracted curve there exists a reduced divisor H for which $B+H$ is LSEPD for f, with the relatively principal divisor of the form $dH + \sum d_i D_i$, where $\operatorname{Supp} B = \{D_i\}$. Moreover, we can assume that $K+B+\varepsilon H$ is log canonical and nef relative to f for some ε with $1 \geq \varepsilon > 0$. As H we can take a number of general elements of a very ample linear system relative to f. By the $\mathbb{Q}$-factorial assumption on X and Corollary 5.19, in a neighborhood of a contracted curve there exists a strictly log terminal extraction $g\colon Y \to X$ for $K+B+\varepsilon H$. By (1.5.7) this is log crepant, that is,

$$g^*(K+B+\varepsilon H)=K_Y+B_Y+\varepsilon g^{-1}H,$$

and it follows from this that $f\circ g$ is a strictly log terminal extraction of Z for $K_Z+f(B+\varepsilon H)$ (in the analytic case, over a neighborhood of W). But by (6.1.1) X has purely log terminal singularities outside $B+H$. Therefore, again by (1.5.7) the exceptional divisors of g lie over $B+H$ and $B_Y+\varepsilon g^{-1}H$ is LSEPD for $f\circ g$ with the components of $g^{-1}H$ having equal multiplicities in the principal divisor. Thus applying Corollary 4.6 as above with H the modified $g^{-1}H$ and $\varepsilon\gg\varepsilon_0>0$, we get the required flip as the log canonical model of f. In this *we need to construct either special flips or limiting flips with chains of distinct fractional multiplicities of the new boundary > the same chains for* B.

Now use the notation of Definition 6.1. By second termination 4.9 everything reduces to special flips for which the chain of fractional multiplicities is maximal, or equivalently, empty: $\{B\}=0$ and $S+B$ reduced. In the analytic case, on shrinking the neighborhood of the contracted curve E we can lose $\mathbb{Q}$-factoriality of X and extremality of f. We can avoid this unpleasantness when $S+B$ is LSEPD for f and the flip exists by Corollary 5.17. In the opposite case, discarding irreducible components of B that are numerically positive relative to f, we can assume that $S+B$ has an irreducible component that is numerically negative relative to f. Hence we only need to consider special flips for which all irreducible components of $S+B$ are numerically negative relative to f (compare the proof of Corollary 5.18). By Corollary 3.16, since we are in the 3-fold case, there are just the following two possibilities.

6.6. **Types of special flips.**

(6.6.1) $B=0$ and S is an irreducible surface negative relative to f.

(6.6.2) $S+B=S_1+S_2$ is the sum of two irreducible surfaces S_1 and S_2 negative relative to f.

Thus the assertions of reductions 6.4–5 reduce to the existence of special flips of these two types. For type (6.6.1), by (1.5.7), and since for suitable H and $0<\varepsilon<1$ as above both $K+S$ and $K+S+\varepsilon H$ are purely log terminal, the strictly log terminal extraction $g\colon Y\to X$ of neighborhoods of the contracted curve E is small and purely log terminal. It follows from this, and from Kawamata's result on the finiteness of $\sigma(X, E)$ (see [7], 1.12, and [19], 3.4), that there exists a small extraction $g\colon Y\to X$ for which Y is $\mathbb{Q}$-factorial for any analytic shrinking of the neighborhood of the curve $g^{-1}E$. Hence according to the preceding construction, the flips occurring in it satisfy the requirements of speciality for any shrinking of the neighborhood of the contracted curve, and in particular are extremal in the analytic sense in a neighborhood of the contracted curve, and this curve is irreducible.

For type (6.6.2), by Corollary 3.8 the curve $E=S_1\cap S_2$ is irreducible and S_1 and S_2 cross normally along it. Hence by (3.2.3) the restriction of $K+S$ to S_1 is purely log terminal and a strictly log terminal extraction g of any neighborhood of E for $K+S_1$ is the identity. In fact by (1.5.7) it is small and log crepant for $K+S$. Moreover, the exceptional curves over E land in the surfaces $g^{-1}S_1$ and $g^{-1}S_2$, which is not possible by the purely log terminal property of the restriction discussed above. Once g is the identity in a neighborhood of E, strict log terminal and extremal are preserved by shrinking to a neighborhood of E. Similar arguments could be carried out for analytic restrictions of any special flips, not only types (6.6.1–2). Q.E.D.

By Theorem 5.12, the special flips of (6.6.1) have 1-, 2-, 3-, 4- or 6-complement; and moreover, if a 1- or 2-complement does not exist, the 3-, 4- or 6-complement is exceptional. Such birational contractions and their flips will be called *special flips* (respectively *exceptional special flips*) of index $n = 1, 2, 3, 4$ or 6 if $K+S$ has an n-complement (respectively exceptional n-complement). For a special flip of index n the log divisor $K+0^+$ has index n and will be written in the form $K+S+B$, where $B = 0^+ - S$ is the n-complement. In distinction to these, the flips of types I–IV of §2 can be called the *basic* types; as we will establish, every log flip can be decomposed in terms of these, resolution of singularities and birational contractions. All the flips used up to now are of this kind; in the course of this, we have shown that the basic flips of types I–III are sufficient. The next result is a typical example of the construction of flips in terms of extractions and contractions.

6.7. **Proposition.** *Flips of type* (6.6.2) *exist.*

Proof. Since $K+S_1+S_2$ is strictly log terminal, by Corollary 3.8 the surfaces S_1 and S_2 are normal and cross normally along an irreducible curve C. By (3.2.3) the restrictions $K_{S_1}+(S_2)_{S_1}$ and $K_{S_2}+(S_1)_{S_2}$ are log terminal and exceptional on S_1 and S_2 respectively. More precisely, C is the unique reduced component of the boundaries $(S_1)_{S_2}$ and $(S_2)_{S_1}$. By the adjunction formula and the proof of (3.2.3) it follows that the restrictions of $K_{S_1}+(S_2)_{S_1}$ and $K_{S_2}+(S_1)_{S_2}$ to C coincide and determine a log terminal divisor $K_C+\sum p_iP_i$ with $p_i<1$, which is negative on C. In particular $C=\mathbb{P}^1$. By (3.2.2) and Proposition 3.9 the index of C at each point P_i is a natural number m_i such that $(m_i-1)/m_i \le p_i$. Therefore, there exists a natural number m depending only on the multiplicities p_i such that the index of C on S_1 divides m. For example, it's enough to take $m=n!$ where $(n-1)/n \ge p_i$ for all i. Since m is universal, the same holds for C on S_2. Hence the negative constants $(C^2)_{S_1}$, $(C^2)_{S_2}$ and their sum

$$\sigma_{12} = (S_1+S_2)\cdot C = (C^2)_{S_1} + (C^2)_{S_2}$$

are rational numbers with denominators dividing m.

I now claim that in a neighborhood of C there is a strictly log terminal extraction $g\colon Y \to X$ with a unique exceptional divisor S_3, and such that S_3 contracts to C and g is the standard blowup of the ideal I_C of C above the generic point of C. First of all, there is a projective extraction g which at the generic point of C is the blowup of the ideal I_C, and such that the exceptional divisors of g, together with the inverse images $g^{-1}S_1$ and $g^{-1}S_2$, are nonsingular and cross normally. Next, acting according to the philosophy of §1, we get a log terminal model of g for $K_Y+(S_1+S_2)_Y$. Since $\operatorname{Supp} g^*(S_1+S_2) \le (S_1+S_2)_Y$ and is LSEPD for g, we can apply termination 4.1. Flips exist by Corollaries 5.17 or 5.15 (when there is a component of $(S_1+S_2)_Y$ not lying over S_1+S_2 and negative on the flipping rays). Write S_3 for the exceptional surface over C; then since the general fiber of $S_3 \to C$ is a $\mathbb{P}^1$ meeting S_1 and S_2, that is $(K_Y+S_1+S_2+S_3)\mathbb{P}^1 = 0 \ge 0$, it follows that S_3 is not contracted back down to C during this process. It can't contract to a point since we're working over X.

By (1.5.7), g is log crepant, and by construction g contracts one irreducible surface S_3 to C. Since $K+S_1+S_2$ restricted to either of S_1 or S_2 is exceptional, the restriction of $K_Y+(S_1+S_2)_Y = g^*(K+S_1+S_2)$ to either of $g^{-1}S_1$ or $g^{-1}S_2$ is exceptional. Hence the surfaces $g^{-1}S_1$ and $g^{-1}S_2$ cut out on S_3 two disjoint irreducible curves C_1 and C_2 respectively that map isomorphically to C under g. Since Y is $\mathbb{Q}$-factorial, the curves C_1 and C_2 intersect only the irreducible components S_3 and S_1 or S_2 (the components of the reduced divisor $(S_1+S_2)_Y$).

Moreover, since $K + S_1 + S_2$ is divisorially log terminal, the boundary $(S_1 + S_2)_Y$ has no other components, that is, $(S_1 + S_2)_Y = g^{-1}S_1 + g^{-1}S_2 + S_3$. In addition, since

$$K_Y + (S_1 + S_2)_Y = K_Y + g^{-1}S_1 + g^{-1}S_2 + S_3$$

is strictly log terminal, the surfaces $g^{-1}S_1$, $g^{-1}S_2$, and S_3 are normal, and since X is $\mathbb{Q}$-factorial the contraction g is extremal. In particular, the surface S_3 is ruled relative to g. Hence the relative Picard number of the composite $f \circ g \colon Y \to X \to Z$ is equal to 2, where $f \colon X \to Z$ is the contraction of C. Therefore the Kleiman-Mori cone $\overline{\mathrm{NE}}(Y/Z)$ is spanned by two extremal (9) rays R_1 and R_2.

Suppose that $R_1 = \mathbb{R}^+[F]$ is spanned by a curve F contracted by g. By construction

$$K_Y + g^{-1}S_1 + g^{-1}S_2 + S_3 = g^*(K + S_1 + S_2)$$

is numerically nonpositive relative to $f \circ g$, 0 on R_1, and negative on some curve not contracted by g. Hence the other extremal ray R_2 is spanned by an irreducible curve $C' \subset S_3$ not contracted by g and negative for $K_Y + g^{-1}S_1 + g^{-1}S_2 + S_3$. If neither C_1 nor C_2 generates R_2 then there is a decomposition $C_1 \equiv aF + bC'$ up to numerical equivalence with positive rational numbers a and b. But this gives a contradiction:

$$0 = g^{-1}S_2 \cdot C_1 = ag^{-1}S_2 \cdot F + bg^{-1}S_2 \cdot C' > 0.$$

Thus for definiteness we can assume $C' = C_2$. Then by the preceding relation, either $a = 0$ and $g^{-1}S_2 \cdot C_2 = (C_2{}^2)_{S_3} = 0$ or $a > 0$ and $g^{-1}S_2 \cdot C_2 = (C_2{}^2)_{S_3} < 0$. In the first case the ray R_2 specifies a divisorial contraction $h \colon Y \to W$ of the surface S_3 to a curve (transversal to the contraction g). I claim that the modification $h \circ g^{-1} \colon X \to W$ is a flip of the contraction f. Indeed, by construction $h \circ g^{-1}$ is a small modification and $h \circ g^{-1}(S_2)$ is positive on the flipped curve $h(S_3)$.

The second case, when R_2 defines a small contraction, reduces to the first by the following arguments. The contraction g induces an isomorphism on $g^{-1}S_2$, under which, by the adjunction formula, the restriction of $K_C + \sum p_i P_i$ is identified with the restriction of the log divisor $K_Y + g^{-1}S_1 + g^{-1}S_2 + S_3$ on C_2. Hence by the argument of the first paragraph of the proof, the 3 negative rational numbers

$$S_3 \cdot C_2 = (C_2{}^2)_{g^{-1}S_2} = (C^2)_{S_2}, \qquad (C_2{}^2)_{S_3} = g^{-1}S_2 \cdot C_2,$$
$$\sigma_{23} = (g^{-1}S_2 + S_3) \cdot C_2 = g^*S_2 \cdot C_2 = S_2 \cdot C = (C^2)_{S_1} > \sigma_{12}$$

have denominators dividing m. Hence R_2 defines a small contraction of the original type with smaller invariant σ. Since the denominator is bounded we can use induction on σ. Thus we can suppose that the curve of R_2 can be flipped $h \colon Y \dashrightarrow W$. But the flip h does not destroy the curve C_1. Thus under this flip the negativity of the intersection of C_1 with $K_Y + g^{-1}S_1 + g^{-1}S_2 + S_3$ is preserved. Therefore by (1.12.4) the cone $\overline{\mathrm{NE}}(W/Z)$ is also generated by two extremal rays R_3 and R_4.

Suppose that R_3 is generated by the flipped curve C_3 (possibly reduced), which is the locus of indeterminacy of h^{-1}, and R_4 is generated by a curve $C_4 \subset h(S_3)$ that is negative against $K_W + (S_1 + S_2)_W$. Since $g^{-1}S_2 \cap S_3 = C_2$ and W is $\mathbb{Q}$-factorial, the surfaces $h(S_3)$ and $h \circ g^{-1}S_2$ can only intersect along the curve C_3. But this is also impossible by (1.12.1), Corollary 3.16 and the fact ([8], 5-1-11) that log discrepancies decrease over C_3. Thus $h \circ g^{-1}S_2$ is numerically 0 on $h(S_3)$. But $h \circ g^{-1}S_2 \cdot C_3 > 0$, more-or-less by definition of flip. Hence R_4 defines a divisorial contraction $l \colon W \to V$ of the surface $h(S_3)$ to a point, and $l \circ h \circ g^{-1}S_2$ is positive

(9) A priori, quasiextremal rays: see my commentary (10.8.2).

against the curve $l(C_3)$. It is not hard to check that the composite $l \circ h \circ g^{-1}: X \to V$ gives the flip of f. Q.E.D.

6.8. **Proposition.** *Index* 1 *special flips exist.*

Proof. By definition, in this case there is a complement of $K+S$, that is, a reduced divisor B such that $K+S+B$ is log canonical and linearly 0 in a neighborhood of the contracted curve. Since $K+S$ is negative on the contraction, B is positive. Now the contraction is extremal, S is negative and B positive on the contracted curve, and hence the boundary $S+B$ is LSEPD. Thus the required flip is of type II and exists by Proposition 2.7. Q.E.D.

Lemma 5.7 does not hold for arbitrary birational contractions. However, from it we deduce the following closely related fact.

6.9. **Theorem.** *Let $f: X \to Z$ be a contraction of a surface X such that the log divisor $K+B$ is numerically 0 relative to f. Then the locus of log canonical singularities in a neighborhood of any fiber is connected except for the following case: f is not birational, the locus of log canonical singularities of $K+B$ has two connected components, and $K+B$ is exceptional in a neighborhood of either of these.*

Proof. By Lemma 5.7, we can assume that f is not birational, that is, Z is a curve or a point. Let $g: Y \to X$ be a strictly log terminal extraction. Since X is normal and the fibers of f are connected, the fibers of $f \circ g$ are also connected, and for $P \in Z$ the loci of log canonical singularities $\mathrm{LCS}(K+B) \subset X$ and $\mathrm{LCS}(K_Y+B_Y) \subset Y$ have the same number of components in a neighborhood of corresponding fibers $f^{-1}P$ and $(f \circ g)^{-1}P$ for $P \in Z$. Since $K_Y + B_Y = g^*(K+B)$ is log crepant, it is numerically 0 relative to $f \circ g$. Thus we can assume that the original $K+B$ is log terminal. Then the number of connected components of $\mathrm{LCS}(K+B)$ equals the number of connected components of the reduced part of the boundary $D = \lfloor B \rfloor$.

It is enough to consider the case that $D \neq 0$ and D is not connected in a neighborhood of the fiber $f^{-1}P$. Then there is a curve that is exceptional for f and negative relative to $K + \{B\} = K + B - D$. By the theorem on the cone there is an extremal contraction $g: X \to Y$ over Z relative to which $K + \{B\}$ is negative. If g is birational then it contracts an irreducible curve, and, by Lemma 5.7, the locus of log canonical singularities of $K+B$ in a neighborhood of the curve is connected. Hence the number of connected components of the locus of log canonical singularities is the same for $K+B$ and $K_Y + g(B)$. All the assumptions of the proposition are preserved, and $K+B$ log terminal is replaced by $K+\{B\}$ log terminal, which is equivalent to saying that no connected component of $\mathrm{LCS}(K+B)$ is an isolated point. In view of this, we can suppose that after a number of such contractions we arrive at a contraction g to a curve or point.

Note that in this case, when Z is a curve, in the preceding contractions in a neighborhood of $f^{-1}P$ we may have contracted an irreducible curve Γ intersecting D but not contained in D. Obviously the ray generated by Γ is extremal, and contracting it preserves log terminal singularities. Therefore in this case, the final extremal contraction g is just f. Thus the reduced part of the original boundary B does not contain curves exceptional for f, that is, the log terminal extraction is the identity and $K+B$ is log terminal in a neighborhood of the fiber $f^{-1}P$. However, in counting the number of connected components of $\mathrm{LCS}(K+B)$ we can assume that f is extremal, that is, the fiber $f^{-1}P$ is an irreducible curve. The number of connected components of the locus of log canonical singularities in this case does not exceed the number of reduced components of the boundary $K+B$ over a general point of Z. From this we easily get what we want. The assertion that $K+B$ is exceptional follows from $K+B$ log terminal in a neighborhood of the fiber.

Now suppose that $Z = pt$. By construction, the number of connected components of $\operatorname{LCS}(K+B)$ equals the number of connected components of D. If g is a contraction to a point, then since it is extremal, X has Picard number 1 and D has only one connected component. If g is a contraction to a curve Y, then X has relative Picard number 1, that is, all fibers of g are irreducible. By construction D is positive against the general fiber of g. It follows that D is connected or consists of at most two connected components D_1 and D_2. In this case D_i are irreducible curves not contracted to points by g. By what we proved above for the contraction f to a curve, $K+B$ is log terminal in a neighborhood of each component D_i. The assertion that $K+B$ is exceptional near D_1 or D_2 follows from this. Q.E.D.

We strengthen Proposition 6.7 for a further reduction.

6.10. **Lemma.** *Suppose that $f\colon X \to Z$ is an extremal birational contraction with boundary $B = S_1 + S_2$ satisfying*

(i) *$K+B$ is log terminal outside B;*

(ii) *$K+B$ is negative against the contracted curves;*

(iii) *$K+S_1$ is log terminal;*

(iv) *the surfaces S_1 and S_2 intersect all contracted curves and are nonpositive on them.*

Then there exists a flip of f relative to any divisor.

Proof. Since f is extremal there is at most one nontrivial flip, namely the flip relative to $K+B$. We can assumed that the contracted curve is connected, and that each of the surfaces S_i is irreducible in a neighborhood of the contracted curve (compare the proof of reductions 6.4–5). According to Corollary 5.17 we can also suppose that one of the surfaces S_i is numerically negative relative to f. By Corollary 3.8, S_1 is a normal surface. The restriction $(K+B)_{|S_1}$ is negative and log canonical on the contracted curve lying on S_1, by (ii) and (iv). Suppose first that this restriction is not exceptional in a neighborhood of the contracted curve. Then it has a 1- or 2-complement, hence by Theorem 5.12 we deduce that the same holds for $K+\varepsilon B$ with $\varepsilon < 1$ and close to 1.

In the 1-complementary case this leads to a flip of type II. However, by (5.2.2), the 2-complementary case can only happen if the intersection of the surfaces S_i is an irreducible curve C, and it is the whole of the contracted fiber. Hence S_i is negative against C. Furthermore, C is the locus of log canonical singularities of the restriction in a neighborhood of this curve, and by Corollary 3.16 the further restriction $(K+B)_{|S_1|C^\nu} = K_{C^\nu} + \sum p_i P_i$ has one multiplicity p_i equal to 1. Since $K_{C^\nu} + \sum p_i P_i$ is negative, the curve C is rational and all the remaining $p_i < 1$. But then by Proposition 3.9 and Lemma 4.2, after renumbering if necessary, $p_1 = 1$ and $p_2 = (m-1)/m$ with $m \geq 1$, and all the remaining $p_i = 0$. Now using Theorem 5.12 it is not hard to construct a surface S_3 in a neighborhood of the point $\nu(P_2)$ passing through $\nu(P_2)$ and providing a 1-complement. Then on some analytic neighborhood of C the divisor $K+B+S_3$ is log canonical and numerically 0 relative to f. Thus the flip f is again of type II.

It remains to consider the case that the restriction $(K+B)_{|S_1}$ is exceptional. Then by (iv) the surfaces S_i again intersect in a unique irreducible curve C forming a fiber of f, and $(K+B)_{|S_1}$ is purely log terminal in a neighborhood of C. Both surfaces S_i are negative against C. The proof of the existence of the required flip is analogous to and based on Proposition 6.7. Indeed, if $K+B$ is log terminal along C then Proposition 5.13 implies that $K+B$ is log terminal in a neighborhood of the contracted curve, and hence the flip exists by Proposition 6.7. In the general case, by Corollary 5.19 we can use the existence of a strictly log terminal model $g: Y \to X$. By Proposition 5.13 again, the exceptional surfaces with log discrepancy 0 are contracted to C. Thus all the exceptional surfaces of g lie over C, and the existence of the flip can be proved by induction on the number of them. For this we find an extremal extraction g of C whose exceptional surface $E = g^{-1}C$ has log discrepancy 0, the log canonical divisor $K_Y + B_Y = g^*(K + B)$ can only have singularities that are not log terminal along the curve $E \cap f^{-1}S_2$, and in a neighborhood of this curve the restriction $(K_Y + B_Y)_{|E}$ is exceptional. This extraction is the final contraction obtained when constructing the model of X in a neighborhood of C, starting from a strictly log terminal model for $H = g^{-1}S_2$ and $\varepsilon = 1$. The flip occurring here is of type II, since the boundary B is LSEPD for g. By the assumption that the restriction $(K + B)_{S_1}$ is exceptional, the surfaces $g^{-1}S_1$ and $g^{-1}S_2$ are disjoint.

On the other hand, by construction, on subtracting $g^{-1}S_2$ from the boundary of the log divisor $K_Y + B_Y$ it becomes log terminal. Hence the singularities of $K_Y + B_Y$ that are not log terminal are contained in $g^{-1}S_2$. In particular, $K_Y + B_Y$ is log terminal in a neighborhood of $E \cap g^{-1}S_1$. Again by the exceptional assumption, the restrictions, $(K_Y + B_Y)_{|g^{-1}S_1}$ and $(K_Y + B_Y)_{|E}$ are exceptional in a neighborhood of $E \cap g^{-1}S_1$. But then, by Theorem 6.9 for the birational contraction g, the restriction $(K_Y + B_Y)_{|E}$ is exceptional in a neighborhood of $E \cap g^{-1}S_2$. From then on we can argue as in Proposition 6.7; in this process, we use Proposition 6.7 if the support of an extremal ray R with $(K_Y + B_Y)R < 0$ is $E \cap g^{-1}S_1$, or induction if it is $E \cap g^{-1}S_2$. Q.E.D.

6.11. **Lemma.** *Suppose that $f: X \to Z$ is an extremal birational contraction, and the boundary $S + B$ satisfies*

(i) *$K + S + B$ is log terminal outside $S + \lceil B \rceil$;*
(ii) *$K + S + B$ is numerically 0 on the contracted curves;*
(iii) *$K + S$ is log terminal;*
(iv) *the surface S is negative on the contracted curve;*
(v) *$S + B$ is LSEPD for f;*
(vi) *B has a reduced component that intersects all the contracted curves.*

Then there exists a flip of f with respect to any divisor.

Proof. If the reduced component S' in (vi) is nonpositive, then first discard the fractional components of B, which are not negative by Corollary 3.16 and the log canonical property of $K + S + B$ on the contracted curve; then by (v) we get the flip of Lemma 6.10. If S' is positive on the contracted curve, then the flip exists by Corollary 5.20. Q.E.D.

6.12. **Proposition.** *Special flips of type* (6.6.1) *exist if $K + S$ is n-complementary in such a way that the complemented log divisor $K + S + B$ has locus of log canonical singularities strictly bigger than S in a neighborhood of the contracted curve.*

Proof. Suppose that the locus of log canonical singularities of $K + S + B$ is bigger than S in a neighborhood of the contracted curve. If there is a surface S' of singularities outside S, then it must occur as a reduced component of B intersecting the contracted curves. Thus in this case the flip exists by Lemma 6.11. Note that $S + B$ is LSEPD, by (6.1.4) with $B = 0$ and by (6.6.1).

It remains to consider the case that there is only a curve of log canonical singularities outside S. Suppose that $g: Y \to X$ is a strictly log terminal extraction in a neighborhood of the flipping curve, which exists by Corollary 5.19. Define the multiplicities d_i from the relation $g^*B = g^{-1}B + \sum d_iD_i$. Then by log discrepancy, $g^*(K+S+B) = K_Y + (S+B)_Y$, and by assumption there exists an exceptional prime component $E = E_i$ with multiplicity 1 of the boundary $(S+B)_Y$ over the general point of the curve of log canonical singularities of $K+S+B$ outside S intersecting the flipping curve. Set $D = g^{-1}B + dE$ where $d = d_i$ is the multiplicity defined above. To construct an extremal extraction that blows up only the curve of log canonical singularities of $K+S+B$, we use Corollary 4.6 with $f = g$ and $H = \varepsilon D$ with a sufficiently small positive ε and $\varepsilon_0 = \varepsilon - 0$ very close to ε.

Note that the divisor g^*B is numerically 0 relative to g and is $\geq D$. Hence the supports of all flipping 0-contractions are contained in the reduced part of the boundary $(S+B)_Y$ and lie over S (this last, possibly after shrinking the neighborhood of the flipping curve). They exist by Lemma 6.11, since one of the reduced components of the boundary $\neq E$ and is negative by construction relative to the flipping contraction, and another reduced component is nonpositive, since S is $\mathbb{R}$-Cartier. By termination 4.1 and the connectedness of the fiber over the point of intersection with the curve of singularities, we eventually contract all the exceptional surfaces except for E. We again denote this birational contraction by g; it is extremal. Hence the relative Picard number of the composite $f \circ g$ equals 2, and the Kleiman-Mori cone $\overline{\mathrm{NE}}(Y/Z)$ has two extremal rays R_1 and R_2.

Suppose that the first of these R_1 corresponds to the contraction g. Now $D = g^*B = g^{-1}B + dE$ and this divisor is numerically 0 relative to g. But by (6.1.4) with $B = 0$ and by definition of complement, B is positive relative to f. Hence the second extremal ray R_2 is positive against D. Note that R_2 is a flipping ray, since the composite $f \circ g$ contracts only one surface E and a curve. More precisely, it lies in the fiber $(f \circ g)^{-1}P$, where P is the image of the curve contracted by f. Both this fiber and $f^{-1}P$ are connected. Therefore some component of the fiber C is not contained in E but intersects it, which implies that R_2 is positive relative to E. Since $S+B$ is LSEPD for f, also $g^{-1}S+D$ is LSEPD for the composite $f \circ g$, in the sense that $ag^{-1}S + bD$ is locally principal for $f \circ g$ for suitable a, b. Thus R_2 is negative relative to $g^{-1}S$ and the flip of R_2 exists by Lemma 6.11.

From this step on, we seek an extremal 0-contraction on which the modified D is relatively positive. Write R_1 for the new flipped ray; then the new ray R_2 remains negative against the modified $g^{-1}S$. (Here we use *new* and *modified* to avoid introducing more notation: *modified* divisors are birational transforms; the *new* (pseudo-) extremal ray R_1 is the flipped curve, whereas the *new* R_2 is the ray that arises automatically because $\rho = 2$, so $\overline{\mathrm{NE}}$ is a wedge in the plane; R_2 is not obtained from the old rays in any predictable way.) The flip again exists by Lemma 6.11 as long as the modified E is positive against R_2. If, however, after such flips we arrive at a case when $ER_2 = 0$, then E is numerically nonpositive on all curves contracted by $f \circ g$. Indeed, by the preceding flip, R_1 is negative against the modified E. On the other hand, the connectedness of the fiber $(f \circ g)^{-1}P$ is preserved by flips, and hence $(f \circ g)^{-1}P$ is entirely contained in E.([10]) Thus the flip again exists by Lemma 6.11, and it remains to consider the case that $ER_2 < 0$. This flip again exists by Lemma 6.11. After it, R_1 becomes positive against E; but then R_2 will again be negative against E and the flip again exists by Lemma 6.11. Indeed, E has a curve on Z with $ZE < 0$. Thus in conclusion we get that D is numerically nonpositive and nontrivial relative to the modification $f \circ g$. By Corollary 4.6 the corresponding log canonical model on subtracting D contracts E and gives the flip we want. Q.E.D.

([10]) $E \cdot (f \circ g)^{-1}P = 0$ and $E \cap (f \circ g)^{-1}P \neq \varnothing$, therefore $(f \circ g)^{-1}P \subset E$.

§7. Exceptional special flips

7.1. The set-up. In this section $f: X \to Z$ is a *special exceptional contraction* of index $n = 2, 3, 4$ or 6. This means that there exist a boundary B and an irreducible surface S on X such that

(7.1.1) $K+S+B$ is log canonical and $n(K+S+B)$ is linearly 0 in a neighborhood of the contracted curve;

(7.1.2) S is negative on the contracted curve;

(7.1.3) $K+S$ is purely and strictly log terminal;

(7.1.4) $K+S$ is negative on the contracted curve;

(7.1.5) the restriction $(K+S+B)_{|S}$ is exceptional in a neighborhood of the contracted fiber.

This final requirement is well defined, since the surface S is normal by Corollary 3.8 and (7.1.3). Recall (see the paragraph after Theorem 5.6) that by *exceptional* we mean that the restriction of $(K+S+B)_{|S}$ has at most one divisor with log discrepancy 0 in a neighborhood of the contracted fiber. ([11]) For a nonexceptional divisor, this is equivalent to saying it has multiplicity 1 in the boundary B_S. By (7.1.3) X is $\mathbb{Q}$-factorial. Such a contraction f is assumed to be extremal, and the contracted curve is connected.

According to Proposition 6.12, we can restrict to the case when the following holds:

(7.1.6) The log canonical singularities of $K+S+B$ are contained in S.

We have already noted above that B is positive on the contracted curve. By (7.1.1), $K+S+B$ has index n. Thus by (7.1.6) the multiplicities of the components of B are k/n for natural numbers $k<n$. If in addition B has a decomposition $B = B_1 + \cdots + B_t$ with effective divisors B_i for $1 \le i \le t$ which are nef on the contracted curve and intersect it, with multiplicities of the irreducible components B_i equal to k/n for natural numbers $k<n$, then t is called the *type* of this contraction, and of the corresponding flip. Since f is extremal, the B_i are either positive on the contracted curve, or numerically 0 on it, and by connectedness must then contain it. Thus each exceptional contraction and flip has type at least $t=1$. However, some flips may have higher type.

7.2. **Reduction.** *The existence of exceptional special flips of index* n *and type* t *follows from the existence of the same kind of flips in the case that* $K+S+B$ *is purely log terminal in a neighborhood of* S.

Thus in this case $K+S$ is strongly complementary (see (5.2.4)).

Proof. Let $g: Y \to X$ be a strictly log terminal extraction of X, which exists by Corollary 5.19. Since S is log crepant all the exceptional divisors E_i of g have log discrepancy 0, and by (7.1.6) and Corollary 3.16, $gE_i \subset S$. By connectedness of the exceptional set, (7.1.5) and Theorem 6.9 it follows that the exceptional irreducible surfaces E_i for $1 \le i \le m$ form a chain. We deduce that, numbering the components appropriately, the final component E_m intersects $g^{-1}S$ in an irreducible curve, and only intersects one exceptional surface E_{m-1} (when $m>1$, of course); and E_i does

([11]) Set $(K+S+B)_{|S} = K_S + A$. Then K_S+A is log canonical, and by the exceptional assumption, either K_S+A is purely log terminal and A has 1 reduced component, or $\lfloor A \rfloor = 0$ and there is at most one exceptional divisor with $a_i = 0$.

not intersect $g^{-1}S$ for $m > i > 1$, and intersects the two exceptional surfaces E_{i+1} and E_{i-1} in irreducible curves.

It is not hard to check that the original model X is obtained by successive modifications by 0-contractions for $H = g^{-1}B$ and $\varepsilon = 1 > \varepsilon_0$. I claim that for suitable choice of the extraction g, the sequence of transformations forms a chain of successive contractions of the divisors $E_1, \dots, E_m$ in that order; moreover, the same holds for all $Y_{(l)}$ in the chain from Y down to X. (All the surfaces of the chain $E_1, \dots, E_m$ are contracted to a point or to a curve, according as to whether the locus of log canonical singularities of the restriction $(K+S+B)_{|S}$ is a point or curve.) To verify this, consider an intermediate contraction g, and suppose that it has the same properties as the original X, replacing the strictly log terminal assumption on $g^*(K+S+B) = K_Y + g^{-1}S + B_Y$ by log canonical, and $K_Y + g^{-1}S + \sum E_i = K_Y + S_Y$ log terminal, since the modified $g^{-1}B$ are positive relative to the preceding 0-contractions.

We also suppose that there is a unique point or irreducible curve where $K_Y + g^{-1}S + B_Y$ is not log terminal, and it lies on the final exceptional component E_l, or on $E_{m+1} := S$ when there are no exceptional components left. More precisely, the surfaces $g^{-1}S$, $E_m, \dots, E_l$ form a chain, and there is a sequence of extractions of divisors $E_{l-1}, \dots, E_1$ that extend it to a strictly log terminal model. For $l = m+1$, since $Y = X$ is $\mathbb{Q}$-factorial, this gives the assertion we want. When $l \le m$, there exists a 0-contraction $h\colon Y \to W$ on which $g^{-1}B$ is positive. If h is divisorial and contracts a surface E_i with $l+1 \le i \le m$ from the middle of the chain, then E_i contains two disjoint curves $E_{i-1} \cap E_i$ and $E_i \cap E_{i+1}$.

Hence since h is extremal, in this case h is a contraction to a curve. Since the general fiber of h on E_i intersects $g^{-1}B$ positively, both curves of intersection are blown down to points. But then they have negative intersection with E_i, and intersection number 0 with E_{i-1}, E_{i+1}; this contradicts the fact that $g^{-1}S$, E_1, $\dots$, E_l supports an effective principal divisor locally over the $\mathbb{Q}$-factorial variety X. Hence h can only contract the extreme component E_l, which allows us to extend the proof by induction on l. If on the other hand h is a small contraction then again, since X is $\mathbb{Q}$-factorial and $g^{-1}B$ is positive relative to h, there is an exceptional divisor E_i that is negative relative to h. In particular, the curves contracted by h are contained in E_i. By the same argument, either a neighboring exceptional divisor $E_{i\pm1}$ or the divisor $g^{-1}S$ is positive on h. Hence by Lemma 5.7, it follows for the restriction $(K_Y + g^{-1}S + B_Y)_{|E_l}$ for $i = l$ that $K_Y + g^{-1}S + B_Y$ is log terminal in a neighborhood of the curve contracted by h. The flip of h exists by Corollary 5.15 or Corollary 5.20. Since the curves of intersection with neighboring components are not contracted by h, by properties 1.12 the flip preserves the log terminal property of $K_Y + g^{-1}S + B_Y$ in a neighborhood of the curve contracted by h. Here as before a point or curve at which $K_Y + g^{-1}S + B_Y$ is not log terminal is resolved by the extremal extractions of divisors $E_{l-1}, \dots, E_1$. Thus by termination 4.1 we finally get the case $l = m+1$, with $g = \mathrm{id}_X$.

Now we prove the existence of the flip f by induction on m. Note that for $m = 0$ the model g is the identity, $K+S+B$ is log terminal in a neighborhood of the contracted curve, and is purely log terminal by irreducibility of S; and f has a flip by the reduction assumption. For $m \ge 1$ consider the contraction g of the final surface E_m. Then apply Corollary 4.6 for the contraction $f \circ g$, $H = \varepsilon g^*B$ with small $\varepsilon > 0$ and $\varepsilon_0 < \varepsilon$ close to ε, that is, $\varepsilon_0 = \varepsilon - 0$. By construction, on subtracting H from $K_Y + g^{-1}S + B_Y$ it becomes purely log terminal, numerically nonpositive and nontrivial relative to $f \circ g$, so by (1.5.6) the corresponding log canonical model

$f \circ g$ is small. Moreover, by the same arguments, since the modified locus over Z is connected and H is numerically nonzero on it, the model of $f \circ g$ is also small.

From the fact that f is extremal and X is $\mathbb{Q}$-factorial, it's not hard to see that these models coincide with the flip of f. In particular, by properties 1.12 these models of $f \circ g$ are extremal over Z, strictly log terminal, and it is sufficient to construct the log terminal model. But $f \circ g$ has relative Picard number 2, and this is decreased by 1 under a divisorial contraction. In the final case we get a flip immediately, since the modified g^*B is positive relative to such a divisorial contraction.

Thus to construct the log terminal model of $f \circ g$ it is enough to be able to construct flips up to the first divisorial contraction or termination. On making flips we preserve the previous notation for Y, referring occasionally to the modified or flipped situation when this is meant, as explained at the end of §6. The Kleiman-Mori cone $\overline{\mathrm{NE}}(Y/Z)$ always has two pseudoextremal rays R_1 and R_2. Suppose that R_1 is the ray corresponding to the contraction g, and R_2 the subsequent ray that requires flipping. If R_2 has positive intersection with the modified g^*B, it corresponds to the next flip in the chain. If at some step $g^{-1}S$ is nonnegative against R_2, then also $E_m R_2 < 0$ and $g^{-1}BR_2 > 0$. Indeed, $S+B$ is LSEPD for f, that is there exist a, $b > 0$ such that $aS + bB$ is f^* of a principal divisor; in this sense $g^*S + g^*B$ is LSEPD for $f \circ g$ and for its modifications. By construction g^*B is numerically positive, and g^*S is negative on the new flipping ray. Hence we get the above assertion, that is, the supports of g^*S and g^*B are respectively $g^{-1}S + E_m$ and $g^{-1}B + E_m$.

Hence if at some step $g^{-1}S$ becomes positive against R_2, then the flip of R_2 exists by Corollary 5.20. After the flip, the flipped ray R_1 has negative intersection with $g^{-1}S$, so that $g^{-1}SR_2 > 0$. But the general curve on E_m over Z has nonnegative intersection with $g^{-1}S$, hence R_2 is nonnegative against $g^{-1}S$, and can only be numerically 0 against $g^{-1}S$ if R_2 is divisorial. Hence the flip again exists, and termination is guaranteed by termination 4.1 on discarding $g^{-1}B$. If at some step $g^{-1}SR_2 = 0$, then provided that the flip exists, the previous arguments give the existence of subsequent transformations and their termination. Usually exactly at this point the proof of the existence of the flip presents the essential difficulty, and is provided here by the reduction assumption. Thus modulo this flip, it remains to find flips which are negative for $g^{-1}S$, since their termination is guaranteed by termination 4.1 after discarding H. By the above, we also do not need to consider divisorial modifications of R_2.

Thus start with a ray R_2 on the initial Y which is positive relative to g^*B. By assumption, its contraction $h\colon Y \to W$ over Z is small and $g^{-1}S$ is negative relative to h, so that the curve contracted by h is contained in $g^{-1}S$. If moreover E_m is positive relative to h, then, because the curve on $g^{-1}S$ contracted by $f \circ g$ is connected, it follows that the curve contracted by h is not contained in the intersection $g^{-1}S \cap E_m$. In this case, the flip exists by Corollary 5.20 and preserves the log terminal property of $K_Y + g^{-1}S + B_Y$ in a neighborhood of $g^{-1}S$. It contracts a curve on $g^{-1}S$ outside the intersection $g^{-1}S \cap E_m$, and the flipped ray R_1 has negative intersection with the modified E_m.

As a result of such flips we arrive at the following dichotomy: either R_2 has intersection number 0 with the modified $g^{-1}S$, or it has negative intersection number with both the modified $g^{-1}S$ and E_m. In the second case, $E_m R_2 = 0$ is impossible by the exceptional assumption (7.1.5), and the flip exists by Lemma 6.11. Again by the exceptional assumption (7.1.5), a flip in R_2 performs a contraction of curves on

$g^{-1}S$ over Z, and the resulting modified surfaces $g^{-1}S$ and E_m are disjoint. In particular, the flipped ray R_1 has positive intersection with the modified surfaces $g^{-1}S$ and E_m, and the new ray R_2 has intersection number 0 with the modified $g^{-1}S$. Hence the next contraction is the divisorial contraction of the modified E_m to a point, which gives the flip f.

In the first case of the above dichotomy, $g^{-1}SR_2 = 0$, and as we already know, $E_mR_2 < 0$ and $g^{-1}BR_2 > 0$. We also assume that the corresponding contraction h is small. By the exceptional assumption (7.1.5), the restriction of the modified $(K_Y + g^{-1}S + B_Y)_{|E_m}$ is exceptional in a neighborhood of the modified $g^{-1}S$. It follows from this that the curve contracted by h does not intersect $g^{-1}S$. Because the current situation has been obtained by flips in rays intersecting E_m positively, the flipped curves lie on the modified E_m, and the modified $K_Y + g^{-1}S + B_Y$ has locus of log canonical singularities in a neighborhood of the curve contracted by h equal to the modified E_m. By Lemma 5.7, the singularities that are not log terminal are not spoilt by the preceding flips. Hence the contraction h, modulo connectedness of the fibers, is exceptional and of the same index n, with the modified E_m instead of S. And the possible point or curve of singularities that are not log terminal is resolved by fewer than m extremal extractions. Hence the flip exists by the inductive assumption.

Note that h also has type t, since the modified B_i in the decomposition $B = B_1 + \cdots + B_t$ corresponding to the given type t are positive on R_2; indeed, if B_i is positive on a contracted curve, then the modified g^*S and g^*B_i are LSEPD over Z (in the sense that $ag^*S + bg^*B_i$ is locally principal over Z), and the modified g^*S and E_m are negative on R_2. If B_i is numerically 0 on the contracted curve, then it also contains the modified g^*B_i that is numerically 0 on R_2 with a positive multiplicity of E_m, and hence since $E_mR_2 < 0$, we deduce that the modified $g^{-1}B_i$ is positive on R_2. We can carry out a localization to make the fibers connected, as in the proof of Reductions 6.4–5; here the index, the type and the log terminal property of $K + S + B$ are preserved. Q.E.D.

7.3. **Corollary.** *Index 2 exceptional special flips exist.*

Proof. By Reduction 7.2 we need only consider flips for which $K + S + B$ is purely log terminal, and these are flips of type IV from Proposition 2.9. Q.E.D.

7.4. **Proposition.** *Suppose that $K + S + B$ is purely log terminal. Then in a neighborhood of a contracted curve we have the following:*

(7.4.1) *For $n = 3$ either $K + S$ has a 1- or 2-complement, or $K + S + B$ has a 4-complement of type ≥ 2.*

(7.4.2) *For $n = 4$ either $K + S$ has a 1-, 2- or 3-complement, or $K + S + B$ has a 6-complement of type ≥ 2.*

(7.4.3) *For $n = 4$ and $t \geq 2$ either $K + S$ has a 1- or 2-complement, or $K + S + B$ has a 4-complement of type $\geq t + 1$.*

(7.4.4) *For $n = 4$ and $t \geq 4$, $K + S$ has a 1- or 2-complement.*

(7.4.5) *For $n = 6$, $K + S$ has a 1-, 2-, 3- or 4-complement.*

(7.4.6) *For $n = 6$ and $t \geq 2$ either $K + S$ has a 1-, 2- or 3-complement, or $K + S + B$ has a 6-complement of type $\geq t + 1$.*

(7.4.7) *For $n = 6$ and $t \geq 3$, $K + S$ has a 1-, 2- or 3-complement.*

Proof. The assumptions of Theorem 5.12 are satisfied. Hence by its proof it is enough to prove the corresponding assertion for the restrictions. For this we introduce the following changes to the notation: X will be a surface S, $f\colon X \to Z$ a contraction of a connected curve, $K+B'$ will denote the restriction $(K+S)_{|S}$ and $K+B$ the restriction $(K+S+B)_{|S}$. Note that $K+B$ has index n. By (3.2.3) and the assumption that $K+B$ is purely log terminal, all the b_i satisfy $0 \le b_i < 1$, and by Corollary 3.10 are of the form

$$(7.4.8) \qquad b_i = \frac{n_i - 1}{n_i} + \sum_j \frac{k_{ij}}{n_i} d_j ,$$

where n_i, k_{ij} and nd_j are natural numbers. Here d_i are our previous boundary multiplicities, and b_i are multiplicities of the boundary B. The corresponding multiplicities of B' are of the form

$$b_i' = \frac{n_i - 1}{n_i}.$$

Since $K+B$ has index n, all $b_i = k_i/n$, where $0 \le k_i \le n-1$. Let $g\colon Y \to X$ be an extraction over an exceptional curve for f. Then in a neighborhood of its inverse image the corresponding assertions means that $K_Y + B'^Y$ is m-complementary for $K+S$ and $K_Y + B^Y$ for $K+S+B$. Since $K+B$ is numerically 0 relative to f, by Lemma 5.4 we can restrict ourselves for $K+B$ to the case that f is the identity contraction, identifying X and Z, and the contracted curve of f is replaced by a point $P \in Z$. Since the original 3-fold is $\mathbb{Q}$-factorial and the contracted curve f has nonempty intersection with every component B_i, by definition of the type t we get a new interpretation of it: *there are at least t components of the boundary passing through P with nonzero product $k_{ij}d_j \neq 0$.*

The existence of the required complement gives the next result.

7.5. **Lemma.** *Under the preceding restrictions and notations, the following hold in a neighborhood of the inverse image of P:*

(7.5.1) *For $n=3$ either $K_Y + B'^Y$ has a 1- or 2-complement, or $K_Y + B^Y$ has a 4-complement.*

(7.5.2) *For $n=4$ either $K_Y + B'^Y$ has a 1-, 2- or 3-complement, or $K_Y + B^Y$ has a 6-complement.*

(7.5.3) *For $n=4$ and $t \ge 2$ either $K_Y + B'^Y$ has a 1- or 2-complement, or $K_Y + B^Y$ has a nontrivial 4-complement.*

(7.5.4) *For $n=4$ and $t \ge 4$, $K_Y + B'^Y$ has a 1- or 2-complement.*

(7.5.5) *For $n=6$, $K_Y + B'^Y$ has a 1-, 2-, 3- or 4-complement.*

(7.5.6) *For $n=6$ and $t \ge 2$ either $K_Y + B'^Y$ has a 1-, 2- or 3-complement, or $K_Y + B^Y$ has a nontrivial 6-complement.*

(7.5.7) *For $n=6$ and $t \ge 3$, $K_Y + B'^Y$ has a 1-, 2- or 3-complement.*

Proof. We use the arguments of the proof of Theorem 5.6. In the case of a log divisor $K+B$, for this we increase the boundary B to B'' in such a way that in a neighborhood of P the new divisor $K_Z + B''$ is actually log canonical. Note that the monotonicity $B' < B$ on the original X implies the monotonicity $B'^Y < B^Y < B''^Y$. In particular, for 1- or 2-complementary $K_Y + B''^Y$ by Lemma 5.3 the same holds for $K_Y + B'^Y$. Hence by the proof of Theorem 5.6 we can restrict to the case that $K_Z + B''$ is exceptional and the unique irreducible curve with log discrepancy 0 is exceptional.

Now for $K+B$, let $f: X \to Z$ be the blowup of this curve C. We identify the divisor B with its birational transform under f. By the fact that K_Z+B is purely log terminal, and K_Z+B'' log canonical and exceptional, the divisor $K+C+B$ is purely log terminal and negative relative to f. By Lemma 5.4, for the assertion about K_Y+B^Y we can restrict to extractions of g that factor through f. Again by the arguments of the proof of Theorem 5.6, to deduce that K_Y+B^Y is m-complementary it is sufficient to prove that its restriction

$$(K+C+B)_{|C}=K_C+\sum p_iP_i$$

is m-complementary. Since $K+C+B$ is purely log terminal, the curve C is nonsingular; and since the restriction of $K+C+B$ is negative, and the boundary $\sum p_iP_i$ is effective, $C=\mathbb{P}^1$, $0\le p_i<1$ and $\sum p_i<2$. In this set-up there is also defined a divisor $B'\le B$ in which all curves except C appear with multiplicities $b_i'=(n_i-1)/n_i$ and C with multiplicity 0. By monotonicity (1.3.3), $K+B'+C$ is purely log terminal, and hence by Corollary 3.10 and the proof of Lemma 4.2 it follows that

$$(K+C+B')_{|C}=K_C+\sum q_iP_i$$

with $q_i=(m_i-1)/m_i\le p_i$, where $m_i=l_in_i$, l_i is the index of $K+C$ in P_i, and not more that one curve of the boundary B' with multiplicity $b_i'>0$ passes through P_i. However, for the original boundary B', if the restriction $K_C+\sum q_iP_i$ is m-complementary it does not follow that $K_Y+B'^Y$ is m-complementary, since because $K+B'$ is negative relative to f for the original boundary one can reduce the new boundary B'^Y on some components. To cure this, replace first X for $K+B'$ by the minimal resolution X' of X for B, that is, we blow up on X the unique curve C if it is exceptional for X, and take $X'=X$ otherwise; write B' for the previous B' outside C. We hope that there will not be too much confusion in what follows caused by one notation X for two different surfaces, depending on the log divisor.

Now the boundary B' is the image of the boundary B' on X'. Similarly on X' there is defined a boundary $B\ge B'$ outside C with image B on X. By construction the log divisor $K_{X'}+C+B$ is exceptional, nef and numerically nontrivial on each connected fiber of X'/X. Hence by monotonicity $K_{X'}+C+B'$ is exceptional, but possibly positive on some curves of X'/X. Also by construction, on X' there is a boundary $B''>B>B'$ such that $K_{X'}+C+B''$ is exceptional and numerically nonpositive over X. I claim that B' on X' can be increased so that $K_{X'}+C+B'$ will be numerically nonpositive on X with multiplicities $>b_i'$ only for curves that are numerically 0 over X, and is exceptional as before. As a first approximation it is enough to increase B' to B'' for curves of X' over X. In doing this some multiplicities $>b_i'$ may occur for curves that are not numerically 0 over X. But these multiplicities can be decreased, preserving the exceptional property and the numerical nonpositivity of $K_{X'}+C+B'$ over X. Hence the minimal boundary $\ge B'$ with the final properties gives what we want.

Note that in the case $K_Y+B'^Y$, by Lemmas 5.3–5.4 it is enough to find the required m-complements of $K_Y+(C+B')^Y$ for resolutions over X'. Moreover, we can after contracting curves that are numerically trivial over X for $K_{X'}+C+B'$ assume that B' does not increase, that is, has the same multiplicities b_i', and that $K_{X'}+C+B'$ is numerically negative over X. Hence the existence of the required m-complements follows from the m-complements of the boundary

$$(K_{X'}+C+B')_{|C}=K_C+\sum q_i'P_i$$

with $q'_i = (m'_i - 1)/m'_i \geq q_i$, where $m'_i = l'_i n'_i$, l'_i is the index of $K_{X'} + C$ at P_i, and not more that one curve of the boundary B' with multiplicity $b'_i > 0$ passes through P_i. Here the curves C on X and X' are identified, and the equality $q'_i = q_i$ is only possible if there is no curve of X'/X over P_i. By construction $K_{X'} + C + B$ is exceptional, nef and numerically nontrivial on each connected fiber of X'/X. Hence $p_i \geq q'_i \geq q_i$ and the first equality is only possible if there are no curve of X'/X over P_i and $p_i = q'_i = q_i$, that is, there are no components of the boundary B through P_i. Choose the numbering such that $q'_1 \geq q'_2 \geq \cdots$, or equivalently $m'_1 \geq m'_2 \geq \cdots$. By the above, $0 \leq q_i \leq q'_i \leq p_i < 1$ and $\sum q_i \leq \sum q'_i \leq \sum p_i < 2$. Suppose now that $K_Y + B'^Y$ and hence also $K_C + \sum q'_i P_i$ does not have a 1- or 2-complement. Then by (5.2.1) $3 \leq m'_1 \leq 5$, $m'_2 = 3$, $m'_3 = 2$ and $m'_i = 1$ for $i \geq 4$. Hence

$$\sum_{i \geq 4} p_i < 2 - p_1 - p_2 - p_3 \leq 2 - q'_1 - q'_2 - q'_3 \leq 2 - 2 \times \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.$$

But for $i \geq 4$ we have $l'_i = n'_i = 1$ for all curves through P_i and by (7.4.8)

$$b_{i'} = \sum_j k_{i'j} d_j \quad \text{satisfies} \quad \begin{cases} \text{either} & b_{i'} = 0 \\ \text{or} & b_{i'} \geq \frac{1}{n} \geq \frac{1}{6}, \end{cases}$$

and hence $p_i = q_i = q'_i = 0$ for $i \geq 4$. Hence for $i \geq 4$ there are no curves of X'/X over P_i, and both the boundaries B and B' on X and X' can only intersect C in the points P_1, P_2, P_3. If there is a curve of X'/X over P_3 then $p_3 > q'_3 = 1/2 > q_3$, hence $m_3 = 1$, and by (7.4.8) $p_3 = k/n \geq 2/3$ and $\sum p_i \geq q'_1 + q'_2 + p_3 \geq 3 \times (2/3) = 2$, which contradicts the inequality $\sum p_i < 2$. Thus the curves of X'/X can only lie over P_1 or P_2. Consider now the case that there are just no such curves, that is, $X' = X$, all $q'_i = q_i$ and $m'_i = m_i$. Then, again by (7.4.8) and Corollary 3.10,

$$p_1 = q_1 + \frac{1}{l_1} \sum b_i = \frac{m_1 - 1}{m_1} + \frac{k_1}{m_1 n},$$

where the sum consists of multiplicities of the boundary B in a neighborhood of P_1, possibly with repetitions, and $k_1 \geq 0$. In a similar way we get integers k_2, $k_3 \geq 0$ such that

$$p_2 = q_2 + \frac{k_2}{3n} = \frac{2}{3} + \frac{k_2}{3n} \quad \text{and} \quad p_3 = q_3 + \frac{k_3}{2n} = \frac{1}{2} + \frac{k_3}{2n}.$$

Here, by the above interpretation of type, $k_1 + k_2 + k_3 \geq t$. Hence and from preceding inequalities one easily finds the possible nontrivial values of p_i and q_i in terms of n and t (see Table 1). Together with (5.2.1), this completes the proof of the current case. For example, for $n = 3$, according to the first two lines, $K_Y + B^Y$ has a 4-complement, and by the last 5 lines $K_Y + B'^Y$ has a 3-complement.

We consider below only the new cases, when $X' \neq X$. There are not so many of these and they give what we want. Using the previous arguments and relations it is not hard to show that for $n = 3$ there is just one new case $p_1 = 7/9$, $p_2 = 2/3$, $p_3 = 1/2$ with $t = 1$ and $m'_1 = 4$, $m_1 = 3$, $m'_2 = m_2 = 3$, $m'_3 = m_3 = 2$. For $n = 4$ the additional cases are listed in Table 2.

Here $p_3 = q_3 = q'_3 = 1/2$ and we assume that $m_1 \geq m_2$ when $m'_1 = m'_2 = 3$. For $n = 6$ we can restrict to the case that $K_Y + B'^Y$, hence also $K_C + \sum q'_i P_i$ are not 1-, 2- or 3-complementary, that is, $m'_1 \geq 4$. Here there is just one new case: $p_1 = 7/9$, $p_2 = 2/3$, $p_3 = 1/2$ with $1 \leq t \leq 2$ and $m'_1 = 4$, $m_1 = 3$, $m'_2 = m_2 = 3$, $m'_3 = m_3 = 2$. Q.E.D.

TABLE 1

n	t	q_1	q_2	q_3	p_1	p_2	p_3
3	1	2/3	2/3	1/2	7/9	2/3	1/2
		2/3	2/3	1/2	2/3	7/9	1/2
4	1	2/3	2/3	1/2	3/4	2/3	1/2
		2/3	2/3	1/2	2/3	3/4	1/2
		2/3	2/3	1/2	2/3	2/3	5/8
		3/4	2/3	1/2	13/16	2/3	1/2
6	1	2/3	2/3	1/2	13/18	2/3	1/2
		2/3	2/3	1/2	2/3	13/18	1/2
		2/3	2/3	1/2	2/3	2/3	7/12
		2/3	2/3	1/2	13/18	13/18	1/2
		2/3	2/3	1/2	2/3	7/9	1/2
		2/3	2/3	1/2	7/9	2/3	1/2
		2/3	2/3	1/2	13/18	2/3	7/12
		2/3	2/3	1/2	2/3	13/18	7/12
		3/4	2/3	1/2	19/24	2/3	1/2
		3/4	2/3	1/2	3/4	13/18	1/2
6	2	2/3	2/3	1/2	13/18	13/18	1/2
		2/3	2/3	1/2	7/9	2/3	1/2
		2/3	2/3	1/2	2/3	7/9	1/2
		2/3	2/3	1/2	13/18	2/3	7/12
		2/3	2/3	1/2	2/3	13/18	7/12

TABLE 2

$\leq t \leq$		m'_1	m_1	m'_2	m_2	p_1	p_2	p_3
1	3	3	3	3	1	2/3	3/4	1/2
1	2	3	3	3	2	2/3	3/4	1/2
1	1	5	4	3	3	13/16	2/3	1/2

Conclusion of proof of Proposition 7.4. It remains to verify the assertions concerning types. For example, for $n = 3$, a 4-complement for $K + S + B$ has type ≥ 2. Indeed, we can take $B_1 = (3/4)B$, since checking coefficient by coefficient shows that $B_1 \leq (1/4)\lfloor 5B \rfloor$. Similarly for $n = 4$ we can take $B_1 = (2/3)B$ as part of a 6-complement. For $n = 4$, a nontrivial 4-complement of type t for $K + S + B$ has type $\geq t + 1$; nontrivial means that it has $B^+ > B$. Hence $B_{t+1} = B^+ - B \neq 0$ and is numerically 0 relative to f, intersects the exceptional locus of f and extends the decomposition by types to B. The arguments for $n = 6$ are similar. Q.E.D.

7.6. **Reduction.** *The existence of exceptional special flips follows from the existence of nonexceptional index* 2 *special flips.*

Proof. Reduction 7.2 reduces the construction of exceptional special flips to similar flips with purely log terminal complement $K + S + B$ of the same index and type. From there by Proposition 7.4 we can either decrease the index n, or increase the index n and at the same time increase t. This finally reduces the problem to the construction of special flips of index 1 or 2. For example from the case $n = 3$ increasing the index can only lead to $n = 4$ and $t \geq 2$, and from this case can only lead to $n = 1$ or 2 with $t \geq 4$. Special flips of index 1 exist by Proposition 6.8, and exceptional index 2 special flips by Corollary 7.3. Q.E.D.

§8. Index 2 special flips

8.1. **The set-up.** In this section $f\colon X \to Z$ is a *special nonexceptional contraction of index* 2. This means that there exist a boundary B and an irreducible surface S on X such that

(8.1.1) $K + S + B$ is log canonical and $2(K + S + B)$ is linearly 0 in a neighborhood of the contracted curve;

(8.1.2) S is negative on the contracted curve;

(8.1.3) $K + S$ is purely and strictly log terminal;

(8.1.4) $K + S$ is negative on the contracted curve;

(8.1.5) the restriction $(K + S + B)_{|S}$ is not exceptional in a neighborhood of the contracted fiber.

According to Proposition 6.12, we can assume that the following holds:

(8.1.6) The locus of log canonical singularities of $K+S+B$ is contained in S.

In particular, after shrinking the neighborhood of the contracted curve if necessary, the irreducible components of B have multiplicities 1/2 or 0. By assumption (8.1.3), X is $\mathbb{Q}$-factorial. We also suppose that f is extremal, and the contracted curve is connected. In the analytic case, all of this holds in a neighborhood of the flipping curve, that is, with $W = pt.$, the image of the flipping curve, and hence the flipping curve is irreducible.

8.2. **Reduction.** *We can assume that there is exactly one irreducible curve C not contracted by f, with multiplicity* 1 *in the boundary of the restricted log divisor $(K + S + B)_{|S}$, and every connected component of* Supp$(B_{|S})$ *outside C and intersecting the locus of log canonical singularities of $(K + S + B)_{|S}$ is contracted to a point by f.*

Proof. Suppose first that there is at least one irreducible curve C not contracted by f with multiplicity 1 in the boundary of the log divisor $(K+S+B)_{|S}$. If B intersects S in a curve $\neq C$ having multiplicity 1 in the boundary of $(K+S+B)_{|S}$ and not contracted by f, then, since the locus of log canonical singularities of $(K+S+B)_{|S}$ is connected and by (8.1.3), we can change the boundary B_S outside C such that it remains $\geq 0_S$, and $K_S + B_S$ becomes log terminal and numerically negative relative to $f_{|S}$. Then by Corollary 5.11, $g^*(K + S)_{|S}$ will have a 1-complement on any extraction g. Hence by the proof of Theorem 5.12, $K+S$ has a 1-complement and the flip of f exists by Proposition 6.8. In a similar way one proves that there exist a 1-complement of $K + S$ and a flip of f when there is a connected component of Supp$(B_{|S})$ not contained in C intersecting the locus of log canonical singularities of $(K + S + B)_{|S}$ and not exceptional relative to f. Hence it remains to carry out the reduction in the case that *all curves with multiplicity* 1 *in the boundary of the log divisor $(K + S + B)_{|S}$ are contracted by f.*

In the analytic set-up, the above arguments prove the existence of the required complement of $K+S$ in a neighborhood of a flipping ray, since B is positive on the flipping curve, and therefore cuts out on S a connected component which is not contracted by f, and intersects the locus of log canonical singularities of $(K+S+B)_{|S}$. In the algebraic set-up, the locus of log canonical singularities $LCS((K+S+B)_{|S})$ forms a chain of curves on the log terminal resolution, and the required complement B can be found on one end of this chain, but this complement might behave badly on the other. In this case we must reduce to other flips.

Consider a strictly log terminal extraction $g: Y \to X$, which exists in a neighborhood of the flipping curve by Corollary 5.19. Since g is log crepant, by Corollary 3.16 there is an exceptional prime divisor E for f such that $f \circ g(E) = pt$. By (8.1.6) the multiplicity d of E in g^*S is positive. Set $H = \varepsilon(g^{-1}S + dE)$ for small $\varepsilon > 0$. Since the exceptional divisors for g together with $g^{-1}S$ are LSEPD for g, we can apply Corollary 4.6 to get a new extraction g with a single exceptional divisor E. The required flip is of type III, since $K_Y + g^{-1}S + B_Y$ is log terminal outside the reduced part of the boundary, which equals $\operatorname{Supp} g^*S$, and is LSEPD for g. From this and by construction it follows that the support of the modified rays are contained in the exceptional divisors other than E. Hence the modifications terminate.

Thus for the new extraction $(K_Y + g^{-1}S + E)_{|E^\nu}$ is numerically 0 and contains the curve of intersection $C_1 = g^{-1}S \cap E$ in the locus of log canonical singularities. The latter is connected by Theorem 6.9 and from the nonexceptional assumption (8.1.5). Note that by construction $K_Y + g^{-1}S$ is log terminal. But $\rho(Y/Z) = 2$ and $\overline{\mathrm{NE}}(Y/Z)$ has two extremal rays R_1 and R_2. We now apply the arguments of the proof of reduction 7.2. Suppose that R_1 corresponds to the contraction g. Obviously $g^{-1}BR_1 > 0$. From now on, we need to consider modifications of 0-contractions for $H = \varepsilon g^*B$ corresponding to R_2; we need only consider flipping rays R_2 with $g^{-1}SR_2 \leq 0$.

Assume first that $g^{-1}SR_2 < 0$. If $ER_2 > 0$ then the flip of R_2 exists by Corollary 5.20. Moreover, it preserves the log terminal property of $K_Y + g^{-1}S$. If $ER_2 \leq 0$ then $g^{-1}BR_2 > 0$ and the flip exists by Lemma 6.10; it again preserves the log terminal property of $K_Y + g^{-1}S$. If the curve of intersection C_1 is lost as a result of such flips then the surfaces $g^{-1}S$ and E no longer meet, and we get a flip by contracting E to a point, as in reduction 7.2.

Finally, if $g^{-1}SR_2 = 0$ then $ER_2 < 0$ and $g^{-1}BR_2 > 0$. In particular, the support of R_2 is contained in E. We can assume that R_2 is a flipping ray. If one of the connected components G of $\operatorname{Supp} R_2$ intersects $g^{-1}S$, then it is contained in $g^{-1}S$, and the flip in it exists by Lemma 6.10 with $S_1 = g^{-1}S$ and $S_2 = E$. In the opposite case G is disjoint from $g^{-1}S$, and hence from C_1. For the remaining connected components we can replace E by S and $g^{-1}B$ by B. Then assumptions (8.1.1–2) and (8.1.4) will hold. (In the analytic case, after passing to a neighborhood of the component in question we may lose the extremal assumption. Then we must construct a flip with respect to $-B$.) By Theorem 6.9 on the normalization S^ν there is a curve B' (possibly reduced) such that every connected component of the locus of log canonical singularities of $(K+S+B)_{|S^\nu}$ intersects B' and no component of B' is contracted by f. Also by construction $K+S+B$ is log terminal outside the boundary $S+B$.

Suppose that the flipping component is contracted to a point P. Then on any weakly log canonical model of f the locus of log canonical singularities of $K +$

$S+B$ is connected, even over an analytic neighborhood of P. Of course, it always contains the modified S. Since S is $\mathbb{Q}$-Cartier, connectedness holds for any strictly log terminal model of f. Moreover, any two surfaces with log discrepancy 0 are extracted on some strictly log terminal model of f. Hence the locus of log canonical singularities of $g^*(K+S+B)$ is connected for any resolution g over a neighborhood of P, which implies what we want.

Thus it remains to carry out a reduction in the case of a flip of the component under consideration. For this, we do a strictly log terminal extraction $g\colon Y \to X$ for $K+S$. This, as well as all its modifications considered in what follows, is a weakly log canonical model of f. The flip of f can be obtained as a result of modifications of 0-contractions of $f \circ g$ with $H = g^{-1}B$—of course, with successive contraction of curves on which $g^{-1}B$ is numerically 0. This is possible, since $K+S+B$ is log terminal outside the boundary $S+B$, which is LSEPD for $f \circ g$. I claim that the flips required for this satisfy the reduction (modulo flips that already exist).

Since $S+B$ is LSEPD for $f \circ g$ and the support of a 0-flipping extremal ray R is positive against $g^{-1}B$, it is negative against $S' = g^{-1}S$ or against a surface E that is exceptional for $f \circ g$. We localize to connected components C' of $\operatorname{Supp} R$ exactly as in the proof of reductions 6.4–5. If C' intersects another similar component S_Y then the flip exists by Lemma 6.11. In the opposite case $C' \subset S'$, and does not intersect any other component S_Y. Hence this is an index 2 special flip, and it exists if it is exceptional. We can thus restrict attention to the case that it is not exceptional; in other words, that $(K_Y + g^{-1}S + g^{-1}B)_{|S'}$ is not exceptional in a neighborhood of C'. In the case $S' = g^{-1}S = S_Y$ and $(f \circ g)^{-1}P \subset g^{-1}S = S'$ over a neighborhood of P, since the extraction $g^{-1}S \to S^\nu$ is numerically 0 with respect to $(K_Y + g^{-1}S + g^{-1}B)_{|S}$, the required curve C either connects C' and the birational inverse image of $(\nu^{-1}f)B'$, or is equal to the birational inverse image of $(\nu^{-1}f)B'$. In the opposite case, by connectedness of the locus of log canonical singularities of $K_Y + g^{-1}S + g^{-1}B$ over P and Theorem 6.9, we get the required curve C, or the $\operatorname{LCS}(K_Y + g^{-1}S + g^{-1}B) \neq S'$ in a neighborhood of the flipping curve C'. In the final case the flip exists by Proposition 6.12. Q.E.D.

In the preprint of this paper, the following result was incorrectly stated as the nonexistence of certain rays (or of the corresponding configurations of flipping contractions). But, as pointed out by Kollár, these flipping contractions actually exist.

8.3. **Proposition.** *Let $f\colon X \to Z$ be an extremal contraction of an irreducible curve C, and $B = S_1 + S_2$, where S_1, S_2 are surfaces and $C \subset S_1 \cap S_2$ is the exceptional curve for f. Assume that*

(i) *S_1 and S_2 cross normally along C;*
(ii) *f is special of type* (6.6.1) *with $S = S_1$;*
(iii) *$(K + S_1 + S_2)_{|S_1|C}$ is not purely log terminal on C at one point P.*
(iv) *$K + S_1 + S_2$ is numerically 0 in a neighborhood of the contracted curve.*

Then the flip $X \dashrightarrow X^+$ of C exists, and has the following properties. $C \subset S_1$ is contracted to a nonsingular point Q of a normal surface S_1^+. Moreover, the normalization of S_2^+ is nonsingular and single-branched over Q. The flipped curve C^+ is irreducible, and the surface S_2^+ is nonnormal along it. The normalization $S_2^{+\nu} \to S_2^+$ defines a double cover $C^* \to C^+$, of which Q is one branch point. The singularities of X^+ along C^+ are canonical of type A_n, and S_2^+ is Cartier. The intersection $C'^+ = S_1^+ \cap S_2^+$ is normal along C'^+ in a neighborhood of Q and

$(K_{X^+} + S_1^+ + S_2^+)_{|S_2^{+\nu}} = K_{S_2^{+\nu}} + C^* + C'^+$; *the curves* C^* *and* C'^+ *intersect normally at* Q.

Proof. The flip exists by Proposition 2.7. We first describe the properties of f. By (ii) and Corollary 3.8, S_1 is normal and irreducible. Taking an analytic neighborhood of C and replacing f by the contraction of C only, we preserve all the above assumptions except for the $\mathbb{Q}$-factoriality of X. However, S_1 and S_2 remain $\mathbb{Q}$-Cartier. By our assumption it follows that $S_2C > 0$. Hence by (iii), possibly after shrinking the neighborhood of C, the intersection $S_1 \cap S_2$ consists of 2 nonsingular irreducible curves C and C' intersecting in P. Hence by (iii), Corollary 3.10, Lemma 4.2 and our assumption, $C = \mathbb{P}^1$, and

(8.3.1) $$(K + S_1 + S_2)_{|S_1|C} = K_{\mathbb{P}^1} + \tfrac{1}{2}P_1 + \tfrac{1}{2}P_2 + P$$

whereas

$$S_{2|S_1} = C + cC',$$

with $0 < c \leq 1$ by (i) and (3.2.2). Note that $((C + cC') \cdot C)_{S_1} = S_2C > 0$. Let $g\colon T \to S_1$ be a minimal resolution of singularities in a neighborhood of C. Suppose first that the points P_1 and P_2 are singular on S_1. By Corollary 3.10 they are nodes and

$$g^*(C + cC') = g^{-1}C + cg^{-1}C' + \tfrac{1}{2}E^1 + \tfrac{1}{2}E^2 + \sum e_iE_i,$$

where E^1 and E^2 are exceptional curves over P_1 and P_2 respectively, $E_1, \dots, E_n$ is a chain of exceptional curves lying over P, and E_1 intersects $g^{-1}C$. By (3.18.6) $0 < e_1 \leq 1$, hence

$$\begin{aligned} 0 < (C + cC') \cdot C &= (g^*(C + cC')) \cdot g^{-1}C \\ &= (g^{-1}C)^2 + \tfrac{1}{2} + \tfrac{1}{2} + \begin{cases} e_1 & \text{if } n \geq 1, \\ c & \text{if } n = 0 \end{cases} \\ &\leq (g^{-1}C)^2 + 2, \end{aligned}$$

and $(g^{-1}C)^2 \geq -1$. Therefore, since $g^{-1}C$ is contractible, it is a (-1)-curve, that is $(g^{-1}C)^2 = -1$. But then the curve $E^1 \cup E^2 \cup g^{-1}C$ is not contractible. Therefore at least one of the points P_i, say P_1, is nonsingular. Then there is a unique irreducible curve C'' with multiplicity $1/2$ in the boundary $(S_2)_{S_1}$ that passes transversally through P_1. In a similar way one checks that if P_2 is nonsingular then $(g^{-1}C)^2 \geq 0$, and this again contradicts the contractibility of C. Hence P_2 is singular. Hence arguing as before,

$$g^*(C + cC') = g^{-1}C + cg^{-1}C' + \tfrac{1}{2}E^2 + \sum e_iE_i,$$

where E^2 is an exceptional curve of P_2, and $E_1, \dots, E_n$ a chain of exceptional curves over P; and E_1 intersects $g^{-1}C$, hence

$$\begin{aligned} 0 < (C + cC') \cdot C &= (g^*(C + cC')) \cdot g^{-1}C \\ &= (g^{-1}C)^2 + \tfrac{1}{2} + \begin{cases} e_1 & \text{if } n \geq 1, \\ c & \text{if } n = 0 \end{cases} \\ &\leq (g^{-1}C)^2 + \tfrac{3}{2}, \end{aligned}$$

and $(g^{-1}C)^2 > -(3/2)$. Hence by contractibility $g^{-1}C$ is again a (-1)-curve, and

$$\tfrac{1}{2} < \begin{cases} e_1 & \text{if } n \geq 1, \\ c & \text{if } n = 0. \end{cases}$$

Thus if $n = 0$ we get that P is nonsingular on S_1, by Corollary 3.10 that $c = 1$, and the curves C and C' cross normally at P. If $n \geq 1$ then by construction

$(E_i)^2 = -m_i$ with $m_i \geq 2$. Since $E^2 \cup E_1 \cup g^{-1}C$ is not contractible, $m_1 \geq 3$. By (3.18.7) and the inequality $e_1 > 1/2$, we get $m_1 = 3$, $m_2 = \cdots = m_n = 2$ and $c > 1/2$, hence by Corollary 3.10 again $c = 1$, that is, in either case S_1 and S_2 cross normally along C'. Furthermore $f_{|S_1}: S_1 \to S$ contracts C to a nonsingular point $Q \in S$ and $K_S + (1/2)f_{|S_1}(C'')$ is canonical at Q. This means that all the log discrepancies of $K_S + (1/2)f_{|S_1}(C'')$ over Q are ≥ 1 (so that their discrepancies are ≥ 0). The corresponding terminal extraction is obtained from T after contracting the curves E^2 and $g^{-1}C$, with C'' mapping to a nonsingular curve and for $n \geq 1$ the curve E_1 maps to a (-1)-curve having simple tangency with C''; for $n = 0$ the image of C' has simple tangency with C''.

Now we describe the properties of the flip. I claim first that it defines a contraction of C on S_1. By (8.3.1) there is a curve C'' with multiplicity $1/2$ in the boundary $(S_2)_{S_1}$ passing transversally through P_1. Hence, since S_1 and S_2 cross normally along C and C' in a neighborhood of C,

$$(K + S_1)_{|S_1} = K_{S_1} + \tfrac{1}{2}C''$$

is log terminal and negative on C. Hence in a neighborhood of the transformed curves $C_1, \ldots, C_m$ that land on S_1^+,

$$(K_{X^+} + S_1^+)_{|S_1^+} = K_{S_1^+} + \tfrac{1}{2}C''^+ + \sum c_i C_i$$

and is positive, where by the effectiveness (3.2.2) all the $c_i \geq 0$. But by the above the curves C_i contract to a nonsingular point Q on S, at which $K_S + (1/2)f_{|S_1}(C'')$ is canonical. This is only possible for $m = 0$.

Hence $S_1^+ = S$. It follows from this that there do not exist finite covers $\pi: V \to U$ of degree $l \geq 2$, where U is an irreducible neighborhood of Q, V is irreducible, and π is ramified only along curves not lying on S and passing through Q. We also assume that all these properties are preserved on restricting π over irreducible analytic neighborhoods of Q. Indeed, according to Corollary 2.2, $\pi^*(K_U + S) = K_V + \pi^{-1}S$ is purely log terminal, and hence by Lemma 3.6 $\pi^{-1}S$ is normal and the induced finite cover $\pi_{|\pi^{-1}S}: \bigcup D_i \to S$ is unramified outside Q. Hence since $Q \in S$ is nonsingular, π is unramified over Q, which contradicts the possibility of analytic restriction π while preserving the irreducibility of V (compare Corollary 3.7).

Now note that S_2^+ is an irreducible surface, and the normalization $\nu: S_2^{+\nu} \to S_2^+$ is one-to-one over Q. For otherwise there exists an analytic neighborhood of Q in which S_2^+ has components through Q. This is not possible, since a $\mathbb{Q}$-Cartier divisor S intersects each of these components along a curve through Q, and these curves are distinct because $K + S + S_2^+$ is log canonical. However $S \cap S_2^+ = C'^+$ is an irreducible curve in a neighborhood of Q. Thus the point Q can be identified with $\nu^{-1}Q$, and C'^+ with $\nu^{-1}C'^+$. The restriction $(K_{X^+} + S + S_2^+)_{|S_2^{+\nu}}$ has at most two irreducible curves with multiplicity 1 in the boundary $S_{S_2^{+\nu}}$ through Q. Suppose there are just two such curves, C'^+ and some other curve C^*. (We will prove later that C^* is the same curve as in Proposition 8.3.) Then in a neighborhood of Q

$$(K_{X^+} + S_2^+)_{|S_2^{+\nu}} = K_{S_2^{+\nu}} + C^*,$$

and is log terminal.

On the other hand, $K_{X^+} + S_2^+$ has index 1 at all points of a punctured neighborhood of $Q \in S$, so that, by what we have just seen, it has index 1 at Q. Therefore $K_{S_2^{+\nu}} + C^*$ has index 1 at Q, and by (3.9.2), $S_2^{+\nu}$ is a nonsingular surface at Q.

Since S has index 2 along C''^+, it defines a double cover $\pi\colon V \to U$ in a neighborhood of Q, ramified only in C''^+. Hence after shrinking the analytic neighborhood of Q, $\pi^{-1}S_2^+$ consists of two irreducible components each of which has nonsingular normalization. Since $K_V + \pi^{-1}S$ is purely log terminal it has a small strictly log terminal extraction $q\colon W \to V$ with connected fiber M over Q, otherwise as before we get a contradiction from π unramified over Q. But this is impossible if the two components of $\pi^*S_2^+$ intersect in at most points. Indeed, then $(\pi \circ q)^*S_2^+ = (\pi \circ q)^{-1}S_2^+$ is numerically 0 on M, and after shrinking to an analytic neighborhood of Q, it consists of two irreducible components which do not intersect even along M, because $K_V + \pi^{-1}S + \pi^{-1}S_2^+$ is log canonical. Hence S_2^+ is nonnormal along $\nu(C^*)$ and by the above X^+ has a singularity along $\nu(C^*)$ of the required type. The irreducibility of the flipped curve C^+ and the fact that it coincides with $\nu(C^*)$ is easily deduced from the fact that all its components pass through Q and are contained in S_2^+; indeed, suppose that C' is a component of C^+ with $C' \not\supset \nu(C^*)$. Then on the normalization $S_2^{+\nu}$ it is an exceptional curve passing through $Q = C^* \cap C'^+$, and numerically 0 against $(K_{X^+} + S + S_2^+)_{|S_2^{+\nu}} = K_{S_2^{+\nu}} + C^* + C'^+$.

We now suppose that there is no C^*, and derive a contradiction. Then in a neighborhood of Q

$$(K_{X^+} + S_2^+)_{|S_2^{+\nu}} = K_{S_2^{+\nu}}$$

is log terminal and of index 1. Therefore $S_2^{+\nu}$ is a normal surface and $Q \in S_2^{+\nu}$ is a canonical singularity. On the other hand $(K_{X^+} + S + S_2^+)_{|S_2^{+\nu}}$ is log canonical and equal to $K_{S_2^{+\nu}} + C'^+$ in a neighborhood of Q, but not log terminal at Q. Then by classification, Q is a Du Val singularity of type D_n. By (ii) and the original assumption each flipped curve C^+ is negative for S_2^+, and hence contained in S_2^+. By (ii) again, C^+ passes through Q, and by the original assumption the divisor $(K_{X^+} + S + S_2^+)_{|S_2^{+\nu}}$ is numerically 0 against C^+. Hence C^+ is a (-1)-curve on the minimal resolution of S_2^+. But then a multiple of C^+ moves on S_2^+, which contradicts $f^+\colon X^+ \to Z$ small. Q.E.D.

The following standard result is useful in simplifying somewhat the induction in the sequel.

8.4. **Reduction.** *At the expense of passing to the analytic case, we can assume that the flipping curve is irreducible.*

In the reverse direction, we can try to return to the algebraic case by algebraic approximation of the contraction and its polarization, afterwards resolving the singularities that are not log canonical and not $\mathbb{Q}$-factorial outside the flipped curve; however, algebraic approximation is probably not always possible.

Proof. Combine the arguments of the end of the proof of Reductions 6.4–5 and Reduction 8.2. By Definition 6.1 one can restrict to an algebraic situation and shrink to an analytic neighborhood. Q.E.D.

8.5. Classification of rays. We classify rays according to two tests: is $K + S + B$ log terminal along the curve contracted by f? and is the contracted curve contained in B (more precisely, in $\operatorname{Supp} B$)? By (8.1.3) and (8.1.6) negative answers to both tests are not possible. Hence the possible cases are as follows:

(8.5.1) $K + S + B$ is purely log terminal along the curve contracted by f, and B does not contain it;

(8.5.2) $K+S+B$ is not log terminal along the curve contracted by f, and B contains it;

(8.5.3) $K+S+B$ is purely log terminal along the curve contracted by f, and B contains it.

The reason for the chosen order will be clear from the reductions of the sequel. Of course, by Reduction 8.2, in each of the indicated cases it is assumed that *there exists exactly one irreducible curve* C *not contracted by* f *with multiplicity* 1 *in the boundary of the log divisor* $(K+S+B)_{|S}$ *intersecting the contracted curve; each connected component of* $\mathrm{Supp}(B_{|S})$ *outside* C *meeting the locus of log canonical singularities of* $(K+S+B)_{|S}$ *is exceptional*; *and in cases* (8.5.1–2), *the curve contracted by* f *is irreducible.*

In what follows we successively reduce case (8.5.1) to (8.5.2–3) and exceptional index 2 flips, (8.5.2) to (8.5.3) and exceptional index 2 flips, and (8.5.3) to exceptional index 2 flips. However, the contracted curve in (8.5.3) is possibly reducible, and $K+S+B$ log terminal along it means log terminal at the general point of each irreducible component. (It is not hard to check that the contracted curve in (8.5.3) has at most two irreducible components.)

Our general strategy is to choose a good extraction in the sense of the following definition. A *good extraction* g is an extremal extraction $g\colon Y\to X$ having irreducible exceptional divisor E, and having the following properties:

(i) $g^*(K+S+B)=K_Y+g^{-1}S+g^{-1}B+E$, that is, g is log crepant;

(ii) $K_Y+g^{-1}S+E$ is log terminal;

(iii) $B_1=g^{-1}S\cap E=\mathbb{P}^1$ is an irreducible curve, and $g^{-1}S$ and E cross normally along it;

(iv) we have

$$(K_Y+g^{-1}S+g^{-1}B+E)_{|g^{-1}S|B_1}=(K_Y+g^{-1}S+g^{-1}B+E)_{|E|B_1}$$
$$=K_{\mathbb{P}^1}+\tfrac{1}{2}P_1+\tfrac{1}{2}P_2+P,$$

where P is the unique point on B_1 at which $K_Y+g^{-1}S+g^{-1}B+E$ is not log terminal.

Note that by Corollary 3.8, (iii) follows from (ii) and (8.1.3), although they are often proved in the opposite order.

The existence of a good extraction will be provided in cases (8.5.1–2) by Propositions 8.6 and 8.8, and in case (8.5.3) by Proposition 8.8. After this, as in the second half of the proof of Reduction 7.2, we apply Corollary 4.6 to $f\circ g$ with $H=\varepsilon g^*B$. The inductive or reduction step is realized for an extremal ray R_2 that is numerically 0 against the modified $g^{-1}S$. Here the base of the contraction changes by a divisorial extraction, a modification of the current good extraction. In the analytic case, which by Reduction 8.4 is now the main case of interest for us, the current W is replaced by its inverse image; here the original W is a point, the image of the flipping curve. The notions of f extremal and X strictly $\mathbb{Q}$-factorial are assumed over such a W. (In the analytic casé, f extremal and X $\mathbb{Q}$-factorial are not preserved in general on shrinking the neighborhood of the contracted fiber.) As for the rest, the speciality assumptions (8.1.1–6) will hold.

Note also that W will always be projective and contained in the reduced part of the boundary, since this holds for the good extraction and is preserved on subsequent modifications, because the flipping ray is positive against E. Hence in a neighborhood of W there exists a strictly log terminal extraction of $K+S+B$ as in Corollary 5.19. However, in the boundary of the log divisor $K+S+B$ and its restrictions, we usually only write the components in a neighborhood of the new flipping curve. In doing so, Lemmas 8.9–10 will allow us to remain within the framework of cases (8.5.1–3). But we can't avoid allowing the contracted curve to be reducible in (8.5.3). In the overall strategy, the given reduction of index 2 special flips to exceptional flips is carried out at the end of §8, and completes the proof of Theorems 1.9–10 and Corollary 1.11.

Since we are not in the exceptional case, in case (8.5.1) the curve contracted by f intersects C in a singular point Q_1 that is not log terminal for $K+S+B$ or for $(K+S+B)_{|S}$ in a neighborhood of the contracted curve; by Theorem 6.9, Q_1 is the unique point in a neighborhood of the contracted curve where $(K+S+B)_{|S}$ fails to be log terminal. On the other hand, by (iii) and (iv), g cannot be an extraction of a curve C. Thus g is an extraction of a point Q_1. In this case, we say that a good extraction g is an *end extraction* if P is the unique possible point on E where $(K_Y+g^{-1}S+g^{-1}B+E)_{|E}$ is not log terminal. In the opposite case, by Theorem 6.9 and the assumption that f is not exceptional, the reduced part of the boundary of $(K_Y+g^{-1}S+g^{-1}B+E)_{|E}$ is of the form B_1+B_2, where B_2 is a curve intersecting B_1 only in P and containing a point $Q_2 \neq P$ where $(K_Y+g^{-1}S+g^{-1}B+E)_{|E}$ is not log terminal. By the extremal property of g the divisors $g^{-1}S$ and B_1 are ample on E. Hence B_2 is irreducible and Q_2 is the unique point of E where $(K_Y+g^{-1}S+g^{-1}B+E)_{|E}$ is not log terminal, except possibly for P. We say that a good extraction of Q_2 is a *middle extraction*; a finite chain of successive extractions ending in an end extraction is *stopped*. It is convenient to subdivide case (8.5.2) in two:

(8.5.2), *unstarred.* As in (8.5.2) above, and on the curve contracted by f there is a point $Q_1 \notin C$ that is not log terminal for $(K+S+B)_{|S}$.

(8.5.2*) The opposite case.

It is not hard to check that, in case (8.5.2), unstarred, a good extraction g blows up Q_1. As before, it is an *end extraction* if P is the unique possible point of E where $(K_Y+g^{-1}S+g^{-1}B+E)_{|E}$ is not log terminal. A *middle extraction* and a *stopped chain of extractions* are defined similarly.

8.6. **Proposition.** *In cases* (8.5.1–2) *there exists a stopped chain of good extractions.*

Proof. By Corollary 5.19 there exists a strictly log terminal extraction over a neighborhood of W (see the remark after Proposition 8.8). We now apply Corollary 4.6 to g with $H=g^{-1}B$; by (8.1.3) we get the original model X as a result of modifications of 0-contractions. I claim that the final modification g that gives back the neighborhood of the point Q_1 is a good extraction of Q_1. Suppose first that this modification was a flip. (For the first step, this is not possible a priori, since we start with some extraction $Y=Y_0\to X$, and in the chain, $Y_i \dashrightarrow Y_{i+1}$ is specified by a ray of $\overline{\mathrm{NE}}(Y_i/X)$, so each $Y_i\to X$ is a morphism. But the point of the argument is that it also works inductively for Q_2, etc.) By construction $g^{-1}B$ is negative on the flipped curve C', and g^*B is numerically 0, so there is an exceptional divisor E for g with $EC'>0$. Moreover, C' is not contained on any divisor exceptional for g, and is hence nonnegative against either of them. But C' is numerically trivial against g^*S. Hence C' is negative against the divisor $g^{-1}S$ and lies on it. By construction $K_Y+g^{-1}S$ is log terminal, and hence the surface $g^{-1}S$ is normal. Since $EC'>0$, the final divisor cuts out a curve of log canonical singularities of $(K_Y+g^{-1}S+B_Y)_{|g^{-1}S}$ and Q_1 is also contained in the locus of log canonical singularities of this divisor. Thus by Theorem 6.9 C' will be a curve of log canonical singularities of $(K_Y + g^{-1}S + B_Y)_{|g^{-1}S}$. But then because $K_Y + g^{-1}S + E$ was log terminal before the flip, it follows that $g^{-1}B$, hence also B, cuts out locally in a neighborhood of Q_1 more than the locus of log canonical singularities of $(K + S + B)_{|S}$, which is not possible by definition of cases (8.5.1–2).

Thus the final modification, giving the extraction of Q_1, is the contraction of the divisor E. (ii) and (iii) hold by construction, and (i) holds locally in a neighborhood of Q_1. Hence by (8.1.5) and (3.2.3) we get that E is contracted to the point Q_1. Now it is not difficult to check (iv). By the above, if this extraction is not an end extraction, the boundary of $g^*(K + S + B)_{|E}$ has two intersecting irreducible components $B_1 = g^{-1}S \cap E$ and B_2. Now on B_2 there exists a unique point Q_2 outside B_1 where $g^*(K + S + B)_{|E}$ is not log terminal. Since in a neighborhood of Q_1 the support of B intersects S only in the curve of log canonical singularities of $(K + S + B)_{|S}$, it follows that $g^{-1}B$ intersects B_1 only in P. Thus in a neighborhood of Q_2 the support of $g^{-1}B$ intersects E only in B_2. Hence the final modification g giving the neighborhood of Q_2 is again a good extraction of Q_2. This process terminates because the number of our modifications is finite. Q.E.D.

In cases (8.5.2*) and (8.5.3) it is convenient to define a natural invariant δ and then establish the existence of a good extraction by decreasing this invariant. We start in a slightly more general set-up. Let $Q \in S$ be a point and suppose that the locus of log canonical singularities of $K + S + B$ is contained in S, and $K + S$ is log terminal in a neighborhood of Q. For each exceptional divisor E_i, define the *multiplicity* d_i of E_i in S by

$$g^*(S) = g^{-1}S + \sum d_i E_i,$$

where E_i is exceptional for the contraction $g\colon Y \to X$. Obviously d_i does not depend on the choice of g.

8.7. **Lemma.** *In a neighborhood of Q the set of exceptional divisors E_i with log discrepancy $a_i = 0$ and multiplicity $d_i \le 1$ in S is finite.*

Here "in a neighborhood of Q" means that the birational transform of E_i passes through Q.

Proof. It follows at once from the definition of log discrepancy that the distinguished exceptional divisors E_i have log discrepancy ≤ 1 for $K + B$. Thus it is enough to prove that the set of exceptional divisor with log discrepancy ≤ 1 (that is, discrepancy ≤ 0) is finite. But by assumption $K + B$ is purely log terminal, so that it follows that all log discrepancies are $\ge \varepsilon$ for some positive ε. From then on one argues as in [25], (1.1). Q.E.D.

Now define δ by

$$\delta = \#\{E_i \mid a_i = 0 \text{ and } d_i \le 1\},$$

where we consider only exceptional divisors in a neighborhood of Q, that is, over Q or over an irreducible curve of log canonical singularities of $K + S + B$ through Q. Returning to our set-up, we take Q in (8.5.2*) to be a general point of the contracted curve, and in (8.5.3) Q to be the unique point on the contracted curve that is not log terminal for $K + S + B$.

8.8. **Proposition–Reduction.** *In the two cases* (8.5.2*) *and* (8.5.3), *either the flip itself exists, or we get a good extraction as follows.*

Case (8.5.2*). *There exists a good extraction* g *of the curve contracted by* f *such that* either *the image of the exceptional divisor* E *passes through* Q, $a = 0$ *and* $d \leq 1$, or *the restriction* $(K_Y + g^{-1}S + g^{-1}B + E)_{|E}$ *is purely log terminal outside* B_1 *over* Q (*where* Q *is the generic point of the curve contracted by* f).

Case (8.5.3). *There exists a good extraction* g *of* Q *such that* either $a = 0$ *and* $d \leq 1$, or *the restriction* $(K_Y + g^{-1}S + g^{-1}B + E)_{|E}$ *is purely log terminal outside* B_1.

The heading Proposition–Reduction means that we aim throughout the proof either to construct a good extraction with the stated property, or to prove that the flip exists for some other reason.

As already remarked above, in the analytic case we assume that f is extremal and $\mathbb{Q}$-factorial with respect to a projective analytic subspace $W \subset S + \lfloor B \rfloor$; then (8.1.6) holds in a neighborhood of the flipping curve. Hence by Corollary 5.19 there is a strictly log terminal extraction of a neighborhood of W for $K + S + B$.

Proof. We start with the case (8.5.2*). Here I claim first that there exists an exceptional divisor E over a curve contracted by f with $a = 0$ and $d \leq 1$. Taking a general hyperplane section, we reduce the problem to the 2-dimensional situation. Let Q be a surface singularity, at which $K + S + B$ is log canonical but not log terminal, where S is a curve and the support of B passes through Q. Then over Q there is an exceptional curve E with $a_i = 0$ and $d_i \leq 1$. Using Lemma 3.6, it is not hard to check that S is irreducible and nonsingular in a neighborhood of Q. Consider the log terminal extraction $g\colon Y \to X$ of a neighborhood of Q for $K+S+B$. The exceptional curves E_i over Q are numerically 0 against the divisor $g^*(K + S + B) = K_Y + g^{-1}S + g^{-1}B + \sum E_i$. Moreover, from the fact that $\bigcup E_i$ is connected, it follows that the curves $E_i = \mathbb{P}^1$ are nonsingular, rational and together with $g^{-1}S$ form a chain $E_1, \dots E_n, g^{-1}S$. If one of the curves E_i for $i \geq 2$ is a (-1)-curve then we can contract it and again get a log terminal extraction of Q. Hence we can suppose that g is minimal, in the sense that $E_i^2 \leq -2$ for $i \geq 2$. Then $a_i = 0$ and $d_i \leq 1$ for every exceptional curve E_i with $i \geq 2$ (compare Lemma 3.18).

Furthermore, the required surface E with $a = 0$ and $d \leq 1$ always exists, except in the case that the surfaces S and $\operatorname{Supp} B$ are nonsingular and simply tangent along Q. But in this case $n = 1$ and we take $E = E_1$. Then the restriction $(K_Y + g^{-1}S + g^{-1}B + E)_{|E}$ is purely log terminal outside B_1 over Q, of course viewed as the general point. The good extraction property in the proposition for it will be established below, assuming that g is extremal.

In the case when there exists E with $a = 0$ and $d \leq 1$ we apply Corollary 4.6 with $H = E$; using the fact that g^*S is LSEPD, we can modify g to be an extremal contraction of E. It remains to check that it is good. $K_Y + g^{-1}S$ is log terminal by construction. Since on the curve contracted by f (in its intersection with C), $(K + S + B)_{|S}$ has a unique singularity that is not log terminal, and E is contracted to this curve, its birational transform gives a curve $B_1 \subset g^{-1}S \cap E$ with $B_1 = \mathbb{P}^1$ and

$$(K_Y + g^{-1}S + g^{-1}B + E)_{|g^{-1}S|B_1} = K_{\mathbb{P}^1} + \tfrac{1}{2}P_1 + \tfrac{1}{2}P_2 + P.$$

This is property (iv) of a good extraction in 8.5. (i) holds obviously, and (ii) will hold if we take E_n as above. Since f and g are extremal, $\rho(Y/Z) = 2$ and $\overline{\text{NE}}(Y/Z)$ has two extremal rays. As usual, suppose that R_1 is the ray corresponding to the contraction g; then $g^{-1}BR_1 > 0$. If the curve $B_1 \neq g^{-1}S \cap E$ then by

(8.5.2*) $g^{-1}B$ is disjoint from it. Hence $g^{-1}BR_2 \leq 0$, and therefore $ER_2 > 0$ and $g^{-1}SR_2 < 0$. But then $g^{-1}BR_2 = 0$, and now R_2 is a flipping ray whose support contains B_1.

Thus $g^{-1}S \cap E = \bigcup_{i=1}^{n} B_i$, where $B_1, \dots, B_n, g^{-1}C$ is a chain of curves on $g^{-1}S$. Moreover, the curves B_i with $i \geq 2$ are contracted by g to a point. Hence their intersection with $g^{-1}B$ is positive, and therefore $n = 2$. Thus $g^{-1}S \cap E = B_1 \cup B_2$, where B_1 and B_2 are irreducible. Note that $P = B_1 \cap B_2$ is not log terminal for $K_Y + g^{-1}S + E$. Let us check that there is no other such point in a neighborhood of E. Note that the semiampleness of $g^{-1}B$ on E is important for this: $g^{-1}B$ is numerically 0 against B_1, and positive against all the other curves of E. Indeed by Theorem 6.9 and the fact that

$$(K_Y + g^{-1}S + g^{-1}B + E)_{|E^\nu}$$

is not purely log terminal at $Q = B_2 \cap g^{-1}C$, the locus of log canonical singularities is connected. Here it is a chain of curves $\nu^{-1}B_1, C_1, \dots, C_n, \nu^{-1}B_2, \dots, B_m$, with $g^{-1}B_{|E^\nu} = D + \sum_{i \geq 3} b_i B_i$, where $\operatorname{Supp} D$ is outside the locus of log canonical singularities and $b_i > 0$. The final assertion follows from the connectedness of $g^{-1}B_{|E^\nu}$ by semiampleness and the fact that D does not intersect C_i and $\nu^{-1}B_2, \dots, B_{m-1}$ for $m \geq 3$ (we write $B_2 = \nu^{-1}B_2$ if $m = 3$), since the restriction in question is log canonical. But $g^{-1}B$ is numerically 0 on $\nu^{-1}B_1$ only. Hence $n = 0$ and there are no curves C_i. Thus by Proposition 5.13, the points at which $K_Y + g^{-1}S + g^{-1}B + E$ is not purely log terminal on E are contained in the support of $g^{-1}S + g^{-1}B$, and this gives what we want. It also follows from this that E is normal.

The support of R_2 equals B_1. The flip in B_1 exists and is described in Proposition 8.3. After the flip, $K_Y^+ + g^{-1}S^+ + E^+$ fails to be log terminal only along the flipped curve $B_1^+ = \nu(C^*)$. Now the intersection $g^{-1}S^+ \cap E^+ = B_2^+$ is irreducible, and we can argue as in Reduction 8.2. Then the nontrivial case is the flip in a ray that is numerically 0 against $g^{-1}S^+$, negative against E^+ and positive against $g^{-1}B^+$. Thus B_2^+ has positive selfintersection on $E^{+\nu}$. Note that by Proposition 8.3, the normalization map $E^{+\nu} \to E^+$ is one-to-one over B_2^+, and we can hence identify B_2^+ with its inverse image in $E^{+\nu}$. Again by Proposition 8.3, on B_2^+, $E^{+\nu}$ can only be singular at $Q^+ = B_2^+ \cap g^{-1}C^+$. Hence B_2^+ is a curve with selfintersection ≥ 0 on the minimal resolution of singularities of $E^{+\nu}$, and selfintersection ≥ 1 in the case that $E^{+\nu}$ is nonsingular on B_2^+. But

$$(1) \qquad (K_{Y^+} + g^{-1}S^+ + g^{-1}B^+ + E^+)_{|E^{+\nu}}$$

has $C^* \cup B_2^+$ as its locus of log canonical singularities in a neighborhood of C^*.

Hence E is obtained from $E^{+\nu}$ by the following procedure. Start by performing on $E^{+\nu}$ a minimal strictly log terminal extraction of the restricted log divisor (1). As a result of this, we get a chain $B_1 = C_1, \dots, C_m, C^*, B_2^+$ in a neighborhood of C^*. Here by *minimal* we mean that C_i with $i \geq 2$ are not (-1)-curves. Then we contract the curves C_i with $i \geq 2$ and C^*. Hence B_2, just as B_2^+, is a curve with selfintersection ≥ 0 on the minimal resolution of E, and selfintersection ≥ 1 if E is nonsingular at Q in B_2. But such a curve cannot be contained in a fiber of the ruling determined on E by g, which is a contradiction.

We now proceed to case (8.5.3). We first assume that there exists an exceptional divisor E_i over Q or over C with $a_i = 0$ and $d_i \leq 1$. Flipping log terminal extractions for $K + S + B$ and using Corollary 4.6 with $H = \varepsilon(\sum d_i E_i)$, where the sum runs over $d_i \leq 1$, and using the fact that g^*S is LSEPD, we get an extraction

$g: Y \to X$ which pulls out all the E_i with a_i and $d_i \leq 1$, and no other exceptional divisor. All, since by Corollary 3.8 all exceptional divisors with log discrepancy 0 over a log terminal extraction of Y lie over the normal crossings of components of the reduced part of the boundary of $g^{-1}S + \sum E_i$, and by arguments used in the proof of Proposition 6.7, it is not hard to carry out addition subextractions for which the E_i with $a_i = 0$ and $d_i \leq 1$ are not exceptional.

By construction

$$g^*(K+S+B) = K_Y + g^{-1}S + g^{-1}B + \sum E_i,$$

$K_Y + g^{-1}S$ is log terminal and $g^{-1}S$ is a normal surface. From this also, by Theorem 6.9, the intersection $g^{-1}S \cap \bigcup E_i$ is a chain of irreducible curves $B_1, \dots, B_n$, where B_n is the birational transform of C. Setting $H = g^{-1}S$, we get the original model without contracting any curves outside $\bigcup E_i$. Hence S is obtained from $g^{-1}S$ by contracting $B_i = \mathbb{P}^1$ with $i \leq n-1$.

(8.8.1) *I claim that*

$$(K_Y + g^{-1}S + g^{-1}B + \sum E_i)_{|g^{-1}S|B_1} = K_{\mathbb{P}^1} + \tfrac{1}{2}P_1 + \tfrac{1}{2}P_2 + P_0$$

where $P_0 = B_1 \cap B_2$.

For this, assume the contrary. Then by definition of case (8.5.3), B_1 has a point P lying on $g^{-1}B$ at which $(K+X+B)_{|S}$ is not log terminal. Suppose that B_1 (and possibly something else) is cut out by $E = E_1$. Then none of the other E_i pass through P. By construction the following holds:

(8.8.2) *For every exceptional divisor E_i over P or over a curve through P with log discrepancy $a_i = 0$ the multiplicity d_i of E_i in $g^{-1}S + E$ satisfies $d_i > 1$.*

Using Corollary 4.6 we can assume that g is an extremal extraction of Q preserving (8.8.2). As in the proof of Reduction 8.2, we take $H = \varepsilon(g^{-1}S + dE)$. From the stated properties and Lemma 3.18 it follows that in a neighborhood of P the point P is the unique point at which $g^*(K+S+B) = K_Y + g^{-1}S + g^{-1}B + E$ is not log terminal, or equivalently, all three of

$$K_Y + g^{-1}S + E, \qquad K_Y + g^{-1}S + g^{-1}B, \qquad K_Y + g^{-1}B + E$$

are also log terminal in a neighborhood of P. Moreover, P is then $\mathbb{Q}$-factorial, since otherwise the log terminal extraction of $K_Y + g^{-1}S + E$ is automatically small, and the birational transforms of the 3 divisors $g^{-1}S$, E and $g^{-1}B$ all contain the fiber curves; but by Corollary 3.16, this contradict the log canonical property of $K_Y + g^{-1}S + g^{-1}B + E$.

We note that $K_Y + g^{-1}S$ purely log terminal follows from (8.1.3) and the fact that a multiple of E is positive in B. Thus $g^{-1}S$ is normal. The remaining log terminal properties required can be proved using Proposition 5.13.

We now check the following addendum to (8.8.2):

(8.8.3) *For every exceptional divisor E_i over P with log discrepancy $a_i' \leq 1$ for $K_Y + g^{-1}S + E$ the multiplicity d_i of E_i in $g^{-1}S + E$ satisfies $d_i > 1$, except for the case that there is an exceptional surface E_j with $a_j = 0$ over P such that E_i can be obtained by blowing up a curve of ordinary double points on the extremal extraction E_j, $a_i' > a_i = 1/2$ and $d_i = (1/2)d_j > 1/2$.*

By monotonicity $a_i < a_i' \leq 1$. Hence $a_i = 0$ or $1/2$. When $a_i = 0$, the result follows at once from (8.8.2). To verify it in the case $a_i = 1/2$ we use a strictly log terminal extraction $h: W \to Y$ for $K_Y + g^{-1}S + g^{-1}B + E$, the exceptional divisors E_j of which lie over P. Such a model exists by Corollary 5.19, since P is the unique point in a neighborhood of P where $g^*(K+S+B) = K_Y + g^{-1}S + g^{-1}B + E$ is not log terminal, and the intersection $g^{-1}S \cap E$ is normal along B_1. This fails only if $\operatorname{Supp} g^{-1}B$ is tangent to E in a neighborhood of P. But then perturbing $g^{-1}B$ while keeping $P \in g^{-1}B$ does not change $a_i = 1/2$. (If P becomes log terminal for $K_Y + g^{-1}S + g^{-1}B + E$, then P is a nonsingular point and all $d_i \geq 2$.)

As before, the log terminal divisor

$$h^*(K_Y + g^{-1}S + g^{-1}B + E) = K_W + h^{-1}g^{-1}S + h^{-1}g^{-1}B + h^{-1}E + \sum E_j$$

has index 2. Since P is $\mathbb{Q}$-factorial it follows that $h^{-1}P = \bigcup E_j$ and that E_i lies over one of the exceptional divisors E_j. Suppose first that E_i with log discrepancy $a_i = 1/2$ is contracted to a point P'. Then $K_W + h^{-1}g^{-1}S + h^{-1}g^{-1}B + h^{-1}E + \sum E_j$ has index > 1 in a neighborhood of P'. By Corollary 3.8 P' is in the intersection of at most two irreducible components of the boundary $h^{-1}g^{-1}S + h^{-1}E + \sum E_j$; moreover, if it lands on the intersection of two components then P' is $\mathbb{Q}$-factorial and by Corollary 3.7 has index 1 if $h^{-1}g^{-1}B$ passes through P', or index 2 otherwise. By (8.8.2),

$$h^*(g^{-1}S + E) = h^{-1}g^{-1}S + h^{-1}E + \sum d_j E_j, \quad \text{where } d_j > 1.$$

Hence $d_i > 1/2 + 1/2 = 1$, since P' lies on at least one of the exceptional components E_j.

Suppose now that P' lies on only one of the reduced boundary components E_j. Then we can modify h into an extremal extraction of $E' = E_j$ preserving the neighborhood of P' and, in particular, preserving the log terminal property of

$$h^*(K_Y + g^{-1}S + g^{-1}B + E) = K_W + h^{-1}g^{-1}S + h^{-1}g^{-1}B + h^{-1}E + E'$$

in a neighborhood of P'. Since h contracts E' to a point, the reduced part of the boundary of $(K_W + h^{-1}g^{-1}S + h^{-1}g^{-1}B + h^{-1}E + E')_{|E'}$ consists of two irreducible curves $C_1 = h^{-1}g^{-1}S \cap E'$ and $C_2 = h^{-1}E \cap E'$. On the other hand, $K_Y + g^{-1}S + E$ has a 1-complement $\overline{0}$ in a neighborhood of P, such that the log discrepancies of E' and E_i for $K_Y + g^{-1}S + E + \overline{0}$ are all 0. For this, we need to use the proof of Theorem 5.12 with $S = g^{-1}S$ and $B = (1-\varepsilon)E$ for sufficiently small $\varepsilon > 0$. Hence the log canonical divisor

$$h^*(K_Y + g^{-1}S + E + \overline{0}) = K_W + h^{-1}g^{-1}S + h^{-1}E + E' + \overline{0}$$

has index 1, and $(K_W + h^{-1}g^{-1}S + h^{-1}E + E' + \overline{0})_{|E'} = K_E + C_1 + C_2 + C_3$, where the curve $C_3 = h^{-1}\overline{0} \cap E'$ is also irreducible. Note that the curves $C_i = \mathbb{P}^1$ intersect pairwise in one point. Since the log discrepancy of E_i is 0, by construction P' lies on C_3 outside C_1 and C_2. But then P' is a log terminal point of $K_E + C_1 + C_2 + C_3$ and by Proposition 5.13 the log discrepancy of $K_W + h^{-1}g^{-1}S + h^{-1}E + E' + \overline{0}$ at E_i is ≥ 1. Hence this case is impossible.

Note also that the log discrepancy of E_i on curves $\neq C_1$ or C_2 is 0 only if it lies over C_3. Moreover, if $a_i = 1/2$ and $d_i \leq 1$ then W has ordinary double points along C_3, and E_i is its extraction and $d_i = (1/2)d_j > 1/2$. This verifies (8.8.3) in the case that E_i with log discrepancy $a_i = 1/2$ is contracted to a curve lying on only one exceptional surface E_j. This completes the proof of property (8.8.3), since the index of $K_W + h^{-1}g^{-1}S + h^{-1}g^{-1}B + h^{-1}E + \sum E_j$ on the curves of intersection of irreducible components of the boundary of $h^{-1}g^{-1}S + h^{-1}E + \sum E_j$ equals 1.

Now we verify that Y is nonsingular outside $g^{-1}S \cup E$ in a neighborhood of P. Obviously we need only consider the singularities along curves C_i not lying on $g^{-1}S \cup E$ and passing through P. As we already know, $K_Y + g^{-1}S + E$ has a 1-complement in a neighborhood of P and the curves of noncanonical singularities of C_i land on the boundary of the complement. As above, we can construct an extremal extraction $h\colon W \to Y$ over one such curve of an exceptional surface E' with log discrepancy $0 < a' < 1$ for $K_Y + g^{-1}S + E$, that is,

$$h^*(K_Y + g^{-1}S + E) = K_W + h^{-1}g^{-1}S + h^{-1}E + (1 - a')E'.$$

But then the 3 surfaces $h^{-1}g^{-1}S$, $h^{-1}E$ and E' all pass through the fiber curve $h^{-1}P$, which contradicts the log canonical property of $K_Y + g^{-1}S + E$ in a neighborhood of P. In the case of canonical singularities along the curves C_i we can use the arguments of Proposition 4.3 and the log terminal property of $K_Y + g^{-1}S + E$ to construct an extraction h of the exceptional divisors over C_i with log discrepancy 1 (that is, discrepancy 0), and no others. By monotonicity (1.3.3), there are no exceptional surfaces over P (compare (1.5.7)) and $h^{-1}P$ is again a curve, and the same arguments as before give a contradiction.

We now verify the following assertion:

(8.8.4) *If P is an isolated singularity then the index of $K_Y + g^{-1}S + E$ is odd, and equal to $2m + 1$, where $m \geq 2$ is a natural number; moreover, there exists an extremal extraction $h\colon W \to Y$ of P with exceptional divisor E' of multiplicity d' in $g^{-1}S + E$ given by $d' = 1 + (1/(2m + 1))$, log discrepancy $a' = 0$ for $K_Y + g^{-1}S + g^{-1}B + E$ and log discrepancy $a'' = 1/(2m + 1)$ for $K_Y + g^{-1}S + E$. Moreover, E' has a singular point P' locally satisfying the same conditions as P, but with $K_W + h^{-1}g^{-1}S + h^{-1}E + E'$ of even index $2m$.*

First of all, I claim that P is a singular point of $g^{-1}S$. For this, note that there is an extremal 0-contraction $h\colon Y \to W$ of a curve $C_1 \subset g^{-1}S \cap \operatorname{Supp} g^{-1}B$. This is a 0-contraction for $H = \varepsilon g^*B$ over Z. Indeed, $\rho(Y/Z) = 2$ and $\overline{\mathrm{NE}}(Y/Z)$ has two extremal rays R_1 and R_2. Suppose that R_1 corresponds to the contraction g. Obviously R_1 is nef against $g^{-1}B$. On the other hand by Reduction 8.2 the curve $g^{-1}S \cap \operatorname{Supp} g^{-1}B$ is exceptional. Hence there is a curve over Z that is negative against $g^{-1}B$. Thus $g^{-1}BR_2 < 0$. But g^*B is numerically 0 on R_1 and is positive against $g^{-1}S \cap \operatorname{Supp} g^{-1}B$. Hence $ER_2 > 0$ and $g^{-1}SR_2 < 0$, which gives what we want. Moreover, the contracted curve B_0 passes through P. It is not hard to check that when P is nonsingular the curve B_0 is irreducible, nonsingular, crosses B_1 normally only at P and is a (-1)-curve. Moreover, there is a curve B_{-1} through P with $B_{-1} \subset g^{-1}S$ but $B_{-1} \not\subset \operatorname{Supp} g^{-1}B$, with multiplicity $1/2$ in the boundary of $(K_Y + g^{-1}S + g^{-1}B + E)_{|g^{-1}S}$, and its restriction in a neighborhood of B_0 is of the form $K_{g^{-1}S} + B_1 + (1/2)B_0 + (1/2)B_{-1}$. But then Y has ordinary double points along B_{-1}, which contradicts our assumption that P is an isolated singularity. From this it follows in particular that P is actually singular and $K_Y + g^{-1}S + E$ has index > 1.

I claim that P is a terminal singularity. For this it is enough to check that $a_i' + d_i > 1$ for the exceptional divisors E_i over P. This follows directly from (8.8.3). By the above and [7], (5.2), the index r of the singular point P is > 1, hence by Kawamata's theorem (given in the Appendix) there is an exceptional divisor E_i over P of log discrepancy $1 + 1/r$ (that is, discrepancy $1/r$). Hence $a_i' + d_i = 1 + 1/r$. On the other hand ra_i' and rd_i are positive natural numbers. Again by (8.8.3) this is only possible when $r = 2m + 1$ is odd, $a_i = 1/2$, $a_i' = d_i = (m + 1)/(2m + 1)$, and the exceptional divisor of E_i is obtained by blowing up a curve C_3 of quadratic singularities on the exceptional divisor E' over P with log discrepancy 0 for $K_Y + g^{-1}S + g^{-1}B + E$ and $d' = 2d_i = 1 + 1/(2m+1)$. We can assume that E' is exceptional for the extremal extraction $h\colon W \to Y$. Then by construction, and since $g^{-1}B$ passes through P, the restriction to E' of $h^*(K_Y + g^{-1}S + g^{-1}B + E)$ is numerically 0 and of the form

$$K_{E'} + C_1 + C_2 + \tfrac{1}{2}C_3 + \tfrac{1}{2}C_4,$$

where the curves $C_1 = h^{-1}g^{-1}S \cap E' = \mathbb{P}^1$, $C_2 = h^{-1}E \cap E' = \mathbb{P}^1$, $C_3 = \mathbb{P}^1$ and $C_4 = \operatorname{Supp} h^{-1}g^{-1}B \cap E'$ are irreducible. Since W has ordinary double points along C_3, the log discrepancy of E' for $K_Y + g^{-1}S + E$ is of the form $a' = 1 - 2(1 - a_i') = 1/(2m+1)$. Hence

$$h^*(K_Y + g^{-1}S + E) = K_W + h^{-1}g^{-1}S + h^{-1}E + \frac{2m}{2m+1}E'.$$

However, by [7,] (5.2), this divisor has index dividing $2m+1$. Hence by the arguments of Lemma 4.2 and Corollary 3.10, it equals $2m+1$, W is nonsingular along C_1 and C_2, and the crossings along $C_1 = h^{-1}g^{-1}S \cap E'$, $C_2 = h^{-1}E \cap E'$ are normal at generic points. Then by the same arguments the unique point of intersection $Q' = C_1 \cap C_2$ is nonsingular on $h^{-1}g^{-1}S$ and $h^{-1}E$. Hence by Corollaries 3.7–8, W is nonsingular, and $h^{-1}g^{-1}S$ and $h^{-1}E$ and E' cross normally at Q'. In particular E' is nonsingular in a neighborhood of Q'. Again by [7], (5.2). the index of $g^{-1}S$ and E divides $2m+1$. Hence the multiplicities of E' in $g^{-1}S$ and in E are both ≤ 1. From this and from the fact that the boundary $h^{-1}g^{-1}S + h^{-1}E + E'$ has normal crossings at Q' we deduce that C_1 (respectively C_2) cannot be a (-1)-curve on the minimal resolution of singularities of $h^{-1}g^{-1}S$ (respectively $h^{-1}E$).

Now we turn to the surface E' and prove that there is a singular point of E' on C_2. Indeed, if not, then $C_2 = \mathbb{P}^1$ and all the singularities of W in a neighborhood of $P' = C_2 \cap C_3$ lie on the curve C_3, which crosses C_2 normally, since C_2 is ample and hence also meets C_1 and C_4. Then from the fact that E' is nonsingular in a neighborhood of P', the divisor $K_W + h^{-1}E + E'$ has index 2, and therefore the restriction $(K_W + h^{-1}E + E')_{|h^{-1}E}$, which in a neighborhood of P' is of the form $K_{h^{-1}E} + C_2$, also has index 2. Thus since P' is log terminal, it is an ordinary double point on $h^{-1}E$. On the other hand, the restriction

$$\begin{aligned} &h^*(K_Y + g^{-1}S + E)_{|h^{-1}E} \\ &= \left(K_W + h^{-1}g^{-1}S + h^{-1}E + \frac{2m}{2m+1}E'\right)_{|h^{-1}E} \\ &= K_{h^{-1}E} + \frac{2m}{2m+1}C_2 + h^{-1}B_1 \end{aligned}$$

is numerically 0 on C_2, hence C_2 is a nonsingular rational curve with selfintersection $-(m+1)$ on the minimal resolution of $h^{-1}E$, that is, the blowup of the ordinary double point P'. In the same way, since the divisor

$$h^*g^{-1}S_{|h^{-1}E} = (h^{-1}g^{-1}S + d_S E')_{|h^{-1}E} = d_S C_2 + h^{-1}B_1$$

is numerically 0 on C_2, we can calculate the multiplicity $d_S = 2/(2m+1)$ of E' in $g^{-1}S$. But then the multiplicity of E' in E is $2m/(2m+1)$. Therefore

$(2m/(2m+1))C_1 + h^{-1}B_1$ is numerically 0 against C_1. But the restriction

$$\begin{aligned} &h^*(K_Y + g^{-1}S + E)_{|h^{-1}g^{-1}S} \\ &= \left(K_W + h^{-1}g^{-1}S + h^{-1}E + \frac{2m}{2m+1}E'\right)_{|h^{-1}g^{-1}S} \\ &= K_{h^{-1}g^{-1}S} + \frac{2m}{2m+1}C_1 + h^{-1}B_1 \end{aligned}$$

is also numerically 0 against C_1, and hence the canonical divisor $K_{h^{-1}g^{-1}S}$ is numerically 0 against C_1. Here C_1 is not a (-1)-curve on the minimal resolution of singularities of $h^{-1}g^{-1}S$. Hence it follows that C_1 will be a nonsingular rational curve with selfintersection -2 on the minimal resolution of singularities of $h^{-1}g^{-1}S$, and on C_1 there is at most one singular point, which is resolved by a chain of nonsingular rational (-2)-curves. Thus $P \in g^{-1}S$ is a Du Val singularity of type A_{2m}. But by what we have said $g^{-1}S$ has an exceptional curve lying in $g^{-1}S \cap \operatorname{Supp} g^{-1}B$ and numerically 0 against the restriction $(K_Y + g^{-1}S + g^{-1}B + E)_{|g^{-1}S}$, whose boundary in a neighborhood of P is $B_1 + (1/2)(g^{-1}S \cap \operatorname{Supp} g^{-1}B)$. From this we deduce that $B_0 = g^{-1}S \cap \operatorname{Supp} g^{-1}B$ is an irreducible curve, and on the minimal resolution of singularities of $g^{-1}S$ is a nonsingular rational (-2)-curve or (-1)-curve passing through a unique singularity P of $g^{-1}S$.

In the first case the contraction of B_0 transforms P into a Du Val singularity of type D_{2m+1}. It follows from this that on the minimal resolution of $Q \in S$ the curve $g(B_0) = S \cap \operatorname{Supp} B$ will not be a (-1)-curve. But this is not possible, since B is positive on the flipping curve $S \cap \operatorname{Supp} B$. In the second case $m = 1$, the inverse image $h^{-1}B_0$ does not pass through the singularity of $h^{-1}g^{-1}S$ on C_1 and crosses C_1 normally at one point. Hence since $K_Y + g^{-1}S + g^{-1}B + E$ is numerically 0 against B_0, its restriction to $g^{-1}S$ in a neighborhood of B_0 is of the form $K_{g^{-1}S} + B_1 + (1/2)B_0 + (1/2)B_{-1}$, where B_{-1} is a nonsingular curve that crosses B_0 normally at one point distinct from P; Y has an ordinary double point along B_{-1}. Hence B_0 contracts to a nonsingular point which is terminal for the image of $K_{g^{-1}S} + (1/2)B_{-1}$.

But then (8.8.4) holds except for the case $m = 1$, when there may be no subsequent point P'. This case will be excluded later, so that for the moment we assume that $m \geq 2$. Therefore C_2 has a singular point P', again coinciding with $C_2 \cap C_3$. Since C_2 on the surface E' is ample and has a unique singularity on E', it becomes a nonsingular rational curve with nonnegative selfintersection on the minimal resolution of singularities of E'. Now applying Theorem 6.9 to the minimal resolution of the singularity $P' \in E'$, one can prove that the selfintersection number is 0, and P' is the unique singularity of E', and is an ordinary cone point, that is, E' is a cone with vertex P' over the nonsingular rational curve C_1. Note also that C_3 and C_4 intersect C_1 in distinct points P_1 and P_2 respectively.

Hence, as above, we get that P_1 is an ordinary double point of $h^{-1}g^{-1}S$, the selfintersection of C_1 on the minimal resolution of $h^{-1}g^{-1}S$ is $-(m+1)$, the selfintersection of C_2 on the minimal resolution of $h^{-1}E$ is -2, and P' is a Du Val singularity of type A_{2m-1}. It follows that the index of $K_W + h^{-1}E + E'$ in a neighborhood of P' concides with that of the restriction $(K_W + h^{-1}E + E')_{|h^{-1}E} = K_{h^{-1}E} + h^{-1}B_1$ and is $2m$. (One can check moreover that P is a quotient singularity of type $\frac{1}{2m+1}(2, -2, 1)$.)

We verify that P' satisfies (8.8.2). Indeed, otherwise there is an exceptional divisor E_i over P' with $a_i = 0$ for $K_W + h^{-1}g^{-1}B + h^{-1}E + E'$ and multiplicity d_i in

$h^{-1}E+E'$ satisfying $d_i \le 1$. Then E_i is exceptional over P with $a_i = 0$ for $K_Y + g^{-1}S+g^{-1}B+E$, and multiplicity $d_i < 1+1/(2m+1)$ in $h^*(g^{-1}S+E) = h^{-1}g^{-1}S+h^{-1}E+E'+(1/(2m+1))E'$ and in $g^{-1}S+E$. But since $g^{-1}S+E$ has index $2m+1$ at P, it follows that $d_i \le 1$, which contradicts (8.8.2). However, we may possibly lose the existence of a contracted curve in the intersection $g^{-1}S \cap \operatorname{Supp} g^{-1}B$, which was important in distinguishing the component on which the singularity P' appeared when $m \ge 2$.

(8.8.5) *If P is a nonisolated singularity, then the index of $K_Y + g^{-1}S + E$ is even, equal to $4m+2$ for a natural number $m \ge 1$, and there exists an extremal extraction $h\colon W \to Y$ of P with exceptional divisor E' having multiplicity $d' = 1 + 1/(4m+2)$ in $g^{-1}S + E$ and log discrepancy $a' = 0$ for $K_Y + g^{-1}S + g^{-1}B + E$. Moreover, on E' there is a singular point P' locally satisfying the same assumptions as P but with the index of $K_W + h^{-1}g^{-1}S + h^{-1}E + E'$ odd and equal to $4m + 1$.*

The only possibility for a curve of singularities through P is a curve Γ of ordinary double points on $g^{-1}S + E$; then $K_Y + g^{-1}S + E$ is log terminal and has index 2 at the general point of Γ. Thus the index of $K_Y + g^{-1}S + E$ is even, and there is a double cover $\pi\colon \widetilde{Y} \to Y$ in a neighborhood of P ramified only in such curves Γ.

Let's check that the birational transforms $\pi^{-1}g^{-1}S$, $\pi^{-1}g^{-1}B$ and $\pi^{-1}E$ preserve the previous properties in a neighborhood of $\pi^{-1}P$. On lifting by π the log terminal property of $K_Y + g^{-1}S + E$ is preserved outside P by construction, and at P by Corollary 2.2. Thus, as above, by (1.3.3) and Corollary 2.2 applied to $g^{-1}S + g^{-1}B + E$, we get that $\pi^{-1}P$ is $\mathbb{Q}$-factorial, and $\pi^{-1}g^{-1}S$ and $\pi^{-1}E$ are irreducible and normal in a neighborhood of $\pi^{-1}P$. By the proof of Corollary 2.2 the log discrepancy $\widetilde{a}_i$ of the exceptional divisor $\widetilde{E}_i$ over E_i for Y and over $\pi^{-1}P$ in $K_{\widetilde{Y}} + \pi^{-1}g^{-1}S + \pi^{-1}g^{-1}B + \pi^{-1}E$ is 0 only if E_i for $K_Y + g^{-1}S + g^{-1}B + E$ has log discrepancy 0. This implies that property (8.8.2) is preserved. Hence by (8.8.4) $\pi^{-1}P$ is a terminal point of odd index $2m + 1$ and the index of $K_Y + g^{-1}S + E$ is of the form $4m+2$. If $m = 0$ then $K_Y + g^{-1}S + E$ and its restriction $(K_Y+g^{-1}S+E)_{|g^{-1}S}$ both have index 2. More precisely, if $g^{-1}S$ does not contain a curve of singularities of Y then $P \in g^{-1}S$ is an ordinary double point and, as above, in a neighborhood of P we get

$$(K_Y + g^{-1}S + g^{-1}B + E)_{|g^{-1}S} = K_{g^{-1}S} + B_1 + \tfrac{1}{2}B_0,$$

where the curve $B_0 \subset g^{-1}S \cap \operatorname{Supp} g^{-1}B$ generates a flipping extremal ray (denoted R_2 in the above). The (-2)-curve resolving $P \in g^{-1}S$ has log discrepancy 1 for $K_{g^{-1}S}+B_1+(1/2)B_0$. Thus on resolving P, we arrive at a contradiction, in the same way as when proving that P is singular in (8.8.4). Therefore the point $P \in g^{-1}S$ is nonsingular, it lies on a curve B_{-1} of ordinary double points, and in a neighborhood of P

$$(K_Y + g^{-1}S + g^{-1}B + E)_{|g^{-1}S} = K_{g^{-1}S} + B_1 + \tfrac{1}{2}B_0 + \tfrac{1}{2}B_{-1},$$

where $B_0 = g^{-1}S \cap \operatorname{Supp} g^{-1}B = |R_2|$ is an irreducible curve. Note that $g^{-1}S$ is nonsingular in a neighborhood of B_0, B_0 is a (-1)-curve, and the restriction $(K_Y + g^{-1}S)_{|g^{-1}S} \doteq K_{g^{-1}S}+(1/2)B_{-1}$ is numerically negative on B_0, since B_0 and B_{-1} intersect only in P, and normally there.

Let us prove that the flip exists in this case. We check first that the intersection $g^{-1}S \cap E = B_1 \cup \dots \cup B_n$ is irreducible. All the B_i with $i < n$ are contracted to Q, and are hence positive against $g^{-1}B$. Hence $n \le 2$. Suppose that $n = 2$. The curve B_0 is the support of the next extremal ray R_2. Moreover, the surfaces $g^{-1}S$ and $g^{-1}B$ are negative, and E is positive against B_0. Hence the flip in B_0 exists by Corollary 5.20. We can prove that it satisfies the properties of Proposition 8.3. For this it is sufficient that the image of $K_{g^{-1}S} + (1/2)B_{-1}$ under the contraction of B_0 is log terminal, which we know. In particular, the only point of $g^{-1}S^+$ at which B_1^+ can be singular is $Q' = B_1^+ \cap B_2^+$. But now $g^{-1}B^+$ is numerically 0 on B_1^+, from which it follows that it is extremal. Therefore B_1^+ is the support of the next extremal ray. Hence E is positive on it. The intersection of $g^{-1}S^+$ and E^+ along B_1^+ is normal by Proposition 8.3. As in Proposition 8.3, we deduce from this using Lemma 3.18 that B_1^+ moves, which gives a contradiction.

Thus the intersection $B_1 = g^{-1}S \cap E$ is irreducible. Suppose now that $g(E) = C$. Then g identifies $g^{-1}S$ and S. Here by Proposition 3.9 and (8.1.4) we have

$$(K+S)\cdot B_0 = \left(K_S + \frac{n-1}{n}C + \frac{1}{2}B_{-1}\right)\cdot B_0 = \frac{n-1}{n} - \frac{1}{2} < 0,$$

where n is the index of $K+S$ along C. Hence $n=1$ and X has in a neighborhood of B_0 only ordinary double points along $g(B_{-1})$, and $K+S$ has index 2. Therefore there is a purely log terminal complement of $K+S$ of index 2 in a neighborhood of B_0, and the flip of f exists by Proposition 2.9.

The case $g(E) = Q$ is similar. Arguing as above, we have

$$(g^*(K+S))\cdot B_0 = \left(K_{g^{-1}S} + \frac{n-1}{n}C + \frac{1}{2}B_{-1} + a'B_1\right)\cdot B_0 = a' - \frac{1}{2} < 0,$$

so that $a' < 1/2$. But B_1 is not a (-1)-curve on the minimal extraction E, and

$$K_{g^{-1}S} + \frac{n-1}{n}C + \frac{1}{2}B_{-1} + a'B_1 = g^*\left(K_S + \frac{n-1}{n}C + \frac{1}{2}B_{-1}\right).$$

It follows from this that B_1 is a (-2)-curve on the minimal extraction E, that X is nonsingular along C, and $P_0 = B_1 \cap g^{-1}C$ is a canonical singularity of type A_n. Hence in a neighborhood of $g(B_0)$ on S there is a purely log terminal complement of $K_S + (1/2)g(B_{-1})$ of index 2. To extend this to X for $K+S$, by the proof of Theorem 5.12, it is enough to have a resolution $Y' \to X$ with normal crossings $S_{Y'}$ minimal over S. For this we need to use a partial resolution of g and extend it. Since Y has ordinary double points along B_{-1}, resolving it does not change $g^{-1}S$. Thus it is sufficient to find a similar resolution of P_0. Now P_0, just as P, is a $\mathbb{Q}$-factorial point. Furthermore, by Corollary 3.7, it is a quotient singularity of index n. If P_0 is not an isolated singularity, then the curve C' of singularities lies on E. Moreover, X has a canonical singularity of type $A_{n'}$ with $n' \mid n$. Performing the resolution of C' as in Proposition 4.3, we again preserve the minimal assumption and reduce the resolution to the isolated singularities of the same type; the surfaces extracted in this will be irreducible. In the case that P_0 is isolated it will be a terminal quotient singularity of type $\frac{1}{n}(k, -k, 1)$, the economic resolution of which gives what we want. This can also be deduced by induction on n from the theorem of the Appendix.

Thus, $m \geq 1$. Hence by (8.8.4) there exists an extremal extraction $\widetilde{h}\colon \widetilde{W} \to \widetilde{Y}$ with exceptional divisor $\widetilde{E}'$ having

multiplicity $\widetilde{d}'$ in $\pi^{-1}g^{-1}S + \pi^{-1}E$ given by $\widetilde{d}' = 1 + 1/(2m+1)$,

log discrepancy $\widetilde{a}'$ for $K_{\widetilde{Y}} + \pi^{-1}g^{-1}S + \pi^{-1}g^{-1}B + \pi^{-1}E$ given by $\widetilde{a}' = 0$,

log discrepancy $\widetilde{a}''$ for $K_{\widetilde{Y}} + \pi^{-1}g^{-1}S + \pi^{-1}E$ given by $\widetilde{a}'' = 1/(2m+1)$.

Therefore by (8.8.2) the ramification index of π at $\widetilde{E}'$ equals 1. Suppose that $\widetilde{E}' \subset \widetilde{Y}$ lies over $E' \subset Y$, which is an irreducible exceptional surface over P. Then the log discrepancy of E' for $K_Y + g^{-1}S + g^{-1}B + E$ equals 0. Let $h: W \to Y$ be the extremal contraction of E'. Using Theorem 6.9, it is not hard to verify that $K_W + E'$ is purely log terminal. Hence, as in the proof of Proposition 8.3, if $\pi^{-1}E'$ is reducible then we deduce from Corollary 2.2 and Corollary 3.8 that π is unramified everywhere over E', and hence also over P. Therefore $\widetilde{E}' = \pi^{-1}E'$ is irreducible, that is, the covering involution of the double cover π acts biregularly on the extremal extraction $\widetilde{h}$. Since h and $\widetilde{h}$ are extremal, the curves $\widetilde{C}_1 = \pi^{-1}h^{-1}g^{-1}S \cap \widetilde{E}'$ and $\widetilde{C}_2 = \pi^{-1}h^{-1}E \cap \widetilde{E}'$ are irreducible and lie over $C_1 = h^{-1}g^{-1}S \cap E'$ and $C_2 = h^{-1}E \cap E'$. On the other hand, by the proof of (8.8.4) there exists a curve $\widetilde{C}_i$ for which

$$\begin{aligned}(K_{\widetilde{Y}} + \pi^{-1}h^{-1}g^{-1}S + \pi^{-1}h^{-1}g^{-1}B + \pi^{-1}h^{-1}E + \widetilde{E}')_{|\widetilde{E}'|\widetilde{C}_i}\\ = K_{\widetilde{C}_i} + \widetilde{Q} + \frac{1}{2}\widetilde{P}_1 + \frac{1}{2}\widetilde{P}_2,\end{aligned}$$

where $\widetilde{Q} = \pi^{-1}h^{-1}g^{-1}S \cap \pi^{-1}h^{-1}E \cap \widetilde{E}'$, and

$$\widetilde{P}_2 = \pi^{-1}h^{-1}g^{-1}S \cap \operatorname{Supp}\pi^{-1}h^{-1}g^{-1}B \cap \widetilde{E}'$$

or

$$\widetilde{P}_2 = \pi^{-1}h^{-1}E \cap \operatorname{Supp}\pi^{-1}h^{-1}g^{-1}B \cap \widetilde{E}'.$$

This implies that π is ramified along the curve $\widetilde{C}_i$ and by the purity theorem W is singular along the corresponding curve C_i. Since π is unramified along $\widetilde{E}'$ the log discrepancy of E' for $K_Y + g^{-1}S + E$ is $1/(2m+1)$. By construction and [7], (5.2), the index of $K_Y + g^{-1}S + E$ divides $4m+2$, and hence arguing as in the proof of Lemma 4.2 we get that W has an ordinary double point along C_i. The corresponding exceptional divisor extracted out of this singularity has log discrepancy 0 for $K_Y + g^{-1}S + g^{-1}B + E$, log discrepancy $1/(4m+2)$ for $K_Y + g^{-1}S + E$, and multiplicity $1 + 1/(4m+2)$ in $g^{-1}S + E$.

Now write h for its extremal extraction. Arguing as in (8.8.4), we get from this that the curves $C_1 = h^{-1}g^{-1}S \cap E'$ and $C_2 = h^{-1}E \cap E'$ are irreducible and that these intersections are normal crossings at their general points. Also the curves C_1 and C_2 on the respective surfaces $h^{-1}g^{-1}S$ and $h^{-1}E$ are not (-1)-curves. Suppose first that $i = 2$ above. Then by the proof of (8.8.4) the multiplicity of the previous E' in E equals $2m/(2m+1)$, hence the multiplicity of the current E' in E equals $(4m+1)/(4m+2)$. Then arguing as in (8.8.4) we can check that C_1 will be a (-2)-curve on the minimal resolution, $g^{-1}S$ does not have a curve of double points of Y, and C_1 passes through a unique singularity, a Du Val singularity of type A_{4m} on $h^{-1}g^{-1}S$. This contradicts that $g^{-1}S$ contains a curve of double points B_{-1}, in view of the argument of (8.8.4) for $m \geq 2$. Thus $i = 1$. By the previous arguments the multiplicity of the current E' in $g^{-1}S$ equals $(4m+1)/(4m+2)$, E does not have a curve of double points of Y, P is a Du Val singularity of type A_{4m+1} on E, C_2 is a (-2)-curve on the minimal extraction of $h^{-1}E$, and C_2 has a unique singularity P' of the surface $h^{-1}E$, a Du Val singularity of type A_{4m}. On the other hand, by construction

$$(K_Y + h^{-1}g^{-1}S + h^{-1}g^{-1}B + h^{-1}E + E')_{|h^{-1}S|C_1} = K_{C_1} + P + \tfrac{1}{2}P_1 + \tfrac{1}{2}P_2,$$

where $\operatorname{Supp}\pi^{-1}h^{-1}g^{-1}B$ passes through P_2, and P_2 is nonsingular on $h^{-1}g^{-1}S$. But $h^{-1}g^{-1}S$ must contain the curve of double point $h^{-1}B_{-1}$.

Obviously $P_1 = C_1 \cap h^{-1}B_{-1}$ is a nonsingular point of $h^{-1}g^{-1}S$. Hence $h^{-1}g^{-1}S$ is nonsingular in a neighborhood of C_1 and $C_1 = \mathbb{P}^1$ is a curve with selfintersection $-(2m+1)$. On the other hand, in a neighborhood of C_1, by nonsingularity of $h^{-1}g^{-1}S$, the surface E' has P_1 as an ordinary double point, and E' does not have double points along curves of W. Hence P' is an isolated singularity. But P' is a Du Val singularity of type A_{4m}. It follows from this that the index of $K_W + h^{-1}E + E'$ in a neighborhood of P' is the same as that of the restriction $(K_W + h^{-1}E + E')_{|h^{-1}E} = K_{h^{-1}E} + h^{-1}B_1$, equal to $4m+1$. From then on, as in the proof of (8.8.4) we verify that P' satisfies (8.8.2).

Now it follows from (8.8.4–5) that in (8.8.4) the index of P' is of the form $4m'+2$, hence that of P is of the form $4m'+3$ with $m' \geq 1$. Hence the case (8.8.5) is impossible altogether, from which it follows that (8.8.4) is impossible. This is now all proved, except for the one case not yet considered, that of $m = 1$ in (8.8.4). We will prove that in this case the flip exists, or reduces to the same type (8.5.3) with $d_i > 1$ for $a_i = 0$.

For this we need the following two lemmas, that are also needed below in the proof of the main results, where they are used to preserve the type of flips in subsequent inductive steps.

8.9. **Lemma.** *Let S be a normal projective surface with boundary B, and C_1, C_2 contractible (possibly reducible) curves such that*

(i) $2(K+B) \sim 0$;

(ii) $\lfloor B \rfloor = B_1 + B_2$, *where B_1 and B_2 are irreducible*;

(iii) $B_1^2 > 0$ (*with B_1 as in* (ii));

(iv) *B_2 becomes ample after contracting C_2*;

(v) *C_1 is disjoint from B_1*;

(vi) *the point $P = B_1 \cap B_2$ is the unique point of B_1 where $K+B$ is not purely log terminal*;

(vii) *the components of C_2 intersect B_1 and B_2 in P.*

Then either C_1 is disjoint from B_2, or P is the unique point of B_2 at which $K+B$ is not purely log terminal.

By Theorem 6.9, the singularities of S are rational, and remain so after contracting C_2. Hence the ampleness in (iv) coincides with numerical positivity by the Nakano-Moishezon criterion (compare [8], 6-1-15 (2)).

Proof. Suppose that C_1 intersects B_2. Then using standard arguments of the theory of extremal rays for the contraction of C_1, together with (i), we can find an irreducible contractible curve $C' \subset C_1$ intersecting B_2. Hence without changing the assumptions we can restrict to the case that C_1 is irreducible and intersects B_2. By (i), (ii) and Theorem 6.9, the locus of log canonical singularities of $K+B$ is just $B_1 \cup B_2$. Suppose now that there exists an irreducible component $C'' \subset C_2$ intersecting C_1. Then by Corollary 3.16 and the log canonical assumption on $K+B$ the curve C'' has multiplicity 0 in B. Hence by (i) C'' is a (-1)-curve on the minimal resolution of S. As in Proposition 8.3 it is easy to check that $B_1 = \mathbb{P}^1$ and that

$$(K + B_1 + B_2)_{|B_1} = K_{\mathbb{P}^1} + \tfrac{1}{2}P_1 + \tfrac{1}{2}P_2 + P.$$

Now let $g\colon T \to S$ be a strictly log terminal model of $K+B$, minimal over P. Then $g^{-1}C''$ does not intersect $g^{-1}B_1$, but crosses normally at Q a component with multiplicity 1 in the boundary B_T that is exceptional over P. Therefore

$g^*(K+B)$ has only canonical singularities on $g^{-1}C''$ outside Q. It follows that C_1 also has multiplicity 0 in B and is a (-1)-curve on the minimal resolution of S, since it intersects B_2. Then on the minimal resolution T a multiple of the total inverse image of the curve $g^{-1}(C_1 \cup C'')$ is mobile. But $g^{-1}(C_1 \cup C'')$ is disjoint from $g^{-1}B_1$ and its intersection with B_T does not lie over P only. Hence $g^{-1}B_1$ is exceptional. From now on, arguing as in Proposition 8.3, we get from (iii) that exactly one of the points P_i is nonsingular.

Suppose that P_1 is nonsingular. Then there is an irreducible curve B_3 with multiplicity $1/2$ in the boundary B passing through P_1. By what we have said C'' does not intersect B_3. Also each irreducible component of C_2 does not intersect B_3, since it passes through P. Hence by (iv) B_3 intersects B_2. Moreoveor, it is not hard to check that $g^{-1}(C_1 \cup C'')$ is disjoint from $g^{-1}(B_1 \cup B_3)$. Thus arguing as above, we get that $g^{-1}B_3$ is exceptional. But $g^{-1}B_3$ intersects the locus of log canonical singularities of $(B_1+B_2)_T$ at 2 points, which contradicts the connectedness lemma, Lemma 5.7. Thus we have proved that every irreducible component of C_2 does not intersect C_1. Contracting C_2 we can assume that $C_2 = \varnothing$, when assumption (iv) means that B_2 is ample. By (iii) $\overline{\mathrm{NE}}\, S$ has an extremal ray R that is positive against B_1. If cont_R contracts a curve, then by ampleness of B_2 and Lemma 5.7 it intersects B_1 and B_2 at P. Hence we can take this last curve as C_2, and then contract. The contraction decreases the Picard number $\rho(S)$. Hence after a finite number of such contractions we can assume that the extremal contraction cont_R is not birational. Since C_1 does not intersect B_1, cont_R must be a morphism to a curve, and B_1, B_2 are not contained in its fibers. Then by Theorem 6.9, P is the unique point of B_2 at which $K+B$ is not purely log terminal. Q.E.D.

8.10. **Lemma.** *Let $f: S \dashrightarrow T$ be a birational map of normal projective surfaces, D an effective ample divisor on S such that D_T is irreducible.* (*Here D_T is the log birational transform as in* §1 *and in* (10.3.2), *that is, all the blown up curve are contained in it with multiplicity* 1.) *Then D_T is numerically positive.*

Proof. Consider a resolution of indeterminacies of f, for example a Hironaka hut

$$\begin{array}{ccc} & U & \\ {}^{h}\swarrow & & \searrow^{g} \\ S & \overset{f}{\dashrightarrow} & T. \end{array}$$

Since D is ample $\operatorname{Supp} D$ is connected, and because S is normal its inverse image $\bigcup C_i$ on U is connected. I claim that $g(\bigcup C_i) \neq pt$. Indeed, otherwise there exist $a_i > 0$ such that

$$\left(\sum a_i C_i\right) \cdot C_j < 0$$

for every irreducible component C_j. In particular

$$0 \leq \left(\sum a_i h(C_i)\right) \cdot D = \left(\sum a_i C_i\right) \cdot h^*D = \left(\sum a_i C_i\right) \cdot \left(\sum b_j C_j\right) < 0,$$

since $h^*D = \sum b_j C_j$ with $b_j \geq 0$ and at least one $b_j > 0$; this is a contradiction. From the claim, because D_T is assumed to be irreducible (that is, $\operatorname{Supp} D_T$ is irreducible), we get that $D_T = g(\bigcup C_i)$, and all the exceptional curves of h not intersecting $\bigcup C_i$ are exceptional for g. Thus

$$g^* D_T = \sum c_i C_i,$$

where all $c_i > 0$. If B is a curve on T disjoint from D_T, then $g^{-1}B$ is disjoint from $\bigcup C_i$ and $h \circ g^{-1}B$ is disjoint from $\operatorname{Supp} D$. Therefore $h \circ g^{-1}B = pt.$, and by what we said above this is impossible. It remains to check that $D_T^2 > 0$. Indeed, otherwise $g^*D_T \cdot C_j \le 0$ for all curves C_j, which gives a contradiction:

$$0 < \left(\sum c_i h(C_i)\right) \cdot D = \left(\sum c_i C_i\right) \cdot h^* D = \left(\sum c_i C_i\right) \cdot \left(\sum b_j C_j\right) \le 0. \quad \text{Q.E.D.}$$

Proof of Proposition 8.8, *continued.* Thus we return to the case (8.8.4) with $m = 1$. By what we have already proved, there exists an extremal extraction of a surface E' over P with $a = 0$ for which (8.8.1) holds. However, E' has multiplicity 2/3 in $g^{-1}S$ and in E, and hence multiplicity $(2/3)(1+d)$ in S, where $d \le 1$ is the multiplicity of E in S. By assumption we have $1 < (2/3)(1+d) \le 4/3$, so that $d > 1/2$.

Consider now an extremal extraction $g\colon Y \to X$ of the new surface $E = E'$. We check that it is a good extraction (see 8.5 for the definition). One checks first, exactly as before, ([12]) that the intersection $g^{-1}S \cap E$ consists of at most two irreducible curves B_i; and that if $g^{-1}S \cap E = B_1 \cup B_2$, then f has a flip. For this, in view of $H = g^*B$, we should first carry out a flip in the birational transform of curves of the flip of f. By definition of the current type, these coincide with the intersection $g^{-1}S \cap g^{-1}B$, and $g^{-1}B$ is negative on them. Hence E is positive on them and $g^{-1}S$ negative. A flip in them does not change the log terminal property of $K_Y + g^{-1}S + E$ outside $P_0 = B_1 \cap B_2$, which is established as before. From this it follows that E and its modification are normal. (There are at most two such flips, and they modify at most two curves.) The flipped curves do not intersect B_2. After this one must carry out the flip in B_1 described in Proposition 8.3, since E is positive on B_1. The modified surface E^+ is nonnormal along the flipped curve $B_1^+ = \nu(C^*)$. ([13]) Now the intersection $B_2^+ = g^{-1}S^+ \cap E^+$ is irreducible. As before we are interested in the subsequent extremal and flipping ray R_2, which is numerically 0 against $g^{-1}S^+$, positive against $g^{-1}B^+$ and negative against E^+.

According to Theorem 6.9, if some connected component of the support of R_2 intersects the locus of log canonical singularities of the log divisor

$$(K_{Y^+} + g^{-1}S^+ + g^{-1}B^+ + E^+)_{|E^{+\nu}}, \tag{2}$$

and does not intersect C^*, then it intersects an (irreducible) curve C^{**} in such a way that

$$\mathrm{LCS}((K_{Y^+} + g^{-1}S^+ + g^{-1}B^+ + E^+)_{|E^{+\nu}}) = C^* \cup B_2^+ \cup C^{**}.$$

However, as in the treatment of the case (8.5.2*), by Proposition 8.3, the curve B_2^+ has at most one singular point at $Q^+ = B_2^+ \cap g^{-1}C^+$. It has selfintersection ≥ 0 on the minimal resolution of singularities, and even ≥ 1 in the singular case. Performing partial resolutions at Q^+ that are log crepant for the restricted log divisor (2) until B_2^+ becomes a 0-curve, we get a contradiction to Theorem 6.9 for the contraction along the modified B_2^+. Hence by Theorem 6.9 again, components of the support of R_2 can only have log canonical singularities of the restricted log divisor (2) only in C^*. Moreover, if the restricted log divisor (2) is purely log terminal outside B_2^+ in a neighborhood of C^* then its divisors with log discrepancy 0 for $K_{Y^+} + g^{-1}S^+ + g^{-1}B^+ + E^+$ lie over the general point of B_1^+. Hence by Proposition 8.3, after perturbing the surface $g^{-1}S$ with base locus in the support of R_2 we arrive at a purely log terminal divisor near C^*. By the above, the restricted

([12]) Compare 1 page after the start of the proof of Proposition 8.8, and the start of the proof of (8.8.5). The same argument is referred to several times in the rest of §8.

([13]) C^* was introduced in the statement of Proposition–Reduction 8.8.

log divisor (2) on $E^{+\nu}$ does not have log canonical singularities outside C^*. Hence after perturbing we get a flip of type IV. Thus in this case a flip of f exists.

Hence from now on we can assume that the restricted log divisor (2) has a point $Q' \in C^*$ outside B_2^+ that is not purely log terminal. We show that this is impossible. By Theorem 6.9 again, the log divisor (2) does not have nontrivial log crepant resolutions with modified 0-curve B_2^+. But it is only trivial when Q^+ is nonsingular on $E^{+\nu}$, and near Q^+,

$$(K_{Y^+} + g^{-1}S^+ + g^{-1}B^+ + E^+)_{|E^{+\nu}} = K_{E^{+\nu}} + B_2^+ + \tfrac{1}{2}D,$$

where D is an irreducible curve that is simply tangent to B_2^+ at Q^+, and B_2^+ has selfintersection on $E^{+\nu}$ equal to 1. Note that in a neighborhood of Q^+ there is an identification $E = E^+ = E^{+\nu}$. The curve D is cut out transversally by $g^{-1}B$, and B_2 by $g^{-1}S$. Therefore by Corollary 3.7, $Q = B_2 \cap g^{-1}C$ is nonsingular on Y.

Thus the surface $E^{+\nu}$ is nonsingular on B_2^+, the selfintersection of B_2^+ equals 1, and

$$(K_{Y^+} + g^{-1}S^+ + g^{-1}B^+ + E^+)_{|E^{+\nu}} = K_{E^{+\nu}} + C^* + B_2^+ + D',$$

has a point $Q' \in C^*$ that is not purely log terminal; here $D' \geq (1/2)D$, and D is an irreducible curve tangent to B_2^+ at Q^+. Thus B_2^+ determines a contraction $h\colon E^{+\nu} \to \mathbb{P}^2$, in such a way that $h(B_1^+)$ and $h(B_2^+)$ are lines, and $h(D') = h(D)$ is a conic touching these lines. h contracts all curves of $E^{+\nu}$ not intersecting B_2^+. In particular all the flipped curves are contracted, since the last flip modifies E into $E^{+\nu}$ in divisors with log discrepancy 0 over C^* for $K_{E^{+\nu}} + C^* + B_2^+$, and the indicated curves only intersect the final component B_1 of the log terminal extraction of Q'. Preserving all the conditions and notations we have mentionned, we contract all the flipped curves before B_1.

Then the original E is obtained as a result of the procedure described in (8.5.2*) above. We have to make a minimal log terminal extraction of Q' for $K_{E^{+\nu}} + C^* + B_2^+ + D'$, and then contract C^* and all the extracted curves B_i apart from the end one B_1. I claim that they are preserved at the point of tangency $Q'' = h(Q') = h(B_1^+) \cap h(D)$. Indeed, all the curves C_i contracted by h intersect B_1 on the log terminal extraction, without touching the other components of the extraction. Otherwise C_i would be a (-1)-curve on a subsequent minimal resolution of $E^{+\nu}$. Since $K_{E^{+\nu}} + C^* + B_2^+ + D'$ is numerically 0 and the minimal extraction is log crepant it does not intersect B_1 and the modified D'. Hence its modification on E passes through P_0 and does not intersect the modified $D' \geq g^{-1}B_{|E}$. This last conclusion contradicts $g^{-1}B$ ample on E. Thus we have established what we needed, and we see that a minimal log terminal extraction of Q' consists of a single curve. By the same arguments C^* must be a (-1)-curve on such an extraction. Hence P_0 is nonsingular on E, and by Corollary 3.7, also on X. Hence P_0 is a canonical singularity of $g^{-1}S$. Its type on $g^{-1}S$ is known from the proof of Proposition 8.3, and from this P_0 is also nonsingular on $g^{-1}S$. Hence, and from the fact that E has multiplicity in S greater than 1, it follows that the same holds for the multiplicity in S of all divisors with $a_i = 0$ over a neighborhood of $Q \in X$, which contradicts the construction of E.

Thus the intersection $B_1 = g^{-1}S \cap E$ is irreducible. Then, as in the case (8.5.2*), $K_Y + g^{-1}S + E$ log terminal follows from the ampleness of $g^{-1}B$ on E. Thus g is a good extraction. In this case, when

$$(3) \qquad (K_Y + g^{-1}S + g^{-1}B + E)_{|E}$$

is purely log terminal outside D_1, we get what we want. In fact then the flip of f exists for the following reasons. By Theorem 6.9 and the ampleness of $g^{-1}S$ on E, the remaining case is when the locus of log canonical singularities of the restricted log divisor (3) is equal to $B_1 \cup C'$, where C' is an irreducible curve on E intersecting B_1 in $Q = B_1 \cap g^{-1}C$. We reduce this case to flips of type (8.5.3) with $d_i > 1$ for all $a_i = 0$.

For this, in view of $H = g^*B$ we should first carry out a flip in the birational transform of curves of the flip of f. As before, a flip in them does not change $K_Y + g^{-1}S + E$ log terminal. From this it follows that E and its modification are normal. (In the case under consideration there is exactly one such flip.) The flipped curves do not intersect C'. As usual we are interested in the subsequent extremal flipping ray R_2, which is numerically 0 against $g^{-1}S$, positive against $g^{-1}B$ and negative against E. Now $g^{-1}B$ intersects B_1 only in Q. If the restricted log divisor (3) is purely log terminal outside B_1 then the required flip is exceptional (up to the connectedness of the flipping curves) and of index 2, and therefore exists. Hence we can assume that the restricted log divisor (3) has a point $Q' \in C'$ outside B_1 that is not purely log terminal. Since the modified $g^{-1}B$ is positive on R_1 and R_2, it is ample on E. By Lemma 8.10, after the birational contraction of the components of $\operatorname{Supp}(g^{-1}B_{|E})$ other than C', it transforms C' into an ample curve. Hence by Lemma 8.9 the support of R_2 equals the contracted curves, and hence the flip of R_2 is again of type (8.5.3).

It remains to check that $d_i > 1$ for all $a_i = 0$ over a neighborhood of Q'. We suppose the contrary, and show that (8.81) holds. For this, note that, by construction, on E we have an irreducible curve C_3 of double points of Y that passes through Q', and is not contracted by flips. Thus by what we have already proved, there exists a flip of R_2, and hence of f, or an extremal extraction $h\colon W \to Y$ of a surface E' with multiplicity $d' \leq 1$ in E and $a = 0$ for $K + S + B$. Moreover, it lies over Q' or over the general point of C' and satisfies (8.8.1). The flips do not touch these extractions, and hence they can be constructed for the original g. However, in the case under consideration,

$$(K_Y + g^{-1}S + g^{-1}B + E)_{|E} = K_E + B_1 + C' + \tfrac{1}{2}C_3 + \tfrac{1}{2}C_4,$$

where both $C_4 = E \cap \operatorname{Supp} g^{-1}B$ and C_3 pass through Q'. Here $h^{-1}C_4$ is irreducible and moves on $h^{-1}E$. This can be deduced from the existence of a 1-complement of $K_Y + g^{-1}S + E$ in a neighborhood of E with log discrepancy 0 for E' (compare the proof of (8.8.3)). Hence $h^{-1}C_4$ and the curve $h^{-1}C_3$ that does not meet it define a ruling on $h^{-1}E$, since $h^{-1}C_3$ is not exceptional by Theorem 6.9. This ruling is induced by a birational contraction of W contracting the surface $h^{-1}E$ to a curve, possibly after a flip in $h^{-1}C'$. This last flip takes place only when E' lies over Q' and intersects $h^{-1}E$ in two curves, one of which belongs to a fiber of the ruling. This implies the relation $d = (1/n)(1 + dd')$ where d is the multiplicity of E in S, dd' that of E' in S, and $n = -(h^{-1}E)\cdot(h^{-1}C_4) > 0$ is an integer. When $n = 1$ we have $dd' = d - 1 \leq 4/3 - 1 < 1$, which gives (8.8.1) after contracting E to a curve as before. But if $n \geq 2$ then $d = (1/n)(1 + dd') \leq (1/2)(1 + d)$, since $d' \leq 1$. Therefore $d \leq 1$. This final contradiction finishes our treatment of the case (8.8.4) with $m = 1$; more precisely, it reduces this case to flips (8.5.3) with $d_i > 1$ for all $a_i = 0$ over a neighborhood of Q. The existence of these flips is discussed in what follows.

Thus (8.8.2) does not hold if the index of $K_Y + g^{-1}S + E$ is ≥ 4, and the existence of flipping curves then does not play any role if we do not worry about the choice of the component on which the new singularity P' appears. They are only required in treating cases with index ≤ 3.

Thus, returning to the start of the proof, we have checked (8.8.1) modulo the existence of flips of type (8.5.3) with $d_i > 1$ for all $a_i = 0$, which gives an extremal extraction g satisfying (i) and (iv) of a good extraction (see 8.5). To check the other properties of a good extraction we first restrict ourselves to the case $g(E) = Q$. Recall that, as before, the multiplicity d of E in S is ≤ 1. As above, the intersection $g^{-1}S \cap E$ consists of at most two irreducible curves B_i. We show that if $g^{-1}S \cap E = B_1 \cup B_2$ then either f has a flip or it reduces to a flip of type (8.5.3) with $d_i > 1$ for all E_i over Q with $a_i = 0$.

For this, as in the treatment of the case (8.8.4) with $m = 1$ we reduce things to the following. The divisor $K_Y + g^{-1}S + E$ is log terminal outside $P_0 = B_1 \cap B_2$, Y is nonsingular at P_0 and $Q = B_2 \cap g^{-1}C$, the surface E is normal, nonsingular along the curve B_2, which has selfintersection $(B_2{}^2)_E = 2$. This follows since the $+1$-curve B_2^+ is obtained by one blowup of B_2 at P_0 (a standard blowup). The surface $g^{-1}S$ is also nonsingular along B_2. Suppose that the selfintersection of B_2 on $g^{-1}S$ is $(B_2{}^2)_{g^{-1}S} = -n$. Then we can calculate the multiplicity of E in S from

$$\begin{aligned} 0 &= (g^{-1}S + dE)\cdot B_2 = (g^{-1}S)\cdot B_2 + dE\cdot B_2 \\ &= \big((B_1 + B_2)\cdot B_2\big)_E + d\big((B_1 + B_2)\cdot B_2\big)_{g^{-1}S} = 3 - d(n-1), \end{aligned}$$

hence $d = 3/(n-1)$, and $n \geq 4$, since $d \leq 1$ under the current case assumption. Also by the above, E is the unique surface over $g(Q) = Q$ with multiplicity ≤ 1 in S and $a = 0$. However, the surface E', the standard blowup of B_1, has multiplicity $1 + d$ in S and $a = 0$, and the same for the surface E'', the standard blowup of B_2. Every other surface over $Q \in X$ with $a = 0$ has multiplicity $> 1 + d$ in S; note that the standard blowup of P_0 gives a surface with multiplicity $1 + d$ in S, but with $a = 1$, and the standard blowup of $Q \in Y$ gives a surface with multiplicity $1 + d$ in S, but with $a = 1/2$.

Now consider the extremal extraction $g' \colon Y' \to X$ of the divisor E'. It intersects $g'^{-1}S$ only in the single curve $B_1 \subset Y'$, the birational transform of the curve $B_1 \subset Y$ of the same name. This is proved just as the corresponding assertion in the case (8.8.4) with $m = 1$. Thus the intersection $B_1 = g'^{-1}S \cap E'$ is irreducible. Moreover, $Q = B_1 \cap g'^{-1}C$ is a point of type A_1 on $g'^{-1}S$ resolved by a curve B_2 with selfintersection $-n \leq -4$. By construction the intersection is normal along B_1. Then as before we check that $K_{Y'} + g'^{-1}S + E'$ is log terminal, and the other properties of a good extraction. We get what we want if

$$(K_{Y'} + g'^{-1}S + g'^{-1}B + E')_{|E'} \tag{4}$$

has B_1 as a curve of log terminal singularities.

Thus it remains to deal with the case that the locus of log canonical singularities of the log divisor (4) contains another curve C'. Just as before, we can moreover assume that the log divisor (4) is not purely log terminal at a point $Q' \in C'$ outside B_1. I assert that all $d_i > 1$ for components over Q' with $a_i = 0$. Indeed, if Y' is nonsingular along C' then over $Q = C' \cap g'^{-1}C$ there is a surface E with $a = 0$ for $K + S + B$, and multiplicity ≤ 1 in $g'^{-1}S + E'$. For this we need to remark that $K_{Y'} + g'^{-1}S + E$ has index ≥ 4 at $Q \in Y'$, and we can perturb $g'^{-1}B$ to preserve Q not log terminal, while $K_{Y'} + g'^{-1}S + E$ is log terminal on a punctured neighborhood of Q. The multiplicity of E in S equals its multiplicity in $g'^{-1}S + (1+d)E'$, which equals $a + b(1+d) < 1 + d$, where $0 < a < 1$ is the multiplicity of E in $g'^{-1}S$, and $0 < b < 1$ that of E in E', so that $0 < a + b \leq 1$ is the multiplicity of E in $g'^{-1}S + E'$. Thus the surface in question is bimeromorphic to E and has multiplicity d in S. Extracting it gives a curve B_2 in the intersection of the blowup with the blowup of $g'^{-1}S$, and over this there is a surface $\neq E'$ also lying over $Q \in X$ with $a = 0$ and multiplicity $1 + d$, hence bimeromorphic to E''. Hence all the multiplicities for $(1+d)E'$ over Q' are $> 1 + d$, hence $d_i > 1$ for $a_i = 0$ over Q'.

Now assume that Y' is singular along C'. By the same arguments this is a singularity of type A_1, and it is resolved by E. Here E'' lies over $Q \in Y'$, or more precisely over the curve in the inverse image of Q for an extremal resolution of E. This completes the treatment of the cases when the intersection $g^{-1}S \cap E$ is reducible. In the contrary case g satisfies (iii), which implies (ii) by the fact that $g^{-1}B$ is ample on E. Thus g is a good extraction, modulo the reduction to the cases with $d_i > 1$ for $a_i = 0$ over Q.

The following case also reduces to these cases, when $g(E) = C$, (8.8.1) holds and E has multiplicity $d \leq 1$ in S. We now carry out the reduction to flips of type (8.5.3) with $d_i > 1$ for all $a_i = 0$ over a neighborhood of Q. By (8.8.1) $g(B_1) = Q \in X$. Further, from Theorem 6.9 and the fact that $g^{-1}B$ is ample relative to g we deduce that the intersection $g^{-1}S \cap E = B_1 \cup B_2$ consists of two irreducible curves B_1 and B_2 over a neighborhood of $Q \in X$. Here $g(B_2) = C$. Then we check that $K_Y + g^{-1}S + E$ is log terminal outside $P_0 = B_1 \cap B_2$. This implies that E is normal. Note that the curves $D \subset Y$ over Q lie on E and intersect $g^{-1}B$, and thus do not pass through P_0 except for $D = B_1$. The contraction of a curve $D \neq B_1$ does not violate (8.8.1) and does not change the singularity of P_0 on E. Hence P_0 either is nonsingular on E or is an ordinary double point of E. In the first case Y is also nonsingular at P_0. We check that the same also holds in the second case. For this note that under our choice $H = g^*B$, first come flips in the birational transforms of curves of the flip of f. Flips in these do not destroy B_2, nor, in particular, P_0. After these comes the flip in B_1 as described in Proposition 8.3. By the proof of Proposition 8.3, P_0 either is nonsingular on $g^{-1}S$ or has index ≥ 3, so that the index of $K_Y + g^{-1}S$ is also ≥ 3 in the second case. But a cover of E ramified only in P_0 has degree ≤ 2. The required nonsingularity follows from this.

Hence E is the unique surface over a neighborhood of $g(Q) = Q$ with multiplicity $d \leq 1$ in S and $a = 0$. Moreover, the multiplicity of other divisors over a neighborhood of Q with $a = 0$ is at least $1 + d$, and this minimum value is only achieved by E', the first blowup of B_1. From now on, as before, we consider an extraction $g'\colon Y' \to X$ of the divisor E' and verify that it intersects $g'^{-1}S$ in the single curve B_1 only, the birational transform of the curve with the same name. By construction the intersection of E' and $g'^{-1}S$ is normal along B_1. Then we verify the log terminal property of $K_{Y'} + g'^{-1}S + E'$ and the other properties of a good extraction. We get what we need if

$$(5) \qquad (K_{Y'} + g'^{-1}S + g'^{-1}B + E')_{|E'}$$

has log terminal singularities along B_1.

As before, it remains to treat the case that the locus of log canonical singularities of the log divisor (5) has another curve component C'. Morever, as before, we can assume that there exists a point $Q' \in C'$ outside B_1 at which the restricted log divisor (5) is not purely log terminal. But then $d_i > 0$ for all a_i over a neighborhood of Q'. For otherwise there exists a surface over Q' or over the general point of C' with $d_i \leq 1$ and $a_i = 0$. Hence it lies over Q and its multiplicity in S is $\leq 1 + d$ if $a = 0$, which is impossible.

This completes the reduction of Proposition 8.8 to flips of type (8.5.3) with $d_i > 1$ for all $a_i = 0$ over a neighborhood of Q in the cases that the required good extraction does not exist. It remains to establish the existence of the flip of f in these exceptional cases.

For them there exists a surface E over a neighborhood of Q with $a = 0$ and having minimal multiplicity d in S; by assumption $d > 1$. On the other hand $d \leq 2$, since $\operatorname{Supp} B$ touches S along C, and there is a surface over the general point of C with $d = 2$ and $a = 0$. Also, there exist only a finite number of surfaces E over a neighborhood of Q with $a = 0$ and with the given multiplicity d in S. They are all extracted by a log terminal extraction of $K + S + B$. Hence, as before, we can choose an extremal extraction g of one such surface E satisfying (8.8.1) or (8.8.2) for $P \in B_1 \subset g^{-1}S \cap E$. For this, note that $g^{-1}S$ and E cross normally along B_1 and along the other components of $g^{-1}S \cap E$; for otherwise by (3.18.6) over a general point of B_1 there would be a surface E' with $a = 0$ having multiplicity $a + b \leq 1$ in $g^{-1}S + E$, where $0 < a, b$ are the multiplicities of E' in $g^{-1}S$ and E respectively. Hence the multiplicity of E' in $g^{-1}S + dE$, equal to its multiplicity in S, is $a + bd < (a+b)d \leq d$, which contradicts the choice of d.

Next we establish that case (8.8.2) is only possible if P is an isolated singularity of Y from (8.8.4) with $m = 1$. But this case again reduces to flips of type (8.5.3) with $d_i > 1$ for all $a_i = 0$ over a neighborhood of Q. Indeed, the multiplicity $d' := (2/3)(1 + d)$ of the new surface E' satisfies $d' \leq 2$, like d itself, hence if $n = 1$ then $dd' = d - 1 \leq 2 - 1 = 1$, which again contradicts the choice of d. However, now Q has a curve C_3 of double points passing through it, and hence we have reduced the existence of the required flips to the case that g satisfies (8.8.1).

Assume first that $g(E) = Q$. As before, the intersection $g^{-1}S \cap E$ consists of at most 2 irreducible curves B_i. We show that if $g^{-1}S \cap E = B_1 \cup B_2$ then the flip of f exists. For this, as in the treatment of the similar case with $d \leq 1$ we reduce to the following set-up. The divisor $K_Y + g^{-1}S + E$ is log terminal outside $P_0 = B_1 \cap B_2$, Y is nonsingular at P_0 and at $Q = B_2 \cap g^{-1}C$, the surfaces E and $g^{-1}S$ are normal, nonsingular on B_2, and B_2 has selfintersection $(B_2{}^2)_E = 2$ on E and $(B_2{}^2)_{g^{-1}S} = -3 = -n$. The last assertion follows because $d = 3/(n-1)$ and $n = 3$, since $1 < d \leq 2$, so that $d = 3/2$. Note also that B_1 has selfintersection ≤ 0 on the minimal resolution of E, since B_2 intersects B_1 transversally and only at the single point P_0. On the other hand, by the ampleness of $g^{-1}S$ on E we have

$$0 < (g^{-1}S) \cdot B_1 = \big((B_1 + B_2) \cdot B_1\big)_E = (B_1{}^2)_E + 1.$$

Hence $(B_1{}^2)_E = 0$ and E is everywhere nonsingular on B_1. Otherwise E has a unique ordinary double point, P_1 say, and B_1 has selfintersection 0 or -1 on the minimal resolution of E. Hence $(g^{-1}S) \cdot B_1 = 1$, $3/2$ or $1/2$ in the 3 cases. Moreover

$$0 = (g^{-1}S + \tfrac{3}{2}E) \cdot B_1 = (g^{-1}S) \cdot B_1 + \tfrac{3}{2}EB_1,$$

and thus $EB_1 = -2/3$, -1 or $-1/3$ in the 3 cases. The fractional cases are not possible, since P_0 is a nonsingular point of $g^{-1}S$, and $g^{-1}S$ can have at most one ordinary double point P_1 on B_1. Therefore B_1 has an ordinary double point P_1 on E, and B_1 has selfintersection 0 on the minimal resolution of E, and $g^{-1}S$ is nonsingular on $B_1 \cup B_2 = g^{-1}S \cap E$. Moreover B_1 is a (-2)-curve and B_2 a (-3)-curve. There is curve of double points B_{-1} through the point P_1 on $g^{-1}S$. It follows from this that $B_0 = g^{-1}S \cap \operatorname{Supp} g^{-1}B$ is irreducible, has a unique singularity Q' (not over $Q \in X$) of type A_1, resolved by a (-3)-curve, and is a (-1)-curve on the minimal resolution of $g^{-1}S$. But in this case $K + S$ has a purely log terminal complement of index 2, and hence the flip of f exists. Note for this that by Proposition 5.13 and Corollary 5.19, $K + S + 2B$ is strictly log terminal at

Q' and has index 3; one half of its 1-complement at Q' gives the required index 2-complement. Furthermore, $K+S$ has index 2 at $Q \in X$, since E is a quadratic cone with vertex at P_1 and $g^*(K+S) = K_Y + g^{-1}S + (1/2)E$.

Now consider the case that $g(E) = Q$ and $g^{-1}S \cap E = B_1$ is irreducible. Then as above we check that g is a good extraction. Thus we can assume that the locus of log canonical singularities of

$$(K_Y + g^{-1}S + g^{-1}B + E)_{|E}$$

contains another curve C'. But then $Q = B_1 \cap g^{-1}C$ is at worst an isolated singularity of Y. Neither $g^{-1}S$ nor E contain curves of singularities of Y through Q by choice of d and by (3.18.4). The fact that there are no other curves of singularities through Q follows as before from $K_Y + g^{-1}S + g^{-1}B + E$ log canonical. Moreover, if we perturb $g^{-1}B$ in a neighborhood of Q, while fixing $Q \in g^{-1}B$, then Q will satisfy (8.8.2). In the opposite case, arguing as in the proof that $g^{-1}S$ and E cross normally along B_1, we get a surface over Q with $a = 0$ and with multiplicity $< d$ in S, which contradicts the choice of d. Thus (8.8.2) holds, so that Q is a singularity of (8.8.4) with $m = 1$ or $m = 0$. Then if $m = 1$, with the previous choice of $g^{-1}B$ we get that the locus of log canonical singularities of $(g \circ h)^*(K+S+B)$ contains a curve $C_3 \subset E'$ of double points. Hence the surface resolving C_3 has $a = 0$ and its multiplicity in $g^{-1}S + E$ equals $2/3$, which is < 1 and $< d$ for S. Therefore $m = 0$ and Q is nonsingular. In this case we can construct an exceptional 2-complement, blowing up the points P_i if necessary.

We proceed to the final case $g(E) = C$. Then $d = 2$. As before, g satisfies (8.8.1). Further, as in the similar case above with $d \le 1$, we check that the intersection $g^{-1}S \cap E = B_1 \cup B_2$ consists of two irreducible curves over a neighborhood of $Q \in X$. Here $g(B_1) = Q$ and $g(B_2) = C$. Then we check that $K_Y + g^{-1}S + E$ is log terminal outside $P_0 = B_1 \cap B_2$ and E is normal. Again P_0 is nonsingular on X and on $g^{-1}S$, and is either nonsingular on E, or is an ordinary double point of E. The latter case is impossible this time, since the general fiber D of the surface E over C satisfies $(g^{-1}B) \cdot D = 1/2$ and $(g^{-1}B) \cdot B_1 \ge 1/2$. Hence $(g^{-1}B) \cdot B_1 = 1/2$ and the curve B_1 is numerically equivalent to D. In particular

$$-\tfrac{1}{2} = ED = EB_1 = (B_1{}^2)_{g^{-1}S} + (B_2B_1)_{g^{-1}S} = (B_1{}^2)_{g^{-1}S} + 1,$$

and thus $(B_1{}^2)_{g^{-1}S} = -3/2$. Therefore $g^{-1}S$ has just one ordinary quadratic singularity on B_1, say P_1. Then B_1 is a (-2)-curve on the minimal resolution of $g^{-1}S$. By the same arguments $(B_1{}^2)_E = 0$, so that E is nonsingular on B_1 and B_1 is a complete fiber of E over C. Moreover, there is a curve of double points of Y through P_1. Hence $K+S$ has index 1 at $Q \in X$, since $g^*(K+S) = K_Y + g^{-1}S$ has index 1 on B_1. On the other hand, the curve $B_0 = g^{-1}S \cap \operatorname{Supp} g^{-1}B$ is irreducible, does not pass through singularities of $g^{-1}S$, and is a (-1)-curve on $g^{-1}S$ intersecting the curve B_{-1} of double points transversally. Thus the index of $K+S$ in a neighborhood of the flipping curve $g(B_0)$ equals 2, and $(K+S) \cdot g(B_0) = -1/2$. Hence one half of the general hyperplane section of B_1 gives a purely log terminal complement of index 2 and the flip exists by Proposition 2.9. This completes the proof of Proposition–Reduction 8.8. Q.E.D.

Proof of Theorems 1.9–10 *and Corollary* 1.11. According to reductions 6.4–5, reduction 7.6 and Propositions 6.7–8 it is enough to establish the existence of nonexceptional index 2 flips. By reduction 8.2, Proposition 8.3 and reduction 8.4, we can restrict ourselves to flips of type (8.5.1–3). In what follows $h\colon Y \to X$ denotes the good extraction of Propositions 8.6 and 8.8, E the unique exceptional divisor of h

and $B_1 = g^{-1}S \cap E = \mathbb{P}^1$ the irreducible curve of property (iii) in 8.5. Since h is extremal we have $\rho(Y/Z) = 2$ and $\overline{\mathrm{NE}}(Y/Z)$ has two extremal rays R_1 and R_2, with $\mathrm{cont}_{R_1} = h$. The proof continues from this point as in Reductions 7.2 and 8.2. The flips of R_2 are considered separately depending on their types.

We start with type (8.5.1). Suppose first that $B_0 \subset g^{-1}S$, the inverse image of the flipping curve, does not pass through $P = B_1 \cap g^{-1}C$. Then by construction B_0 is irreducible and not contained in $g^{-1}B$. Hence $g^{-1}BR_2 \leq 0$, $ER_2 > 0$ and $g^{-1}SR_2 < 0$. Therefore the support of R_2 coincides with B_0, since $B_1 \in R_1$. Note that the flip in B_0 exists by Corollary 5.20. (Contracting or flipping a ray of $\overline{\mathrm{NE}}(Y/Z)$ of course preserves the morphism to Z.) Here since $K_Y + g^{-1}S + g^{-1}B + E$ is log terminal in a neighborhood of B_0, the flip throws B_0 over into a curve B_0^+ lying on the modified surface E, and preserves the index 2 or index 1 and log terminal property of the given divisor on E in a neighborhood of B_0^+. It is not hard to check that the transformed curve B_0^+ intersects the modified B_1 and is irreducible. The subsequent ray R_2 can be negative against $g^{-1}S$ only when it is generated by the modified $B_1 = g^{-1}S \cap E$ and is hence negative against E. As in reduction 7.2, in this case the flip of f exists. Thus, except for the case of a divisorial contraction, it remains to deal with the case that the next flipping curve C_1 lies on E and does not intersect B_1. Here $B_1^2 > 0$.

Since C_1 is numerically 0 against $g^{-1}S$, it must be negative against E and positive against $g^{-1}B$. If the restriction $(K_Y + g^{-1}S + g^{-1}B + E)_{|E}$ is log terminal in a neighborhood of the support of R_2, the flip of R_2 exists by Corollary 7.3, since it is an exceptional flip of index 2 for every connected component of the flipping curve. (In the analytic case, passing to connected components while preserving the assumptions that the contraction is extremal and the space is $\mathbb{Q}$-factorial can be carried out either by changing the base by an extraction outside a fixed fiber or by localizing as in the proof of reductions 6.4–5.) In the opposite case, by Theorem 6.9, E contains a curve B_2 that intersects the support of R_2 in a unique point Q at which $(K_Y + g^{-1}S + g^{-1}B + E)_{|E}$ is not log terminal, and the reduced part of the boundary of the most recent restriction is of the form $B_1 + B_2$.

The curves B_1 and B_2 intersect in a unique point P, hence since B_1 is ample on the original E, it follows that B_2 is irreducible. But after flipping, the curve B_2 is nef and numerically 0 only on B_0^+. After contracting any of the irreducible components of C_1, since B_1 is ample on the original E, we get that B_2 is ample. By Lemma 8.9 the flipped curve is totally contracted, that it, it is irreducible. Consequently a flip in C_1 again has type (8.5.1), and exists by Proposition 8.6 because the number of good extractions has decreased.

We now proceed to the case that B_0 passes through P. Then the extremal ray R_2 generated by B_0 is positive against E and $g^{-1}B$, but negative against $g^{-1}S$. The flip in B_0 exists by Corollary 5.20 and the flipped curve B_0^+ lies in the intersection of the modified $E \cap \operatorname{Supp} g^{-1}B$, and $B_1 = E \cap g^{-1}S$. Again it is enough to consider the case that the flipping curve C_1 is on E and does not intersect $g^{-1}S$. If the locus of log canonical singularities of the restriction $(K_Y + g^{-1}S + g^{-1}B + E)_{|E}$ is disjoint from C_1 then the flip exists and is of type IV by Proposition 5.13. Otherwise, by Theorem 6.9, B_0^+ is irreducible and is contained in the reduced part of the boundary of the most recent restriction.

On the other hand, the divisor $g^{-1}B_{|E}$ was ample before the flip, and its support intersects B_1 only in P. Hence the support of the modified $g^{-1}B_{|E}$ is contained in C_1 and is a contractible curve. After its contraction, by Lemma 8.10 the curve

B_0^+ becomes numerically ample and by Lemma 8.9 the image of C_1 must be 0, that is, the support of the modified $g^{-1}B_{|E}$ is exactly C_1. If the divisor $K_Y + g^{-1}S + g^{-1}B + E$ is log terminal along every component of C_1, we arrive at a flip of type (8.5.3); otherwise C_1 is irreducible and defines a flip of type (8.5.2). This completes the reduction in case (8.5.1).

Consider now the unstarred case (8.5.2), when the good extraction g has an exceptional divisor over a point. By construction, in this case, the birational transform of B_0, the curve contracted by f, generates R_2. Thus R_2 satisfies $ER_2 > 0$ and $g^{-1}SR_2 < 0$. Hence the flip of B_0 exists by Corollary 5.20.

After the flip the curve $g^{-1}S \cap E$ may be reducible. However, this can only happen when $g^{-1}B$ is numerically negative against B_0. By our choice $H = g^*B$, $K_Y + g^{-1}S$ remains log terminal. Moreover the flipped curves on $g^{-1}S$ land in the intersection with E. As in the proof of Proposition 8.8, the intersection $g^{-1}S \cap E$ has at most two curves, B_1 and a flipped curve B_2. In particular, B_2 is exceptional on E^ν. Now B_1 becomes the support of the subsequent extremal ray, which is numerically 0 against $g^{-1}B$, positive against E and negative against $g^{-1}S$. It follows that $g^{-1}B$ is positive on all the remaining curves of E. As in the proof of Proposition 8.8, using this one can verify that $K_Y + g^{-1}S + E$ is log terminal in a neighborhood of E except at the point $P_0 = B_1 \cap B_2$. Hence E is normal.

We carry out the flip in B_1 described in Proposition 8.3. The arguments from the proof of Proposition 8.8 in the case (8.5.2*) allow us to prove either that the flip of f exists, or that B_2 is nef on the minimal resolution of E. But the last case is impossible since B_2 is exceptional on E. Hence we can assume that the intersection $g^{-1}S \cap E = B_1$ is irreducible. The log terminal property of $K_Y + g^{-1}S + E$ continues to hold on $g^{-1}B$, which is nef on B_0, as follows since otherwise $g^{-1}B$ is ample on the modified E.

Thus, again the new flipping curve C_1 is contained in E and does not intersect $g^{-1}S$. If the restriction $(K_Y + g^{-1}S + g^{-1}B + E)_{|E}$ has log terminal singularities along C_1 then the flip exists by Corollary 7.3 as above. Otherwise by Theorem 6.9 there exists an irreducible curve B_2 that is contained together with B_1 in the boundary of $(K_Y + g^{-1}S + g^{-1}B + E)_{|E}$ after modification and intersecting B_1 in P. I claim that B_2 is contained in the support of the new ray R_1, that is, the ray obtained after flipping B_0. For otherwise all the components of the flipped curve B_0^+ would intersect B_1 only at P. After contracting B_0^+ we return to the situation before flipping, when the curve $B_2 = \operatorname{Supp} g^{-1}S \cap E$ is ample on E. Therefore by Lemma 8.9 there is no C_1. Thus B_2 is contained in B_0^+, the remainder of B_0^+ is contracted to a point P and its components intersect B_1 and B_2 only at P. The surface E is normal, because $K_Y + E$ is log terminal.

By Theorem 6.9 and Lemma 8.9, after contracting the support of $g^{-1}B_{|E}$ outside B_2 we get that C_1 coincides with the given contracted curve. By Lemma 8.10, B_2 becomes ample after contracting C_1 and the components of B_0^+ other than B_2. (The components of $g^{-1}B_{|E}$ other than B_2 are contained in C_1, since they do not intersect B_1 and are numerically 0 on $g^{-1}S$.) If the support of $g^{-1}B_{|E}$ outside B_2 contains a curve along which $K_Y + g^{-1}S + g^{-1}B + E$ does not have log terminal singularities, then it coincides with it, and the contraction of the curve in question has type (8.5.2). Here in the unstarred case (8.5.2), by Proposition 8.6, the number of good extractions is decreased and the flip exists by induction. In the opposite case we get a reduction to type (8.5.2*). Type (8.5.3) arises if $K_Y + g^{-1}S + g^{-1}B + E$ is log terminal along C_1.

In case (8.5.2*), the ray R_2 with $g^{-1}SR_2 < 0$ leads at the first step to a flip in B_1 and separates the surfaces E and $g^{-1}S$. After this the contraction of E to a point gives a flip of f. Thus the case that is essential for us is when the flipping curve C_1 is contained in E and disjoint from $g^{-1}S$. As above, we need only consider the case that C_1 passes through a point at which the restriction $(K_Y + g^{-1}S + g^{-1}B + E)_{|E}$ is not log terminal. Then the fiber B_2 of the ruled surface E over $P = B_1 \cap g^{-1}C$ is irreducible and contained in the boundary of $(K_Y + g^{-1}S + g^{-1}B + E)_{|E}$. Since $g^{-1}B$ is positive against R_1 and R_2 it is positive on E, and by contracting the components of $\operatorname{Supp} g^{-1}B_{|E}$ other than B_2 we transform B_2 into an ample curve.

Thus again by Lemma 8.9, C_1 coincides with the given contracted curve. If C_1 contains a curve of the locus of log canonical singularities of $K_Y + g^{-1}S + g^{-1}B + E$ then it is equal to it, and the contraction of the given curve is of type (8.5.2*). Here by choice of a good extraction in Proposition 8.8, δ decreases. Indeed, the exceptional divisors of E_i over C_1 have log discrepancy 0 for $K_Y+g^{-1}S+g^{-1}B+E$ precisely when $a_i = 0$, and the multiplicity of E_i in E equals its multiplicity in $g^{-1}S+E$, and is greater than or equal to its multiplicity in $g^{-1}S+dE = g^*S$. This gives strict monotonicity for δ. In the remaining cases we get a reduction to type (8.5.3).

In case (8.5.3), we first perform flips in curves of the intersection $E \cap \operatorname{Supp} g^{-1}B$. These curves intersect B_1 in points P_1 and P_2 that are log terminal for $K_Y+g^{-1}S+g^{-1}B+E$. Hence after such flips the intersection $B_1 = g^{-1}S \cap E$ remains irreducible. However, now $g^{-1}B$ only intersects B_1 in a point $P = B_1 \cap g^{-1}C$ which is possibly not log terminal, and the curve B_1 becomes the unique curve on $g^{-1}S$ over Z in a neighborhood of the flipping fiber. Again it remains to deal with the case that the subsequent flipping curve C_1 is contained in E and is disjoint from $g^{-1}S$. As before, we need only consider the case that C_1 passes through a point Q' at which the restriction $(K_Y + g^{-1}S + g^{-1}B + E)_{|E}$ is not log terminal. Then there exists an irreducible curve $B_2 \neq B_1$ such that $B_1 + B_2$ is the reduced part of the boundary of the restriction and $Q' \in B_2$. Since $g^{-1}B$ is positive against R_1 and R_2, it is ample on E.

By Lemma 8.10, after contracting the components of $\operatorname{Supp} g^{-1}B_{|E}$ other than B_2 we transform B_2 into an ample curve. Thus again by Lemma 8.9 C_1 coincides with the given contracted curve. But by construction C_1 is not contained in the locus of log canonical singularities of $K_Y + g^{-1}S + g^{-1}B + E$. Here by our choice of the good extraction in Proposition 8.8, δ decreases, or more precisely δ' for Q' is less than δ. Q.E.D.

§9. APPLICATIONS

We give here some consequences of the main results.

9.1. **Corollary.** *An algebraic (or analytic) 3-fold X has a strictly log terminal model $f: Y \to X$ for $K+B$ (in the analytic case, in a neighborhood of a projective subspace $W \subset X$), even if X is not $\mathbb{Q}$-factorial and $K+B$ is not log canonical. Moreover, there exists such a model $f: Y \to X$ that is nontrivial only over the points of X at which X is not $\mathbb{Q}$-factorial or $K+B$ is not log terminal. (In the analytic case, we should include in the non-log-terminal locus the singular curves of irreducible and reduced components B in a neighborhood of W.)*

Proof. The singular locus of X and of components of B, together with the nonnormal crossings of components of B, form a closed algebraic (or analytic) subset of dimension ≤ 1, so that the same holds for the locus M of points at which $K+B$ is not log terminal. Hence through M (in the analytic case, in a neighborhood of W) we can choose a general hyperplane section H such that, outside M, $K+B+H$ is log terminal and $H+\lceil B\rceil$ has normal crossings. After this, the proof of reductions 6.4–5 is applicable. However, now flips exist by (1.3.5) and the corollary in §0, and the log terminal model satisfies the properties we want. By (1.5.7), it is enough to verify this for the strictly, but not purely, log terminal points of $K+B$. Since $K+B$ is divisorially log terminal, these are either triple points, or points on double curves. Now H does not pass through triple points or contain double curves. But the log terminal extraction over a point of a double curve is the identity, since by Corollary 3.8 the log discrepancy of $K+B$ over such a point is >0. Q.E.D.

The next result follows from this in the same way that Corollary 1.11 follows from Theorem 1.10.

9.2. **Corollary.** *An algebraic (or analytic) 3-fold X has a log canonical model for $K+B$, even if X is not $\mathbb{Q}$-factorial and $K+B$ not log canonical, provided that $K+B$ is log terminal outside B and B locally supports a Cartier divisor.*

The last clause means that B is LSEPD for id_X. If $K+B$ is numerically negative relative to f, Theorem 1.10 can be strengthened:

9.3. **Corollary.** *Let $f\colon X\to Z$ be a contraction of an algebraic (or analytic) 3-fold X, and suppose that $K+B$ is log terminal and numerically negative relative to f. Then the flip of f exists.*

Proof. By Corollary 9.1, and the fact that log terminal singularities are $\mathbb{Q}$-factorial in codimension 2, there exists a strictly log terminal extraction $g\colon Y\to Z$ for $K_Z+f(B)$, that blows up only the image of the exceptional set M for f in a neighborhood of M. As in (1.5.6) one can check that because $K+B$ is negative relative to f, the log discrepancies of $K_Y+f(B)_Y$ are positive over M. In particular g is small over M and is log terminal. Thus it remains to contract a finite number of curves over M on which $K_Y+f(B)_Y$ is numerically 0. These obviously span an extremal face of $\overline{\mathrm{NE}}(Y/Z)$. But by the above, in a neighborhood of the connected fibers of g containing those curves, it is not hard to find an effective Cartier divisor D such that $K_Y+f(B)_Y+\varepsilon D$ is log terminal for small $\varepsilon>0$ and is negative on the whole fiber. On localizing around a fiber one may lose the strictly log terminal property, but weakly log terminal is preserved. Hence by [8], 3-2-1, and rational approximation (1.3.5), these curves are contractible; by the same arguments, $K_Y+f(B)_Y$ pushes down as a log terminal divisor (compare the proofs of (1.3.2) and 4.5). Q.E.D.

9.4. **Corollary.** *Let $f\colon X\to Z$ be a projective morphism of algebraic 3-folds (or analytic spaces), and suppose that $K+B$ is weakly log terminal. Then every extremal face R of the Kleiman–Mori cone $\overline{\mathrm{NE}}(Y/Z)$ (in the analytic case, $\overline{\mathrm{NE}}(Y/Z\,;W)$, where W is a compact subset of Z) on which $K+B$ is negative defines either a nontrivial fiber space cont_R of log Fanos, or a flip tr_R of the contractions cont_R (respectively, over a neighborhood of W).*

The proof is similar to that of the corollary in §0, replacing the theorem of §0 by Corollary 9.3. The contraction cont_R exists by [8], 3-2-1, and rational approximation (1.3.5).

The conditions that $K+B$ is weakly log terminal and X projective over Z is preserved under flips (in the analytic case, over a neighborhood of W). Hence the termination of flips would either give a nontrivial fiber space cont_R of log Fanos, or a log terminal model of f (in the analytic case, over a neighborhood of W). However, *termination remains a conjecture.* ([14])

Corollary 9.1 and Inversion 3.4 give the following result:

9.5. **Corollary.** *Problem* 3.3 *on the inversion of adjunction has an affirmative solution in dimension* 3.

Appendix by Y. Kawamata: The minimal discrepancy of a 3-fold terminal singularity

Theorem. *Let* (X, P) *be a* 3*-dimensional terminal singularity of index* $r > 1$, *and* $\mu\colon Y \to X$ *a resolution of singularities. Write* E_j *for the exceptional divisors, with* $1 \le j \le t$, *and set*

$$K_Y = \mu^* K_X + \sum_{j=1}^{t} d_j E_j.$$

Then $d_j = 1/r$ *for some* j, *and hence* $\min\{d_j\} = 1/r$.

Proof. ([15]) It is enough to construct some partial resolution $\nu\colon X' \to X$ where X' is a normal variety having an exceptional divisor E of discrepancy $1/r$, in other words, such that

$$K_{X'} = \nu^* K_X + \frac{1}{r} E + \text{other components}.$$

For this, we use Mori's classification of terminal singularities ([Mori] or [22], (6.1)); up to local analytic isomorphism, (X, P) is the singularity at the origin of a hypersurface $X : (\varphi = 0) \subset W = \mathbb{C}^4/G$ in a quotient of $\mathbb{C}^4$ by $G = \mathbb{Z}/(r)$. The quotient W is of type $\frac{1}{r}(a, b, c, d)$ if a generator of G acts on $\mathbb{C}^4$ by

$$(x, y, z, w) \mapsto (\zeta^a x, \zeta^b y, \zeta^c z, \zeta^d w),$$

where ζ is a primitive rth root of 1. The hypersurface X is defined by a semi-invariant $\varphi(x, y, z, w)$. We prove the result separately in the following six cases.

Case 1. W is of type $\frac{1}{r}(a, -a, 0, 1)$ and $\varphi = xy + f(z, w^r)$, where $0 < a \le r$ are coprime integers. Give z and w weights $\mathrm{wt}(z, w) = (1, 1/r)$, and write $k = \operatorname{ord} f$; we consider the weighted blow-up $\sigma\colon W' \to W$ with weights

$$\mathrm{wt}(x, y, z, w) = (a/r + i, k - i - a/r, 1, 1/r)$$

for an arbitrary fixed i with $0 \le i < k$. Then W' has an affine open subset U, where U is a quotient of $\mathbb{C}^4$ of type

$$\frac{1}{a + ir}(-r, (k - i)r - a, r, 1),$$

and σ is the map given on U by

$$(x, y, z, w) \mapsto (x^{a/r+i}, x^{k-i-a/r} y, xz, x^{1/r} w).$$

On U, the birational transform X' of X is defined by $y + f(xz + xw^r)x^{-k} = 0$, the exceptional divisor F of σ by $x = 0$, and $E = X' \cap F$ is reduced. Set $K_{W'} = \sigma^* K_W + \alpha F$, $\sigma^* X = X' + \beta F$ and $K_{X'} = \nu^* K_X + dE$, where $\nu = \sigma_{|X'}$. Then

$$\alpha = (a/r + i) + (k - i - a/r) + 1 + 1/r - 1 = k + 1/r,$$

([14]) See [Kawamata3, 4] and [Utah], Theorems 6.10–11 and 6.15, for more recent information.

([15]) According to the Kollár, a student of Mori has proved that the discrepancies take all the values $\{1/r, 2/r, \dots, (r-1)/r\}$. (See also [28], 4.8).

and $\beta = k$, so that E is the required component with discrepancy $d = \alpha - \beta = 1/r$. In this case it can be proved that X' has only terminal singularities and E is irreducible. The calculation in all the other cases is similar.

Case 2. W is of type $\frac{1}{2}(1, 0, 1, 1)$ and $\varphi = x^2 + y^2 + f(z, w)$. Suppose $\operatorname{ord} f = 2k$, and take the weighted blow-up $\sigma\colon W' \to W$ with weights

$$\operatorname{wt}(x, y, z, t) = \begin{cases} (\frac{k}{2}, \frac{k+1}{2}, \frac{1}{2}, \frac{1}{2}) & \text{if } k \text{ is odd,} \\ (\frac{k+1}{2}, \frac{k}{2}, \frac{1}{2}, \frac{1}{2}) & \text{if } k \text{ is even.} \end{cases}$$

Then W' has an affine open set U of type $\frac{1}{k+1}(k, -2, 1, 1)$ if k is odd or of type $\frac{1}{k+1}(-2, k, 1, 1)$ if k is even, and σ is the map

$$(x, y, z, w) \mapsto \begin{cases} (xy^{k/2}, y^{(k+1)/2}, y^{1/2}z, y^{1/2}w) & \text{if } k \text{ is odd,} \\ (x^{(k+1)/2}, x^{k/2}y, x^{1/2}z, x^{1/2}w) & \text{if } k \text{ is even.} \end{cases}$$

Here X' is given on U by $x^2 + y + f(y^{1/2}z, y^{1/2}w)y^{-k} = 0$, and the exceptional divisor F by $y = 0$ if k is odd (respectively, the same with x and y interchanged if k is even), and $E = X' \cap F$ is reduced. Since $\alpha = k + 1/2$ and $\beta = k$ we have $d = 1/2$.

Case 3. W is of type $\frac{1}{2}(1, 1, 0, 1)$ and $\varphi = w^2 + f(x, y, z)$, with $\operatorname{ord} f = 3$; we take σ with weights

$$\operatorname{wt}(x, y, z, t) = (1/2, 1/2, 1, 3/2).$$

Then the open set U has type $\frac{1}{3}(1, 1, 2, -2)$, and σ is given by

$$(x, y, z, w) \mapsto (xw^{1/2}, yw^{1/2}, zw, w^{3/2}).$$

X' and F are respectively given by $w + f(xw^{1/2} + yw^{1/2}, zw)w^{-2} = 0$ and $w = 0$. Thus E has a reduced irreducible component E_1. If we set $K_{X'} = \nu^* K_X + dE_1 +$ other components, then $\alpha = 5/2$, $\beta = 2$, hence $d = 1/2$.

Case 4. W is of type $\frac{1}{3}(1, 2, 2, 0)$ and $\varphi = w^2 + f(x, y, z)$, where $\operatorname{ord} f = 3$; moreover, if we write $f = f_3 +$ higher order terms, then $f_3 = x^3 + y^3 + z^3$ or $x^3 + yz^2$ or $x^3 + y^3$. In the first case (respectively, the remaining cases) we take σ with weights

$$\operatorname{wt}(x, y, z, t) = (2/3, 1/3, 1/3, 1) \quad (\text{respectively } (2/3, 4/3, 1/3, 1).$$

Then U has type $\frac{1}{3}(2, 1, 1, 0)$, and σ is given by

$$(x, y, z, w) \mapsto (xw^{2/3}, yw^{1/3}, zw^{1/3}, w)$$
$$(\text{respectively } (xw^{2/3}, yw^{4/3}, zw^{1/3}, w)).$$

Then in the three cases, X' is given by $w + x^3w + y^3 + z^3 + \cdots$ or by $1 + x^3 + yz^2 + \cdots$ or by $1 + x^3 + y^3w^2 + \cdots$, while F is given by $w = 0$. So F is reduced, $\alpha = 4/3$, $\beta = 1$ (respectively $\alpha = 7/3$ and $\beta = 2$), and $d = 1/3$.

Case 5. W is of type $\frac{1}{2}(0, 1, 1, 1)$ and $\varphi = w^2 + x^3 + xf(y, z) + g(y, z)$, where $\operatorname{ord} f \geq 4$ and $\operatorname{ord} g = 4$. Write $g = g_4 +$ higher order terms; if g_4 is a square we can assume that $g_4 = y^4$ or y^2z^2. If g_4 is not a square (respectively, is a square), we take σ with weights

$$\operatorname{wt}(x, y, z, w) = (1, 1/2, 1/2, 3/2) \quad (\text{respectively } (1, 3/2, 1/2, 3/2)).$$

Then U has type $\frac{1}{2}(0, 1, 1, 1)$, and σ is given by

$$(x, y, z, w) \mapsto (x, x^{1/2}y, x^{1/2}z, x^{3/2}w)$$
$$(\text{respectively } (x, x^{3/2}y, x^{1/2}z, x^{3/2}w)).$$

X' is given by $xw^2+x+f(x^{1/2}y, x^{1/2}z)x^{-1}+g(x^{1/2}y, x^{1/2}z)x^{-2}=0$ (respectively $w^2+1+f(x^{3/2}y, x^{1/2}z)x^{-2}+g(x^{3/2}y, x^{1/2}z)x^{-3}=0$) and F by $x=0$. So F is reduced, $\alpha=5/2$ and $\beta=2$ (respectively $\alpha=7/2$ and $\beta=3$), hence $d=1/2$.

Case 6. W is of type $\frac{1}{4}(1, 3, 2, 1)$ and $\varphi=x^2+y^2+f(z, w^2)$. Give z and w weights $\mathrm{wt}(z, w)=(1, 1/2)$, and suppose $\operatorname{ord} f=k$; then take σ with weights

$$\mathrm{wt}(x, y, z, t)=\begin{cases}(\frac{k}{4}, \frac{k+2}{4}, \frac{1}{2}, \frac{1}{4}) & \text{if } k\equiv 1 \bmod 4,\\ (\frac{k+2}{4}, \frac{k}{4}, \frac{1}{2}, \frac{1}{4}) & \text{if } k\equiv 3 \bmod 4.\end{cases}$$

Then U is of type $\frac{1}{k+2}(k, -4, 2, 1)$ if $k\equiv 1 \bmod 4$ (respectively $\frac{1}{k+2}(-4, k, 2, 1)$ if $k\equiv 3 \bmod 4$), and σ is given by

$$(x, y, z, w)\mapsto\begin{cases}(xy^{k/4}, y^{(k+2)/4}, y^{1/2}z, y^{1/4}w) & \text{if } k\equiv 1 \bmod 4,\\ (x^{(k+2)/4}, x^{k/4}y, x^{1/2}z, x^{1/4}w) & \text{if } k\equiv 3 \bmod 4.\end{cases}$$

X' is given by $x^2+y+f(y^{1/2}z, y^{1/2}w^2)y^{-k/2}$ and F by $y=0$ if $k\equiv 1 \bmod 4$ (respectively the same with x and y interchanged if $k\equiv 3 \bmod 4$). Thus E is reduced, $\alpha=(2k+1)/4$ and $\beta=k/2$, hence $d=1/4$. Q.E.D.

§10. Commentary by M. Reid

> Nothing is easier than for a man to translate, or copy, or compose a plausible discourse of some pages in technical terms, whereby he shall make a shew of saying somewhat, although neither the reader nor himself understand one tittle of it. ([16])

10.1. **General review of contents of paper.** The paper proposes a program for constructing 3-fold flips (including Mori's flips) and log flips, and claims to carry it out. The rough idea is an inductive approach along the following lines: suppose $f\colon X\to Z$ is a flipping contraction, with exceptional curve C. We attempt to construct a partial resolution $Y\to X$ that either blows up one point on the flipping curve, or blows up C at the general point, and such that the composite $Y\to Z$ has $\rho(Y/Z)=2$, and $\overline{\mathrm{NE}}(Y/Z)$ has two extremal rays (or log rays) R_{old} and R_{new}. One of these gives the old contraction to X, and R_{new}, if it exists and is divisorial, gives the new contraction to the flipped X^+. Then $X \dashrightarrow X^+$ is the flip.

In carrying out this construction, we need to use auxiliary flips for two purposes:

(1) To establish the model Y: Start from a more-or-less arbitrary resolution of singularities $\widetilde{X}\to X$ that includes either a blowup of a point of C or a blowup of C itself, then proceed to climb down from $\widetilde{X}$ to the controlled model Y by the minimal model program.

(2) To deal with the possibility that R_{new} is not a divisorial contraction.

In order for this to provide a proof of the existence of flips, we need to know that the auxiliary flips can be done. This might be achieved in one of two ways: either (a) by induction, because we can assert that the auxiliary flipping contractions are simpler than $f\colon X\to Z$ (for example, some invariant is smaller); or (b) because we know the auxiliary flipping contractions from some other point of view, for example, as fibers of semistable families of surfaces, for which the epic theorem of Tsunoda-Shokurov-Kawamata-Mori (see [7], [23], and [28]) is applicable.

Shokurov's attempt at this in the 3-fold case is extremely serious, and it seems to me almost certain that it is correct and complete (after all, he is the master of the

([16]) George Berkeley, Bishop of Cloyne, "A defence of free-thinking in mathematics" (1735), in his *Works*, Vol. IV, T. Nelson, London, 1951, p. 140.

spaghetti proof), although the presentation cannot exactly be described as attractive. The Utah seminar [Utah] seems to guarantee the results (at least in general terms) up to the middle of §8. It seems at least possible to me that we may eventually fully understand the inductive workings of Mori theory, and that we will then be able to make this program work purely by induction, maybe even in higher dimensions when the more concrete approach in terms of classifying singularities seems doomed.

In addition to his main theorems, Shokurov introduces several important new ideas, including

(1) the LSEPD trick (see before Example 1.6, (10.5) below, and compare [Utah], Definition 2.30);
(2) the ideas of §5 on complements of a log divisor and the 1-, 2-, 3-, 4- and 6-complements that are characteristic to dimensions 2 and 3 (compare [Utah], §19);
(3) the insight in §4 that invariants of log canonical singularities and varieties such as discrepancy, index and so on have "spectral" properties such as a.c.c. (see [Utah], Theorem 1.32, for a discussion);
(4) the ideas and results on "inverting adjunction" of Problem 3.3.

Shokurov's theorem on log flips already has very substantial applications in the literature, most notably Kawamata's solution of the abundance conjecture ([Kawamata 2] and [Utah], Chapters 10–15).

10.2. **The Utah seminar.** A preprint of this translation was circulated as a Warwick and Utah preprint in May 1991, and formed the basis for the second Utah summer seminar on Mori theory, August 1991. A number of corrections by participants in the seminar have been included in the final edition of the translation. Most importantly, the seminar discovered the mistake in the preprint version of Proposition 8.3. The book of the seminar [Utah] works out (and straightens out) practically all the ideas of results of this paper.

It is clear that the Utah book represents a very major step in Mori theory, and 3-fold geometry more generally. However, I find regrettable their attitude towards the mistake in §8 of the preprint, which they try to make out as a terminal crash. It is not hard to point to similar mistakes in the papers and preprints of several of the top specialists in the subject. If Shokurov's patch (in 7 or 8 pages) of Proposition 8.3 turns out to be correct, then it is surprising that a *truly joint effort* of 30 seminar participants failed to look for it or to find it. It was a traditional complaint of Soviet mathematicians (Arnol'd and Shafarevich were outstanding specimens) that their work would be implicitly rubbished by Westerners working under infinitely better conditions, and I would like to echo the sentiment in this case.

10.3. **The log category.** This section is an appendix to (1.1). The *genuine*, that is, nonlog category of classification theory has varieties X, birational morphisms $f\colon X \to Y$, a notion of resolution, birational transform of divisors; then canonical divisors K_X, discrepancy $K_Y = K_X + \Delta_f$ (in essence the Jacobian determinant of f); and of course other ingredients such as irregularity $H^1(X, \mathscr{O}_X)$, which we're not dealing in at present. The moral backbone of the theory is that invariants such as irregularity $H^1(X, \mathscr{O}_X)$ and the plurigenera are biregular invariants of X, and are birational invariants when restricted to varieties with canonical singularities.

The log category was developed by Iitaka in the 1970s. This deals with pairs X with B, where B is a divisor (usually reduced); and the intention is to study differentials with *log poles* along B. The basic starting point of the theory is Grothendieck and Deligne's theory of Hodge structures; Deligne proves that if X with B is nonsingular with normal crossings, then the log de Rham complex $\Omega^{\bullet}{}_X(\log B)$ of differential forms on X with log poles along B defines a Hodge theory that is a biregular invariant of $X \setminus B$. Starting from this, Iitaka went on to discuss log plurigenera $H^0(m(K_X + B))$, log irregularity, etc. as "proper birational" invariants of X with B; this means that in addition to being biregular invariant of the "open" variety $X \setminus B$, they are also invariant under some operations that change $X \setminus B$ by blowups or blowdowns without losing exceptional divisors. More recently, log varieties have achieved prominence in classification theory as a kind of intermediate step between dimension n and $n+1$.

It is most unfortunate that Iitaka's students, Fujita, Kawamata and others, while making formidable technical extensions of this notion, have lost sight of the simplicity of Iitaka's original conception. If you are faithful to the original guiding principles, and regard the log category as part of primeval creation, then there is practically no argument about the correct log generalisation of the notions of the genuine category, and all the fine points about when multiplicity 1 is allowed are just irrelevant. In fact the only substantial argument is whether to extend the category of log varieties to allow nonnormal varieties with ordinary double points in codimension 1, to which the whole apparatus of differentials with log poles extends very naturally, and with many compatibilities, for example, invariance under log normalisation and under restriction to a component. This is the *semilog category* of the Utah seminar, [Utah], Chapter 12.

(10.3.1) **Definition.** A *log variety* X with B is a normal variety X with a $\mathbb{Q}$-Weil divisor B with multiplicities $0 \le b_i \le 1$.

The initial case is that all $b_i = 1$, but one might reduce a value of b_i, if we are absolutely certain that no log pluricanonical differential $\in H^0(X, (\Omega^1(\log B))^{\otimes m})$ will ever have a higher order pole along B_i. This is Kawamata's approach to minimal models of log surfaces [Kawamata], when you have to reduce the b_i in a Zariski decomposition, at the same time as contracting certain (-1)-curves, to get $K_X + B$ nef on a log surface.

(10.3.2) **Definition.** A *log morphism* $f\colon X$ with $B_X \to Y$ with B_Y is a morphism $f\colon X \to Y$ such that $f(B_X) \subset B_Y$. It is *log proper* if f is proper and B_X contains the set-theoretic inverse image $f^{-1}(B_Y)$, and all exceptional divisors of f (including those not mapping anywhere near B_Y) with multiplicity 1. If $f\colon X \to Y$ and B_Y are given, there's a unique B_X that fits the bill, the *log birational transform* $(f^{-1})^{1,\log}(B_Y)$. (Compare after (1.1).) That is, if $X \to Y$ is a morphism, and D a divisor on Y, the *log birational transform* of D on Y is the birational transform (see (10.8.3) below) plus all the exceptional divisors of f with multiplicity 1.

The exceptional components of f must be included in B_X in order to ensure that $f\colon X \setminus B_X \cong Y \setminus B_Y$, so that invariants of X with B and Y with B_Y defined by log differentials in codimension 1 coincide.

(10.3.3) **Definition.** X with B is *log nonsingular* if X is nonsingular and $\operatorname{Supp} B$ is a divisor with (local) normal crossings. $f\colon X$ with $B_X \to Y$ with B_Y is a *log resolution* of Y with B_Y if X with B is log nonsingular and f is log proper.

(10.3.4) **Exercise.** Find your own definition of log discrepancy, log canonical and log terminal singularities, and check that they agree with those of §1 (compare [8]).

Log canonical and log terminal surface singularities were completely classified in [Kawamata] (although the definition was not explicitly known until around 1981, possibly first occurring in work of S. Tsunoda). It's an exercise to prove from first principles that a log terminal surface singularity $P \in S$ with $B = 0$ is a quotient singularity (several different proofs are possible). Thus the codimension 2 behaviour of log varieties with log canonical singularities is completely known.

(10.3.5) There are important respects in which the log category is simpler to work in than the genuine category. The behaviour under cyclic covers is an obvious case. Another case is the toric description of plurigenera and log plurigenera of hypersurfaces singularities: if f is a polynomial that is nondegenerate for its Newton polygon Newton(f), then the log plurigenera of the hypersurface singularity $(f = 0)$ are given in the simplest possible way in terms of the interior of Newton(f), whereas the genuine plurigenera are very much more subtle (see [22], Remark 4.14). I have repeatedly failed to impress the importance of this point on singularity theorists.

10.4. **Eventual freedom.** A point that requires care is the correct statement of Kawamata's eventual freedom theorem for log varieties; in some quarters, it's been taken for granted for almost 10 years that the log version is false as stated, and that to get a good statement requires mutilating the log category by imposing restrictions of the form $b_i < 1$. (This is the source of an incredible lot of mess in the theory, for example all the different technical flavours of log terminal.) In reality, you just have to give the right statement of what "log big" means, and the theorem goes through exactly as in the case of the genuine category of varieties.

I explain. Recall Zariski's famous counterexample to finite generation: a nonsingular rational surface S, an elliptic curve $E \sim -K_S$ with $E^2 = -1$, and a divisor class L on S such that L is nef and big, $LE = 0$, but $L_{|E}$ is a nontorsion divisor of degree 0 (for example, this arises by blowing up $k \geq 10$ general points P_i on a plane cubic E_0, and considering plane curves of degree $k \geq 10$ having these as triple points). Then $|mL|$ has scheme-theoretic base locus E for every $m \gg 0$, so that $R(S, L)$ is not finitely generated; or again, E is contractible in the category of analytic spaces, but not projectively contractible. If you think the log version of eventual freedom is

$$L \text{ nef and } L - \varepsilon(K_S + E) \text{ nef and big} \quad \Longrightarrow \quad L \text{ is eventually free},$$

then of course Zariski's example is a counterexample, since L is nef and big and $K_S + E = 0$. What's really going on here is that L is numerically 0 on a boundary component E with $K_E = (K_S + E)_{|E} = 0$ of (log) Kodaira dimension ≥ 0; to make further progress with this particular L involves the log classification theory of E. In other words, minimal model theory (which should only involve K not nef, and the aim is to exploit the vanishing of cohomology groups) has got mixed up with classification theory (K nef, the point is to prove that cohomology groups do not vanish). In other words, you must stay away from $b_i = 1$ if you intend to restrict $K + E$ to a nonexceptional component E of B on which L is numerically 0 and hope to be able to continue to use Kawamata-Viehweg vanishing.

Thus define *log big* to mean that $L^k\Gamma > 0$ for every k-dimensional stratum $\Gamma = B_1 \cap \cdots \cap B_{n-k} \subset X$ of the log divisor with $b_1 = \cdots = b_{n-k} = 1$. Then eventual freedom can be stated and proved in exactly the same form as in the genuine case. Shokurov's LSEPD trick of subtracting a relatively principal divisor is one way of reducing this to the standard Kawamata result.

10.5. **LSEPD.** If $f \colon X \to Z$ is a proper morphism, and B a boundary on X, the notion here is LSEPD, that is, B *supports a principal divisor locally over* Z (or relative to f). That is, for any point $z \in Z$ there is a rational (respectively meromorphic) function h on Z defined locally at z such that one connected component of the principal divisor $f^*(\operatorname{div} h)$ contains $\lfloor B \rfloor$ and is contained in $\operatorname{Supp} B$. In other words, this means that there exists a divisor B' whose support contains all the components with $b_i = 1$ and is contained in $\operatorname{Supp} B$ and which is a fiber relative to f: there exists a morphism $Z \to C$ of Z to a curve such that B' equals a union of fibers of the composite $X \to C$. Working locally over Z, which is enough for the construction of the log canonical model, this is equivalent to the existence of an effective Cartier divisor D with $\operatorname{Supp} D = \operatorname{Supp} B'$, that is, linearly 0 relative to f, that is, principal on X (locally over Z). Moreover, by what we have said, we can replace linearly 0 with numerically 0 relative to f, even when f is *weakly log canonical*, that is, $K + B$ is nef relative to f (compare (1.5.7)), B is a $\mathbb{Q}$-divisor and $K + B$ is big relative to f.

10.6. **Shokurov's different.** The different is well known (without the name). By the adjunction formula 3.1, it measures the failure of the $\mathbb{Q}$-divisor adjunction formula $(K+S)_{|S} - K_S$ for a prime divisor S arising from S passing through codimension 2 terminal singularities (transverse surface quotient singularities). For example, consider the adjunction formula for a line generator of the ordinary quadratic cone in $\mathbb{P}^3$.

3.1 of the Russian says assume $S \not\subset \operatorname{Supp}(K + S + D)$, which is illiterate; for example, it could mean that $K+S+D$ is linearly equivalent to a divisor not containing S. But the Grothendieck duality adjunction formula $\omega_S = \mathscr{E}xt^1_{\mathscr{O}_X}(\mathscr{O}_S, \omega_X)$ gives an exact sequence

$$0 \to \omega_X \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{I}_{X,S}, \omega_X) \to \omega_S \to 0$$

that coincides with the Poincaré residue $\omega_X(S) \to \omega_S$ in codimension 1, so that K_{S^ν}, K_S and $(K_X + S)_{|S^\nu}$ can be intrinsically compared.

For a technically more sophisticated treatment of the different, see [Utah], Chapter 16.

10.7. **Comment on Reduction 8.4.** The final remark on algebraic approximation seems to me to be nonsense: an analytic flipping singularity is more-or-less isolated (quotient singularities have no moduli), so analytically equivalent to an algebraic singularity. The analytic flipping contraction over it is projective. Complete any-oldhow to a projective variety and resolve singularities with a couple of B52 loads of blowups. Then the algebraic situation is $\mathbb{Q}$-factorial, nonsingular outside a codimension 2 locus of transverse quotient singularities, and the single contracted curve is an extremal ray of a projective variety. What's the problem? (Response from Shokurov: The problem is to give the complete proof.)

10.8. A treatise on terminology.

(10.8.1) *To hyphenate or not to hyphenate?* The expressions log differentials, log terminal singularities, and so on are etymologically *logarithmic differentials*, that is, differentials with log poles, or *logarithmically terminal* singularities; (Iitaka's original papers are full of logarithmic irregularity, logarithmic Albanese map, log of general type, etc., all written out in full). Therefore the word log stands either for an adjective or for an adverb, and in neither case is it grammatical or desirable to hyphenate it or join it as a single word. So there!

(10.8.2) *Pseudoextremal rays.* Shokurov and others sometimes use *extremal rays* only in the sense of convex body theory, so that the rays are only boundaries of the cone, not necessarily extremal rays in the sense of Mori theory. A pseudoextremal ray R_2 is not a priori rational or spanned by curves, although this will be the case whenever $(K+D)R_2 < 0$ against a log canonical divisor $K+D$ by the log version of the theorem on the cone. Without some such assumption, I don't think anything useful can be said about it.

(10.8.3) *Birational transform.* If $f\colon X \dashrightarrow Y$ is birational and D a Weil divisor on X, there is a well-defined divisor obtained as the Zariski closure of the divisorial part of the image $f_0(D_0)$, where $f_0\colon X_0 \to Y$ is the biggest morphism in f (compare the section after (1.1)). This is traditionally called *proper transform* by Russians and in my papers, for example, and *strict transform* by people in resolution of singularities, for example, although there is no logic in either term. I propose *birational transform* as self-documenting terminology, and use this throughout the translation.

When $f\colon Y \to X$ is a birational morphism, and D an effective divisor of X, Shokurov writes $f^{-1}D$ for the birational transform of D, meaning that he takes f^{-1} of the generic points of D (where of course f^{-1} is well defined). By definition $f^{-1}D$ is an effective divisor without any exceptional terms, so it's nothing to do with the set-theoretic inverse image $f^{-1,\mathrm{sets}}D$, which contains all exceptional divisors. More logical notation would be g^1D to mean the forward image under a rational map g of D as a codimension 1 cycle, so $f^{-1,1}D$ or $(f^{-1})^1D$ for the birational inverse image.

(10.8.4) *Blowup or extraction.* It's traditional to think of a birational map $f\colon Y \to X$ either as a *birational contraction* (of something on Y, for example a divisor or an extremal ray) or as a *partial resolution* (of something on X, for example singularities or indeterminacies of a rational map). Another traditional name for the same object is a *model* (for example, the relative minimal or canonical model $f\colon Y \to X$ of a singularity X). It seems to me to be wrong to call the last type of construction a *blowup* of X, since it is not usually constructed as a blowup of a sheaf of ideals I in the sense of Grothendieck and Hironaka, and even if it happens to be so (every projective birational morphism is a blowup of some sheaf of ideals), there may be no sensible way of saying what I is, or of proving that it makes sense, in terms of X. In surface singularities, it is traditional to make a partial resolution which *extracts* (or "pulls out") a subset of the curves of the minimal resolution. Therefore I launch *extraction* as a bottomup counterpart to *birational contraction*, and may God bless all who sail in her. (Actually, Shokurov deliberately uses the pair of words "blowup" or "blowdown" (Russian *razdutie* and *sdutie* from *dut'*, to blow) to mean a birational morphism $f\colon X \to Z$ viewed bottom-up or top-down, so my translation is deliberately going against his clearly expressed preference.)

(10.8.5) *Multiplicities d_i of a divisor $D = \sum d_iD_i$.* The Russian has *coefficients* throughout, but the same word is used later in the extraordinary circumlocution *coefficient of taking part in* divisors such as $g^{-1}D + E$ on an extraction $g\colon Y \to X$, so multiplicity is a better term. Multiplicities d_i are often opposed to discrepancy coefficients a_i.

(10.8.6) *Exceptional.* The key notion of exceptional complement (ray, flip, etc.) is defined in Theorem 5.6 and the following paragraph (involving only one component with log discrepancy 0). Unfortunately, exceptional divisor for f seems also to be used in the Russian preprint in the ordinary meaning of divisor contracted by a birational map f. It is possible that the context makes clear (in the author's mind) which is intended, but with the limited time available for the translation, I may not always have succeeded in untangling it.

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