3-FOLD LOG MODELS

V. V. Shokurov UDC 512.776

0. Introduction

The purpose of this paper is to demonstrate the power of geometrical methods which allow one to substantially improve the log minimal model program (LMMP) for 3-folds. This means that the basic facts such as the cone, contraction, flip, termination, and abundance theorems hold under much more general and, perhaps, ultimate conditions:

- boundaries are $\mathbb{R}$-divisors and
- singularities are log canonical.

Of course, we do not pretend to give a direct proof of them, except for the termination theorem. It is only a derivation of these facts from the well-known situation where the boundaries are $\mathbb{Q}$-divisors and singularities are log terminal [11, 15]. In the latter case, the proofs use essentially cohomological methods: the vanishing and nonvanishing theorems.

The improvement pursues the standard mathematical goal of being close to the cutting edge, but it gives some new applications as well. The most important of them are the two main theorems in Sec. 6, which describe the behavior of log models with respect to their boundaries. In particular, the second of them answers in the affirmative regarding [24, Problem 6] in dimension 3 and improves substantially the first attempt [25, Relative Model Theorem].

Another application presents results on the Kleiman–Mori cone in the critical zone where the log canonical divisor is trivial.

As we see in Sec. 6, the core of the methods is based on the *standard* LMMP with $\mathbb{Q}$-boundaries and log terminal singularities and it works in any dimension, except for the log termination, which is given for 3-folds in Sec. 5. The log termination for 3-folds in the standard case is credited to Kawamata [10].

Other sections contain preparatory and related materials.

Acknowledgment. Some gaps in the first draft were pointed out by T. Hayakawa and S. Mori in Theorem 3.2, as well by A. Bruno and K. Matsuki in the contraction results (6.16 and 6.16.1). The author would like to express thanks to all of them.

Partial financial support was provided by the NSF under grant number DMS-9200933.

The geometric objects we work with are either normal complex analytic spaces or normal algebraic varieties over an algebraically closed base field k of characteristic 0 (even sometimes algebraic spaces). In the analytic case the objects X are almost algebraic. This means that X is equipped with a proper morphism $f: X \to S$, which in most cases are Moishezon and projective respectively.

The meromorphicity is considered in the sense of Remmert, i.e., a *meromorphic* map $f: X \dashrightarrow Y$ of normal complex spaces is identified with its graph, which is an analytic subset Γ_f of the product $X \times Y$, satisfying the following conditions:

(i) Γ_f is locally irreducible in $X \times Y$, and
(ii) the natural projection $\Gamma_f \to X$ is proper, surjective, and generically one-to-one.

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory. Vol. 33, Algebraic Geometry–4, 1996.

1072-3374/96/8103-2667$15.00 ©1996 Plenum Publishing Corporation

Then a bimeromorphic *modification* $X \dashrightarrow Y$ is assumed to be an invertible meromorphic map, invertible in the category of meromorphic maps. Thus, the modification can be presented as a Hironaka hut

$$\begin{array}{ccccc} & & W & & \\ & f \swarrow & & \searrow g & \\ X & & & & Y \end{array}$$

where f and g are proper and 1-to-1 over generic points.

We assume that the reader is aware of the following notation:

$a(D, C_X, X)$, the log discrepancy of a log divisor $K_X + C_X$ for a prime divisor D in a modification of X;

$d(D, C_X, X)$, the discrepancy of a log divisor $K_X + C_X$ for a prime divisor D in a modification of X;

$\operatorname{center}_X D$, the center in X for a prime divisor D in a modification of X;

$\operatorname{Div}(X)$, the group of Weil divisors of X;

$\operatorname{Div}_K(X) = K \otimes_{\mathbb{Z}} \operatorname{Div}(X)$, the group of Weil K-divisors of X, where K is a commutative ring (below we need only $K = \mathbb{R}$ or $\mathbb{Q}$);

$\sim_A$, for a ring A, denotes an A-linear equivalence of A-divisors (see 2.5);

$\operatorname{mult}(E, D, X) = \operatorname{mult}_E D$, the multiplicity of an $\mathbb{R}$-Cartier divisor $D \in \operatorname{Div}_{\mathbb{R}}(X)$ for a prime divisor E in a modification of X.

We agree to drop some of the variables in discrepancies and multiplicities whenever they are assumed to be fixed, e.g., we write $a(D)$ instead of $a(D, B_X, X)$ if a log variety (X, B_X) is fixed.

1. Bi-Divisors, Discrepancies, and Singularities

Fix a birational (bimeromorphic) class $\mathcal{X}$ of an algebraic variety (respectively complex space) X.

Now it is time to consider the group $\mathcal{D}iv(\mathcal{X}) = \mathcal{D}iv(X)$ of (*birational* or *bimeromorphic* respectively, or simply) *bi-divisors* of X. This is an Abelian group which is generated by prime Weil divisors on the modifications of X. Two such prime divisors, e.g., $D \subset Y$ and $D' \subset Y'$, are identified if the induced modification $Y \dashrightarrow Y'$ birationally (bimeromorphically) transforms D into D'. In the algebraic case, according to the theory of valuations [30], these prime divisors are in one-to-one correspondence with $\mathbb{Z}$-valuations of the field of rational functions of X.

Prime divisors generate $\mathcal{D}iv(X)$ in the following sense. Every divisor D in $\mathcal{D}iv(X)$ is an *integral* linear combination

$$D = \sum d_i D_i$$

with distinct prime divisors D_i. "Integral" means that the coefficients d_i are integral. The coefficient d_i is called the *multiplicity* of D_i in D. Note that it should not be confused with the usual multiplicity $\operatorname{mult}(E, D, X)$ for $\mathbb{R}$-Cartier divisor $D \in \operatorname{Div}_{\mathbb{R}}(X)$, especially whenever a prime $E \in \mathcal{D}iv(X)$ is exceptional on X. Nonetheless, the latter defines an important example of a bi-divisor (see Example 1.1.1 below).

Let Y be another modification of X. Any Weil divisor of Y is a bi-divisor of X or $\mathcal{X}$. Thus we have an inclusion $\operatorname{Div}(Y) \subseteq \mathcal{D}iv(X)$. Moreover, it gives a split

$$\mathcal{D}iv(X) = \operatorname{Div}(Y) + \mathcal{E}(Y),$$

where $\mathcal{E}(Y)$ corresponds to the so-called *exceptional* divisors of Y which do not present in Y. In the algebraic case an exceptional divisor of Y may be a divisor in a complement $\overline{Y} \setminus Y$.

In fact, the bi-divisor D as a linear combination is defined if it has such a split for some Y. More precisely, let

$$D = D_Y + {}_Y D$$

be the corresponding decomposition, where D_Y denotes the part of the linear combination $D = \sum d_i D_i$ with the prime divisors D_i running in Y, and ${}_Y D \in \mathcal{E}(Y)$ denotes the remainder. Then

$$D = \sum d_i D_i$$

is a divisor of $\mathcal{D}iv(X)$ if D_Y is a divisor of $\mathrm{Div}(Y)$.

So, in the algebraic case the divisor $D_Y \in \mathrm{Div}(Y)$ is a *finite integral* linear combination $D_Y = \sum d_i D_i$, where the prime divisors D_i lie in Y. "Finite" means that there is only a finite number of multiplicities $d_i \neq 0$. In the analytic case, we assume that the last condition takes place locally.

If we have two modifications X and $Y \in \mathcal{X}$, then the prime Weil divisors of Y, but not of X, form prime (divisorial) components of the exceptional divisor for the modification $Y \dashrightarrow X$. Here and everywhere we identify a prime Weil divisor with its generic point. Since Y is normal, the modification is well defined for the Weil divisors and, in particular, for the exceptional divisor.

Thus for any modification Y of X and a bi-divisor $D \in \mathcal{D}iv(X)$, the component D_Y is a divisor of Y. Moreover, we can extend any divisor $D_Y \in \mathrm{Div}(Y)$ to a bi-divisor D by an arbitrary exceptional part ${}_Y D$.

The most important invariants of a prime bi-divisor D are its *center* $\mathrm{center}_X D$ in X and discrepancies (see Example 1.1.4 below). We would like to remind the reader that if $Y \dashrightarrow X$ is a modification of X and D is not exceptional in Y, then $\mathrm{center}_X D$ is an irreducible subvariety, being the birational (bimeromorphic) transform $f(D)$ (in the analytic case, which is its proper inverse image on the graph Γ_f and a subsequent projection to X). So, as prime divisors we identify the center with its generic point in the Grothendieck sense.

In the algebraic case, it may happen that $\mathrm{center}_X D = \emptyset$, e.g., D is in $\overline{X} \setminus X$. This leads to complications in the definition of discrepancies. There are two ways to avoid uncomfortable divisors with centers outside of X. First, we may simply require the condition $\mathrm{center}_X D \neq \emptyset$, whenever we need it [15, log canonical models]. We prefer the second way. We restrict our consideration to complete varieties (compact spaces) or to an appropriate relative version.

Fix an algebraic variety (complex space) S as a base. Let $\mathcal{X}/S$ denote a birational (bimeromorphic) class (or a category) of proper morphisms $f\colon X \to S$. Here the equivalence is defined by the modifications/S (which are the morphisms of the category). Then by a *relative* group of bi-divisors $\mathcal{D}iv(X/S) = \mathcal{D}iv(\mathcal{X}/S)$, we mean a subgroup of $\mathcal{D}iv(X)$ generated by the prime bi-divisors D with $\mathrm{center}_X D$ in S, i.e., it is *nonempty* in S. Note that in the analytic case $\mathcal{D}iv(X/X) = \mathcal{D}iv(X)$ by its very definition, but not in the algebraic case.

For a commutative ring K, put

$$\mathcal{D}iv_K(X/S) = \mathcal{D}iv(X/S) \otimes_{\mathbb{Z}} K.$$

Elements of this group are called (relative Weil) K-bi-divisors. In particular, $\mathcal{D}iv_{\mathbb{Z}}(X/S) = \mathcal{D}iv(X/S)$. The most important for us are $\mathbb{R}$-bi-divisors, the elements of $\mathcal{D}iv_{\mathbb{R}}(X/S)$. The same terminology is used for divisors as well.

So, every K-bi-divisor is a linear combination $D = \sum d_i D_i$ with multiplicities $d_i \in K$. As in the case of bi-divisors, for any modification Y of X, we have a split

$$\mathcal{D}iv_K(X/S) = \mathrm{Div}_K(Y) + \mathcal{E}_K(Y/S),$$

where $\mathrm{Div}_K(Y) = \mathrm{Div}(Y) \otimes_{\mathbb{Z}} K$, or $D_Y = \sum d_i D_i$ with $D_i \in \mathrm{Div}(Y)$ is assumed to be finite (locally finite).

In the algebraic case, according to Hironaka and since k is of characteristic 0, we can describe $\mathcal{D}iv_K(X/S)$ as the projective limit of $\mathrm{Div}_K(Y)$ with nonsingular projective modifications $Y \to X/S$. This is the so-called *foam space* of X. The same holds in the analytic case over a neighborhood of a projective subspace in S.

1.1. Examples.

1.1.1. Let D be an $\mathbb{R}$-Cartier divisor of X. Then we may define its *completion* in $\mathcal{D}iv(X/S)$ as a bi-divisor

$$\overline{D} = \sum m_i D_i,$$

where $m_i = m(D_i, D)$ is the multiplicity of D_i in D.

The completion may be used in the explanation of a Zariski decomposition.

1.1.2. *Bi-boundaries.* A *birational (bimeromorphic)* boundary or, simply, bi-boundary of X/S or $\mathcal{X}/S$ is an $\mathbb{R}$-bi-divisor $B = \sum b_i D_i$ of X/S such that

(i) all $b_i \in [0, 1]$;

(ii) for some modification Y of X and for any D_i exceptional in Y, the multiplicity of D_i in B is equal to 1.

In the algebraic case, this means that, for any modification $X \in \mathcal{X}$, $b_i = 0$ for the nonexceptional divisors D_i of X, and $b_i = 1$ for the exceptional divisors D_i of X, except for a finite set of bi-divisors D_i, for which $b_i \in [0, 1]$. The same holds over a neighborhood of a compact subset of S in the analytic case.

So, replacing X by its modification, we can assume that the bi-divisors D_i with $b_i < 1$ are nonexceptional in X, or $b_i = 1$ for the exceptional divisors D_i of X. In this case, the divisor B_Y for any other modification Y of X coincides with that for the boundary $B = B_X$ in [26].

By a *log* birational class of algebraic varieties (spaces) we mean a pair $(\mathcal{X}/S, B)$ with a bi-boundary $B \in \mathcal{D}iv_{\mathbb{R}}(\mathcal{X}/S)$. It may be considered as a birational class (or category) of log varieties (spaces) or *log pairs* $(X/S, B_X)$, where $X/S \in \mathcal{X}/S$. Then the modifications $f\colon X \dashrightarrow Y$ are compatible with boundaries, i.e., the modified boundary $f(B_X)$ has the same multiplicities as B_Y in all common prime divisors (as bi-divisors) of X and Y. Note that $B_Y = f(B_X)$ whenever f^{-1} has no exceptional divisors, i.e., whenever f is essentially a *flip*.

1.1.3. *Canonical bi-divisors.* Another astonishing example is a *canonical* bi-divisor K in $\mathcal{D}iv_{\mathbb{R}}(X/S)$ which can be defined by any nontrivial rational (meromorphic) differential form of the highest degree. For each $X \in \mathcal{X}$, K_X is the canonical divisor defined by the form on X. Thus, it will always be a $\mathbb{Z}$-divisor.

In the analytic case, it may be defined only locally. For instance, this holds locally/S, when X/S is locally/S Moishezon or projective.

By a log canonical bi-divisor we mean a shift $K + C$, where $C \in \mathcal{D}iv_{\mathbb{R}}(\mathcal{X}/S)$. Besides introducing discrepancies below, we assume that $C = B$ is a boundary.

1.1.4. *Discrepancies.* Let $K + C$ be a log bi-divisor of X such that $K_X + C_X$ is $\mathbb{R}$-Cartier. Then

$$K + C = \overline{K_X + C_X} + R,$$

where $R = R(C, X)$, is a *relative discrepancy* bi-divisor. It is nontrivial only on exceptional prime divisors D_i of X, and its multiplicity for every such D_i is the *relative* discrepancy $r_i = r(D_i, C, X)$ of $K_X + C_X$ in D_i.

By a *discrepancy* bi-divisor we will mean that $D = D(C, X) = R(C, X) - C$. Thus $K = \overline{K_X + C_X} + D$. Its multiplicities $d_i = d(D_i, C, X) = d(D_i, C_X, X) = r_i - c_i$ are *Reid's* discrepancies for the exceptional prime divisors D_i of X, and $d_i = d(D_i, C, X) = -c_i$ is the negative of the multiplicity c_i of C in every nonexceptional D_i of X.

If we define the log discrepancies in a naive way, $a_i = a(D_i, C, X) = d_i + 1$, then they will not correspond to a bi-divisor. The correct approach relies on a normalization depending on X. Namely, let $Supp_X C$ be a *log* bi-support of C *with respect* to X, i.e., a bi-divisor with multiplicity 1 for each D_i exceptional on X or with $c_i \neq 0$, and multiplicity 0 otherwise.

Then a *log* discrepancy bi-divisor will be $A = A(C,X) = D(C,X) + Supp_X C$. It has log discrepancies $a_i = a(D_i, C, X) = a(D_i, C_X, X)$ as multiplicities, whereas $a_i = d_i + 1$ for each D_i exceptional on X or with $c_i \neq 0$, and multiplicity 0 otherwise. In particular, if $C = B$ is a bi-boundary such that $b_i = 1$ for the exceptional bi-divisors D_i of X, then $a_i = d_i + 1$ if D_i is exceptional on X, $a_i = 1 - b_i$ if D_i is nonexceptional and $b_i \neq 0$, and $a_i = 0$ otherwise.

The discrepancies and their bi-divisors are independent of the choice of K. Therefore, even in the analytic case, they are well defined because K exists at least locally on X.

1.2. Definition. A pair (X, B_X) or simply X has only log canonical singularities *with respect* to B, if

1.2.1. $K_X + B_X$ is $\mathbb{R}$-Cartier, and

1.2.2. the relative log discrepancy $R(B,X)$ is effective.

The latter means that all $r_i \geq 0$. But they are nontrivial only for prime bi-divisors D_i exceptional on X. On the other hand, for every such D_i, the log discrepancy is easily recovered in terms of B and the relative discrepancy:

$$a_i = r_i + 1 - b_i.$$

Thus 1.2.2 is equivalent to

1.2.3. $a_i \geq 1 - b_i$ for each D_i exceptional on X (cf. [26, p. 101]).

Moreover, we say that (X, B_X) or simply X has only (respectively strictly) log terminal singularities *with respect* to B when (respectively X is $\mathbb{Q}$-factorial, projective/S, and)

1.2.4. $a_i > 1 - b_i$ for each D_i of a log resolution for (X, B_X) exceptional on X (cf. [26, p. 101]).

If (X, B_X) has only log canonical singularities with respect to B, then the log discrepancies a_i are well defined and are nonnegative, because $1 - b_i \geq 0$. So, (X, B_X) has only log canonical singularities (respectively log terminal or strictly log terminal singularities of with respect to B). The only difference from the former is in condition 1.2.3 (respectively 1.2.4).

It may happen that (X, B_X) will have better singularities such as log terminal, strictly log terminal, etc. We would like to modify some of them.

1.3. Singularities. Let (X, B_X) be a log pair such that $K_X + B_X$ is $\mathbb{R}$-Cartier in a generic point η, i.e., in its neighborhood. Fix nonnegative real ε.

1.3.1. We say that (X, B_X) has only ε-log canonical (respectively ε-log terminal) singularities in η if $a(D) \geq \varepsilon$ for every D (respectively $> \varepsilon$ for every exceptional D on a log resolution of X) with $\mathrm{Supp}_X D = \eta$.

1.3.2. We say that (X, B_X) has only ε-canonical (respectively ε-terminal) singularities in η if $d(D) \geq \varepsilon$ (respectively $> \varepsilon$) for every D (respectively exceptional on X) with $\mathrm{Supp}_X D = \eta$.

1.3.3. The singularities of (X, B_X) are only ε-log canonical, etc., in *codimension* d, if they are such for each $\eta \in X$ with $\mathrm{codim}_X \eta = d$. Similarly, we may define singularities which are only ε-log canonical, etc., in *codimension* $\geq d$ or somewhere else. Respectively (X, B_X) has only ε-log canonical, etc., singularities, if they are such for each $\eta \in X$. The only difficulties may occur in the log terminal case and for $\varepsilon = 0$, where we consider some log resolution of X for all given points η at once.

For $\varepsilon = 0$, we say simply log canonical instead of 0-log canonical, etc. This agrees with the previously used terminology [11, 26].

We need the following results only in special cases, for which they are easily checked. They will be treated in a full generality in [29].

1.4. Lemma. *Let η and θ be generic points in X such that η is a specialization of θ, and $K_X + C_X$ is $\mathbb{R}$-Cartier in η. Then $K_X + C_X$ is log canonical in θ, whenever it is so in η.*

1.5. Corollary. *If $K_X + C_X$ is log canonical in η, then it is so in a neighborhood of η. In particular, C_X is a subboundary there.*

1.6. Lemma ([23, 1.1]). *Suppose that, for a real $\varepsilon > 0$, $K_X + C_X$ is ε-log canonical in a neighborhood of η. Then over a neighborhood of X, there exists a log resolution $f : Y \to X$ such that $K_Y + C_Y = f^*(K_X + C_X)$* [26, Sec. 3] *is $1 + \varepsilon$-log canonical in codimension ≥ 2.*

1.7. Corollary. *Under the assumptions of Lemma* 1.6, *there exists a finite number of bi-divisors E/X with $a(E, C_X) < 1 + \varepsilon$ and η is a specialization of* $\mathrm{center}_X E$. *Moreover, we can drop the last restriction whenever $K_X + C_X$ is ε-log canonical everywhere (and in the analytic case X is a neighborhood over a compact set W).*

2. Log Models

As above, let $f: X \to S$ be a proper morphism and B be a bi-boundary of X.

2.1. Definition. A birational (bimeromorphic) modification $g: Y \to S \in \mathcal{X}/S$ of f is called a (respectively *weakly*) *log canonical model of f* or *$\mathcal{X}/S$ with respect to B* if the following conditions hold:

(LCS) (Y, B_Y) has only log canonical singularities with respect to B; and

(LCN) $K_Y + B_Y$ is ample (respectively nef) relative to g.

Similarly, g is a (respectively *strictly*) *log terminal model of f with respect to B* when

(LTS) (Y, B_Y) has only (respectively strictly) log terminal singularities with respect to B; and

(LTN) $K_Y + B_Y$ is nef relative to g.

The strictly log terminal case is also referred to as *log minimal*.

Note that each type of log model above depends only on the class $(\mathcal{X}/S, B)$.

The numerical properties (LCN) and (LTN) of the log models make sense in Definition 2.1 because $K_Y + B_Y$ is $\mathbb{R}$-Cartier by (LCS) and (LTS) respectively. As we know, the latter also implies that $K_Y + B_Y$ has only log canonical and (strictly) log terminal singularities depending on the type of model. So, $(Y/S, B_Y)$ is simply a *weakly log canonical* (respectively *log terminal, log minimal,*) etc. model, i.e., it has only that type of singularities and satisfies (LTN).

According to [26, 1.5.1 and 1.4.3], a log canonical model is unique and projective (in the analytic case, over a neighborhood of any compact subset $W \subseteq S$), if it exists. Indeed, we can assume that $b_i = 1$ for all exceptional divisors D_i of X. Then the reference works.

2.2. Conjecture on log minimal models. *(Suppose that in the analytic case X is Moishezon over a neighborhood of a compact (Moishezon) subspace $W \subseteq S$, for instance, quasi-projective). Then for any bi-boundary B of X, there is a modification $g: Y \to S$ (respectively over a neighborhood U of $W \subseteq S$), which has only strictly log terminal singularities for B and one of the following types:*

(LMM) *g is a log minimal model with respect to B;*

(LFF) *g is a nontrivial fiber space of log Fano over S (respectively over U).* More precisely, there is an extremal and non-birational (non-bimeromorphic) contraction $f: Y \to S'$ over S (respectively over U) such that $K_Y + B_Y$ is negative on the fibers of f.

Moreover, we may assume that a log minimal model Y/S has a resolution in 1.2.4 *in which each of a given finite set of prime bi-divisors$/S$ is nonexceptional.*

2.3. Theorem on log minimal models for 3-folds. *For any bi-boundary B of a 3-fold X, the conjecture on log minimal models holds.*

This is an improved version of [15, Theorem 1.4] where the multiplicities b_i are assumed to be rational. However, the proof follows along the same line: the LMMP with respect to B (see Sec. 6 below and cf. [26; 15, 2.26]). In particular, for this we need log flips [26] and a log termination, which we discuss later.

Log Fano fiberings belong to Sarkisov's theory [22, 21, 5], and they are beyond the scope of this article. We will concern ourselves with log terminal and log canonical models. Note that if f has a weakly log canonical model, then it cannot be a log Fano fibering with respect to the same boundary B even birationally (bimeromorphically). More generally, we check below that the weakly log canonical models of $(\mathcal{X}/S, B)$ have the same numerical dimension.

However, first we need to fix some notions and notation. By a prime cycle C of X/S we mean a *relative* prime cycle of X over S and denote this by $C \subseteq X/S$. So, C is a proper irreducible subvariety (subspace) of X with the image $f(C) = s$ being a (closed) point of S. In other words, C belongs to a fiber $X_s = f^{-1}s$. We say that C is a d-cycle if C is a prime cycle of dimension d. For any $\mathbb{R}$-Cartier divisor D of X and a prime d-cycle C, the intersection

$$(C.D^d) \stackrel{\text{df}}{=} (C.(D|_C)^d)$$

is well-defined, as in the case where $d = 1$ and C is an irreducible curve. This can be extended to any *d-cycles* (even *cycles*) of X/S, which are finite linear combinations of prime d-cycles (respectively cycles) of X/S. Note that C is a point and $(C.D^d) = 1$ when $d = 0$.

For a modification $h: X - \to Y$ of f into $g: Y \to S$ and a prime cycle C, let $h(C) \subseteq Y/S$ denote the image of the generic point of C if h is defined in the latter, and 0 (or $\emptyset$) otherwise (cf. definition of a birational image in [26, p. 97]). This can be extended to a homomorphism h of all *cycles*.

Let $\mathrm{WLCM}(X/S) = \mathrm{WLCM}(\mathcal{X}/S)$ denote a subclass in $(\mathcal{X}/S, B)$ of the weakly log canonical models $g: Y \to S$ with respect to B.

2.4. Proposition. *Suppose that* $\mathrm{WLCM}(X/S) \neq \emptyset$, *i.e., f has a weakly log canonical model with respect to B. Then*

2.4.1. *f has no model which has a log Fano fibering as in* (LFF) *of* 2.2;

2.4.2. *the discrepancies $d_i = d(D_i, Y)$ (as well as the relative ones $r_i = r(D_i, Y)$) of the bi-divisors D_i are independent of $Y/S \in$* $\mathrm{WLCM}(X/S, B)$;

2.4.2′. *the log discrepancies $a_i = a(D_i, Y)$ of the bi-divisors D_i are independent of $Y/S \in$* $\mathrm{WLCM}(X/S, B)$, *whenever D_i is exceptional (or respectively nonexceptional) on Y;*

2.4.3. *for any $X/S \in \mathcal{X}/S$, natural d, and cycle C of X/S, the intersection number $(C'.(K_Y + B_Y)^d)$ is independent of $Y/S \in$* $\mathrm{WLCM}(X/S, B)$, *where C' denotes birational (bimeromorphic) image of C in Y;*

2.4.4. *for any point $s \in S$ the* numerical dimension

$$\nu(K_Y + B_Y)_s = \max\{d | (C.(K_Y + B_Y)^d) > 0 \text{ for a prime } d\text{-cycle } C \subseteq X_s = f^{-1}s\}$$

is independent of $Y/S \in$ $\mathrm{WLCM}(X/S, B)$;

2.4.5. *the dimension $\nu(K_Y + B_Y)_s$ has the upper semi-continuity property (if X is Moishezon locally/S in the analytic case); in particular, this function in s attains the minimum in the generic point of S.*

Thus, we can define the *relative numerical log Kodaira dimension of* X/S or $\mathcal{X}/S$ with respect to B as the numerical dimension $\nu(K_Y + B_Y)_s$ for the generic point $s \in S$ when $X/S \in \mathrm{WLCM}(X/S, B) \neq \emptyset$, and $-\infty$ otherwise. We denote the dimension by $\nu(X/S, B) = \nu(\mathcal{X}/S, B)$.

Note also that 2.4.3 implies that any two weakly log canonical models of $(\mathcal{X}/S, B)$ are equivalent in the sense of Definition 6.1 below.

Proof. The following arguments use the negativity of birational contractions [26, 1.1] and are similar to that of [26, 1.5]. Since the statements are local, we may assume that S is a neighborhood of its point s. Here we prefer also relative log discrepancies.

Let $g: Y \to S$ and $g': Y' \to S$ be two models of X/S: g is weakly log canonical and g' has only log canonical singularities for B. To compare them we consider the Hironaka hut

$$\begin{array}{ccccc} & & W & & \\ & h \swarrow & & \searrow h' & \\ Y & & & & Y' \\ & g \searrow & & \swarrow g' & \\ & & S & & \end{array}$$

where h and h' are proper birational (bimeromorphic). Then

$$K_W + B_W = h^*(K_Y + B_Y) + \sum r(D_i, Y)D_i$$

and

$$K_W + B_W = h'^*(K_{Y'} + B_{Y'}) + \sum r(D_i, Y')D_i,$$

where $r(D_i, Y)$ and $r(D_i, Y')$ are nontrivial only for divisors D_i of W which are exceptional for h and h' respectively. By the choice of g, all $r(D_i, Y)$ and $r(D_i, Y') \geq 0$, $h^*(K_Y + B_Y)$ is nef/S, and $h'^*(K_{Y'} + B_{Y'})$ is numerically trivial on $/Y'$. Thus

$$\sum (r(D_i, Y) - r(D_i, Y'))D_i = h'^*(K_{Y'} + B_{Y'}) - h^*(K_Y + B_Y)$$

is semi-negative/Y' and is effective by [26, 1.1]. (Note that it is better to use the Russian original this time, which states that we need only the non-positivity of D/Z, in [26, 1.1 (ii)], for the effectiveness of D.)

If g' were a log Fano fibering as in (LFF) of 2.2, then effective $\sum(r(D_i, Y) - r(D_i, Y'))D_i' = h'^*(K_{Y'} + B_{Y'}) - h^*(K_Y + B_Y)$ would be non-negative on rather general curves/S. This gives a contradiction for a nontrivial log Fano fibering.

Suppose now that g' is also a weakly log canonical model. Then by the symmetry of the above arguments, $\sum(r(D_i, Y) - r(D_i, Y'))D_i = 0$. This gives 2.4.2 and 2.4.2′.

The last implies 2.4.3 since, for any prime d-cycle $C \subsetneq W/S$,

$$\begin{aligned}[C : h(C)](h(C).(K_Y + B_Y)^d) &= (C.h^*(K_Y + B_Y)^d) \\ &= (C.(K_W + B_W - \sum r_i D_i)^d),\end{aligned}$$

where $[C : h(C)] = \deg(h|_C)$ if $\dim h(C) = d$, and 0 otherwise.

Now 2.4.4 is obvious, and 2.4.5 can be obtained by a Noetherian induction. In the analytic case, we may replace Y/S by W/S which is locally projective/S and check the semi-continuity of $\nu(h^*(K_Y + B_Y))_s$. This follows from the compactness of each connected component of the Hilbert scheme for projective spaces. ∎

Note that $\nu(X/S, B) \leq \dim X/S = \dim X_s$, where $s \in S$ is the generic point. In the case $\nu(X/S, B) = \dim X/S$ we say that f or X/S has the *numerically* general type with respect to B. When B is a $\mathbb{Q}$-divisor, this coincides with a well-known notion: $K_X + B_X$ is big over S.

2.5. Definition. An $\mathbb{R}$-divisor D of X/S is called *semi-ample/S* if there is a contraction $g: X \to T/S$ such that $D \sim_{\mathbb{R}} g^*H$ for an ample $\mathbb{R}$-divisor H of T/S, where $\sim_{\mathbb{R}}$ denotes $\mathbb{R}$-linear equivalence, i.e., $D - g^*H$ is an $\mathbb{R}$-linear combination of principal divisors (in the analytic case over a neighborhood of $W \subsetneq S$ under the Moishezon conditions of 2.2).

Similarly, for any ring A, we may define an A-linear equivalence $\sim_A$. Since g is a contraction, $\sim_A$ is always equivalent to that of locally/S (cf. 6.17).

Since X/S is proper, g and H are uniquely defined whenever they exist, i.e., D is semi-ample. More precisely, g contracts the curves C of X/S with $(C.D) = 0$. Note that D is semi-ample only if it is nef. The converse does not hold in general.

For a nef $\mathbb{Q}$-divisor D, the semi-ampleness implies that its Iitaka dimension equals the numerical one which is the abundance [11, 6.1.1]. Moreover, they are equivalent for nef log divisors having only Kawamata log terminal singularities [11, 6.1.13]; presumedly, this should be true for log canonical singularities as well.

2.6. Log semi-ampleness conjecture. *If $(X/S, B_X)$ is a weakly log canonical model, then $K_X + B_X$ is semi-ample (over a neighborhood of any compact subset of S in the analytic case). The corresponding contraction $I: X \to S'/S$ will be called an* Iitaka *morphism of $(X/S, B_X)$.*

Since the Iitaka morphism I is unique, it is enough to construct I locally/S. Note also the following property of the Iitaka morphism. For every $s \in S$, $\dim S'_s = \nu(K_X + B_X)_s$. In particular, $\dim S'/S = \nu(X/S, B)$.

2.7. Theorem on log semi-ampleness for 3-folds. *(Suppose that in the analytic case X is Moishezon locally/S.) A weakly log canonical model $(X/S, B_X)$ possesses an Iitaka morphism whenever* $\dim X \leq 3$.

When B_X is a $\mathbb{Q}$-divisor, the theorem has been essentially proved by Kawamata [9] and Miyaoka [17] in the case $B_X = 0$, and then generalized by Fong, Keel, McKernan, and Matsuki [12] (cf. [15, 8.4]). In Sec. 6 we derive Theorem 2.7 from this and investigate moduli of log canonical models depending on B.

2.8. Corollary on log canonical models for 3-folds. *A 3-fold X/S with a bi-boundary B has a log canonical model if and only if it has the numerically general type with respect to B (and Moishezon locally/S in the analytic case). Moreover, this is equivalent to $K_X + B_X$, being big/S when $K_X + B_X$ is a* $\mathbb{Q}$-divisor *having only log canonical singularities for B.*

Proof. If X/S has the numerically general type with respect to B, then by Theorem 2.3 and Definition 2.1, X/S has a weakly log canonical model $(Y/S, B_Y)$, and $\nu(Y/S, B) = \dim Y/S$. So, by Theorem 2.7 we have an Iitaka morphism $Y \to S'/S$ which is birational (bimeromorphic) and gives the required log canonical model.
The converse statement follows directly from definitions. ■

3. Blow-ups for 3-Folds

Let X be a 3-fold neighborhood of a point P with a boundary B_X, and let Δ be an effective $\mathbb{R}$-Cartier divisor. Suppose that, for $0 < \varepsilon \ll 1$, $B_X - \varepsilon\Delta$ is a boundary and $K_X + B_X - \varepsilon\Delta$ is purely log terminal. Then $K_X + B_X$ is log canonical. By divisors *over* a neighborhood of P, we mean *bi*-divisors with prime components having centers passing through P. Fix a finite set of exceptional divisors E_i over a neighborhood of P with log discrepancies $a_i = a(E_i, B_X, X) < 1$. All these divisors are nonexceptional on a log resolution of X. Of course, the resolution may not be sufficiently economical. Theorem 2.3 allows us to improve this.

3.1. Theorem on blow-ups. *Under the above assumptions and over a neighborhood of P, there exists a blow-up $f: Y \to X$ such that the divisors E_i are nonexceptional in Y and form the set of all exceptional divisors of f. Moreover,*

$$f^*(K_X + B_X) = K_Y + B_Y,$$

Y is $\mathbb{Q}$-factorial, log canonical with respect to $B_Y = f^{-1}B_X + \sum(1-a_i)E_i$ and with the same log discrepancies as for $K_X + B_X$. Respectively, for $0 < \varepsilon \ll 1$,

$$f^*(K_X + B_X - \varepsilon\Delta) = K_Y + B_Y - \varepsilon f^*\Delta$$

is purely log terminal.

In the theorem we deliberately identify X with the neighborhood.

Replacing B_X by $B_X - \varepsilon\Delta$, we can assume that $K_X + B_X$ is purely log terminal and, for chosen divisors E_i, $a_i < 1$ still. Let us extend B_X to a bi-boundary B such that its multiplicity b_i in exceptional E_i is $1 - a_i$ for chosen E_i and 1 otherwise. Note that X itself is a weakly log canonical model of $(X/X, B)$. So, by 2.4 we know discrepancies of any log minimal model. We contend that a log minimal model Y/X, which has a resolution with nonexceptional chosen E_i's, satisfies the requirements. Indeed, by 2.4.2′ and 1.2.4, each such E_i is also nonexceptional in Y. All other E_i's, exceptional in X, are exceptional on Y. Otherwise $a_i = a_i - 1 + b_i = r(E_i, B_X, X) = r(E_i, B_Y, Y) = 0$. This contradicts the purely log terminal property of (X, B_X).

Unfortunately, this proof uses the existence of log minimal models stated in Theorem 2.3. We know the existence of flips [26]. So, the main obstacle is their *termination*, which has been proved for $\mathbb{Q}$-boundaries B by Kawamata [10]. Here we develop another approach, which improves his induction [10, Lemma 6] and works as well for $\mathbb{R}$-boundaries. Finally, this gives in Sec. 5 the termination for such boundaries.

Proof. Note that the $\mathbb{Q}$-factorialization of X corresponds in the above arguments to the case where we choose an empty set of exceptional divisors. It needs only the special termination [26, 4.1 and 9.1]. Since we have a unique required resolution in Theorem 3.1 when X is a surface or over the generic point of any curve in X, we may assume that P is $\mathbb{Q}$-factorial.

By [26, 9.1] and induction, it is enough consider the case of a single exceptional divisor $E = E_{i_0}$. Put $a = a_{i_0}$.

Let C be a center of E. Thus $C = P$ or it is a curve through P. To construct the required blow-up, we consider a log resolution $f: Y \to X$ of a neighborhood of P, such that Y/X is projective, and E is nonexceptional in Y. As above, we choose the bi-boundary B with ${}_XB = (1-a)E + \sum_{i \neq i_0} E_i$, where E_i denotes exceptional in X prime bi-divisors of X/X, or with $B_Y = f^{-1}B + (1-a)E + \sum_{i \neq i_0} E_i$, where the E_i's are prime divisors of Y/X exceptional in X. As we know the corresponding log minimal model gives the required blow-up of E.

We can construct the model using the LMMP [26, general philosophy in Sec. 1; 16] (see also Sec. 5 below). Since we have log flips [26, 9.3-4], we need only log termination. I contend that a slight modification of the special termination is enough in this case. Indeed, let R be a flipping ray in Y after some extremal transformations of Y/X with respect to $K_Y + B_Y$. Then its curves intersect $\sum_{i \neq i_0} E_i$, a reduced part of the boundary. Otherwise, f is a divisorial blow-down and R has a curve C'/P on E which does not intersect other E_i's with $i \neq i_0$. Since P is $\mathbb{Q}$-factorial, $(C'.E) < 0$. On the other hand, we can find a nef divisor D which is big on E and does not intersect C'. For example, take an inverse image of a hyperplane section of the blow-down of R on Y. Thus $C = f(E) \subset f(D)$ and $f^*f(D)$ has a positive multiplicity μ in E. Then $0 = (C'.f^*f(D)) = (C'.\mu E + D) = \mu(C'.E) < 0$ is a contradiction. Therefore, each flipping curve intersects the reduced part of the boundary B_Y. As has been noted by Kawamata, this is enough for the termination in [26, 4.1] (cf. [15, 7.1]). ∎

3.1.1. Remark. Of course, after $\mathbb{Q}$-factorialization in the proof, we may perturb boundary coefficients and assume that they are rational. Then the theorem follows from [10]. However, the above proof is important even in this case because it uses the existence of flips and only the special termination (but not like Theorem 3.2 below). It may be useful in higher dimensions.

Another application concerns the following theorem.

3.2. Theorem ([28, 4.8]). *Let P be a 3-fold singular terminal point of X having the index r. Then, for any integer $1 \leq i \leq r-1$ there exists an exceptional divisor/P with discrepancy i/r.*

Perhaps it is a good idea to use deformation arguments in the proof of this fact. But in general it needs the LMMP for 4-folds and an appropriate deformation. Nonetheless, there exists a loophole with a more precise realization of the idea in the main case which was assumed in [28, 4.8]. In special cases, it is better to use direct calculations. An independent proof, perhaps in the latter style, is given by T. Hayakawa.

First, the theorem holds for the main type (1) in Mori's classification [20]. Indeed, this singularity is a quotient of type $1/r(a, -a, 1, 0)$ of a hypersurface $xy + F(z^r, w) = 0$, where $0 \le a \le r-1$ and $(a, r) = 1$. By [19] we may assume that P is $\mathbb{Q}$-factorial, which is equivalent to the irreducibility of F.

The quotient of $w = 0$ will be a reduced and irreducible Cartier divisor B with P of type $1/kr^2(1, bkr-1)$, where $k = \operatorname{ord} F$, $0 \le b \le r-1$ and $ab \equiv 1 \mod r$. According to the inversion of adjunction [26, 9.5], $K_X + B$ is purely log terminal in a neighborhood of P (of course, in codimension ≥ 2).

By Kawamata [8], there exists an exceptional divisor E/P with $d(E, 0) = a(E, B) = 1/r$ and $\operatorname{mult}(E, B) = 1$. Moreover, the quotient of $z = 0$ gives a 1-compliment B', i.e., $K_X + B + B'$ is log canonical of index 1. So, $a(E, B + B') = 0$, and by Theorem 3.1 we can make a blow up $f : Y \to X$ such that Y is $\mathbb{Q}$-factorial, E is the only exceptional divisor and the exceptional locus of f, whereas $f^*(K_X + B + B') = K_Y + f^{-1}(B + B') + E$ is log canonical. Moreover, $f^*(K_X + B) = K_Y + f^{-1}B + (r-1)/rE$ is purely log terminal of index r, and the intersection $E \cap f^{-1}B$ is connected and transversal in the generic points by [26, 3.10]. We would like to remind the reader that $1/r = a(P, B, X)$ is the minimal log discrepancy in P (see Sec. 4 below).

According to [26, 3.6], $f^{-1}B$ is normal, and by the adjunction f blows up on B curves with the minimal log discrepancy $1/r$. Let Q and R be two points of $f^{-1}B'$ in $E \cap f^{-1}B$. (They exist and there are exactly two of them due to the structure of a 1-complement for $K_{f^{-1}B} + E|_{f^{-1}B}$.) We check that they are semi-stable with respect to $f^{-1}B + E$ of type $V_2(a + \alpha r, -r)$ and $V_2(r - a + \beta r, -r)$, respectively, with α and $\beta \in \mathbb{N}$.

It is enough to check that Y and E are nonsingular in a punctured neighborhood of points Q and R. Then $K_Y + f^{-1}B + E$ will be log terminal and even purely log terminal in Q and R, whereas $f^*B = f^{-1}B + E$ is Cartier. Thus, by the covering trick the singularities are of type $V_2(-, -)$ [26, 3.9, 3.7]. The invariants are the same as for a semi-stable singularity of similar type $V_1(r, a; n_1, \dots, n_\sigma)$ [28, 4.7], because they depend only on type $1/kr^2(1, bkr - 1)$ of $P \in B$. (This is a relic of deformation arguments.) Moreover, we may assume that $n_1 = \dots = n_\sigma = 1$, that is, the discrepancies over Q and R are the same as for $V_1(r, a; 1, \dots, 1)$. These give i/r with $2 \le i < r$, because after α Kawamata's blow-up of Q we reduce the problem to the case where Q has type $V_2(a, -r)$. Similarly, we may assume that R has type $V_2(r-a, -r)$. Therefore, the required discrepancies can be obtained by the generalized flower pot [28, 4.5].

If Y or E is singular in a punctured neighborhood of points Q or R, say Q, then there is a curve $C \subset E \setminus f^{-1}B$ of such singularities through Q. Moreover, if $K_X + f^{-1}B + E$ is not purely log terminal in Q, then there exists an exceptional divisor E' with $\operatorname{center}_Y E' = C$ and $a(E', f^{-1}B + E) = 0$. By Theorem 3.1 we can blow up E', which contradicts the purely log terminal property of Q for $K_{f^{-1}B} + E|_{f^{-1}B}$ [26, 3.11].

Hence, $K_X + f^{-1}B + E$ is purely log terminal in Q, and, according to the classification of log canonical singularities in dimension 2, E is nonsingular in a punctured neighborhood of a point Q. By [26, 3.9], X is also nonsingular in C, because $f^{-1}B + E$ and E are Cartier in C. This concludes type (1) in Mori's classification.

In the remaining cases, $r \le 4$. More exactly, types (3), (5), and (6) of Mori's classification [20] are examined by Kawamata [8]. In cases (2) and (4), $r = 4$ and 3 [20], respectively. In these cases we use subsequent or other weighted blow-ups.

Type (2) is the quotient singularity $1/4(1, 3, 2, 1)$ of a hypersurface $x^2 + y^2 + F(z, w^2) = 0$. As above, $k = \operatorname{ord} F$. Then the first blow-up [8, Case 6] is given from a domain of type $1/(k+2)(k, -4, 2, 1)$ if $k \equiv 1 \mod 4$, and of type $1/(k+2)(-4, k, 2, 1)$ if $k \equiv 3 \mod 4$, by

$$(x, y, z, w) \mapsto \begin{cases} (xy^{k/4}, y^{(k+2)/4}, zy^{1/2}, wy^{1/4}) & \text{if } k \equiv 1 \mod 4, \\ (x^{(k+2)/4}, yx^{k/4}, zx^{1/2}, wx^{1/4}) & \text{if } k \equiv 3 \mod 4. \end{cases}$$

Y is given as a quotient of $x^2 + y + F(zy^{1/2}, w^2y^{1/2})y^{-k/2} = 0$ and exceptional E by $y = 0$ if $k \equiv 1 \mod 4$; respectively, the same with x and y interchanged if $k \equiv 3 \mod 4$.

After interchanging coordinates, point $Q \in Y$, corresponding to $(0,0,0,0)$, will be the quotient singularity of type $1/(k+2)(2,-2,1)$, and E will be given as the quotient of $y^2 + G(x, z^2) = 0$, where $G(x, z^2)$ is the homogeneous component of $F(x, z^2)$ of degree k assuming z with weight $1/2$.

The next blow-up $g: Z \to Y$ of Q has weights

$$\mathrm{wt}(x,y,z) = \left(\frac{2}{k+2}, \frac{k}{k+2}, \frac{1}{k+2}\right).$$

Then Z has an affine open subset of a quotient type $1/2(1,1,1)$ and the map g is given by

$$(x,y,z) \mapsto \left(x^{2/(k+2)}, yx^{k/(k+2)}, zx^{1/(k+2)}\right).$$

The birational transform of E, denoted again by E, is given by $y^2 + G(1, z^2) = 0$, whereas the new exceptional divisor E' is given by $x = 0$. Let $R \in Z$ be a point corresponding to $(0,0,0)$. Therefore, in a neighborhood of R, $g^*E = 2k/(k+2)E' + E$, $g^*K_Y = K_Z - 1/(k+2)E'$ [8, Case 1], and

$$d(E', 0, X) = d(E', 0, Y) + \mathrm{mult}(E', E, Y) d(E, 0, X) = \frac{1}{k+2} + \frac{2k}{k+2}\frac{1}{4} = \frac{1}{2}$$

(cf. calculations in [26, 8.8.4]).

If $G(1,0) \neq 0$, then $R \notin E$, and the next blow-up in R gives the next exceptional divisor E'' with

$$d(E'', 0, X) = \frac{1}{2} + \frac{1}{2}\frac{1}{2} = \frac{3}{4}.$$

Otherwise the required divisor lies over another affine chart. It can also be found directly: by a different first blow-up $f: Y \to X$ with weights

$$(x,y,z,w) \mapsto \begin{cases} \left(\frac{k}{4}, \frac{k+2}{4}, \frac{1}{2}, \frac{3}{4}\right) & \text{if } k \equiv 3 \mod 4, \\ \left(\frac{k+2}{4}, \frac{k}{4}, \frac{1}{2}, \frac{3}{4}\right) & \text{if } k \equiv 1 \mod 4, \end{cases}$$

where $k = \mathrm{ord}\ F$ assuming w with weight $3/2$ (hinted at by Mori). Then f is given from a domain of type $1/(k+2)(k,-4,2,3)$ if $k \equiv 3 \mod 4$, and of type $1/(k+2)(-4,k,2,3)$ if $k \equiv 1 \mod 4$, by

$$(x,y,z,w) \mapsto \begin{cases} \left(xy^{k/4}, y^{(k+2)/4}, zy^{1/2}, wy^{3/4}\right) & \text{if } k \equiv 3 \mod 4, \\ \left(x^{(k+2)/4}, yx^{k/4}, zx^{1/2}, wx^{3/4}\right) & \text{if } k \equiv 1 \mod 4. \end{cases}$$

Y is given as a quotient of $x^2 + y + F(zy^{1/2}, w^2y^{3/2})y^{-k/2} = 0$ and exceptional E by $y = 0$ if $k \equiv 3 \mod 4$; respectively, the same with x and y interchanged if $k \equiv 1 \mod 4$. Thus, as for the Kawamata's blow-up above, E is reduced, and we can easily calculate that $d = d(E, 0, X) = 3/4$.

The case (4) with $r = 3$ can be treated similarly. Here we have a quotient singularity of type $1/3(2,1,1,0)$ of a hypersurface $w^2 + F(x,y,z) = 0$, where

$$F(x,y,z) = \begin{cases} x^3 + y^3 + z^3, & \text{or} \\ x^3 + yz^2 + xG(y,z) + H(y,z), & \text{or} \\ x^3 + y^3 + xG(y,z) + H(y,z) \end{cases}$$

with ord $G \geq 4$ and ord $H \geq 6$.

In the case $F(x,y,z) = x^3 + y^3 + z^3$, the first blow-up [8, Case 4] is given from a domain of type $1/3(2,1,1,0)$ by

$$(x,y,z,w) \mapsto \left(xw^{2/3}, yw^{1/3}, zw^{1/3}, w\right).$$

Y is given as a quotient of $w + x^3w + y^3 + z^3$ and exceptional E by $w = 0$. Thus, point $Q \in Y$, corresponding to $(0,0,0,0)$, will be the quotient singularity of type $1/3(2,1,1)$, and E will be given as the quotient of $y^3 + z^3$.

The next weighted blow-up in Q is given from a domain of type $1/2(1,1,1)$ by

$$(x,y,z) \mapsto (x^{2/3}, yx^{1/3}, zx^{1/3}).$$

It gives the next exceptional divisor E' with $\mathrm{mult}(E',E,Y) = 1$ and

$$d(E',0,X) = \frac{1}{3} + 1\frac{1}{3} = \frac{2}{3}.$$

In the other two cases, the first blow-up [8, Case 4] is given from a domain U of type $1/2(0,1,1,1)$ by

$$(x,y,z,w) \mapsto \left(xy^{2/3}, y^{4/3}, zy^{1/3}, wy\right).$$

Y is given as a quotient of $w^2+x^3+z^2+xG'(y,z)+H'(y,z)=0$ if $F(x,y,z)=x^3+yz^2+xG(y,z)+H(y,z)$, and of $w^2+x^3+y^2+xG'(y,z)+H'(y,z))=0$ if $F(x,y,z)=x^3+y^3+xG(y,z)+H(y,z)$, with ord $G' \geq 4$ and ord $H' \geq 6$, and exceptional E by $y=0$. Thus, E is reduced.

The next weighted blow-up $g\colon Z \to Y$ in a point $Q \in Y$, corresponding to $(0,0,0,0)$, has weights

$$\mathrm{wt}(x,y,z,w) = \left(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

It is given from a domain V of type $1/1(0,0,0,0)$ by

$$(x,y,z,w) \mapsto \left(xz, yz^{1/2}, z^{1/2}, wz^{1/2}\right).$$

Thus, Z is given as a quotient of $w^2+x^3z^2+1+xG'(yz^{1/2},z^{1/2})+H'(yz^{1/2},z^{1/2})z^{-1}=0$ if $F(x,y,z)=x^3+yz^2+xG(y,z)+H(y,z)$, and of $w^2+x^3z^2+y^2+xG'(yz^{1/2},z^{1/2})+H'(yz^{1/2},z^{1/2})z^{-1}=0$ if $F(x,y,z)=x^3+y^3+xG(y,z)+H(y,z)$, whereas the new exceptional divisor E' is given by $z=0$. Let $R \in Z$ be a point corresponding to $(0,0,0,0)$. Since $z|G'(yz^{1/2},z^{1/2})$ and $H'(yz^{1/2},z^{1/2})z^{-1}$, E' is reduced. As above, the birational transform of E is denoted again by E. Then, in a neighborhood of R, $d(\{z=0\},0,U) = 1+1/2+1/2+1/2-1 = 3/2$ or $K_V = g^*K_U + 3/2\{z=0\}$, $g^*Y = Z + \{z=0\}$, and by the adjunction $g^*K_Y = K_Z - 1/2E'$, or $d(E',0,Y) = 1/2$ [8, Case 1]. Now $g^*E = 1/2E' + E$ and

$$d(E',0,X) = d(E',0,Y) + \mathrm{mult}(E',E,Y)d(E,0,X) = \frac{1}{2} + \frac{1}{2}\frac{1}{3} = \frac{2}{3}.$$

■

3.2.1. Remark. Note that f in the first part of the proof is extremal, because P is $\mathbb{Q}$-factorial. Thus it is uniquely defined as a log canonical model. An analytic construction of f gives a Kawamata's weighted blow-up [8]. More precisely, f is such a blow-up with $0 \leq \alpha = i < k$ as above, $\beta = k-i-1$, $E \cap f^{-1}B$ is an irreducible curve in a neighborhood of which $K_X + f^{-1}B + E$ is semi-stable [16] and has at most two singularities Q and R.

Theorem 3.2 cannot be improved.

3.2.2. Example. For a quotient singularity P of type $1/r(a,-a,1)$, discrepancies of the exceptional divisors$/P$ form a set

$$\{i/r \mid i \in \mathbb{Z}, i \geq 1, \text{ and } i \neq r\}.$$

Thus, 1 is *missing* in this case.

Indeed, this characterizes such singularities.

3.3. Corollary. *Let P be a 3-fold singular terminal point of X having the index r. Then, for any integer $i \geq 1$ there exists an exceptional divisor$/P$ with discrepancy i/r except for the case where P is a quotient singularity of type $1/r(a,-a,1)$ with $0 \leq a \leq r-1$, and $(a,r)=1$.*

Proof. If we take subsequent blow-ups of general curves in an exceptional divisor$/P$ with discrepancy i/r, we may find exceptional divisors$/P$ with discrepancies in $i/r + \mathbb{Z}$. Thus, by Theorem 3.2 it is enough to construct an exceptional divisor$/P$ with discrepancy 1.

We use a covering trick. Let $\widetilde{X}/X$ be the Reid–Wahl canonical covering [26, 2.4.3]. It is a cyclic Galois covering of degree r with $\widetilde{X}$ having the only terminal Gorenstein singularity $\widetilde{P}/P$. Suppose that X does not belong to the exceptions. Then $\widetilde{P}$ is really singular, and, according to Markushevich, we have an exceptional divisor $\widetilde{E}/\widetilde{P}$ with $d(\widetilde{E}, 0, \widetilde{X}) = 1$. Then the corresponding exceptional divisor (image bi-divisor) E/P has discrepancy 1 too.

It is better to handle log discrepancies. By [26, 2.1], for a natural multiplicity $m \geq 1$,

$$2 = a(\widetilde{E}, \widetilde{X}) = ma(E, X) = m(1 + i/r),$$

where $i \geq 1$. Thus $m = 1, i = r$, and $d(E, X) = 1$. ∎

3.3.1. Remark. In the higher-dimensional case, the situation may be more complicated. First, it is possible that for a terminal singularity $p \in X$ of dimension $n \geq 4$ and of index r, there is no good interval where we can predict discrepancies $i/r, i \in \mathbb{N}$, in exceptional divisors$/p$, for instance, in the case of $n - 2 > i/r \geq n - 3$. But this may hold for $(n-3)$-terminal singularities.

As in Corollary 3.3, we can construct an exceptional divisor D/p with discrepancy $d(D) \leq n - 1$ if we have such in the Gorenstein case by [24, Problem 5]. Moreover, if p is not a cyclic quotient (see Example 3.2.2 above), we should have D with $d(D) \leq n - 2$. It will be integer when p is $(n-3)/2$-terminal. So, for $n = 3$, $d(D) = 1$ if p is a terminal but not a cyclic quotient. For $n = 4$, $d(D) = 1$ or 2 if p is a 1/2-terminal but not a cyclic quotient, and so on. The latter is conjectured.

On the other hand, for each toric *singularity* p, we have D with $d(D) \leq (n-2)/2$ (Borisov).

3.3.2. Remark. In general, by a quotient singularity X of dimension n, we mean a finite quotient of a smooth neighborhood Y, i.e., there exists a finite Galois morphism $Y \to X$. Such singularities are purely log terminal, but may not be terminal even for cyclic quotients. We may ask whether this category is *closed* for some partial resolutions, e.g., they have only quotient singularities or a subclass of them. For example, does it hold for log crepant blow-ups as in Theorem 3.2? The last holds for toric singularities in any dimension, and, in particular, for all quotient singularities in dimension 2.

In dimension 3, it is not clear even for canonical quotient Gorenstein singularities and their crepant blow-ups. Presumedly, in dimension 3 the *terminal* quotient Gorenstein points are nonsingular (cf. [2, Question 1]). Then all together it gives a crepant resolution for the quotient Gorenstein singularities. Alas, this is the Dixon–Harvey–Vafa–Witten conjecture.

3.4. Corollary. *Let $P \in X$ be a 3-fold singularity with a boundary B. Let ε and N be a positive real and natural number, respectively. Suppose that*

(i) *B is $\mathbb{R}$-Cartier,*
(ii) *$K_X + B$ is terminal in codimension ≥ 2, and*
(iii) *there exists at most N exceptional divisors$/P$ with discrepancy $\leq \varepsilon$.*

Then each Weil $\mathbb{Q}$-Cartier divisor has at most index $\left\lceil \dfrac{N+2}{\varepsilon} \right\rceil - 1$ in P.

Proof. By the monotonicity [26, 1.3.3], we may assume that $B = 0$ and replace (ii) by

(ii)′ P is a terminal singularity of X.

Then by Kawamata [6, 5.2] we want to check that P has at most index $\left\lceil \dfrac{N+2}{\varepsilon} \right\rceil - 1$. Kawamata's statement concerns $\mathbb{Q}$-factorial singularities. However, a covering trick reduces it to Gorenstein canonical singularities where it follows from the existence of a $\mathbb{Q}$-factorialization.

So, if K_X has index r, then by Corollary 3.3 and Example 3.2.2, there exist exceptional divisors E_i with discrepancies $i/r, i \geq 1$, except for $i = r$. Therefore, $N+1 \geq \lceil \varepsilon r \rceil > \varepsilon r - 1$, which gives the required estimation of r. ■

3.4.1. Remark. We know that $r \geq 1$. Thus $\left\lceil \frac{N+2}{\varepsilon} \right\rceil \geq 2$. In particular, this includes that $N \geq 1$ whenever $\varepsilon \geq 2$.

Moreover, for singular P, $\varepsilon < 1 + 1/r$ unless $N = \infty$. By a construction in the proof of Corollary 3.2.2, we have an infinite set of exceptional divisors with discrepancies $1 + 1/r$.

4. A.c.c. and Semi-Discontinuity of Discrepancies for 3-Folds

If A is linearly ordered, then the d.c.c. (descending chain condition) is equivalent to the well-ordering. In particular, if $A \subset \mathbb{R}$ is a subset of reals with the induced order, then A is well ordered for the opposite order if and only if it satisfies the a.c.c. (ascending chain condition). Equivalently, it is bounded from above and has no condensation points from below. Thus, it looks discontinuous from above, and we say then that it is *semi-discontinuous from above* [24]. Similarly, if $A \subset \mathbb{R}$ is well-ordered, it will be bounded and *semi-discontinuous from below*. A set of reals will be discontinuous if and only if it is semi-discontinuous from both directions. It is finite if and only if it is well ordered in both directions.

4.1. Examples.
4.1.1. If $A \subset \mathbb{R}$ satisfies the d.c.c. (respectively the a.c.c.), then so does any subset of A.
4.1.2. If $A \subset \mathbb{R}$ satisfies the d.c.c. (respectively the a.c.c.), then

$$-A = \{-a \mid a \in A\}$$

satisfies the a.c.c. (respectively the d.c.c.).
4.1.3. If $A \subset \mathbb{R}$ satisfies the d.c.c. (respectively the a.c.c.) and $B \subset [0, +\infty)$ is finite, then, for any natural N

$$A_N = \left\{ \sum_{i=1}^{N} a_i b_i \mid a_i \in A \text{ and } b_i \in B \right\}$$

satisfies the d.c.c. (respectively the a.c.c.) (cf. [26, 4.9]).
4.1.4. (*Standard*)

$$\Gamma = \{1\} \cup \left\{ \frac{n-1}{n} \mid n \in \mathbb{Z} \text{ and } n \geq 1 \right\}$$

satisfies the d.c.c..

Let η be a generic point of X. Then we define the m.l.d. (minimal log discrepancy) of (X, C_X) or $K_X + C_X$ in η as

$$a(\eta, C, X) = a(\eta, C_X, X) = \min \left\{ a(D_i, C, X) \mid D_i \in \mathcal{D}iv(X) \text{ with } \operatorname{center}_X D_i = \eta \right\}.$$

It is well defined when $C = B$ is a bi-boundary and X has only log canonical singularities with respect to B, or even when $C_X = B_X$ is a boundary and (X, B_X) has only log canonical singularities. A m.l.d. is attained on a log resolution.

Below b_i denotes a multiplicity of B_X in a prime divisor of X.

4.2. Conjecture on discrepancies (cf. [24]). *Let $\Gamma \subseteq [0,1]$ be a well-ordered set of reals, i.e., it satisfies the d.c.c. (for example, finite). Then the set of m.l.d.'s*

$$A(\Gamma, n) = \left\{ a(\eta, B_X, X) \in \mathbb{R} \middle| \eta \text{ of codimension } n, \text{ and all } b_i \in \Gamma \right\}$$

(since $a(\eta, B_X, X)$ is defined and finite, it is assumed that $K_X + B_X$ is log canonical in η) satisfies the a.c.c., i.e., the semi-discontinuous and bounded from above.

The conjecture is proved in dimension two ([1] for rational boundary coefficients) [29], and for $\Gamma = \{0\}$ in the case of toric varieties [2].

4.2.1. Example ([29]). Let η be a generic nonsingular point of codimension n in X, $B_X = \sum b_i D_i$ be a boundary in its neighborhood, and $a = a(\eta, B_X)$ be the m.l.d. in η. Then

(a) $a \leq n$.
(b) $a = n$ if and only if $B_X = 0$ in η.
(c) $n - 1 \leq a \leq n$ if and only if $\operatorname{mult}_\eta B_X \leq 1$. Moreover, $a = n - \operatorname{mult}_\eta B_X$ in this case; it is attained in the monoidal transform of η, and only in it if $a > n - 1$.

If $b_i \in \Gamma$ as in 4.2, then the set of m.l.d.'s

$$A = \left\{a \in [0,1] \middle| n - 1 \leq a = n - \sum n_i b_i \text{ with } n_i \in \mathbb{N}\right\} \subseteq A(\Gamma, n) \cap \{a \geq n-1\}$$

satisfies the a.c.c. (cf. [10, Lemma 3]). Indeed, for $b_i \neq 0$, n_i are bounded by $1/\gamma$, where $\gamma = \min\{b_i \mid b_i \neq 0 \in \Gamma\}$. Thus, we may apply 4.1.

Note that presumedly [24] the above should hold if $K_X + B_X$ is $\mathbb{R}$-Cartier in η and $a > n-1$. It is known when $n \leq 3$.

Since a finite union of a.c.c. subsets of $\mathbb{R}$ satisfies the a.c.c. too, we get the following results.

4.3. Corollary. *If 4.2 holds for natural numbers $n_1, \dots, n_m$, then we have the a.c.c. in* mixed codimensions, *i.e., for*

$$A(\Gamma, n_1, \dots, n_m) = \left\{a(\eta, B_X, X) \in \mathbb{R} \middle| \eta \text{ of codimension } n_i, \text{ and all } b_j \in \Gamma\right\}.$$

In, particular, if 4.2 holds for the natural numbers $\leq n$, then the a.c.c. takes place in the dimensions $\leq n$, *i.e., for*

$$A^n(\Gamma) = \left\{a(\eta, B_X, X) \in \mathbb{R} \middle| \dim X \leq n, \text{ and all } b_i \in \Gamma\right\}.$$

Here we partially prove 4.2 for codimension $n \leq 3$ under one additional assumption which is natural in applications.

4.4. Proposition. *Fix $\varepsilon > 0$, $d \leq 3$, and $N \in \mathbb{N}$ and a well-ordered subset $\Gamma \subset [0,1]$. Let $A = A(\Gamma, d, \varepsilon, N)$ be the set of log discrepancies ≤ 1 for $K_X + B_X$ in the prime bi-divisors D of X/X with* $\operatorname{center}_X D$ *of codimension d, where (X, B_X) are all such pairs that*

(i) *the multiplicities b_i of B_X belong to Γ;*
(ii) *log divisor $K_X + B_X$ is $\mathbb{R}$-Cartier; and*
(iii) *at most N prime exceptional bi-divisors of X/X with center of codimension d have log discrepancies $a_i < 1 + \varepsilon$.*

Then A satisfies the a.c.c., and, moreover, A is finite whenever so is Γ.

Moreover,

4.4.1. Lemma. *If (X, B_X) satisfies* (ii)–(iii) *of Proposition* 4.4, *then the indices in the generic points of codimension $d \leq 3$ are bounded for the $\mathbb{Q}$-Cartier Weil divisors in such points.*

The bound depends on ε, N, and d.

Since the m.l.d.'s under the above restrictions form a subset of $A(\Gamma, d, \varepsilon, N)$, we get Conjecture 4.2 under these restrictions.

4.5. Corollary. *Fix $\varepsilon > 0$, $d \leq 3$ and $N \in \mathbb{N}$. Then the set of m.l.d.'s*

$$\left\{a(\eta, B_X, X) \in [0,1] \middle| \eta \text{ of codimension } d, \text{ and all } b_i \in \Gamma\right\}$$

satisfies the a.c.c. whenever we consider only pairs (X, B_X) which satisfy (i)–(iii) *as in Proposition* 4.4.

4.5.1. Remark. 4.2–4.5 hold for $a(\eta, B_X, X) \geq 1$ without assumptions (i)–(iii) of Proposition 4.4 [29].

Proof of Proposition 4.4 and Lemma 4.4.1. Taking hyperplane sections, we may assume that $\dim X = d \leq 3$.

We are taking into consideration only the log discrepancies $a(E, B_X) \leq 1$ and with $\mathrm{center}_X\, E = P$ being a closed point of X and assuming 4.4(iii) for the center P. Then, by a construction in the proof of Corollary 3.3 (cf. Remark 3.4.1), $K_X + B_X$ is ε-log canonical in P. Hence according to Corollary 1.5, $K_X + B_X$ is log canonical in a neighborhood of P.

We may also assume that X is $\mathbb{Q}$-factorial and even strictly log terminal. For this we replace X by its strictly log terminal model of Y with respect to $K_Y + B_Y$. By [26, 1.5.7, Corollary 9.1] it exists and has the same log discrepancies$/P$ for a new boundary B_Y which is the log transform of B_X (see also Proposition 2.4). According to the classification of log canonical singularities in codimension 2, the $\mathbb{Q}$-factorialization is small over a neighborhood of P, and also $K_X + B_X$ is purely log terminal (and ε-log canonical in codimension 2) there. Hence, after changing the boundary B_Y we may present the $\mathbb{Q}$-factorialization as an extremal contraction negative with respect to a Kawamata log terminal divisor $K_Y + B_Y$. By the contraction theorem, the indices of the $\mathbb{Q}$-Cartier Weil divisors are the same on the $\mathbb{Q}$-factorialization.

Note also that conditions 4.4 (ii)–(iii) will hold automatically on Y. If, for a new model, $\mathrm{center}_Y\, E = P$ is not a closed point, we can use induction on d.

We prove both statements by induction on

$$M = \#\{E/P \mid a = a(E, B_X) \leq 1\} \leq N.$$

If $M = 0$, then $A = \emptyset$, and 4.4.1 holds by 3.4 for $d = 3$. For $d = 2$, X will be nonsingular.

If $M \geq 1$, we have an exceptional divisor E/P with log discrepancy $a \leq 1$. Let $f\colon Y \to X$ be a resolution of it as in Theorem 3.1 with $\Delta = \lceil B_X \rceil$. Literally, we may apply the theorem when $a < 1$. In our case, we may add to B_X a small multiple of a Cartier divisor through P which decreases $a = 1$.

First, we prove Proposition 4.4. The case $d = 1$ states that the set of discrepancies $1 - b_i$ satisfies the a.c.c. (see Examples 4.1). So, we may assume that $d = 2$ or 3.

To calculate a we fix a covering family of curves on E with the generic curve C, and then use the linear relation

$$a \cdot (E.C) = (K_Y + B_Y.C)$$

with $B_Y = f^{-1}B_X + E$.

Thus, it is enough to check that $(E.C)$ and $(K_Y + B_Y.C)$ belong respectively to a finite and d.c.c. set of reals. Note that f is extremal and $(E.C) < 0$ by the $\mathbb{Q}$-factorial property of P.

Indeed, it is enough to check that the intersection numbers $(E.C), (D_i.C)$'s and $(K_Y.C)$ belong to a finite set of rationals, where the divisors D_i form $\mathrm{Supp}\, f^{-1}B_X$. Indeed,

$$(K_Y + B_Y.C) = (K_Y.C) + (E.C) + \sum b_i(D_i.C)$$

with $(D_i.C) \geq 0$. So, by Examples 4.1, these satisfy the d.c.c., because so does Γ. Moreover, intersections $(K_Y + B_Y.C)$ form a finite set whenever Γ does so.

By induction, the indices of E, D_i's, and K_Y are bounded. So, we need to check that $(E.C), (D_i.C)$'s and $(K_Y.C)$ are bounded. Since the a, b_i's are in $[0, 1]$ and bounded, we may check this only for $(E.C)$ and $(D_i.C)$'s. According to our construction,

$$f^*(K_X + B_X) = K_Y + cE + \sum b_i D_i,$$

where $c = 1 - a$ is the *codiscrepancy* in E, is numerically trivial on E, and ε-log canonical in codimension ≥ 2, whereas $0 \leq c \leq 1 - \varepsilon$. By the extremal property of f, all $(D_i, C) \geq 0$, and $(E.C), (K_Y + E.C) < 0$. On the other hand, by [27, Theorem and Remark (3)] there exists a generic curve C such that $0 > (K_Y + E.C) \geq -3$, So, $0 > (E.C) \geq -3/\varepsilon$ and $0 \leq (D_i.C) \leq 3/\gamma$ with $\gamma = \min\{b_i \mid b_i \neq 0 \in \Gamma\}$ (or $= 1$ when $\Gamma \setminus \{0\} = \emptyset$). Note that γ is positive according to the d.c.c. property of Γ (cf. [27, Corollary 1]).

Now we are ready to prove 4.4.1. According to the contraction theorem or rationality of P, it is enough to check that, for the $\mathbb{Q}$-Cartier Weil divisors, the multiplicities $m = \operatorname{mult}(E, D, X)$ have bounded denominators. This follows by induction from the finiteness and rationality of $(E.C)$, because

$$m = -(f^{-1}D.C)/(E.C).$$

In turn, the finiteness has been proved for $\Gamma = \emptyset$, according to the monotonicity [26, 1.3.3] of discrepancies. ■

5. Log Minimal Model for 3-Folds

First, we recall what log termination means (in the analytic case over a projective subset). More generally, consider an $\mathbb{R}$-Cartier divisor D. Then a *contraction with respect to* D or a D-*contraction* of X/S is a contraction of $g: X \to Y/S$ such that $-D$ is numerically f-ample. By a *modification of* X/S *with respect to* D or by a D-*modification* of X/S we mean a flip of a birational (bimeromorphic) D-contraction g with respect to D; we also call it a D-*flip*. It is defined as a modification

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow_{g} \quad \swarrow_{g^+} & \\ & Y & \end{array}$$

over S, where g^+ is a small contraction/S for which the modified divisor D^+ is numerically ample [26, Sec. 1]. The *termination with respect to* D or D-*termination* means that any infinite chain of D-modifications stabilizes, i.e., all of these are trivial, except for a finite number of them. Precisely, after a finite number of D-modifications of X/S, either we have only fiber contractions with respect to D or we have a birational (bimeromorphic) one which has no D-flip. If $D = K_X + B_X$ is a log divisor, we talk about log termination, log flips, and so on. In the case of good singularities, we may anticipate the following ultimate form of the LMMP.

5.1. Conjectures on the LMMP. Suppose that X/S is projective and $K_X + B_X$ is log canonical or even log canonical with respect to a bi-boundary B. Then

5.1.1. If $K_X + B_X$ is not nef $/S$ (over a compact subset $W \subseteq S$ in the analytic case), then there exists a nontrivial $(K_X + B_X)$-contraction$/S$ (respectively, $/W$). As in the MMP we can split this into two problems.

5.1.1a. (The log cone.) For $K_X + B_X$, the negative part $\overline{\mathrm{NE}}^{K_X+B_X}(X/S)$ of the Kleiman–Mori cone (respectively, $\overline{\mathrm{NE}}^{K_X+B_X}(X/S; W)$) is locally polyhedral and rational by the next property.

5.1.1b. (The log contraction.) Any face F of $\overline{\mathrm{NE}}^{K_X+B_X}(X/S)$ (respectively of $\overline{\mathrm{NE}}^{K_X+B_X}(X/S; W)$) can be contracted, i.e., there is a contraction $\mathrm{cont}_F: X \to Y/S$ $(/W)$ such that for a curve C/S $(/W)$, $\mathrm{cont}_F C = \mathrm{pt.}$ if and only if the numerical class of C belongs to F. This is a $(K_X + B_X)$-contraction and is nontrivial by Kleiman's criterion. Obviously, this problem may be reduced to the semi-ampleness (see the proof and cf. 6.16).

5.1.2. The log flips with respect to $K_X + B_X$ exist (cf. 6.13). Moreover, it easy to check that the flips preserve the ε-log canonical (ε-log terminal and, for the extremal flips, strictly log terminal) property with respect to B (cf. [23, 2.13]). In the log terminal case we may also assume that the flipped log minimal models of X/S have resolutions in 1.2.4 in which each of a given finite set of prime bi-divisors$/S$ is nonexceptional.

5.1.3. The log termination holds with respect to $K_X + B_X$. With the existence of a projective log resolution and 5.1.1 and 5.1.2, this implies that after a finite number of modifications of X/S $(/W)$ with respect to $K_X + B_X$ we obtain either a fiber contraction (a log Fano fibering) of a modified X/S $(/W)$, or a modified X/S $(/W)$ will be a log minimal model of X/S $(/W)$ with respect to B.

However, some results of the log terminal case and MMP do not hold in such generality. For instance, we may have nonrational singularities (cf. [11, 1.3.6]).

5.2. Theorem. *The conjectures on the LMMP hold in dimension* 3, *i.e., when* $\dim X \le 3$.

As in the MMP for 3-folds [23], we start with the log termination. Indeed, we reduce the log termination to that of the terminal case. This is a slight modification of [10], where we replace [8] by a stronger statement, namely, by Theorem 3.2.

Proof of 5.1.3 for 3-folds.

Step 1. Reduction to the strictly log terminal and extremal flipping case. Let $X - \to X^+$ be a flip of a nontrivial contraction $g: X \to Y/S$. According to [26, 9.1], we have a log minimal resolution $h: \widetilde{X} \to X$, i.e., $K_{\widetilde{X}} + B_{\widetilde{X}} = h^*(K_X + B_X)$ is strictly log terminal. In the 3-fold strictly log terminal case, we know 5.1.1 and 5.1.2. Thus, using the LMMP for $\widetilde{X}/Y$ we can decompose the contraction $g \circ h: \widetilde{X} \to Y$ into a nonempty sequence of nontrivial flips and a contraction $\widetilde{X}^+ \to X^+$ which is a minimal log resolution of X^+ [26, 1.5]. This lifts log flips of X into nontrivial sequences of log flips of $\widetilde{X}$ with a strictly log terminal divisor $K_{\widetilde{X}} + B_{\widetilde{X}}$ (cf. [15, 8.2]).

Since divisorial modifications decrease the Picard number, we may assume that all modifications are flipping. Such modifications in the $\mathbb{Q}$-factorial and 3-fold case are modifications in curves. Also we may assume THAT THEY ARE *extremal*, i.e., of extremal contractions.

Step 2. Reduction to the ε-log canonical case. By a modified version of the special termination [26, 4.1; 15, 7.1], after a finite number of log flips with respect to $K_X + B_X$ all the next flips do not intersect the reduced part $\lceil B_X \rceil$ of the boundary. Thus, if we decrease boundary coefficients in this part we may assume that $\lceil B_X \rceil = 0$, i.e., in addition, $K_X + B_X$ is ε-log canonical for a positive real ε.

Step 3. Reduction to the terminal case in codimension ≥ 2. Let N be a natural number which bounds the number of exceptional divisors of X having log discrepancies $< 1 + \varepsilon$. By Corollary 1.7, such N exists and depends only on (X, B_X) (cf. [23, 2.15]). It fits as well for the next log modifications by [23, 2.13.3].

Thus, according to Proposition 4.4, the log discrepancies of $K_X + B_X$ in codimension 2 and 3 for X and its log modifications belong to a finite set

$$A = A(\Gamma, 2, \varepsilon, N) \cup A(\Gamma, 3, \varepsilon, N),$$

where ε have been fixed in Step 2, $\Gamma = \{b_i\}$ consists of the multiplicities of B_X and is finite. (In the analytic case, Γ is finite over a compact set W.)

Note now that each modification increases the discrepancies and strictly over a modified locus [23, 2.13.3]. Therefore, after a finite number of modifications, the next modifications will be performed in curves which do not contain log terminal singularities.

The latter means centers of the exceptional divisors E with $a(E, X, B_X) \le 1$. By Theorem 3.1, we can resolve such singularities and assume that $K_X + B_X$ is terminal in codimension ≥ 2. In particular, new $X = \widetilde{X}$ will have only (isolated) terminal singularities. Such a transformation $h: \widetilde{X} \to X$ does not touch generic points of the flipping curves and has no exceptional divisors over them. Then, as in Step 1, we can decompose $g \circ h$ into a nontrivial sequence of flips and a contraction that is a minimal resolution of X^+. According to our construction, all flips of $\widetilde{X}$ are extremal and *small*, i.e., in curves. Hence, they preserve the terminal property in codimension ≥ 2.

Now we put $\varepsilon = \min \{a_i = 1 - b_i\}$. That is the m.l.d. for prime divisors of X. Thus $\varepsilon = 1 - b$, where $b = \max \{b_i\} = \max \Gamma$ is the maximal multiplicity of B_X for these divisors. The corresponding components of B_X will be called *maximal*.

Step 4. Reduction to the terminal case. Suppose that a flipped curve has a component C in a prime divisor with maximal boundary multiplicity b. Then, by Step 3 and Example 4.2.1, X is nonsingular along C and the monoidal transform $E(C)$ in the generic point of C has the m.l.d. $a(C, B_X) = a(E(C), B_X) =$

$2 - \text{mult}_C B_X \le 2 - b = 1 + \varepsilon$. Since the set of such discrepancies $1 - \text{mult}_C B_X$ and the set of exceptional divisors with discrepancies $< \varepsilon$ is finite, this is impossible after a finite number of modifications (see Example 4.2.1). Thus, we may assume that the maximal prime divisors D_i^+ do not pass through the flipped curves.

If a flipped curve C^+ intersects one such divisor $D_i^+ \subset X^+$, then by the extremal property $(D_i^+, C^+) > 0$ for each such curve C^+. Therefore $(D_i, C) < 0$, where C is a flipping curve and D_i is the divisor in X corresponding to D_i^+. Each such transformation contracts a curve C on D_i. So, after a finite number of modifications we may assume that flipping curves do not intersect the maximal components D_i (cf. [26, 4.1]). As in Step 2, we can decrease boundary multiplicities to 0 in such components.

Since we have only a finite set of boundary coefficients, we may finally assume that $B_X = 0$. Then, by Step 3, K_X is terminal.

The termination in the last case has been established in [23]. ■

Proof of Theorem 2.3. According to [12, 4.2.1 and 3.2.1; 26, 1.3.5], we have 5.1.1, i.e., the cone and contraction Theorem, when $K_X + B_X$ is strictly log terminal. So, the existence of a log minimal model follows directly from this, the established termination and existence of log flips in the strictly log terminal case [26, 9.4].

The existence of an initial strictly log terminal model can be obtained by the Hironaka work (in the analytic case we suppose that X is Moishezon over a neighborhood of W). We may also assume that a given finite set of prime bi-divisors$/S$ is nonexceptional in the resolution. ■

6. Geography of Log Models

6.1. Definitions. Let $h: X - \to X'$ be a modification. Recall that a birational transform $C' = h_* C$ of a cycle C is a homomorphic extension of this for prime cycles. Note that for a prime cycle C, $h(C) = 0$ whenever h is not regular (analytic) in the generic point of C. Cycles C and $h_* C$ as well as C' and $h^{-1}{}_* C'$ are called *corresponding*.

Fix two *comparable* families $\{C\}$ and $\{C'\}$ of curves respectively in X/S and X'/S, i.e., to any curve C of the first family corresponds a curve of the second one or 0-cycle, and vice versa.

Two log pairs $(X/S, B)$ and $(X'/S, B')$ are called *(numerically) equivalent with respect to the given families* $\{C\}$ *and* $\{C'\}$ if they have the same signature in the corresponding curves C/S. This means that, for every $C \in \{C\}$ and $C' \in \{C'\}$, $(K_X + B.C)$ and $(K_{X'} + B'.C')$ are defined, whereas for any pair of corresponding curves C and C', $(K_X + B.C)$ is positive, negative, or 0 if and only if $(K_{X'} + B'.C')$ is positive, negative, or 0 respectively.

We say simply that pairs $(X/S, B)$ and $(X'/S, B')$ are *equivalent* when both families are maximal, i.e., all curves$/S$.

By Proposition 2.4, two weakly log canonical models $(X/S, B_X)$ and $(Y/S, B_Y)$ with the same bi-boundary B are equivalent. In addition, two weakly log canonical models $(X/S, B_X)$ and $(Y/S, B'_Y)$ are equivalent if and only if each of them is an equivalent model of another, i.e., $(X/S, B'_X)$ and $(Y/S, B_Y)$ are equivalent respectively to $(X/S, B_X)$ and $(Y/S, B'_Y)$, and weakly log canonical too. The latter is the only nontrivial part. This is easy to derive from the log semi-ampleness. Another approach is to use the arguments in the proof of 2.4.

First, we suppose X/S to be a fixed 3-fold with a fixed finite set of distinct prime divisors D_i of X. According to [26, 1.3.2], we have a (closed) convex rational polyhedron

$$\mathcal{P} = \left\{ B = \sum b_i D_i \,\middle|\, \text{all } b_i \in [0,1], \text{ and } K_X + B \text{ is log canonical} \right\}$$

in the cube $\bigoplus [0,1] D_i$. Each model $(X/S, B)$, with $B \in \mathcal{P}$, is identified with its boundary, i.e., point $B \in \mathcal{P}$.

Second, we fix a system of extremal rays $R_j = \mathbb{R}^+[C_j] \subset \overline{\mathrm{NE}}(X/S)$ (respectively in $\overline{\mathrm{NE}}(X/S;W)$ in the analytic case) which are *generated* by curves C_j. For boundaries $B \in \mathcal{P}$, equivalent classes of $(X/S, B)$ with fixed X and with respect to the curves C_j give a decomposition of $\mathcal{P}$ into convex subsets.

6.2. First Main Theorem. *The given decomposition is rationally polyhedral and locally finite in the interior of $\mathcal{L} \cap \mathcal{P}$, where $\mathcal{L}$ is a rational affine plane. The decomposition is given in $\mathcal{L}$ by hyperplanes $(K_X + B.C_j) = 0$ and maximal faces.*

Moreover, the set $\mathcal{N}$ of $B \in \mathcal{P}$, such that $(K_X + B.C_j) \geq 0$ for all j, is a (closed) convex rational polyhedron with finite decomposition into equivalent classes with respect the curves C_j. The faces and the decomposition of $\mathcal{N}$ are given in $\mathcal{P}$ by a finite set of rational hyperplanes $(K_X + B.C_j) = 0$ and maximal faces of $\mathcal{P}$.

The curves C_j in the defining equations above may be chosen to be rational, and they generate extremal rays $R_j \subset \overline{\mathrm{NE}}^{K_X+B}(X/S;W)$ (respectively in $\overline{\mathrm{NE}}^{K_X+B}(X/S;W)$ in the analytic case) for some $B \in \mathcal{P}$.

We need the following elementary geometric fact to reduce the theorem to the 1-dimensional case.

6.3. Lemma. *Let V be a finite-dimensional affine space over $\mathbb{Q}$ and H_i an infinite sequence of distinct hyperplanes convergeing to a hyperplane $H \subset V \otimes_{\mathbb{Q}} \mathbb{R}$. The set of rational lines $\mathcal{L}$, such that intersections $p_i = \mathcal{L} \cap H_i$ give a convergent (of course, to $\mathcal{L} \cap H$) sequence of distinct points $p_i \in \mathcal{L}$, is everywhere dense.*

In this statement, we may replace $\mathbb{Q}$ by any number field $K \subseteq \mathbb{R}$.

Proof-commentary. The hyperplanes in an affine space A form a Zariski open subset of a projective space (a certain Grassmannian). If A is defined over reals, this will be a real affine set.

The convergence is considered with respect to the real topology in that set.

Similarly, we consider the set of lines in $V \otimes_{\mathbb{Q}} \mathbb{R}$ equipped with the real topology.

Of course, the statement means that we consider only well-defined p_i, i.e., points. More precisely, it means that for a given $\mathcal{L}$ there exist an *infinite subsequence* $H_{i'}$ such that all $p_{i'} = \mathcal{L} \cap H_{i'}$ are distinct points convergent to point $p = \mathcal{L} \cap H$. So, it is enough to prove the statement for an infinite subsequence.

Suppose that an infinite subsequence $H_{i'}$ has a common point. According to the above, we may assume that all H_i's have nonempty intersection $I = \cap H_i$. The plane I is defined over $\mathbb{Q}$, and the projection from I reduces the statement to a lower-dimensional case, because a rational lifting of any rational line under the projection possesses the required properties.

If $\dim V = 1$, $\mathcal{L} = V$ satisfies the required properties by our assumption. Thus, by induction we may assume that any infinite subsequence $H_{i'}$ does not have common points.

In that case, any rational line $\mathcal{L}$, such that $p = \mathcal{L} \cap H$ is a point, satisfies the required properties. Indeed, it implies that $p_i = \mathcal{L} \cap H_i$ is a point except for a finite set of indices i. This also means that H_i is not parallel to $\mathcal{L}$ for the former i. Note that, according to our assumption, points p_i coincide only for a finite subset of indices i. ■

Proof of Theorem 6.2, when X is strictly log terminal. If we have a finite system of curves C_j, then hyperplanes H_j, given by $(K_X + B.C_j) = 0$, and maximal faces define the decomposition. This means that each equivalent class is given as the intersection of a finite set of these hyperplanes or its half-planes. Since each of them is rational, we have the required properties.

In general, we want to check that H_j has no cluster hyperplanes inside $\mathcal{L} \cap \mathcal{P}$, i.e., a neighborhood of any such point is intersected by a finite set of hyperplanes H_j. Then Lemma 6.3 reduces the proof to the case where $\mathcal{L}$ is a rational line.

In that case, $\mathcal{L} \cap \mathcal{P}$ is a rational segment. We may assume that it is nontrivial. So, one end of the segment corresponds to a $\mathbb{Q}$-boundary B and the other to a $\mathbb{Q}$-boundary $B' = B + \Delta$, where $\Delta \neq 0$ is a $\mathbb{Q}$-divisor too.

The log divisors $K_X + B$, $K_X + B'$ are $\mathbb{Q}$-Cartier. Therefore, there exist natural N such that $N(K_X + B), N(K_X + B')$, and $N\Delta$ are Cartier.

Hyperplanes H_j give, in intersection, points $p_j = B + \lambda_j\Delta$, where $\lambda_j \in [0,1] \cap \mathbb{Q}$ (except for those with $\mathcal{L} \| H_j$, which we drop).

We want to check that B and B' are the only possible cluster points of $\{p_j\}$. Equivalently, 0 and 1 are the only possible cluster points of $\{\lambda_j\}$.

Suppose that $\lambda \in (0,1)$ is such a cluster point.

First, we consider the case where $\lambda = p/q$ is rational. Let B_λ be the corresponding boundary. Then $Nq(K_X + B_\lambda)$ is Cartier and, for *any* curve C_j in every R_j, $(K_X + B_\lambda.C_j) = 0$ or $|(K_X + B_\lambda.C_j)| \geq 1/Nq$. Note that $(K_X + B_{\lambda_j}.C_j) = 0$. Let us take R_j such that $|\lambda_j - \lambda| \neq 0$ and $\ll 1$. So, the slope of the linear function $l(\mu) = (K_X + B_\mu.C_j)$ is very steep. Thus $(K_X + B.C_j)$ or $(K_X + B'.C_j) \ll 0$, which contradicts [27, Theorem].

Note that R_j is contracted because X is strictly log terminal. The latter implies that strictly (and even purely) log terminal $K_X + B$ are dense in $\mathcal{P}$ [26, 1.3.5]. So, we may choose rational C_j [7] generating R_j. The latter satisfies the required properties by the construction.

Suppose now that λ is irrational. Then according to the approximation theorem there exists a rational number p/q such that $|p/q - \lambda| < 1/q^2$ and $q \gg 1$ [3]. Now we may choose R_j as above. The slope will be bounded from below by a multiple of $1/q$ (depending only on N).

The nef property with respect to C_j's is closed and convex. So, we will check only the rationally polyhedral property. Again, we show that the last statement holds for any $\mathcal{L} \cap \mathcal{N}$.

If $\mathcal{L}$ is 1-dimensional, then the latter holds. In that case $\mathcal{L} \cap \mathcal{P} = [B, B']$ is a segment. We know that set

$$\mathcal{L} \cap \mathcal{N} = \left\{ D \in [B, B'] \mid \text{ with } (K_X + D.C_j) \geq 0 \text{ for all } j \right\}$$

is rationally polyhedral inside $[B, B']$. We now check this near edge points, say, near B. Note that $(K_X + B.C_j) = 0$ or $\geq 1/N$, where N is the index of $K_X + B$, i.e., $N(K_X + B)$ is Cartier. Thus, as above, there exists $0 < \varepsilon \ll 1$ (depending only on N) such that $\mathcal{L} \cap \mathcal{N} = B$ or $[B, B + \varepsilon\Delta] \subset \mathcal{L} \cap \mathcal{N}$. In both cases $\mathcal{L} \cap \mathcal{N}$ is rationally polyhedral near B.

Thus we can use induction.

In general, $\mathcal{N}$ is rationally polyhedral inside $\mathcal{P}$, and by induction each face of $\mathcal{P}$ intersects $\mathcal{N}$ in such a polyhedron too. So, the set of rational B is dense in $\partial(\mathcal{N})$. Here ∂ denotes the topological boundary. Since $\mathcal{P}$ and $\mathcal{N}$ are compact, again by [3] it is enough to check the rationally polyhedral property in a quite large neighborhood U of such points B. More precisely, we take

$$U = U(B, \varepsilon) = \{B + \varepsilon\Delta \mid \Delta \in \mathcal{P}\},$$

where $\varepsilon = c/N$ with index N of $K_X + B$ and $c = 1/7$.

Passing a certain number of rational hyperplanes through B, we can decompose $\mathcal{P}$ into a finite set of rational convex polyhedra $\mathcal{P}_i$ with vertex B. Thus it is enough to show the required properties for each $\mathcal{N} \cap \mathcal{P}_i$. Above and below we may replace $\mathcal{P}$ by its rational convex polyhedral part.

So, we assume that B is a vertex of $\mathcal{P}$.

Then we contend that $\mathcal{N}$ near B is a cone with the same vertex B. Moreover, for $0 < \varepsilon = 1/7N$, as above, $\mathcal{L} \cap \mathcal{N} = B$ or $[B, B + \varepsilon\Delta] \subset \mathcal{L} \cap N$ for any line through B [27, Theorem]. As we know, this is true for rational lines. (The same arguments work for all lines through B.) The latter are dense in all. Note that the nontrivial segments $\mathcal{L} \cap \mathcal{P}$ form a continuous (piecewise linear) family, because $\mathcal{P}$ is polyhedral.

Finally, by induction, a rational hyperplane section $\mathcal{L}$ of this cone is a finite rational polyhedron. It is given by the intersection of $\mathcal{L} \cap \mathcal{P}$ and rational half-hyperplanes $(K_X + B.C_j) \geq 0$. Locally near B, they cut $\mathcal{N}$ in $\mathcal{P}$ whenever $\mathcal{N} \cap \partial\mathcal{P}$ is a cone over $\mathcal{N} \cap \partial(\mathcal{L} \cap \mathcal{P})$. By induction we may assume that $\mathcal{N} \not\subseteq \partial\mathcal{P}$. ■

6.4. Remark. We need the strictly log terminal condition only to prove the existence of a contraction and because [27, Theorem] needs it. Actually, for the latter fact, we need only the LMMP in the strictly log terminal case in dimensions ≤ 3 [27, Heuristic arguments]. Moreover, 5.1.1 is sufficient (cf. the proofs of Theorem 2.3 in Sec. 5 and Corollary 6.6 below).

Therefore, if X is strictly log terminal, the last proof works in any dimension. We should take constant $c = 1/(2\dim X + 1)$ in the proof (see Remark 6.23.5 below).

6.5. Example. Let X be a smooth projective surface with an infinite set of exceptional curves C_j of the first kind [4, 4.6.4]. Take its smooth hyperplane section H as a single prime divisor. Then $\mathcal{P} = \{\lambda H \mid \lambda \in [0,1]\}$. The first main theorem states that numbers $\lambda_j = 1/(H.C_j)$, such that $(K_X + \lambda_j H.C_j) = 0$, form a subset of (rational) points in $[0,1]$ with the only possible clusters 0 and 1 in $[0,1]$. In our case only 0 is really a cluster.

Let d be the minimal degree $(H.C_j)$ of the curves C_j. Then $\mathcal{N} = \{\lambda H \mid \lambda \in [1/d, 1]\}$. If these C_j's are the only exceptional curves of the first kind on X, then, according to the next result, $\mathcal{N}$ corresponds to the nef log divisors $K_X + B$ with $B \in \mathcal{P}$.

6.6. Corollary. *The set*

$$\mathcal{N} = \left\{B \in \mathcal{P} \middle| \text{ with } (K_X + B.C) \geq 0 \text{ for all curves of } X/S\right\}$$

forms a convex rational polyhedron cut out by hyperplanes $(K_X + B.C_j) = 0$, given by maximal faces of $\mathcal{P}$ and a finite set of rational curves C_j/S, generating extremal rays $R_j \subset \overline{\mathrm{NE}}^{K_X+B}(X/S; W)$ (respectively in $\overline{\mathrm{NE}}^{K_X+B}(X/S; W)$ in the analytic case) for some $B \in \mathcal{P}$.

The interior points of each of its faces are equivalent, and the decomposition on equivalent classes is finite in $\mathcal{N}$.

Proof in the strictly log terminal case. This means that there exists a boundary $B \in \mathcal{P}$ such that $K_X + B$ is nef and strictly log terminal. In particular, this assumes that $\mathcal{N} \neq \emptyset$. Otherwise, $\mathcal{N} = \emptyset \subseteq \mathcal{P}$ can be given as a finite intersection of half-planes $(K_X + B.C_j) \geq 0$ in $\mathcal{P}$, because $\mathcal{P}$ is compact.

Since the strictly log terminal property is open by definition, for the interior points B of $\mathcal{N}$, $K_X + B$ is strictly log terminal too by the convexity of $\mathcal{N}$ and the linearity of the discrepancies with respect to B.

It is enough to check that $\mathcal{N}$ is defined by curves C_j/S which generate the required extremal rays R_j of X/S. Denote the corresponding subset by $\mathcal{N}'$. We know that $\mathcal{N} \subseteq \mathcal{N}'$, and want to check $=$. In the interior points B of $\mathcal{N}'$, $K_X + B$ is also strictly log terminal.

Thus, we should check that if $(K_X + B.C_j) \geq 0$ for all such C_j, then $(K_X + B.C) \geq 0$ for any curve C of X/S. The former means that $B \in \mathcal{N}'$. It implies the latter by 5.1.1, whenever $K_X + B$ is strictly log terminal (see the proof of Theorem 2.3 in Sec. 5) and, in general, because $\mathcal{N}$ is closed.

Finally, none of the hyperplanes $(K_X + B.C) = 0$ crosses a face of $\mathcal{N}$ through its interior points. Thus, these points are equivalent. ■

6.7. Remark. The equivalent classes over $\mathcal{N}$ are fine unions of interiors for faces of $\mathcal{N}$. Moreover, in such a semi-closure we add only open faces belonging to $\partial\mathcal{P}$.

According to Remark 6.4 and [26, Heuristic proof], we get the following results.

6.8. Theorem. *Let $f: X \to S$ be a proper morphism of a 3-fold X (Moishezon over a neighborhood of a compact subspace $W \subseteq S$ in the analytic case), and D be an effective $\mathbb{R}$-divisor in X such that $K_X + D$ is $\mathbb{R}$-Cartier and log canonical in the generic points of a subvariety E, consisting of components of the* degenerate locus

$$\mathcal{E}xc(f) := \{x \in X \mid g \text{ is not finite at } x\}.$$

Then E is covered by a family of effective 1-cycles $\{C_\lambda\}/S$ with $(-K_X - D.C_\lambda) \leq 2n$ (possibly disconnected and over a neighborhood of $W \subseteq S$ in the analytic case), where $n =$ $\dim E/S$ (and even $< 2n$ if $K_X + D$ is Kawamata log terminal in the generic points of E and $X \neq E$). Moreover, we could assume that the generic

1-cycles C_λ are curves, i.e., *reduced and irreducible, when* $K_X + D$ *is numerically definite, and the curves* C_λ *with* $(K_X + D.C_\lambda) < 0$ *(resp.* ≤ 0*) are rational.*

6.9. Corollary. *If, in addition,* F *is an* $\mathbb{R}$*-Cartier divisor such that* $K_X + D + F$ *is log canonical in the generic points of* E*,* $D + F$ *is effective, and* $K_X + D$ *is numerically semi-negative with respect to* f*, then* E *is covered by a family (possibly disconnected) of effective* 1*-cycles* $\{C_\lambda\}/S$ *(resp. with the curves as generic members when* $K_X + D + F$ *is numerically definite) with*

$$(F.C_\lambda) \geq -2n$$

(resp. $> -2n$ *if* $K_X + D + F$ *is Kawamata log terminal in the generic points of* E*, and* $X \neq E$*).*

6.10. Corollary ([13], cf. also [7]). *Let* $f\colon X \to S$ *be a projective morphism of a* 3*-fold* X*, and* D *be an effective* $\mathbb{R}$*-divisor in* X *such that* $K_X + D$ *is* $\mathbb{R}$*-Cartier, let* H *be an* f*-ample* $\mathbb{R}$*-Cartier divisor and* $\varepsilon > 0$*. Then the number of extremal contractions* cont_R *and corresponding rays* R *such that* $K_X + D$ *is*

(*) *log canonical in a generic point of the degenerate fibers of* cont_R*,*

and such that $(K_X + D + \varepsilon H.R) < 0$ *is finite.*

Thus, the half-cone $\overline{\mathrm{NE}}^{K_X+D}(X/S)$ $(\overline{\mathrm{NE}}^{K_X+D}(X/S;W)$ *with a compact subset* $W \subseteq S$ *in the analytic case) is locally polyhedral when* (*) *holds for all extremal rays in it, including the* existence *of the extremal contractions* cont_R.

6.11. Corollary (cf. [7, Theorem 2]). *For* $f\colon X \to S$ *and* D *as in Theorem* 6.8*, let* E *be a subvariety, consisting of components of*

$$\mathcal{E}xc(f) := \{x \in X \mid \text{an irreducible component of a fiber of } f \text{ through } x \\ \text{having dimension greater than } d = \dim X/S\}.$$

Then E *is covered by a family (possibly disconnected) of effective* 1*-cycles* $\{C_\lambda\}/S$ *with* $(-K_X - D.C_\lambda) \leq 2(n-d)$*, where* $n = \dim E/S$ *(and even* $< 2(n-d)$ *if* $K_X + D$ *is Kawamata log terminal in the generic points of* E*). Moreover, we could assume that the generic* 1*-cycles* C_λ *are curves when* $K_X + D$ *is numerically definite, and these curves* C_λ *with* $(K_X + D.C_\lambda) < 0$ *(resp.* ≤ 0*) are rational.*

6.12. Corollary. *If, in addition to Corollary* 6.11*,* F *is an* $\mathbb{R}$*-Cartier divisor such that* $K_X + D + F$ *is log canonical in the generic points of* E*,* $D + F$ *is effective, and* $K_X + D$ *is numerically semi-negative with respect to* f*, then* E *is covered by a family (possibly disconnected) of effective* 1*-cycles (curves when* $K_X + D + F$ *is numerically definite)* $\{C_\lambda\}/S$ *with*

$$(F.C_\lambda) \geq -2(n-d)$$

(resp. $> -2(n-d)$ *if* $K_X + D + F$ *is Kawamata log terminal in the generic points of* E*)*

In the above statements 6.8–6.12, we have $0 \leq d \leq n \leq 3$. Thus, we get rough boundaries if we replace n and d respectively by 3 and 0, i.e., we will have ≥ -6 (> -6).

Proof of Theorem 2.7.

Step 1. Reduction to the strictly log terminal case (with a finite boundary in the analytic case). Since the Iitaka fibration is unique, we can localize our problem and suppose that S is a neighborhood of a point. (Then in the analytic case the boundary has a finite support in X, i.e., a union of finite divisors.) Denote by D_i prime divisors in the support of the boundary.

Note also that, by Theorem 2.3 and Proposition 2.4, we may assume that X/S is a log minimal model.

In particular, X is $\mathbb{Q}$-factorial. So, by Corollary 6.6, we have a convex rational polyhedron

$$\mathcal{N} = \{B = \sum b_i D_i \mid \text{ all } b_i \in [0,1],\ K_X + B \text{ is nef and log canonical}\}$$

in the cube $\oplus[0,1]D_i$.

If B is rational, we know the semi-ampleness as an abundance from [12].

Step 2. Reduction to a $\mathbb{Q}$*-boundary.* If B is not rational, it will be an internal point of a face of $\mathcal{N}$. The interior points of the face are equivalent. So, they have the same Iitaka morphism $I: X \to Y/S$ for the rational interior points of the face. I contend that the same holds for other points. Indeed, any other points can be presented as a weighted linear combination $\sum r_i B_i$, where $r_i \geq 0$, $\sum r_i = 1$, and B_i are equivalent rational boundaries in the given face. By definition, $K_X + B_i \sim_{\mathbb{R}} I^* H_i$, where $\mathbb{R}$-divisors H_i of Y are numerically ample/S. Therefore

$$K_X + B = \sum r_i(K_X + B_i) \sim_{\mathbb{R}} I^*(\sum r_i H_i),$$

and $\sum r_i H_i$ is numerically ample/S. ■

Proof of 5.1.2 for 3-folds (cf. [15, 8.1]). Suppose that $f: X \to Y/S$ is a birational (bimeromorphic) contraction. Then the log canonical model of X/Y is a flip of X/S with respect to $K_X + B_X$ [26, 1.7]. Now we know that for a 3-fold X such a model exists by Corollary 2.8.

This also proves the following generalization of [26] and [15, 8.1]. ■

6.13. Log Flip Theorem. *Let $g: X \to Y/S$ be a birational (bimeromorphic) contraction, and B be a boundary such that*

(i) *$K_X + B$ is log canonical, and*

(ii) *$-(K_X + B)$ is nef /S.*

Then the flip of g with respect to $K_X + B$ exists.

Proof of 5.1.1 for 3-folds. According to Corollary 6.10, the contractible extremal rays R_j with $(K_X + B_X.R_j) < 0$ are discrete in $\overline{\mathrm{NE}}^{K_X+B_X}(X/S)$. (The analytic case is similar.)

Thus, it is enough to check that they and $\overline{\mathrm{NE}^{-(K_X+B_X)}(X/S)}$ generate $\overline{\mathrm{NE}}(X/S)$.

If this is not true, there exists an extremal ray $R \subset \overline{\mathrm{NE}}^{K_X+B_X}(X/S)$ such that $R \neq R_j$ for all j. We derive a contradiction if we check that it is contractible.

Indeed, we may choose R in such a way that it has a supporting hyperplane $\mathcal{S}$ with $\mathcal{S} \cap \overline{\mathrm{NE}}(X/S) = R$.

By Kleiman's criterion, there exist an ample and effective $\mathbb{R}$-divisor H and a real $r \in (0,1)$ such that $K_X + B_X + rH$ is nef and $\mathcal{S} = (K_X + B_X + rH)^{\perp}$. Moreover, we may assume that $B_X + rH$ is a boundary and $K_X + B_X + rH$ is log canonical. Indeed, if H is a $\mathbb{Q}$ divisor, we may take $H = (1/m)H'$, where H' is a generic very ample divisor. For an $\mathbb{R}$-divisor H, we may use a weighted linear combination, as in the proof of Theorem 2.7.

Now by semi-ampleness and Kleiman, we have a nontrivial (Iitaka) contraction $g: X \to Y/S$ of R, i.e., R is contractible and generated by a curve. ■

6.14. Remark. The proof of the LMMP conjectures of 5.1 in dimension n can be based on the LMMP for the strictly log terminal singularities and the semi-ampleness in dimension $\leq n$. We may even assume that the boundaries are rational except for the termination.

Of these, only 5.1.1 is known for $n \geq 4$ [11].

Proof of the First Main Theorem and of Corollary 6.6. Now they are implied by the LMMP in the log canonical case and Theorem 6.8. ■

6.15. Contraction Theorem. *Suppose that $K_X + B$ is Kawamata log terminal on a 3-fold X (locally Moishezon/S in the analytic case) and nef/S. Let D be an effective and nef $\mathbb{R}$-divisor/S, such that $K_X + B$ is numerically trivial on the face $F = D^{\perp} \cap \overline{\mathrm{NE}}(X/S)$ ($\cap\overline{\mathrm{NE}}(X/S; W)$ respectively). Then F is contractible/S (over a neighborhood of W).*

Proof. Note that $K_X + B + rD$ also defines a supporting hyperplane for F and any $r > 0$. However, for $0 < r \ll 1$, $K_X + B + rD$ is log canonical and even Kawamata log terminal. Thus F is contractible by 2.7. ■

In the same manner we prove (see also Corollary 6.18) the following.

6.16. Base Point Free Theorem. *Suppose that $K_X + B$ is log canonical on a 3-fold X/S (under the Moishezon condition of 2.2 in the analytic case). Let locally/S H be a nef/S $\mathbb{R}$-Cartier divisor, such that $H - \varepsilon(K_X + B)$ is semi-ample/S for some nonnegative real ε. Then H is semi-ample/S.*

6.16.1. *If $K_X + B$ is Kawamata log terminal, it is enough to suppose that $H - \varepsilon(K_X + B)$ is nef/S and $\mathbb{R}$-linear equivalent to an effective divisor D. The latter holds, for instance, when $H - \varepsilon(K_X + B)$ is nef/S and of a maximal numerical dimension (see Lemma 6.17 below).*

Proof. First, it is a local statement/S. Next, we may assume that $\varepsilon > 0$ and is rational, as well as $H - \varepsilon(K_X + B) \sim_{\mathbb{Q}} D$ for an effective divisor D (see Lemma 6.17) such that $K_X + B + D/\varepsilon$ is log canonical. So, $H \sim_{\mathbb{Q}} K_X + B + D/\varepsilon$ and the latter is semi-ample by 2.7. ∎

6.16.2. Example. (Cf. 6.18.1.) Under the assumptions of 6.16, suppose that $K_X + B$ is numerically trivial/S. Let D be a semi-ample divisor. Then we may contract exceptional E's in X/S, which do not have a maximal numerical dimension with respect to D: $(E.D^{\dim E}) = 0$. Just take $H = K_X + B + D$ in Theorem 6.16.

Locally/S and in the analytic case, such D is easily constructed for an exceptional sub-locus with the 1-dimensional complement in its fiber. In particular, if X/S is small we can contract analytically any of its exceptional sub-loci. In this case, $\overline{\mathrm{NE}}(X/S; \mathrm{pt.})$ is simplicial with extremal rays generated by the irreducible curves of X/pt.

Further applications of Corollary 6.15 need the following result.

6.17. Lemma. *Let D be a nef $\mathbb{R}$-Cartier divisor of X/S (locally Moishezon/S in the analytic case) such that $D^d > 0$ on the generic fiber of X/S, where $d = \dim X/S$. Then D is $\mathbb{Q}$-linearly and locally/S equivalent to an effective divisor in X.*

Proof. D has/S the maximal numerical dimension. If D is a $\mathbb{Q}$-divisor, then the lemma is true due to the Riemann–Roch–Hirzebruch theorem and vanishings [11, 6.1.2].

In general, we use similar arguments.

First, we assume that X is nonsingular, and that the prime components of $\operatorname{Supp} D$ are nonsingular too, with normal crossings.

Second, we also suppose that K_X has no common components with $\operatorname{Supp} D$, and that all multiplicities of K_X are ± 1.

Third, we may assume that D has ample prime components with noninteger and even irrational multiplicity. (We need a noninteger rational for a $\mathbb{Q}$-linear equivalence.)

Fourth, we may assume that $S = \mathrm{pt}$.

Then by the Kawamata–Viehweg vanishings [11, 1.2.3]

$$h^0(X, K_X + \lceil ND \rceil) = \chi(X, D)$$

for any natural N. More precisely, this is true for a $\mathbb{Q}$-divisor ND. If ND has irrational coefficients, we can change them into rational coefficients, preserving the nef and big property of ND, as well as $\lceil ND \rceil$ itself. This is due to the ample component of D with an irrational multiplicity. (If it has a noninteger rational coefficient we may take N's relatively prime to a denominator.)

Thus, by the Riemann–Roch theorem

$$h^0(X, K_X + \lceil ND \rceil) \approx \frac{1}{d!} N^d D^d.$$

In particular, $|K_X + \lceil ND \rceil| \neq \emptyset$ for $N \gg 0$.

We contend further that, for $N \gg 0$,

$$|\lfloor D \rfloor| = |\lceil ND \rceil - \operatorname{Supp}\{ND\}| \neq \emptyset,$$

where $\{ND\} = \sum f_i D_i$ is the fractional part of ND. Then $ND \sim D' + \sum f_i D_i \geq 0$, where $D' \in |\lfloor D \rfloor|$, $\sim$ means linear equivalence, and by the construction all $f_i \in [0,1)$. This proves the lemma.

Take $F = \mathrm{Supp}(K_X + D)$. Thus, it is enough to check that, for $N \gg 0$, $|K_X + \lceil ND \rceil - F| \neq \emptyset$. Moreover, we may extend F to a very ample divisor, and then replace it by its linear equivalent divisor which is smooth, irreducible (for $d \geq 2$), not a component of $\mathrm{Supp}(K_X + D)$, and intersects normally $\mathrm{Supp}(K_X + D)$. Then

$$(K_X + \lceil ND \rceil)|_F = K_X|_F + \lceil N(D|_F) \rceil.$$

According to the restriction sequence on F,

$$\begin{aligned} h^0(X, K_X + \lceil ND \rceil - F) &\geq h^0(X, K_X + \lceil ND \rceil) - h^0(F, (K_X + \lceil ND \rceil)|_F) \\ &= h^0(X, K_X + \lceil ND \rceil) - h^0(F, K_X|_F + \lceil N(D|_F) \rceil) \\ &\geq h^0(X, K_X + \lceil ND \rceil) - h^0(F, K_F + \lceil N(D|_F) \rceil) \\ &\approx \frac{1}{d!} N^d D^d \gg 0 \end{aligned}$$

for $N \gg 0$, because

$$h^0(F, K_F + \lceil N(D|_F) \rceil) \approx \frac{1}{(d-1)!} N^{d-1} (D|_F)^{d-1}.$$

Of course, $D|_F$ is nef, and we may assume that it is big. ∎

6.18. Corollary. *Under the assumptions of* 6.15, *let F be a face of $\overline{\mathrm{NE}}(X/S)$ ($\overline{\mathrm{NE}}(X/S; W)$ in the analytic case), such that $K_X + B$ is numerically trivial on F, and F has a supporting hyperplane H of maximal (numerical) dimension,* i.e., on the generic fiber of X/S, $D^d > 0$ for a divisor D in $H^\perp$, where $d = \dim X/S$. *Then F is contractible/S (over a neighborhood of W).*

6.18.1. *$\overline{\mathrm{NE}}(X/S)$ (respectively $\overline{\mathrm{NE}}(X/S; W)$) is rationally polyhedral in a* conical *neighborhood of F when X/S is projective/S. In particular, the former holds when X/S is quasi-finite (for instance, a modification) and projective.*

A conical neighborhood means a neighborhood containing a conical open neighborhood U of F, i.e., $F \subseteq U$ and U is conical (not including 0). Note also that in the nonprojective case we may check nearby F a rational polyhedral property but not of the finite type. It may sometimes appear in the projective case, which will be discussed elsewhere.

So, if $K_X + B$ is numerically trivial/S, the dual cone of nef divisors is locally rationally polyhedral beside $H^d = 0$. Sometimes, the "beside" part is empty. For instance, if X has no contractions of birational (bimeromorphic) type and $K_X + B$ is again numerically trivial/S, then the dual cone is given by the equation $H^d = 0$. The latter holds for the Abelian 3-folds with $B = 0$. In higher dimension we should have a similar picture.

Proof. The statement is essentially local/S. According to the assumptions, there exists an $\mathbb{R}$-divisor D such that D is nef, and $H = D^\perp$, i.e., D belongs to the class of linear functions defining H. Thus $D^d > 0$, and we may choose effective D. So, F is contractible by Corollary 6.15.

6.18.1 follows from the existence of an effective Cartier D negative on F. Then $K_X + \varepsilon D$ is also negative on F and in a conical neighborhood of F. ∎

Before the next application of the LMMP, we check the following consequence of the semi-ampleness.

6.19. Lemma. *Let $(Y/S, B_Y)$ be a log pair with a 3-fold Y, having nonnegative Kodaira dimension. Then there exists a positive real $\hbar$, such that, for any weakly log canonical model X/S of $(Y/S, B_Y)$ and for any curve C/S of X,*

$$\hbar = \min\{K_X + B_X.C) > 0 \mid C \text{ is a curve of } X/S\}$$

(over a neighborhood of W in the analytic case). So, $(K_X + B_X.C) = 0$ whenever $(K_X + B_X.C) < \hbar$.

Proof. By Proposition 2.4, we may fix a weakly log canonical model X/S. According to the abundance theorem, we may replace X/S by its minimal model, and $K_X + B_X$ by an ample $\mathbb{R}$-divisor H. (As we see later, we need only a linear combination below, which exists according to the proof of Theorem 2.7.)

If H is a $\mathbb{Q}$-divisor, $\hbar \in \mathbb{N}/r$, where r is the index of H.

In general, H is a weighted linear combination of ample $\mathbb{Q}$-divisors, and we take $\hbar$ as the weighted linear combination of multiples of $\hbar$'s for the components (see 4.1.3). ■

Fix a finite set of prime bi-divisors $D_i, i \in I$, of X/S and consider

$$\mathcal{C} = \left\{ B = \sum b_i D_i \mid \text{all } b_i \in [0,1], \text{ and } b_i = 0 \text{ or } 1 \text{ for } i \notin I \right\},$$

which is essentially the cube $\bigoplus_{i \in I} [0,1] D_i$ of *boundaries*. We say that D_i with $i \in I$ is *chosen*. Of course, for $i \notin I$, we assume also that $b_i = 0$ or 1 fixed, and $b_i = 0$ and 1 for almost all nonexceptional and exceptional D_i respectively. (Another option is totake any $b_i \in [0,1]$ for such D_i.)

The next result gives a positive answer to [24, Problem 6], and improves substantially [25, Relative Model Theorem].

6.20. Second Main Theorem. *For a 3-fold X/S (over a neighborhood of W in the analytic case, and under the assumptions of 2.2), the set $\mathcal{M}$ of $B \in \mathcal{C}$ with nonnegative Kodaira dimension of $(X/S, B)$ is a convex, closed, rational polyhedron. $\mathcal{M}$ is divided into a finite set of equivalent classes of log minimal models which are convex (maybe nonclosed) rational polyhedra.*

6.20.1. Remark. Theorem 6.6 implies that each equivalent class of $\mathcal{M}$ is the interior of a polyhedron in the decomposition with a fine union of interiors for some of its faces (cf. Remark 6.7). So, the decomposition is also polyhedral in the sense that any face of its polyhedron is its polyhedron.

Note that the structure of the polyhedra depends on the same for the Kleiman–Mori cones and positions of supports of its extremal rays. In some sense locally the decompositions fail to be given by hyperplanes in $\mathcal{M}$ (cf. the first main theorem and the next example) when the supports of extremal rays have points in common. Flips in such instances are typically noncommuting.

6.20.2. Example. Let X/S be a contraction of two curves C_1 and C_2 in a surface X such that C_1 and C_2 are smooth and rational with $C_1^2, C_2^2 \leq 3$.

First, suppose that C_1 and C_2 do not have points in common. Then

$$\mathcal{C} = \{B_X = b_1 C_1 + b_2 C_2 \mid b_1, b_2 \in [0,1]\}$$

is divided into four quadrangles by lines

$$(K_X + b_1 C_1 + b_2 C_2 . C_1) = 0 \text{ and } (K_X + b_1 C_1 + b_2 C_2 . C_2) = 0.$$

An interior point of each quadrangle corresponds to one of the following models: X, X_1 with contracted C_1, X_2 with contracted C_2, and $S = X_{1,2}$.

Suppose now that $(C_1 . C_2) = 1$. Then similar equations divide $\mathcal{C}$ into four quadrangles, and only one of them gives a correct domain, namely

$$(K_X + b_1 C_1 + b_2 C_2 . C_1) > 0 \quad \text{and} \quad (K_X + b_1 C_1 + b_2 C_2 . C_2) > 0.$$

Indeed, take (b_1, b_2) such that

$$(K_X + b_1 C_1 + b_2 C_2 . C_1) = 0 \quad \text{and} \quad (K_X + b_1 C_1 + b_2 C_2 . C_2) < 0.$$

Then it will be in the interior of model S. Indeed, in contraction X_2, $(K_{X_2} + b_1 C_1 . C_1) < 0$; the same holds in a neighborhood of (b_1, b_2).

6.20.3. Problem. It is interesting to construct a similar example for two flips. Essentially this means that there exists a 3-fold with a small contraction which is $\mathbb{Q}$-factorial, numerically trivial for $K_X + B_X$, and ample for $K_X + B_X + \varepsilon D$ but not $\mathbb{Q}$-factorial after the flop with respect to $-D$. (Cf. [23, 2.13.5; 14, 13.7].)

Proof. The convex property of $\mathcal{M}$ is a local property$/S$ and follows from that for divisors having effective linearly equivalent multiples. Indeed, if $B \in C$ has a nonnegative Kodaira dimension for $(X/S, B)$, then $K_X + B_X$ on a weakly log canonical model has an effective $\mathbb{R}$-linear$/S$ equivalent presentation. This follows from the log semi-ampleness and Lemma 6.17. According to the very definition of the log singularities, the same holds for $K_Y + B_Y$ on any model Y/X.

The closed property of $\mathcal{M}$ is implied by the open property of B with the negative Kodaira dimension of $(X/S, B)$. In that case we have a model $(X/S, B)$ such that all D_i with multiplicities $b_i < 1$ are resolved, $\operatorname{Supp} B_X$ is log nonsingular, and we have a covering family of curves C/S with $(K_X + B_X.C) < 0$ for its generic curve. This is derived from the existence of a nontrivial Fano fibering on an appropriate model of X/S by Theorem 2.3. The inequality $(K_X + B_X.C) < 0$ is preserved for small perturbations of $B \in \mathcal{C}$.

So, it is enough to check the last statements on the decomposition given by the equivalent classes. Essentially, it is enough to check finiteness, because the classes are rational polyhedra by Corollary 6.6. Since the entire domain is compact, we may do it locally and use induction, as in the proof of the first main theorem 6.6.

Let B^0 be a fixed boundary with a nonnegative Kodaira dimension, and X/S be a log minimal model with respect to B^0. We may assume that all chosen D_i's with $a(D_i, B^0, X) = 1 - b_i$ (or with $r(D_i, B^0, X) = 0$) are nonexceptional (cf. Conjecture 2.2 and Theorem 3.1). Moreover, we may discard other chosen D_i, preserving the variety of the models in a neighborhood of B^0. Then *all* chosen D_i's will be nonexceptional in X. Indeed, let us start with a log resolution X/S on which all chosen divisors D_i are nonexceptional. Then $K_X + B_X$ is strictly log terminal for every $B \in \mathcal{C}$. If $K_X + B_X^0$ is nef$/S$, this is the required model.

Otherwise, by 5.1.1 we have an extremal contraction $X \to Y/S$ with respect to $K_X + B_X^0$. This holds in a neighborhood of B^0 in $\mathcal{C}$. If it contracts a chosen divisor D_i we discard it or assume that its boundary multiplicity is 1. This gives an orthogonal projection of the cube on its face. By induction on the number of chosen divisors, we may assume that the extremal contractions are small. They are not of the fiber type, because B^0 has a nonnegative Kodaira dimension.

Therefore, we replace X/S by its flip of X/Y. Then, in the same neighborhood of B^0, $K_X + B_X$ are strictly log terminal. Finally, by the termination we get the required model.

Now we fix an ε-neighborhood of B^0 in $\mathcal{C}$ where $K_X + B_X$ is strictly log terminal. In the sequel we may replace $\mathcal{C}$ with any subpolyhedron. Since $\mathcal{C}$ is a polyhedron, this neighborhood will be a polyhedral cone for $0 < \varepsilon \ll 1$, in particular, it intersects any ray from B^0 by B^0 or by a semi-interval of length ε. In the latter case R will be called nontrivial.

We contend that there exists $0 < \varepsilon \ll 1$ such that in any nontrivial ray R from B^0 either the Kodaira dimension of every $B \in B^0 + (0, \varepsilon)R$ is negative, or the minimal models for all $B \in B^0 + (0, \varepsilon)R$ are equivalent. Here we identify R with its unit vector.

For this, it is enough to find $0 < \varepsilon \ll 1$ such that

(OP) $K_X + B_X^0$ is numerically trivial on an extremal ray R' of $\overline{\mathrm{NE}}(X/S)$ whenever $(K_X + B_X.R') < 0$ somewhere in an ε-neighborhood of B^0.

By Lemma 6.19 and Theorem 6.8, we can take $\varepsilon := \hbar\varepsilon/(6 + \hbar)$ (in higher dimensions it will be slightly different). Moreover, we see that ε depends only on $\hbar$, which in turn is independent of the weakly log canonical model of $(X/S, B^0)$ or direction R. The property OP is preserved for flips of R' on fixed R, because $(K_X + B_X.R') < 0$ on R (except for B^0) in the old ε-neighborhood too, and the simultaneous flip in R' preserves the log terminality.

So, let $B = B^0 + rR$ with $r \in (0, \varepsilon)$. Suppose that $K_X + B_X$ is not nef. Then there exists an extremal ray R' of $\overline{\mathrm{NE}}(X/S)$ with $(K_X + B_X.R') < 0$. According to the OP, $(K_X + B_X^0.R') = 0$, $K_X + B_X$ is strictly log terminal, and $(K_X + B_X.R') < 0$ for any $r \in (0, \varepsilon)$. If the corresponding contraction is of the fiber type, then all such B will have negative Kodaira dimension. Otherwise it defines a birational contraction. We can make a simultaneous flip with respect to all $K_X + B_X$. Note that it will be a flop with respect to $K_X + B^0$.

As a result, we may obtain a weakly log canonical model for B_0, but not a log terminal model. We can repeat the same construction for X^+ (cf. [26, 4.5]). By the termination we get simultaneously either a negative Kodaira dimension for the given boundaries B, or a log minimal model. The latter models are equivalent by Corollary 6.6.

Thus, in an ε-neighborhood of B^0, all boundaries with a nonnegative Kodaira dimension form a cone, as well as the equivalent classes.

By induction, in any section of the cone we have a finite decomposition into the models which give that for an ε-neighborhood of B^0. By Corollary 6.6. it is rational and polyhedral. ■

6.21. Corollary. *The numerical dimension of $(X/S, B)$ is a step upper convex function, and it is lower semi-continuous with respect to B on $\mathcal{M}$.*

Proof. It is constant inside the polyhedra of the decomposition given in Theorem 2.19.

The semi-continuity follows from 6.20.1, 6.6, and the fact that for any $\mathbb{R}$-Cartier divisor D and d-cycle C/S, the inequality $(D^d.C) > 0$ is preserved for small perturbations of D.

The convexity can be derived in the style of Theorem 6.20 due to the convexity of the Iitaka dimension for divisors. ■

6.22. Corollary. *Let $(X/S, B)$ be a pair of general type with 3-fold X having a Kawamata log terminal minimal model. Then it has a finite set of the* projective *weakly log canonical models and their flops. In particular, this holds for the log minimal models and their flops.*

Each of them is reconstructed from another by a finite chain of elementary flops.

By elementary flops we mean blow-ups and blow-downs $X \dashrightarrow Y/S$ with relative Picard number 1, being numerically trivial with respect to the log divisor $K_X + B_X$, but not the usual flops. The latter are decomposable into two elementary small flops. Flops are directed with respect to some divisor.

6.22.1. Problem. Of course in dimension 2, we can prove the theorem for all weakly log canonical models and their flops. However, it appears that projectivity is very important for dimension 3 and higher.

Find a 3-fold of general type having infinitely many complete weakly log canonical models (at least in a category of algebraic spaces).

Proof. Since $(X/S, B)$ has a log canonical model, we may assume that it is X/S. Let H be its very ample/S divisor.

Now we choose a set of bi-divisors D_i. They are, up to the numerical equivalence/S, generators of Weil divisors of X/S, prime components of B_X, and exceptional divisors D_i of X with $a_i = a(D_i, B, X) \leq 1$. This set is finite by our assumption, Proposition 2.4, and Corollary 1.7. We can assume that $H = \sum h_i D_i$ in X, where the sum runs over chosen divisors in X and even the multiplicities $h_i > 0$ for every chosen divisor, as well as $a_i > 0$ for those D_i's.

We contend that, for chosen D_i's, the classes of equivalent models in a neighborhood of B are in one-to-one correspondence with the isomorphism classes of weakly log canonical models of $(X/S, B)$.

We take a neighborhood intersecting only the polyhedra with B in the boundary. Thus, by the very definition of them, for B' in such a neighborhood, the log canonical model Y/S of $(X/S, B')$ is a weakly log canonical model of $(X/S, B)$. This gives the required correspondence

$$(Y/S, B') \mapsto (Y/S, B),$$

and it is injective on equivalent classes.

Therefore, we want to check that it is surjective, i.e., any weakly log canonical model Y/S of $(X/S, B)$ is a canonical model for B' in a neighborhood of B.

First, note that if H' is very ample, then, for any $0 < \varepsilon \ll 1$, $K_Y + B + \varepsilon H'$ is (Kawamata) log terminal and ample. So, it is a canonical model for $B + \varepsilon H'$. The latter makes sense whenever we consider H' as a bi-divisor and define its multiplicities as for the complete inverse image for the chosen divisors and 0 on the other exceptional divisors of Y.

Then we may find $H' = \sum \alpha_i D_i$, where the sum runs over chosen divisors nonexceptional in Y and $\alpha_i > 0$. Indeed, by the construction we have the same presentation with integer α_i. Now note that there exists a projection $g: Y \to X/S$, and we may replace H' by $H' + Ng^*H$, where N is any natural number, and $g^*H = \sum h_i D_i$ in Y is an effective divisor with positive multiplicities on each D_i.

The last statement is implied by the structure of the polyhedral decomposition (see Remark 6.23.3).

There exists a more direct approach. Let X/B be the initial model and D be the log birational transform of a very ample divisor from the reconstructed model. Then the reconstructed model will be a model for $B + \varepsilon H$ and $0 < \varepsilon \ll 1$.

It can be constructed as follows. First, we make a $\mathbb{Q}$-factorialization of X, which can be decomposed into elementary small blow-ups. Then X will be $\mathbb{Q}$-factorial, and we can apply the LMMP to $B + \varepsilon H$. As in the proof of Theorem 2.19, the next extremal modifications will be the usual flops and divisorial contractions with respect to $K_X + B_X$. The former are decomposable into two elementary flops, and so are the latter themselves. Finally, we decompose an Iitaka contraction into elementary blow-downs.

The decomposition of regular flops, i.e., contractions, into elementary flops is possible because the relative Kleiman–Mori cone for them is polyhedral by Corollary 6.18.1. ■

6.23. Remarks.

6.23.1. The generic points B' in a neighborhood of B are *stable* in the following sense. Any small variation of B' gives the same model. We discuss this concept only in the case of nonnegative Kodaira dimension.

Then the stability of B implies that B is of general type. The proof of Corollary 6.22 also implies that it has only one weakly log minimal model which is simultaneously log canonical and log minimal. Thus it is projective, Kawamata log terminal, and relatively terminal in codimension ≥ 2, i.e., for all exceptional divisors D_i of such models X/S, $a(D_i, B, X) > 1 - b_i$.

Every B has such an approximation whenever it has a weakly log canonical model X/S. Indeed, after the replacement of X/S by a log minimal model, we can replace B by $B + \varepsilon H$, where H is very ample on X/S. Thus, log minimal models are *semi-stable*, i.e., stable for a small variation of B.

6.23.2. From this point of view, all basic morphisms of the LMMP are stable — Fano fiberings, flips, and divisorial contractions with respect to B — in the category of models with strictly log terminal singularities. This means that if we have such a morphism $X \to Y/S$, then $K_X + B_X$ is strictly log terminal and it is negative with respect to $K_X + B_X$, and this also holds for any small variation of the boundary B.

According to the termination, even the construction of a log minimal model is stable.

However, the flops and Iitaka contractions are not stable. We can make them stable by changing the boundary which leads to directed flops.

6.23.3. Thus, the inside points in the maximal polyhedra of $\mathcal{M}$, in the proof of Corollary 6.22, are stable and they give the $\mathbb{Q}$-factorial and log minimal models of $(X/S, B)$. The maximal faces for such polyhedra correspond to the elementary flops of such models, i.e., they are contractions with relative Picard number 1.

Moreover, the log minimal models of $(X/S, B)$ correspond to an open convex maximal subpolyhedron and the walls between them correspond to the small elementary contractions of log minimal models. So, each log minimal model can be reconstructed from another by the usual flops (in curves).

Therefore, in dimension 3, the typical new phenomenon is that $(X/S, B)$ has only log minimal models as $\mathbb{Q}$-factorial projective weakly log canonical, and they are related by flips (cf. [19]).

In an opposite case which reflects dimension 2, all weakly log canonical models may be $\mathbb{Q}$-factorial. It is possible to consider such 3-folds singularities, and this is actually done by V. Nikulin [18].

6.23.4. *Category of flops*. The objects of it are projective weakly log canonical models of $(X/S, B)$ and morphisms are their regular flops. It may be considered as an order. (We assume that $X \geq Y$ whenever we have a regular flop $X \dashrightarrow Y$.) If $(X/S, B)$ is of general type, it will be finite with the log canonical model as the least element. It is always connected and the nearest elements are related by elementary flops.

The maximal elements are the log minimal models.

In Nikulin's case, we have only one maximal element, which is the greatest.

6.23.5. Everything in this section should work in any dimension if the LMMP holds. Moreover, the LMMP is sufficient for $\mathbb{Q}$-boundaries, except for the termination.

REFERENCES

1. V. Alexeev, "Two two-dimensional terminations," *Duke Math. J.*, **69**, No. 3, 527–545 (1993).
2. A. Borisov, *Minimal discrepancies of toric singularities*, Algebraic Geometry E-prints.
3. J. W. S. Cassels, *An Introduction to Diophantine Approximations*, Cambridge University Press (1957).
4. H. Clemens, J. Kollár, and S. Mori, "Higher dimensional complex geometry," *Astérisque*, **166**, Soc. Math. France (1988).
5. A. Corti, *Factoring Birational Maps of Threefolds After Sarkisov*, preprint.
6. Y. Kawamata, "Crepant blowing-up of 3-dimensional canonical singularities and its application to the degeneration of surfaces," *Ann. Math.* (2), **127**, 93–163 (1988).
7. Y. Kawamata, "On the length of an extremal rational curve," *Invent. Math.*, **105**, No. 3, 609–611 (1991).
8. Y. Kawamata, "The minimal discrepancy coefficients of terminal singularities of dimension 3," Appendix to [26], *Izv. Akad. Nauk SSSR. Ser. Mat.*, **56**, No. 1, 201–203 (1992).
9. Y. Kawamata, "Abundance theorem for minimal threefolds," *Inv. Math.*, **108**, 229–246 (1992).
10. Y. Kawamata, "Termination of log flips for algebraic 3-folds," *Int. J. Math.*, **3**, No. 5, 653–659 (1992).
11. Y. Kawamata, K. Matsuda, and K. Matsuki, "Minimal model problem," In: *Adv. Stud. Pure Math.*, Vol. 10, Kinokuniya Company (1987), pp. 283–360.
12. S. Keel, K. Matsuki, and J. McKernan, "Log abundance theorem for threefolds," *Duke Math. J.*, **75**, No. 1, 99–119 (1994).
13. J. Kollár, "The Cone theorem: Note to Kawamata's 'The cone of curves of algebraic varieties'," *Ann. Math.*, **120**, 1–5 (1984).
14. J. Kollár and S. Mori, *Classification of Three-Dimensional Flips*, preprint.
15. J. Kollár et al., "Flips and abundance for algebraic threefolds," A Summer Seminar at the University of Utah, Salt Lake City, 1991, *Asterisque*, **211** (1992).
16. T. Luo, *On the Divisorial Extremal Contractions of Threefolds: Divisor to a Point*, preprint.
17. Y. Miyaoka, "Abundance conjecture for 3-folds: $\nu = 1$ case," *Comp. Math.*, **68**, 203–220 (1988).
18. V. V. Nikulin, *Diagram Method for 3-Folds and Its Application to Kähler Cone and Picard Number of Calabi–Yau 3-Folds*. I, preprint alg-geom/9401010.
19. M. Reid, "Minimal models of canonical threefolds," In: *Algebraic Varieties and Analytic Varieties, Adv. Stud. Pure Math.*, Vol. 1, Kinokuniya and North Holland (1983), pp. 131–180.
20. M. Reid, "Young person's guide to canonical singularities," *Proc. Symp. Pure Math.*, **46:1**, 345–414 (1987).
21. M. Reid, *Birational Geometry of 3-Folds According to Sarkisov*, preprint (1991).
22. V. G. Sarkisov, *Birational Maps of Standard $\mathbb{Q}$-Fano Fiberings*, I. V. Kurchatov Institute Atomic Energy preprint (1989).
23. V. V. Shokurov, "A nonvanishing theorem," *Izv. Akad. Nauk SSSR. Ser. Mat.*, **49**, 635–651 (1985).

24. V. V. Shokurov, "Problems about Fano varieties," In: *Birational Geometry of Algebraic Varieties: Open problems. The* XXIII*rd International Symposium, Division of Mathematics, The Taniguchi Foundation. Aug. 22–27, 1988*, pp. 30–32.
25. V. V. Shokurov, *Special* 3-*Dimensional Flips*, preprint, MPI/89-22.
26. V. V. Shokurov, "3-Fold log flips," *Izv. Akad. Nauk SSSR. Ser. Mat.*, **56**, No. 1, 105–201 (1992).
27. V. V. Shokurov, "Anticanonical boundedness for curves," Appendix to [18].
28. V. V. Shokurov, "Semi-stable 3-fold flips," *Izv. Akad. Nauk SSSR. Ser. Mat.*, **57**, No. 2, 162–224 (1993).
29. V. V. Shokurov, *A.c.c. of m.l.d.*, preprint.
30. O. Zariski and P. Samuel, *Commutative Algebra* I, II, Van Nostrand, Princeton (1958, 1960).