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ABSTRACT. The main result of the paper is a nonvanishing theorem that is a sufficient condition for nontriviality of the zeroth cohomology group of inverse sheaves. In addition, applications of this theorem to multidimensional projective geometry are indicated and problems illuminating further insight into the theory of Mori extremal rays are formulated.
Bibliography: 14 titles.

§0. Terminology and the main result

(0.0) Let X be a normal projective variety over the complex number field. By a **Q**-*Cartier divisor* we mean an element $D \in \operatorname{Div}_{\mathbf{Q}} X = \operatorname{Div} X \otimes \mathbf{Q}$, where $\operatorname{Div} X$ is the group of Cartier divisors of X; in other words a **Q**-Cartier divisor is a linear combination of Cartier divisors with coefficients in **Q**. The group $\operatorname{Div}_{\mathbf{Q}} X$ also contains certain Weil divisors of X, namely those D such that $rD \in \operatorname{Div} X$ for some $0 \neq r \in \mathbf{Z}$; for then $D = (1/r)(rD)$. Recall that a variety X is **Q**-*factorial* if every Weil divisor of X is **Q**-Cartier in this sense.

Throughout what follows we assume that X is **Q**-*Gorenstein*. This means that any canonical Weil divisor K_X is **Q**-Cartier; that is, $K_X \in \operatorname{Div}_{\mathbf{Q}} X$. In addition, we assume that the singularities of X satisfy the following condition: there exists a resolution $f: Y \to X$ whose exceptional locus is made up of divisors F_i which are nonsingular and have normal crossing, and such that in the relation

$$K_Y = f^*K_X + \sum a_i F_i,$$

all the a_i are greater than -1. In §1 we show that the definition of these singularities does not depend on the resolution f with the stated properties. We will say that a **Q**-Gorenstein variety X satisfying the above conditions has *routine singularities*.* Recall that

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**Translator's note*. This notion of routine singularities also appears in the current literature as *log-terminal* singularities: if we consider $Y \supset F$ as a resolution of X in the log category (X is considered as marked with the empty divisor $D \neq \varnothing$) then

$$(K_Y + F) = f^*(K_X + D) + \Delta_{f,\log}$$

and the condition $a_i > -1$ is the usual terminal condition that every exceptional component of f should occur in Δ with strictly positive multiplicity.

X has *canonical* (respectively *terminal*) singularities if all the a_i are nonnegative (respectively positive). We remark that from the point of view of linear systems the routine singularities should perhaps be called canonical, and canonical singularities pluricanonical.

(0.1) There is a natural intersection theory for **Q**-divisors. A **Q**-Cartier divisor $D \in \operatorname{Div}_{\mathbf{Q}} X$ is said to be *nef* if $(D \cdot C) \geqslant 0$ for evey curve $C \subset X$. We write $\approx$ for numerical equivalence of cycles. To each nef divisor D we associate its numerical dimension

$$\nu(D) = \kappa_{\text{num}}(D) = \max\{\, k | D^k \not\approx 0\};$$

that is, the maximal k such that $D^k \cdot C \neq 0$ for some k-cycle C of X. Obviously,

$$\max\{0, \kappa(D)\} \leqslant \nu(D) \leqslant n = \dim X,$$

where $\kappa(D)$ is the Iitaka D-dimension of X.

Let D be a nef **Q**-divisor. Then one easily verifies that the following five conditions are equivalent:

(0.1.1) $\nu(D) = \eta$.

(0.1.2) $D^n > 0$.

(0.1.3) $h^0(X, mD) \sim m^n$ as $m \to \infty$ with $mD \in \operatorname{Div} X$.

(0.1.4) For any ample divisor $H \in \operatorname{Div} X$ there exists an $m > 0$ such that $mD \sim H + M$, where M is an effective Cartier divisor ($\sim$ denotes linear equivalence).

(0.1.5) $\kappa(D) = n$.

A nef **Q**-divisor D satisfying any one these conditions is said to be *big*.

As in [8], we introduce the "round-up" symbol $\ulcorner\ \urcorner$: if $x \in \mathbf{R}$ then $\ulcorner x \urcorner$ is the smallest integer $\geqslant x$ (compare the Gauss symbol $[x]$). The corresponding notion for a **Q**-divisor $D = \sum d_i F_i$ is

$$\ulcorner D \urcorner = \ulcorner\left(\sum d_i F_i\right)\urcorner = \sum \ulcorner d_i \urcorner F_i.$$

The key result of the present article is the following theorem.

(0.2) NONVANISHING THEOREM. *Let X be a variety with routine singularities, D a nef Cartier divisor and $A = \sum d_i D_i$ a **Q**-Cartier divisor on X. Suppose that the following conditions hold*:

(a) *The **Q**-divisor $aD + A - K_X$ is nef and big for some $a \in \mathbf{Q}$.*

(b) *The D_i are prime divisors of X, and are nonsingular, have normal crossings, and lie in the nonsingular part of X if $d_i < 0$.*

(c) *Each $d_i > -1$.*

Then for all $b \gg 0$,

$$H^0(X, \mathcal{O}_X(bD + \ulcorner A \urcorner)) \neq 0,$$

or, in other words, $|bD + \ulcorner A \urcorner| \neq \varnothing$.

§1 is devoted to the proof of the theorem, and §2 to its applications, which were the original motivation for the theorem.

§1. Proof of the nonvanishing theorem

We first spend some time on the invariance of the definition of routine singularities. For this it is enough to check the following assertion:

(1.1) LEMMA. *Let X be a nonsingular variety, and $A = \sum d_i D_i$ a **Q**-divisor such that*

(a) *$d_i > -1$, and*

(b) *the D_i are nonsingular divisors with normal crossings.*

Consider an arbitrary birational morphism $f\colon Y \to X$; then letting F_j be the exceptional prime divisors for f, and writing $K_Y = f^*K_X + \sum a_j F_j$, we get

$$f^*A + \sum a_j F_j = \sum d_j^* F_j = A^*$$

where $d_j^* > -1$; that is,

$$K_Y + f^*A = f^*K_X + A^*,$$

where the divisor A^* satisfies (a). *Moreover precisely, if $d_i \geqslant -1 + \delta$ for all i and some $0 < \delta < 1$, then $d_j^* > -1 + \delta$ for all j.*

PROOF. The lemma can be checked directly in the case that f is a blow-up with center contained in an intersection of components D_i; moreover, in this case if $d_i \geqslant -1 + \delta$ for each i, then the new components have $d_i^* \geqslant -1 + 2\delta$. Thus after making a finite number of such blow-ups we can get to a divisor A^* for which the components F_i with $d_i^* < 0$ are disjoint. Since by Hironaka's results any morphism f can be dominated by a sequence of blow-ups, we see that the problem reduces to proving that for any linear system L on X there exists a resolution $f\colon Y \to X$ of the locus of indeterminacy of L satisfying the statement of the lemma.

By our previous remark, we can assume that the divisors D_i with $d_i < 0$ are disjoint. The lemma can also easily be checked for a blow-up in any nonsingular center, provided that the divisors D_i with $d_i < 0$ are nonsingular. In order to ensure that the requisite divisors D_i remain nonsingular, it is enough to carry out blow-ups whose centers are either contained in D_i with $d_i < 0$ (since such D_i are disjoint), or disjoint from all such D_i. But such blow-ups are sufficient to resolve the locus of indeterminacy of any linear system: we proceed as follows. We throw away the fixed components, then work separately with the restriction of L to any component D_i with $d_i < 0$ to resolve the locus of indeterminacy and bring the fixed components to normal form; then we use blow-ups with centers in these fixed components to separate the locus of indeterminacy from the divisors D_i with $d_i < 0$. After this we resolve the indeterminacy outside such D_i. Note that all the new components appearing have $d_j^* > 0$.

The final assertion of the lemma can be obtained by an easy scrutiny of the abvoe proof. ∎

(1.2) LEMMA. *Let X be a projective variety, H an ample divisor and D a nef $\mathbf{Q}$-divisor on X. Then either*

$$(H + bD)^n \to \infty \quad \text{as } b \to \infty \tag{1.2.1}$$

or

$$D \not\equiv 0. \tag{1.2.2}$$

PROOF. Since D is nef, by Kleiman's criterion [5], $H^iD^{n-i} \geqslant 0$ for all i; suppose that (1.2.1) is false. Then the binomial expansion of $(H + bD)^n$ gives

$$H^iD^{n-i} = 0 \quad \text{for } i = 0, 1, \ldots, n-1,$$

and in particular $H^{n-1}D = 0$. This implies that $D \not\equiv 0$; for given any curve C, we can complement it by an effective 1-cycle C to a complete intersection of $(n-1)$ divisors of $|mH|$ for sufficiently large m, that is

$$C + C' \not\equiv (mH)^{n-1}.$$

Thus

$$(C + C' \cdot D) = (mH)^{n-1}D = 0;$$

now since D is nef, it follows that $(C \cdot D) = 0$ for all C, so that $D \approx 0$. ■

(1.3) Lemma. *For any natural number n and real k there exists a real constant d such that for any projective n-fold X, any ample $\mathbf{Q}$-divisor H on X with $H^n \geqslant d$ and any general point $x \in X$ the following holds: for all $m \gg 0$ such that $mH \in \operatorname{Div} X$, the linear system $|mH|$ contains a divisor D with multiplicity $\geqslant mk$ at x; more precisely, we take x to be a nonsingular point, and ask that the total transform of D under the blow-up at x contain the exceptional divisor with multiplicity $\geqslant mk$.*

Proof. The exceptional divisor of the blow-up at x is $\mathbf{P}^{n-1}$. Therefore, passing through x imposes 1 condition, passing through x with multiplicity 2 imposes a further n conditions, ..., and passing through x with multiplicity l imposes a further $\binom{l+n-2}{n-1}$ conditions. Therefore passing through x with multiplicity l imposes a total of $\binom{l+n-1}{n}$ conditions, which is $\leqslant l^n$ for $l \gg 0$. Hence the required divisor exists provided that $m \gg 0$ and $\dim|mH| > m^n(k+1)^n$. Now using Serre vanishing for higher cohomology and the leading term of the Hirzebruch Riemann-Roch formula we get $\dim|mH| \geqslant \text{const} \cdot m^n H^n$, where the constant depends only on n. Hence the lemma holds when

$$\text{const} \cdot m^n H^n \geqslant m^n(k+1)^n,$$

or equivalently

$$H^n \geqslant (k+1)^n/\text{const} = d. \quad ■$$

Proof of Theorem 0.2.

Reduction to the case of a nonsingular X. Consider a resolution $f\colon Y \to X$ such that the proper transforms of the divisors D_i and the exceptional divisors are nonsingular with normal crossings. By Lemma 1.1,

$$K_Y + f^*A = f^*K_X + \sum d_j^* F_j,$$

with all $d_j^* > -1$. As above, we write A^* for the $\mathbf{Q}$-divisor $\sum d_j^* F_j$. Then the divisor

$$af^*D + A^* - K_Y = f^*(aD + A - K_X)$$

is nef and big. On the other hand, if a divisor F_j is not exceptional, then its coefficient d_j^* is the same as the coefficient d_i of the divisor $D_i = f(F_j)$. Hence normality gives the implication

$$|bf^*D + \ulcorner A^* \urcorner| \neq \varnothing \Rightarrow |bD + \ulcorner A \urcorner| \neq \varnothing.$$

Thus in the statement of Theorem 0.2, we can assume that X is nonsingular.

Reduction to the case of ample $aD + A - K_X$. Now suppose that X is a nonsingular projective variety, H a very ample divisor on X, and L a nef and big $\mathbf{Q}$-divisor. Then by (0.1.4),

$$L = (1/m)H + \sum p_i F_i,$$

where $\sum p_i F_i$ is an effective $\mathbf{Q}$-divisor. Now for any $\varepsilon > 0$ there exist coefficients $0 < q_i \ll \varepsilon$ such that $L - \sum q_i F_i$ is an ample $\mathbf{Q}$-divisor; in fact, we could take q_i to be p_i/N for some large natural number N, since the $\mathbf{Q}$-divisor

$$NL - \sum p_i F_i \approx (N-1)L + L - \sum p_i F_i \approx (N-1)L + (1/m)H$$

is ample by Kleiman's criterion [5]. By blow-ups we can make all of the divisors F_i nonsingular and with normal crossings, and we will need to include the exceptional components among the F_i; furthermore, we can assume that all the divisors of the previous construction are included among the F_i. If we now take L to be the divisor $aD + A - K_X$, then we find that for any $\varepsilon > 0$ there exists $0 < p_j \ll \varepsilon$ such that the divisor

$$af^*D + A^* - Ky - \sum p_j F_j = f^*(L) - \sum p_j F_j$$

is ample. Now we can replace A^* by $A^* - \Sigma p_j F_j$, provided that $d_i^* \geqslant -1 + \delta$, and $0 < p_j \ll \delta$. Hence in what follows we assume that $aD + A - K_X$ is ample.

The theorem as reformulated will be proved by induction on $n = \dim X$. Formally, the induction could start at $n = 0$, when $D = A = 0$ and $|bD + \ulcorner A \urcorner| = |0| \neq \varnothing$. However, the case $n = 1$ is also instructive: if $D \not\equiv 0$ then everything is clear, since $\deg D > 0$. If $D \not\equiv 0$ then the reader can check the assertion independently, or can glance to the end of the present proof.

We choose a divisor $M \in |m(aD + A - K_X)|$ and construct a resolution $f\colon Y \to X$ which has a system $\{F_j\}$ of nonsingular divisors with normal crossings, and numbers d_j^*, r_j, $p_j \in \mathbf{Q}$ such that

(a) $K_Y + f^*A = f^*K_X + \Sigma d_j^* F_j$ with $d_j^* > -1$;

(b) $f^*M = L + \Sigma r_j F_j$ with $r_j \in \mathbf{Z}$, $r_j \geqslant 0$, and $\operatorname{Bs} L = \varnothing$ (that is, $|L|$ is a linear system without base points); and

(c) $\frac{1}{2}f^*(aD + A - K_X) - \Sigma p_j F_j$ is ample, with $0 < p_j \ll \delta$ and $d_j^* \geqslant -1 + \delta$.

Condition (a) is always satisfied by Lemma 1.1; (b) can be obtained from Hironaka's results, and condition (c) can easily be checked for an ample $\mathbf{Q}$-divisor $aD + A - K_X$ provided that the resolution is achieved using only blow-ups in nonsingular centers, which is also possible from Hironaka's results.

The method of proof develops the technique of Kawamata [2], Benveniste [1], Reid [8], and especially the author's first draft [10].

Consider the $\mathbf{Q}$-divisor

$$\begin{aligned} N = N(b,c) &= f^*(bD) + \sum\left(-cr_j + d_j^* - p_j\right)F_j - K_Y \\ &\underset{\cdot}{\approx} cL + f^*\left(bD + A - K_X - cm(aD + A - K_X)\right) - \sum p_j F_j. \end{aligned}$$

This is an ample divisor for $c \geqslant 0$, $b \geqslant a$ and $cm \leqslant \frac{1}{2}$. We will discuss below the question of when these inequalities hold, so just assume ampleness for the time being. Then, by the Kawamata-Viehweg vanishing criterion [3], [12],

$$H^i(X, \mathcal{O}_X(\ulcorner N \urcorner + K_Y)) = 0 \quad \text{for } i > 0. \tag{$*$}$$

For c we will choose

$$c = \min\left\{\left(d_j^* + 1 - p_j\right)/r_j\right\},$$

where the minimum is taken over all j with $r_j > 0$. By (a) and (c) of the construction, $c > 0$ provided that there exists some j for which $r_j > 0$. Furthermore, by small perturbation of the p_j, we can assume that the minimum value $c = (d_j^* + 1 - p_j)/r_j$ is achieved for just one value of j, for example $j = 0$. Write $B = F_0$. Then

$$N + K_Y = f^*(bD) + \tilde{A} - B, \tag{$**$}$$

where $\tilde{A} = \Sigma_{j \neq 0} \tilde{d}_j F_j$ and $\tilde{d}_j = -cr_j + d_j^* - p_j$.

Let us check that the divisors $D^* = f^*D|_B$ and $A^* = \tilde{A}|_B$ satisfy the conditions of Theorem 0.2 on the nonsingular variety B. Indeed, D^* is a nef Cartier divisor, and the $\mathbf{Q}$-divisor

$$bD^* + A^* - K_B = \left(f^*(bD) + \tilde{A} - B - K_Y\right)|_B = N|_B$$

is ample; moreover, the divisor A^* is concentrated on the divisors $F_j|_B$ with $j \neq 0$, which are nonsingular with normal crossings. It thus remains to check the condition $\tilde{d}_j > -1$ for $j \neq 0$. In fact, if $r_j = 0$ then $\tilde{d}_j = d_j^* - p_j > -1$ by construction; if $r_j > 0$ then for $j \neq 0$ we have $(d_j^* + 1 - p_j)/r_j > c$, and hence

$$\tilde{d}_j = -cr_j + d_j^* - p_j > -1.$$

Now notice that $\dim B = n - 1$; therefore, by induction,

$$H^0\left(B, \mathcal{O}_B(bD^* + \ulcorner A^* \urcorner)\right) \neq 0 \quad \text{for all } b \gg 0.$$

We consider the restriction exact sequence

$$0 \to \mathcal{O}_Y\left(bf^*(D) + \ulcorner \tilde{A} \urcorner - B\right) \to \mathcal{O}_Y\left(bf^*(D) + \ulcorner \tilde{A} \urcorner\right) \to \mathcal{O}_B\left(bD^* + \ulcorner A^* \urcorner\right) \to 0;$$

since by $(**)$ $\ulcorner N \urcorner + K_Y = bf^*(D) + \ulcorner \tilde{A} \urcorner - B$, it follows from $(*)$ that

$$H^1\left(Y, \mathcal{O}_Y\left(bf^*(D) + \ulcorner \tilde{A} \urcorner - B\right)\right) = 0.$$

Hence the map

$$H^0\left(Y, \mathcal{O}_Y\left(bf^*(D) + \ulcorner \tilde{A} \urcorner\right)\right) \to H^0\left(B, \mathcal{O}_B\left(bD^* + \ulcorner A^* \urcorner\right)\right)$$

is surjective. Thus by what we have seen, $H^0(Y, \mathcal{O}_Y(bf^*(D) + \ulcorner \tilde{A} \urcorner)) \neq 0$; that is,

$$\left|bf^*(D) + \ulcorner \tilde{A} \urcorner\right| \neq \varnothing \quad \text{for all } b \gg 0.$$

Now observe that $f_* \ulcorner \tilde{A} \urcorner \leqslant \ulcorner A \urcorner$; indeed, if F_j is a nonexceptional divisor with $j \neq 0$, corresponding to a divisor D_i, then $d_j^* = d_i$ and $\tilde{d}_j = -cr_j + d_j^* - p_j < d_j^* = d_i$, and thus $\ulcorner \tilde{d}_j \urcorner \leqslant \ulcorner d_i \urcorner$. Using this, as at the beginning of the proof, by the projection formula and the nonsingularity of X, we get that $|bD + \ulcorner A \urcorner| \neq \varnothing$ for all $b \gg 0$.

To complete the proof, it remains to determine when the inequality $cm \leqslant \frac{1}{2}$ is satisfied, the inequalities $c \geqslant 0$ and $b \geqslant a$ being trivial. For this it is enough to find a divisor $M \in |m(aD + A - K_X)|$ having multiplicity $\geqslant 2mn$ at some general point $x \in X$. In fact then $c \leqslant (1/2mn)((n-1) + 1 - p_j) < 1/2m$, and $cm < \frac{1}{2}$. According to Lemma 1.3, the required divisor M will exist provided that $(aD + A - K_X)^n \to \infty$ as $a \to \infty$; by Lemma 1.2, this will always be possible unless $D \approx 0$. Note that here we have used the fact that $aD + A - K_X$ is ample for $a \gg 0$.

Now we need to verify the theorem when $D \approx 0$. In this case, by the Kawamata-Viehweg vanishing criterion,

$$H^i\left(X, \mathcal{O}_X(bD + \ulcorner A \urcorner)\right) = H^{n-i}\left(X, \mathcal{O}_X(\ulcorner K_X - bD - A \urcorner)\right) = 0$$

for any $b \in \mathbf{Z}$ and for all $i > 0$. Therefore, by the topological invariance of the Euler characteristic (Riemann-Roch),

$$\begin{aligned} h^0\left(X, \mathcal{O}_X(bD + \ulcorner A \urcorner)\right) &= \chi\left(\mathcal{O}_X(bD + \ulcorner A \urcorner)\right) \\ &= \chi\left(\mathcal{O}_X(\ulcorner A \urcorner)\right) = h^0\left(X, \mathcal{O}_X(\ulcorner A \urcorner)\right) > 0, \end{aligned}$$

which completes the proof. ∎

A posteriori, we get the following result.

(1.4) EFFECTIVE FORM OF NONVANISHING. *Under the hypotheses of Theorem* 0.2, *there exist at most n integers* $b \geqslant a$ *for which*

$$H^0(X, \mathcal{O}_X(bD + \ulcorner A \urcorner)) = 0.$$

In particular, there exists an integer $b \in [a, \ulcorner a \urcorner + n]$ *such that*

$$H^0(X, \mathcal{O}_X(bD + \ulcorner A \urcorner)) \neq 0.$$

PROOF. By what we said at the beginning of the proof of Theorem 0.2, we can assume that X is nonsingular, and that the divisors D_i are nonsingular with normal crossings. Under these circumstances, by the Kawamata-Viehweg vanishing criterion, we have

$$H^i(X, \mathcal{O}_X(bD + \ulcorner A \urcorner)) = H^i(X, \mathcal{O}_X(\ulcorner bD + A - K_X \urcorner - K_X)) = 0$$

for all $i > 0$ and for every integer $b \geqslant a$. But then by the Hirzebruch Riemann-Roch formula, the function

$$H^0(X, \mathcal{O}_X(bD + \ulcorner A \urcorner)) = \chi(\mathcal{O}_X(bD + \ulcorner A \urcorner))$$

is a polynomial function of degree $\leqslant n$ of the integer $b \geqslant a$. By Theorem 0.2, this polynomial is nontrivial, which completes the proof. ■

(1.5) REMARKS. (a) The bound in Theorem 1.4 is sharp. Consider for example $X = \mathbf{P}^n$; then $K_X = -(n+1)H$, where H is a hyperplane in $\mathbf{P}^n$. Take $A = 0$, $D = H$, and $a = -n - \frac{1}{2}$. In this case there are exactly n integers $b = -1, -2, \ldots, -n \geqslant -n - \frac{1}{2}$ for which

$$H^0(X, \mathcal{O}_X(bD)) = 0,$$

and the interval $[-n, 0]$ contains exactly one integer for which $H^0(X, \mathcal{O}_X(bD)) \neq 0$, namely $b = 0$.

(b) However, under certain extra conditions on a, there are probably more precise estimates. For example, if $\dim X = 2$ and $a \geqslant -1$ then $H^0(X, \mathcal{O}_X(bD + \ulcorner A \urcorner)) \neq 0$ for every $b \geqslant a + 1$ (see [8], Proposition 1.5).

(1.6) COROLLARY (boundedness of the index of Fano varieties). *Let X be a Fano variety with routine singularities* (*see* (2.3.5)). *Then* $i(X) \leqslant \dim X + 1$; *here* $i(X)$, *the Fano index of* X, *is the greatest positive rational number such that* $(1/i(X))(-K_X)$ *is numerically equivalent to a big Cartier divisor. The fact that* $i(X)$ *exists follows from* (2.3.5).

PROOF. Otherwise $|k \cdot (1/i)K_X| \neq \varnothing$ for some $k \geqslant 1$, by (1.4). ■

(1.7) REMARKS. (a) Examples are known for which $i(X) = \dim X + 1$ and $i(X) = \dim X$, namely $\mathbf{P}^n$ and a quadric in $\mathbf{P}^{n+1}$ respectively; it seems likely that there are no other examples with $i(X) \geqslant \dim X$.

(b) For $\dim X = 2$, the author has checked that the only del Pezzo surfaces with routine singularities and index > 1 are $\mathbf{P}^1 \times \mathbf{P}^1$ and the cone over a rational normal curve (that is, the surface obtained by contracting the negative section of a rational scroll $\mathbf{F}_m$).

(1.8) COROLLARY ("adjunction terminates"). *If X is a variety such that "adjunction terminates"* (*that is,* $|D + mK_X| = \varnothing$ *for every Cartier divisor D and all* $m \gg 0$), *then the canonical class* K_X *is not nef*.

PROOF. Suppose that K_X is nef, and let H be an ample divisor on X. Then by nonvanishing, $|H + mK_X| \neq \varnothing$ for all $m \gg 0$ for which mK_X is a Cartier divisor. ■

(1.9) REMARK. Corollary 1.8 confirms to some extent the conjecture of Iskovskikh that a variety for which adjunction terminates should be birational to a relative Fano variety in the class of varieties with terminal singularities.

§2. Applications

The following result is a direct generalization of [8], Theorem 0.0, and of [10], Proposition 3.1; in essence it goes back to [2], Theorem 2. It could also be interpreted as a generalization of Kleiman's ampleness criterion [5].

(2.1) CONTRACTION THEOREM.** *Let D be a Cartier divisor on a variety X with routine singularities, and suppose that the* $\mathbf{Q}$*-divisor $D - \varepsilon K_X$ is nef and big for all sufficiently small rational numbers $\varepsilon > 0$. Then D is stably free; that is, for all $m \gg 0$, the linear system $|mD|$ is free, or* $\mathrm{Bs}(mD) = \varnothing$. *Equivalently, there exist a morphism $\varphi\colon X \to Z$ onto a normal projective variety Z such that $\varphi_*\mathcal{O}_X = \mathcal{O}_Z$, and an ample invertible sheaf $H \in \mathrm{Pic}\, Z$ such that $\varphi^*H = \mathcal{O}_X(D)$.*

PROOF. To avoid repetition, we indicate how to modify the proof of [8], Theorem 0.0, leading to the proof of our theorem.

(2.2) We first reformulate the theorem in a slightly different form; for this, note first of all that from the conditions it follows that D is nef and the $\mathbf{Q}$-divisor $(1/\varepsilon)D - K_X$ is nef and big for some rational number $\varepsilon > 0$. If we replace $1/\varepsilon$ by a positive integer a, we get the conditions

(2.2.1) D is a nef Cartier divisor, and

(2.2.2) $aD - K_X$ is a nef and big $\mathbf{Q}$-divisor,

which together are equivalent to the hypotheses of Theorem 2.1. By the nonvanishing theorem there exists an integer $m > 0$ such that $|mD| \neq \varnothing$. From then on we use the proof of [8], §1, with the following changes:

(2.2.3) $a_j \geqslant -1 + \delta$, with $\delta > 0$;

(2.2.4) $0 \leqslant p_j \ll \delta$.

Moreover, we replace Proposition 1.5 by the nonvanishing theorem. ■

We now discuss the properties of the morphism φ of Theorem 2.1.

(2.3) PROPERTIES OF φ.

(2.3.1) $R^i\varphi_*\mathcal{O}_X = 0$ *for all $i > 0$, and in particular,* $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Z)$.

(2.3.2) $H^i(Z, H^{\otimes m}) = 0$ *for all $m \geqslant a$ and all $i > 0$.*

(2.3.3) RELATIVE ANTICANONICAL MODEL. *There exists a decomposition $\varphi = h \circ g$ of φ as a composite of morphisms $X \xrightarrow{g} \overline{X} \xrightarrow{h} Z$, where $\overline{X}$ is a normal projective variety with routine singularities, such that g is a birational morphism with $K_X = g^*K_{\overline{X}}$, and $-K_{\overline{X}}$ is relatively ample for h, that is, $-CK_{\overline{X}} > 0$ for any curve C such that $h(C) =$ pt; if X has only canonical singularities, then so does $\overline{X}$.*

(2.3.4) $\dim Z = \nu(D) = \kappa(D)$.

(2.3.5) *The general fiber $Y = \varphi^{-1}(z)$ for $z \in Z$ is a Fano variety with routine singularities; by this we mean that $-K_Y$ is nef and big. Moreover, $H^i(\mathcal{O}_Y) = 0$ for all $i > 0$, and the group scheme* $\mathrm{Pic}\, Y$ *is discrete and torsion-free. If X has only canonical (respectively terminal) singularities, then so does Y.*

***Translator's note.* The author calls this "stably free theorem" or "theorem on stable freeness", and it also appears in the literature as "base-point free theorem"; the notion of "stably free" appearing in the theorem is sometimes called "semi-ample" (mainly by Japanese authors) and also "eventually free". I rather like the idea of a "theorem on eventual freedom", but "contraction theorem" seems to have become universally accepted.

PROOF. The fact that the general fiber Y in (2.3.5) has only routine singularities is proved by means of the adjunction formula and Bertini's theorem applied to some resolution of X (compare the proof of [7], Theorem 1.13); the final clause of (2.3.5) is proved in the same way, as is the fact that $-K_Y$ is nef and big. The remaining assertions are all either obvious, or are essentially proved in [8], (1.7). ∎

(2.4) REMARK. It seems to be true that Z is a normal variety with only rational singularities. If $\dim X \leqslant 3$ or if $\dim Z \geqslant n-1$ then this is essentially proved in [8], (1.7).

Theorem 2.1 has a series of consequences which are not without interest.

(2.5) COROLLARY. *If the canonical class K_X is nef and big—that is, if K_X is nef and $K_X^n > 0$ (in this case X is said to be a "minimal model of a variety V of general type" birational to X)—then the linear system $|mrK_X|$ is free for all $m \gg 0$, where r is the index of the $\mathbf{Q}$-divisor K_X, or the canonical index of X. In particular, the pluricanonical ring of X is finitely generated.*

Note that if X has canonical singularities and K_X is nef, then K_X is big if and only if X is of general type in the ordinary sense. Moreover, a nef and big divisor K_X is ample if and only if $(K_X \cdot C) > 0$ for every curve $C \subset X$, which generalizes a result of P. M. H. Wilson.

PROOF. Corollary 2.5 is a direct consequence of the contraction theorem applied to $D = rK_X$ and standard results on projective normality. ∎

(2.6) COROLLARY. *Let X be a smooth variety of general type. Then the following conditions are equivalent:*

(a) *The pluricanonical ring $\bigoplus_{n \geqslant 0} H^0(X, \mathcal{O}_X(nK_X))$ is finitely generated.*

(b) *There exists a minimal model of X in the sense of* (2.5) *having canonical singularities.*

PROOF. (b) $\Rightarrow$ (a) follows from (2.5), and (a) $\Rightarrow$ (b) from [7], Proposition 1.2. ∎

For $\dim X = n \leqslant 3$ this result and the previous assertions were established by Benveniste [1] in the terminal case, and by Kawamata [4] in the nonsingular case.

(2.7) COROLLARY. *Let X be a Fano variety with routine singularities; then for any Cartier divisor D on X, the following conditions are equivalent:*

(2.7.1) *D is nef.*

(2.7.2) *D is stably free.*

(2.7.3) *$|mD|$ is free for some $m > 0$.*

PROOF. (2.7.1) $\Rightarrow$ (2.7.2) comes from Theorem 2.1. The implications (2.7.2) $\Rightarrow$ (2.7.3) $\Rightarrow$ (2.7.1) are obvious. ∎

For the other applications we need to recall some terminology which goes back to Mori [6]. We write

$$N^1_{\mathbf{Q}}(X) = \{\text{Cartier divisors} \otimes \mathbf{Q}\}/\approx$$

and

$$N^1X = N^1_{\mathbf{Q}}(X) \otimes \mathbf{R};$$

we also set

$$N_1X = \{\text{1-cyles} \otimes \mathbf{R}\}/\approx .$$

By definition of numerical equivalence $\approx$, the real vector spaces N^1X and N_1X are dually paired under the form induced by intersection pairing. The common dimension of N^1X and N_1X is called the *Picard number* $\rho(X)$ of X.

Let $\overline{NE} = \overline{NE}(X) \subset N_1 X$ be the Kleiman-Mori cone, which by definition is the closure of the cone of effective 1-cycles.

(2.8) COROLLARY (compare [11]). *Let F be a face of the cone $\overline{NE}(X)$ entirely contained in the half-space $\{z \in N_1 | K_X \cdot z < 0\}$. Suppose in addition that there exists a nef element $d \in N^1_{\mathbf{Q}}(X)$ such that $d^{\perp} \cap \overline{NE} = F$. Then F uniquely determines a morphism $\varphi = \operatorname{cont}_F$: $X \to Z$ onto a normal variety Z, such that*

(2.8.1) *$\varphi_* \mathcal{O}_X = \mathcal{O}_Z$, and*

(2.8.2) *for any curve $C \subset X$,*

$$\varphi(C) = \mathrm{pt} \in Z \Leftrightarrow \text{the numerical class of } C \text{ belongs to } F.$$

In particular, if $F \neq \varnothing$ then there exists a curve $C \subset X$ whose numerical class belongs to F. The morphism cont_F is called the contraction of the face F.

PROOF. We can repeat almost word-for-word the argument of [8], (0.3), replacing the reference to Theorem 0.0 of [8] by Theorem 2.1 above. ■

The most interesting case is when F is a ray. Recall that a ray $R \subset \overline{NE}$ is called *extremal* if the following implication holds:

$$a + b \in R \text{ with } a, b \in \overline{NE} \Rightarrow a, b \in R.$$

An extremal ray R is *rational* if $R \cap N_1(X)_{\mathbf{Q}} \neq 0$, or equivalently, if there exists a 1-cycle $C \not\equiv 0$ on X whose numerical class belongs to R. Let us say that an extremal ray R is *acute* if the subset $\{d \in N^1 X | d^{\perp} \cap \overline{NE} = R\}$ contains an open subset of $N^1 X$.

(2.9) COROLLARY. *Let R be an extremal ray which is rational and acute, and generated by a numerical class C with $(K_X \cdot C) < 0$. Then the contraction cont_R of R is defined.*

PROPERTIES OF $\varphi = \operatorname{cont}_R$: $X \to Z$.

(2.9.1) *There is an exact sequence*

$$0 \to \operatorname{Pic} Z \xrightarrow{\varphi^*} \operatorname{Pic} X \xrightarrow{(\cdot C)} \mathbf{Z};$$

thus $\rho(Z) = \rho(X) - 1$. In particular if $Z = \mathrm{pt}$, then $\rho(X) = 1$, and if Z is a curve then $\rho(X) = 2$.

(2.9.2) *If* $\dim Z < \dim X$ *then we do not add any essential new information about φ beyond what was said in* (2.3.5).

(2.9.3) *The following four conditions are equivalent*:

(a) $\dim Z = n$;

(b) *R is not numerically effective; that is, there exists an effective divisor F with* $(F \cdot C) < 0$;

(c) *there exists $d \in N^1_{\mathbf{Q}}(X)$ such that $d^{\perp} \cap \overline{NE} = R$ and $d^n > 0$;*

(d) *there exists a subvariety E of X of dimension $\leqslant n - 1$ such that $\varphi|_{X-E}$ is an isomorphism, and $\dim \varphi(E) < \dim E$; that is, E is an exceptional subvariety.*

Following Kawamata [2], in this situation we call the corresponding birational contraction $\varphi = \operatorname{cont}_R$ and *elementary* contraction.

(2.9.4) *Suppose now that X is $\mathbf{Q}$-factorial; then there are two types of elementary contractions*:

(A) *The exceptional subvariety E is an irreducible Weil divisor (and a $\mathbf{Q}$-Cartier divisor, in view of $\mathbf{Q}$-factoriality). In this case Z is again a $\mathbf{Q}$-factorial variety with routine* *(respectively canonical or terminal) singularities (corresponding to X having canonical or terminal singularities respectively). The divisor E is the exceptional divisor associated to the ray R.*

(B) *The exceptional divisor E has dimension $\leqslant n-2$. In this case Z is not $\mathbf{Q}$-factorial; worse still, it is not $\mathbf{Q}$-Gorenstein.*

PROOF. We can repeat almost word-for-word the arguments of Kawamata [2], §3, complementing Corollary 2.8. ■

(2.10) OUTLOOK. Let us look again at the morphism $\varphi\colon X \to Z$ of Theorem 2.1. If $\dim Z < \dim X$ then φ is a fiber space of Fano varieties. Roughly speaking, the geometry of X reduces in this case to the geometry of Z, the geometry of Fano varieties, and the geometry of φ. This can be considered to be more or less the good case—although up to now neither the geometry of del Pezzo fiber spaces (even over $\mathbf{P}^1$), nor the birational geometry of the quartic 3-fold, a Fano 3-fold, has been satisfactorily studied.

Now we turn to the worse situation when $\dim Z = \dim X$, so that φ is a birational morphism. Note first that there is an inclusion $\varphi^*\colon \operatorname{Pic} Z \hookrightarrow \operatorname{Pic} X$, so that the change in the Picard number can be controlled. Hence we could hope that the geometry of Z is simpler than that of X, and study Z instead. However, already in the 3-fold case ($n=3$) there are examples when Z not only fails to have routine singularities, but is not even $\mathbf{Q}$-Gorenstein. Only for nonsingular varieties X of dimension $\leqslant 3$ is it known by Mori's classification [6] that all elementary contractions lead again to $\mathbf{Q}$-factorial varieties.

There is an entirely natural conjecture that in the general case there should exist a "flip" or adjoint diagram.

(2.11) ADJOINT DIAGRAM. This should be a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\mathrm{tr}_\varphi"] \arrow[dr] & & X^+ \arrow[dl, "\varphi^+"] \\ & Z & \end{tikzcd}$$

consisting of

(2.11.1) a normal projective $\mathbf{Q}$-Gorenstein variety X^+;

(2.11.2) a rational map $\mathrm{tr}_\varphi\colon X \to X^+$ which is an isomorphism except over a subset of X^+ of codimension $\geqslant 2$; and

(2.11.3) a morphism $\varphi^+\colon X^+ \to Z$ such that the canonical class K_{X^+} is relatively ample for φ^+ and $(\varphi^+)_*\mathcal{O}_{X^+} = \mathcal{O}_Z$.

(2.12) PROPOSITION (on a platter on one's head). *The following conditions are equivalent:*

(2.12.1) *There exists an adjoint diagram for φ.*

(2.12.2) *For any ample divisor $H \in \operatorname{Div} Z$ and any integer $m \gg 0$ the pluri-adjoint ring*

$$R(X, m\varphi^*H + K_X) = \bigoplus_{n\geqslant 0} H^0(X, \mathcal{O}_X(n(m\varphi^*H + K_X)))$$

is finitely generated.

(2.12.3) *For some ample divisor $H \in \operatorname{Div} Z$ and any integer $m \gg 0$ the pluri-adjoint ring $R(m\varphi^*H + K_X)$ is finitely generated.*

PROOF. (2.12.1) ⇒ (2.12.2). By definition of the adjoint diagram,

$$R(m\varphi^*H + K_X) = R(mH + K_Z) = R(m(\varphi^+)^*H + K_{X^+}).$$

Now standard arguments with the Kleiman-Mori cone show that the divisor $m(\varphi^+)^*H + K_{X^+}$ is ample for $m \gg 0$. Hence the pluricanonical ring on the right-hand side is finitely generated.

(2.12.2) $\Rightarrow$ (2.12.3) is obvious.

(2.12.3) $\Rightarrow$ (2.12.1). This is a simple modification of the argument of [7], pp. 278–279, or properly speaking of [7], Lemma 1.6. We need to take $X^+ = \operatorname{Proj}(R(m\varphi^*H + K_X))$ for $m \gg 0$ in order that there should exist a projective morphism $\varphi^+\colon X^+ \to Z$. Then the fact that X^+ is **Q**-Gorenstein will follow from the fact that $\mathcal{O}_{X^+}(r(m(\varphi^+)^*H + K_{X^+})) = \mathcal{O}_X + (1)$ for $r \gg 0$. ■

The proof has the following consequences:

(2.13) PROPERTIES OF THE ADJOINT TRIANGLE.

(2.13.1) *If an adjoint triangle exists for φ, then it is unique.*

(2.13.2) *If X has only routine (respectively canonical, terminal) singularities, then the same holds for X^+.*

(2.13.3) *If $-K_X$ is relatively ample for φ, then any "minimal" common resolution W of X and X^+:*

$$\begin{tikzcd} & W \arrow[dl, "g"'] \arrow[dr, "h"] & \\ X & & X^+ \end{tikzcd}$$

has $a_i^+ > a_i$ for all i, where $K_W = g^*K_X + \sum a_iF_i$ and $K_W = h^*K_{X^+} + \sum a_i^+F_i$.

(2.13.4) *There exists a surjective map of Weil divisor groups*

$$(\mathrm{tr}_\varphi)_*\colon \operatorname{WeilDiv}(X) \twoheadrightarrow \operatorname{WeilDiv}(X^+)$$

*which induces a map of **Q**-Cartier divisor groups*

$$(\mathrm{tr}_\varphi)_*\colon \operatorname{Div}_{\mathbf{Q}} X \to \operatorname{Div}_{\mathbf{Q}} X^+$$

if $\varphi\colon X \to Z$ is the contraction of an extremal ray.

(2.13.5) *If moreover $\varphi\colon X \to Z$ is the contraction of an extremal ray and X is **Q**-factorial, then the map*

$$(\mathrm{tr}_\varphi)_*\colon \operatorname{Div}_{\mathbf{Q}} X \twoheadrightarrow \operatorname{Div}_{\mathbf{Q}} X^+$$

*is surjective, and X^+ is again **Q**-factorial.*

PROOF. Consider a minimal resolution

$$\begin{tikzcd} & W \arrow[dl, "g"'] \arrow[dr, "h"] & \\ X & & X^+ \end{tikzcd}$$

that is, a resolution of the base locus of some linear system $a\varphi^*H + bK_X$ which defines the model X^+. Then from the fact that $-K_X$ is relatively ample for φ it follows that

$$g^*\left(a\varphi^*H + bK_X\right) \sim h^*\left(a(\varphi^+)^*H + bK_{X^+}\right) + \sum r_iF_i,$$

where the F_i are the exceptional divisors of g and h, and all $r_i > 0$. Therefore

$$g^*(bK_X) \sim h^*(bK_{X^+}) + \sum r_iF_i.$$

But $bg^*(K_X) \sim bK_W - \sum ba_iF_i$ and $bh^*(K_{X^+}) \sim bK_W - \sum ba_i^+F_i$. Therefore

$$-\sum ba_iF_i \sim -\sum ba_i^+F_i + \sum r_iF_i,$$

and hence $a_i^+ = a_i + (r_i/b)$. This proves (2.13.3). If we do not require that $-K_X$ is relatively ample, then $r_i \geqslant 0$ and $a_i^+ \geqslant a_i$, which gives (2.13.2).

Suppose that $\varphi = \mathrm{cont}_R$, where R is an extremal ray. Suppose that $D \in \mathrm{Div}_{\mathbf{Q}} X$ is a $\mathbf{Q}$-Cartier divisor such that $(D \cdot R) = 0$; then $(\mathrm{tr}_\varphi)_* D = (\varphi^+)^* D'$ (see (2.9.1)). If $D = K_X$ then $(\mathrm{tr}_\varphi)_* K_X = K_{X^+}$. Again by (2.9.1),

$$\mathrm{Div}_{\mathbf{Q}} X = \varphi^* \mathrm{Div}_{\mathbf{Q}} Z \oplus \mathbf{Q} \cdot K_X.$$

This establishes the existence of $(\mathrm{tr}_\varphi)_*$. Now suppose in addition that X is $\mathbf{Q}$-factorial. Consider an arbitrary common resolution

$$\begin{array}{ccc} & W & \\ {}^{g}\swarrow & & \searrow{}^{h} \\ X & & X^+ \end{array}$$

Then for any $D \in \mathrm{Div}_{\mathbf{Q}} X^+$ we have $D = (\mathrm{tr}_\varphi)_* g_* h^* D$, so that in this case $(\mathrm{tr}_\varphi)_*$ has a section $(\mathrm{tr}_\varphi)^* = g_* h^*$. To prove that X^+ is $\mathbf{Q}$-factorial it is enough to check the equality $D = (\mathrm{tr}_\varphi)_* g_* h' D$, where $h'D$ denotes the proper transform of D. The remaining assertions are trivial. ■

(2.14) REMARKS. (a) $\varphi^+ = \mathrm{id}$ if and only if Z is already $\mathbf{Q}$-Gorenstein. This case generalizes case (A) of elementary contractions (see (2.9.4)); in this case we can just take $\mathrm{tr}_\varphi = \varphi$ and $\varphi^+ = \mathrm{id}_Z$. Note also that in this case $\rho(X) > \rho(Z)$, provided that K_X is not trivial relative to φ.

(b) In (2.13.5), either φ is a contraction of type (A), with $X^+ = Z$ and $\rho(X^+) = \rho(Z) = \rho(X) - 1$, or φ is a contraction of type (B), and in this case $(\mathrm{tr}_\varphi)_*$ is an isomorphism, and $\rho(X) = \rho(X^+)$. Thus it is not at all obvious that a sequence of extremal modifications has to terminate after finitely many steps, so that there exists a finite chain of such modifications

$$X \dashrightarrow X^+ \dashrightarrow X^{++} \dashrightarrow \cdots \dashrightarrow X^{(+n)} = Y$$

such that either (1) K_Y is nef, or (2) Y has an extremal contraction $\mathrm{cont}_R\colon Y \to Z$ which is a Fano fiber space, with $\dim Z < \dim X$ (see (2.3.5)).

However, as we can see from the inequalities (2.13.3), the singularities undergo a certain simplification, which is sufficient to imply this in the 3-dimensional canonical case. More precisely:

(2.15) DEFINITION. Let X be a variety with canonical singularities, and let $f\colon Y \to X$ be a resolution of X. Then $K_Y = f^* K_X + \sum a_i F_i$, where the F_i are exceptional divisors, and all $a_i \geqslant 0$. We define the *difficulty* of X by

$$d(X) = \#\{i \mid a_i < 1\}.$$

It is easy to check that $d(X)$ does not depend on the resolution.

(2.16) COROLLARY. *Let X be a $\mathbf{Q}$-factorial variety with canonical singularities, and let $\mathrm{tr}_R\colon X \to X^+$ be an extremal modification in a ray R such that the dimension of the exceptional set of φ^+ is not less than* $\dim X - 2$. *Then either* (A) $\rho(X^+) = \rho(X) - 1$, *or* (B) $\rho(X^+) = \rho(X)$ *but* $d(X^+) < d(X)$.

In particular, the dichotomy (A) *or* (B) *is always true in the* 3*-fold canonical case, with no hypothesis on the exceptional locus, since if the exceptional locus of φ^+ has dimension* 0 *then* $\varphi^+ = \mathrm{id}$.

PROOF. Case (A) corresponds to an extremal contraction of a divisor. Otherwise, φ^+ has an exceptional set Γ of dimension $\geqslant \dim X^+ - 2$, along which X^+ must be nonsingular, by the fact that it has terminal singularities (2.13.3). But then by (2.13.3) again, we get that on a common resolution, some exceptional component over Γ has $a_i^+ = 1$. Hence $d(X^+) < d(X)$. ■

(2.17) COROLLARY. *If an adjoint diagram always exists in the 3-fold canonical case, then the sequence of extremal modifications must terminate.* ■

In conclusion, we remark that if X is a 3-fold with **Q**-factorial canonical singularities such that K_X is not nef, then there always exists an extremal ray R satisfying the conditions of Corollary 2.9; see [8] and [9]. The same thing holds for any nonsingular variety [6].

Thus in the extremological program for the construction of minimal models formulated explicitly by Reid, and implicitly by Mori [6], there remains at present only one open problem in the 3-fold case:

(1) *the existence of the adjoint diagram (the "flip conjecture").*

In case $n \geqslant 4$ two further problems must be added:

(2) *the existence of extremal faces of F (as in (2.8));*

(3) *the finiteness of a sequence of extremal modifications, and more particularly, finding bounds on the minimal a_i.*

ADDED IN PROOF. Towards the end of 1983 an entirely satisfactory solution of problem (2) was obtained by Kawamata [13], and rather more precisely by Kollár [14]. For the problem of bounds in (3), D. Markushevich showed that for any **Q**-Gorenstein singular 3-fold X there always exists components with $a_i \leqslant 1$, and V. Danilov checked that the same holds for toric varieties in any dimensions.

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