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Birational models and flips

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Abstract. This survey treats two chapters in the theory of log minimal models, namely, the chapter on different notions of models in this theory and the chapter on birational flips, that is, log flips, mainly in dimension 3. Our treatment is based on ideas and results of the second author: his paper on log flips (and also on material from the University of Utah workshop) for the first chapter, and his paper on prelimiting flips (together with surveys of these results by Corti and Iskovskikh) for the second chapter, where a complete proof of the existence of log flips in dimension 3 is given. At present, this proof is the simplest one, and the authors hope that it can be understood by a broad circle of mathematicians.

Contents

Der Vogelsprecher bin ich ja,
Stets lustig heißa hopsassa!
Ich Vogelsprecher bin bekannt
bei Alt und Jung im ganzen Land.
Weiß mit dem Locken umzugehn
und mich aufs Pfeifen zu verstehn!

Papageno's aria, Act I, Scene I,
Die Zauberflöte, W. A. Mozart

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Introduction

In this survey we use the approach that seems simplest for many mathematicians to treat two important parts of the Log Minimal Model Programme (LMMP), namely, different notions of models in this theory and properties of models (Chapter I), and birational flips, that is, log flips and log flops, mainly in dimension 3 (Chapter II). The principal ideas and most results here are due to the second author. For the first author, this is a course of lectures he delivered in the Department of Mechanics and Mathematics of the Moscow State University during the 2003–2004 academic year.

In Chapter II we give a complete existence proof for log flips in dimension 3. The proof is based on a new inductive approach proposed by the second author (see [37]). The authors believe that this is simpler and shorter than previous proofs ([26], [34], [21]). The last section discusses semistable 3-fold flips.

The main purpose of the (L)MMP is to apply the programme to geometry, but this is beyond the scope of the present survey. The reader can find typical examples of applications of this kind in [38] and in other papers of that series.

Notation and terminology

0.1. As usual, we denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ the sets of positive integers, integers, rationals, reals, and complex numbers with their standard structures. Unless otherwise stated, the base field k is assumed to be algebraically closed and of characteristic zero (frequently it is simply the field $\mathbb{C}$ of complex numbers). Analyticity is understood the complex sense.

0.2. A morphism (that is, an everywhere defined map of algebraic varieties, spaces, schemes, and so on) is denoted by an arrow ($f\colon X \to Y$) and a rational (or meromorphic) map is denoted by a dashed arrow ($f\colon X \dashrightarrow Y$). By a *modification* we mean a birational (bimeromorphic) transformation of complete varieties (of compact complex analytic spaces, respectively). Correspondingly, a modification of a proper morphism $f\colon X \to Z$ is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd},$$

where $g\colon Y \to Z$ is a proper morphism and φ is a birational (bimeromorphic) map.

A proper morphism $f\colon X \to Y$ is called a *contraction* if $f_*\mathcal{O}_X = \mathcal{O}_Y$ (in particular, in characteristic zero the fibres of f are connected if X and Y are normal). If the morphism f is birational (bimeromorphic), then it is referred to as a *blowdown*. A *blowup* is another name for a blowdown when it is regarded as a morphism constructed from Y. A *normal blowup* is a blowup with a normal object X. A birational contraction (blowup) is said to be *small* if its exceptional locus $\operatorname{Exc} f$ is of codimension $\geqslant 2$.

0.3. The free Abelian group $\operatorname{Div} X$ generated by the irreducible subvarieties of codimension 1 on a normal variety X is called the *group of Weil divisors*, and the generators are called *prime divisors*. The subgroup of $\operatorname{Div} X$ formed by the Cartier divisors consists of all locally principal divisors. The $\mathbb{Q}$-divisors ($\mathbb{R}$-divisors) are the elements of $\operatorname{Div} X \otimes \mathbb{Q}$ (of $\operatorname{Div} X \otimes \mathbb{R}$, respectively). A $\mathbb{Q}$-divisor ($\mathbb{R}$-divisor) of the form $D = \sum d_i D_i$, where the divisors D_i are prime and $d_i \in \mathbb{Q}$ ($d_i \in \mathbb{R}$), is called a $\mathbb{Q}$-*Cartier divisor* (an $\mathbb{R}$-*Cartier divisor*) if $D = \sum r_i C_i$, where the C_i are Cartier divisors and all the numbers r_i are in $\mathbb{Q}$ (in $\mathbb{R}$). A $\mathbb{Q}$-divisor D is a $\mathbb{Q}$-Cartier divisor if and only if mD is a Cartier divisor for some non-zero $m \in \mathbb{Z}$. A $\mathbb{Q}$-divisor is an $\mathbb{R}$-Cartier divisor if and only if it is a $\mathbb{Q}$-Cartier divisor.

Linear equivalence of divisors is denoted by the symbol $\sim$. $\mathbb{Q}$-linear ($\mathbb{R}$-linear) equivalence is denoted by the symbol $\sim_{\mathbb{Q}}$ ($\sim_{\mathbb{R}}$). We recall that $\mathbb{Q}$-divisors $D_1, D_2 \in \operatorname{Div} X \otimes \mathbb{Q}$ (and even $\mathbb{R}$-divisors $D_1, D_2 \in \operatorname{Div} X \otimes \mathbb{R}$) are said to be $\mathbb{Q}$-*linearly equivalent* if their difference $D_1 - D_2$ is $\mathbb{Q}$-principal, that is, $D_1 - D_2 = \sum d_i(f_i)$, where $d_i \in \mathbb{Q}$, the functions $f_i \neq 0 \in k(X)$ are rational (in the analytic situation, meromorphic) on X, and each (f_i) is the corresponding divisor (of zeros and poles). The definition for $\mathbb{R}$-divisors is similar. We note that if D_1 is a $\mathbb{Q}$-Cartier divisor and if $D_1 \sim_{\mathbb{Q}} D_2$, then D_2 is also a $\mathbb{Q}$-Cartier divisor; in addition, the equivalence $D_1 \sim_{\mathbb{Q}} D_2$ amounts to the linear equivalence $mD_1 \sim mD_2$ for some non-zero $m \in \mathbb{Z}$.

We denote by the symbol $\equiv$ the numerical equivalence of $\mathbb{Q}$- and $\mathbb{R}$-divisors (which is defined not only if they are $\mathbb{Q}$- or $\mathbb{R}$-Cartier divisors). Namely, on a complete variety X we have

$$D_1 \equiv D_2 \iff D_1 - D_2 \equiv 0 \iff (D_1 - D_2)C = 0 \text{ for all curves } C \subset X;$$

here the difference $D_1 - D_2$ is assumed to be a $\mathbb{Q}$-Cartier or an $\mathbb{R}$-Cartier divisor, respectively. A divisor $D \in \operatorname{Div} X \otimes \mathbb{R}$ is said to be *nef* (numerically effective) if it is an $\mathbb{R}$-Cartier divisor and the condition $DC \geqslant 0$ holds for any curve $C \subset X$. In the relative situation with $f\colon X \to Z$ the curves $C \subset X$ are taken over Z, that is, $f(C) = \text{pt.} \in Z$. In this case we say that D is *nef over* Z, or f-*nef*.

An $\mathbb{R}$-divisor D is said to be *big* if $h^0(X, mD) > \operatorname{const} m^{\dim X}$ for any $m \gg 0$. For a nef D this is equivalent to the positivity condition $D^{\dim X} > 0$. Relative bigness can be defined in a similar way.

0.4. A *canonical divisor* $K = K_X = (\omega)$ is the Weil divisor (of zeros and poles) of any rational (meromorphic) differential form $\omega \neq 0$ on X of the highest degree. A variety X is called a *Gorenstein* variety if K is a Cartier divisor, and a $\mathbb{Q}$-*Gorenstein* variety if K is a $\mathbb{Q}$-Cartier divisor, that is, if mK is a Cartier divisor for some non-zero $m \in \mathbb{Z}$. For a point $x \in X$ the least positive integer m_x such that $m_x K$ is locally principal in x is called the (*local Gorenstein*) *index* of x, and the least common multiple of the set $\{m_x \mid x \in X\}$ of all indices is called the (*global Gorenstein*) *index* of X.

Similarly, an index of a log pair (X, B) can be defined as the index of the *adjoint* divisor $K + B$. Here B is assumed to be a $\mathbb{Q}$-boundary (or a $\mathbb{Q}$-divisor). In general, an $\mathbb{R}$-*boundary* (or simply *boundary*) is an $\mathbb{R}$-divisor $B = \sum b_i D_i$, where the sum is taken over the prime divisors D_i and the multiplicities satisfy the conditions $b_i \in \mathbb{R}$ and $0 \leqslant b_i \leqslant 1$. For a $\mathbb{Q}$-*boundary* it is assumed that $b_i \in \mathbb{Q}$.

A variety X is said to be $\mathbb{Q}$-*factorial* if any Weil divisor on X is a $\mathbb{Q}$-Cartier divisor.

The following notation is customary for an $\mathbb{R}$-divisor $D = \sum d_i D_i$:

$\lceil D \rceil := \sum \lceil d_i \rceil D_i$ is the upper integral part;
$\lfloor D \rfloor := \sum \lfloor d_i \rfloor D_i$ is the lower (usual) integral part.

For any real number $d \in \mathbb{R}$ the symbol $\lceil d \rceil$ stands here for the least integer $\geqslant d$ and $\lfloor d \rfloor$ for the greatest integer $\leqslant d$.

We write $H^i(X, D) = H^i(X, \mathcal{O}_X(D))$ and $h^i(X, D) = h^i(X, \mathcal{O}_X(D))$ for any $\mathbb{R}$-divisor D, where $\mathcal{O}_X(D) = \mathcal{O}_X(\lfloor D \rfloor)$ stands for the divisorial sheaf associated with D.

The base field need not be algebraically closed if the prime Weil divisors are treated algebraically rather than geometrically. The same holds for cycles.

0.5. We recall now the main notion of the theory. We mean the Kleiman–Mori cone, that is, the closure of the cone of effective 1-cycles. A 1-*cycle* on X is a sum of the form $z = \sum n_i C_i \in \mathrm{Z}_1 X$ taken over prime 1-cycles $C_i \subset X$, that is, curves, where the multiplicities satisfy the condition $n_i \in \mathbb{Z}$. If $n_i \geqslant 0$ for all i, then the cycle is said to be *effective*. The following notions and notation are customary for a complete algebraic variety X.

$\mathrm{N}(X) := \mathrm{N}_1 X = \mathrm{Z}_1 X \otimes \mathbb{R}/(\mathrm{mod} \equiv)$. The numerical equivalence $\equiv$ of 1-cycles is dual to the numerical equivalence of divisors:

$$z_1 \equiv z_2 \Longleftrightarrow z_1 - z_2 \equiv 0 \Longleftrightarrow D(z_1 - z_2) = 0 \text{ for all Cartier divisors } D \text{ on } X.$$

$\rho(X) = \dim_{\mathbb{R}} \mathrm{N}(X)$ is the Picard number (it coincides with the rank of the Neron–Severi group).

$\mathrm{NE}(X)$ is the cone of effective 1-cycles. The cone $\mathrm{NE}(X)$ is generated in the vector space $\mathrm{N}(X) \cong \mathbb{R}^{\rho(X)}$ by the classes of effective 1-cycles, that is, the vectors of the cone are the classes, up to numerical equivalence, of effective real 1-cycles $z = \sum r_i C_i$, $r_i \in \mathbb{R}$, $r_i \geqslant 0$.

$\overline{\mathrm{NE}}(X)$ is the Kleiman–Mori cone, where the closure $\overline{}$ is taken in the standard real topology.

The space $\mathrm{N}(X)$ is also denoted by $\mathrm{N}_1 X$ in contrast to its dual $\mathrm{N}^1 X$ which is the quotient space of the $\mathbb{R}$-Cartier divisors on X modulo the numerical equivalence $\equiv$. Thus, any $\mathbb{R}$-Cartier divisor D can be regarded as an $\mathbb{R}$-valued linear function on $\mathrm{N}(X)$ induced by the intersection $Dz = \sum r_i DC_i$, $r_i \in \mathbb{R}$.

It follows from the Kleiman ampleness criterion that

$$\text{a } \mathbb{Q}\text{-Cartier } D \text{ is ample } \Longleftrightarrow Dz > 0 \text{ on } \overline{\mathrm{NE}}(X) \setminus \{0\}.$$

If D is an $\mathbb{R}$-Cartier divisor, the last condition holds, and X is projective, then D is said to be *numerically ample*. This kind of ampleness is equivalent to the existence of a *semilinear* decomposition of the form $D = \sum h_i H_i$, $h_i \in \mathbb{R}$, $h_i \geqslant 0$, where each divisor H_i is ample.

To define the corresponding relative notions for a proper morphism $f\colon X \to Z$, one assumes that the 1-cycles z are cycles over Z, that is, $f_* z = 0$. For example, an $\mathbb{R}$-Cartier divisor D is (*relatively*) *numerically ample* over Z, or f-*ample*, if the positivity condition $Dz > 0$ holds on $\overline{\mathrm{NE}}(X/Z) \setminus \{0\}$.

A (birational) contraction $f\colon X \to Z$ is said to be *extremal* if the relative Picard number satisfies the condition $\rho(X/Z) = \rho(X) - \rho(Z) = 1$.

0.6. *Extremal rays.* Let (X, D) be a complete log pair, that is, the normal variety X is complete, and let the adjoint divisor $K + D$ of the $\mathbb{R}$-divisor D be an $\mathbb{R}$-Cartier divisor. We set

$\overline{\mathrm{NE}}_{\geqslant 0}(X, D) := \{z \in \overline{\mathrm{NE}}(X) \mid (K + D)z \geqslant 0\}$ (this is the $(K + D)$-positive part of the Kleiman–Mori cone);

$\overline{\mathrm{NE}}_{<0}(X, D) := \{z \in \overline{\mathrm{NE}}(X) \mid (K + D)z < 0 \text{ or } z = 0\}$ (this is the $(K + D)$-negative part of the Kleiman–Mori cone).

The dual hyperplane $H^\perp := \{z \in \mathrm{N}(X) \mid Hz = 0\}$ in $\mathrm{N}(X)$ is defined for each $\mathbb{R}$-Cartier divisor H. A divisor H is nef if and only if $H^\perp$ is a supporting hyperplane of the cone $\overline{\mathrm{NE}}(X)$ and this cone lies entirely in the positive half-space $\{z \in \mathrm{N}(X) \mid Hz \geqslant 0\}$. Here the subset $F = \overline{\mathrm{NE}}(X) \cap H^\perp \subset \overline{\mathrm{NE}}(X)$ is a face of the cone. A 1-dimensional face $R \subset \overline{\mathrm{NE}}(X)$ is called an *extremal ray*. An extremal ray is defined with respect to $K + D$ if $R \subset \overline{\mathrm{NE}}_{<0}(X, D)$ or, as is frequently written, $(K + D)R < 0$. A face (a ray) need not be generated by curves in general.

By the cone theorem ([1], Theorem 2), if (X, B) is a log canonical log pair (for example, $\mathbb{Q}$-factorial and log terminal) with a boundary B and if X is projective, then

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}_{\geqslant 0}(X, B) + \sum R_l,$$

where R_l ranges over the set (which is at most countable) of all extremal rays with respect to $K + B$. Moreover, the cone $\overline{\mathrm{NE}}(X)$ is locally copolyhedral in its negative part $\overline{\mathrm{NE}}_{<0}(X, B)$, and $\overline{\mathrm{NE}}_{<0}(X, B) = \mathrm{NE}_{<0}(X, B) := \{z \in \mathrm{NE}(X) \mid (K + B)z < 0 \text{ or } z = 0\}$, that is, the rays R_l are discrete and the cone of effective 1-cycles is closed in its $(K + B)$-negative part. Here each ray R_l has the form $R_l = \mathbb{R}_+[C_l]$, where $[C_l] \in \overline{\mathrm{NE}}(X)$ is the class of a rational curve $C_l \subset X$, and $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r \geqslant 0\}$.

In the relative situation a complete log pair (X, D) is replaced by a proper relative pair $f\colon (X, D) \to Z$, that is, the map $f\colon X \to Z$ is assumed to be proper and the projectivity is replaced by relative projectivity.

0.7. *Cyclic and hypersurface quotient singularities.* Let $\mathbb{Z}_r \cong \mu_r \subset k$ be a cyclic group of order $r \geqslant 1$ regarded as the group of rth roots of unity. This group acts naturally on the space k^n with integral weights $(a_1, \dots, a_n), a_i \in \mathbb{Z}$:

$$(x_1, \dots, x_n) \mapsto (\zeta^{a_1} x_1, \dots, \zeta^{a_n} x_n),$$

where $\zeta \in \mu_r$. The weights are usually assumed to be coprime: $(a_1, \dots, a_n) = 1$. Then the action is free in codimension 1. The quotient variety $X = k^n/\mu_r = k^n/\mathbb{Z}_r(a_1, \dots, a_n)$ has only cyclic quotient singularities. We denote this variety by $\frac{1}{r}(a_1, \dots, a_n)$, and the same symbol is used for the type of the point 0 corresponding to $(0, \dots, 0) \in k^n$ up to a local isomorphism (analytic or formal).

The case in which $0 \in Y = (F = 0) \subset k^{n+1}$ is a normal hypersurface singularity with an action of μ_r and the point $0 \in X = Y/\mu_r$ is the corresponding quotient singularity can be treated in a similar way. In this case the action of μ_r can be extended to k^{n+1} (as the action on the cotangent space at 0 with appropriate coordinates), as above, with weights $(a_0, \dots, a_n)$. Since the hypersurface Y is invariant under this action, the function $F = \zeta^e F$ is a (semi-invariant) eigenfunction of the action. We denote the type of the corresponding hypersurface quotient singularity $0 \in X$ by the symbol $\frac{1}{r}(a_0, \dots, a_n; e)$ or, more precisely, by $(F = 0) \subset \frac{1}{r}(a_0, \dots, a_n)$.

0.8. *Weighted blowup.* Let us consider a cyclic quotient singularity $0 \in X := k^n/\mathbb{Z}_r(a_1, \dots, a_n)$, $r \geqslant 1$, where $(a_1, \dots, a_n) = 1$, and let $x_1, \dots, x_n$ be the coordinates which are eigenfunctions of the action ($X = k^n$ for $r = 1$). In this case a weighted blowup of X with the weights $\frac{1}{r}(a_1, \dots, a_n)$ means a blowup $\sigma\colon Y \to X$ such that Y has a covering by affine charts $U_1, \dots, U_n$ of the form

$$U_i = k^n/\mathbb{Z}_{a_i}(a_1, \dots, \overset{i}{r}, \dots, a_n).$$

The coordinates x_j on X and y_j on U_i are related by the formulae $y_j = x_j/x_i^{\frac{a_j}{a_i}}$ and $y_i = x_i^{\frac{r}{a_i}}$. The exceptional locus E of σ is a Weil $\mathbb{Q}$-Cartier divisor, $E \cap U_i = (y_i = 0) \subset k^n/\mathbb{Z}_{a_i}(a_1, \dots, r, \dots, a_n)$, and $E = \operatorname{Proj}(a_1, \dots, a_n)$ is a weighted projective space of dimension $n - 1$. (See [8], Chapter II, 3.6, up to the notation for weights.)

Remark 0.9. Most results in this survey can readily be adapted and proved for the analytic and formal categories. For example, the relative LMMP in the analytic category is usually regarded over a neighbourhood of a compact subset in the base space ([11], § 1, and [27]).

CHAPTER I

Models

In this chapter we recall basic notions developed in the Log Minimal Model Programme (LMMP) and reproduce some results used below. The notions of models in the LMMP will be treated more thoroughly, and the famous Noether–Fano inequality will be interpreted in these terms.

§ 1. Basic notions of the LMMP

1.1. Let (X, D) be a pair consisting of a complete normal algebraic variety X (usually over a base Z, omitted in our notation so far) and an $\mathbb{R}$-Cartier divisor D which need not be effective. The *D-Minimal Model Programme* (D-MMP) means a chain of modifications

$$(X_0, D_0) \overset{g_0}{\dashrightarrow} (X_1, D_1) \overset{g_1}{\dashrightarrow} (X_2, D_2) \overset{g_2}{\dashrightarrow} \cdots$$

constructed as follows.

a) (X_0, D_0) is an *initial* pair, for example, $(X_0, D_0) = (X, D)$, where the latter pair has some properties preserved under the transformations in c) and d) below. It is usually assumed that X_0 is projective and that (X_0, D_0) is non-singular. If X is not projective or the pair (X, D) is quite singular, then one can take an appropriate model (X_0, D_0) of (X, D) (in the birational sense); different approaches can be used to define D_0.

b) Suppose that a pair (X_i, D_i) has been constructed. If the $\mathbb{R}$-Cartier divisor D_i is nef, then the algorithm is terminated. By definition, the pair thus obtained is a *D-minimal* model of (X, D).

c) If D_i is not nef, then one must find a contraction $\varphi_i\colon X_i \to Y_i$ such that $-D_i$ is relatively numerically φ_i-ample and $\rho(X_i/Y_i) = 1$, where $\rho(\,\cdot\,/\,\cdot\,)$ stands for the (relative) Picard number as usual. A contraction of this kind is called a *D_i-contraction* (or a *D-contraction*). If φ_i is not birational, that is, the contraction is of fibred type, then the algorithm is terminated again. If φ_i is birational and $\varphi(D_i)$ is an $\mathbb{R}$-Cartier divisor (this usually happens if the exceptional locus of φ_i is irreducible and divisorial), then we set $g_i := \varphi_i$ and $(X_{i+1}, D_{i+1}) := (Y_i, \varphi_i(D_i))$. (The programme does not work if there is no contraction φ_i with the above properties!)

d) If $\varphi_i(D_i)$ is not an $\mathbb{R}$-Cartier divisor, then to continue the process one must find a commutative diagram of the form

$$\begin{tikzcd}[column sep=small] (X_i, D_i) \arrow[rr, dashed, "g_i"] \arrow[dr, "\varphi_i"'] & & (X_i^+, D_i^+) = (X_{i+1}, D_{i+1}) \arrow[dl, "\varphi_i^+"] \\ & Y_i & \end{tikzcd} \tag{1.1}$$

with the following properties:

(1) (i) φ_i^+ is a small birational contraction, that is, the exceptional locus $\operatorname{Exc}\varphi_i^+$ of φ_i^+ is of codimension $\geqslant 2$;

(2) (ii) $D_{i+1} = D_i^+ = g_{i*}D_i$;

(3) (iii) D_i^+ is numerically φ_i^+-ample.

A diagram (1.1) with these properties is called a *D_i-flip* (or a *D-flip*) *of* φ_i. If a D_i-flip exists, then the algorithm for constructing a D-minimal model can be continued. In addition to existence problems we face a natural problem of *termination*, that is, termination of the algorithm. If this happens, then at the output of the algorithm we obtain a *resulting* (*terminal*) pair which is either a D-minimal model as in b), or a fibred morphism as in c).

The termination of divisorial contractions in c) follows because the Picard number $\rho(X_i)$ decreases for such contractions. Thus, one must prove the termination only of flips, and this reduction is taken into account in what follows.

The existence problem for D-minimal models depends mainly on the character of the singularities of the *initial* pair $(X_0, D_0) = (X, D)$. For example, if $D = K = K_X$ is a canonical divisor and X is a projective $\mathbb{Q}$-factorial variety with (at worst) only terminal singularities, then this is the usual Minimal Model Programme (MMP) for algebraic varieties, or the Mori theory. However, if $D = K + B$, where $B = \sum b_i D_i$ is a *boundary*, that is, if $0 \leqslant b_i \leqslant 1$ for all i, the sum runs through prime divisors D_i, and the pair (X, B) has $\mathbb{Q}$-factorial log terminal (or even log canonical) singularities only, then this is the LMMP (see, for example, [22], [17], [34], [36]). If X is not projective or the singularities of (X, B) are worse, then a projective resolution (X_0, B_0) of (X, B) can be taken as an initial model.

In this connection let us recall the basic definitions. In what follows, D stands for a different object, namely, for a part of an adjoint divisor.

Definition 1.2. Let X be a normal variety and let $D = \sum d_i D_i$, $d_i \in \mathbb{R}$, be an $\mathbb{R}$-divisor (not necessarily effective) such that the *adjoint* divisor $K + D$ has the

$\mathbb{R}$-Cartier property, where $K = K_X$ is the canonical divisor of X and each factor D_i is a Weil prime divisor. Let $f\colon Y \to X$ be a normal blowup of X. Then for an appropriate choice of a canonical divisor K_Y one can write

$$K_Y = f^*(K+D) + \sum a(E, X, D)E,$$

where each factor $E \subset Y$ is a Weil prime divisor and each factor $a(E, X, D) \in \mathbb{R}$ is a real number. If $E = D_i$ is not exceptional on X, then the previous formula implies that $a(D_i, X, D) = -d_i$. The numbers $a(E, X, D)$ are called the *discrepancies* of (X, D) at E. They depend only on a divisorial valuation of the field $k(X)$ or, equivalently, on the prime b-divisor $\underline{E} = E$ associated (and identified) with E (see Definition 5.5 below) and do not depend on Y nor on the choice of K and K_Y. The image $f(E)$ is called the *centre* of E on X (or the centre of the associated valuation or the b-divisor) and is denoted by $\mathrm{c}_X E$. A b-divisor E is said to be *exceptional* on X if $\mathrm{c}_X E$ is of codimension $\geqslant 2$. The prime b-divisors of X are the prime b-divisors, or divisorial valuations of the field $k(X)$, with non-empty centres on X.

The choice of a divisor K_Y is determined by a choice of K. Namely, if $K = (\omega)$ is the divisor of zeros and poles of a rational differential form $\omega \neq 0$ on X, then $K_Y = (f^*\omega)$ is the divisor of $f^*\omega$. For an arbitrary choice of K_Y the equality $=$ must be replaced by the linear equivalence $\sim$.

The discrepancy $\mathrm{dis}(X, D)$ and the total discrepancy $\mathrm{tdis}(X, D)$ of a pair (X, D) are defined as follows:

$$\begin{aligned}\mathrm{dis}(X, D) &:= \inf_E \{a(E, X, D) \mid E \text{ ranges over the prime b-divisors of } X \\ &\qquad\qquad \text{that are } \textit{exceptional} \text{ on } X\},\\ \mathrm{tdis}(X, D) &:= \inf_E \{a(E, X, D) \mid E \text{ ranges over the prime b-divisors of } X\}.\end{aligned}$$

Definition 1.3. Let X be a normal variety and let $D = \sum d_i D_i$ be an $\mathbb{R}$-divisor such that the adjoint divisor $K + D$ has the $\mathbb{R}$-Cartier property. We say that the pair (X, D) (or simply the divisor $K + D$) has only the singularities listed below if the corresponding inequalities hold:

$$\left.\begin{array}{l}\text{terminal (trm)}\\ \text{canonical (cn)}\\ \text{purely log terminal (plt)}\\ \text{Kawamata log terminal (klt)}\\ \text{log canonical (lc)}\end{array}\right\} \iff \mathrm{dis}(X, D) \left\{\begin{array}{l} > 0;\\ \geqslant 0;\\ > -1;\\ > -1 \text{ and } d_i < 1\ \forall\, i;\\ \geqslant -1 \text{ and } d_i \leqslant 1\ \forall\, i.\end{array}\right.$$

Remarks 1.4. (i) In addition, the following singularities are also widely used:

a pair (X, D) is said to be *log terminal* (*lt*) if there is a log resolution $f\colon Y \to X$ such that $a(E, X, D) > -1$ for all exceptional divisors E of f;

a pair (X, D) is said to be *divisorially log terminal* (*dlt*) if there is a log resolution $f\colon Y \to X$ with divisorial exceptional locus such that $a(E, X, D) > -1$ for any exceptional divisor E.

These definitions can be used, for example, when working with non-$\mathbb{Q}$-factorial varieties.

By definition,

$$\text{klt} \Rightarrow \text{plt} \Rightarrow \text{lt} \quad \text{and} \quad \text{dlt} \Rightarrow \text{lt}.$$

Simple examples show that the above classes of singularities are distinct.

We recall that by a *log resolution* of a pair (X, D) one means a resolution $Y \to X$ such that the variety Y is non-singular and the prime components of the birational transform D_Y of D, together with the exceptional divisors E_i (the *log birational transform*), are non-singular and have only normal crossings (this is usually denoted by $\operatorname{Supp}(D_Y) \cup (\cup E_i)$ and called a divisor with *simple normal crossings*). The pair $(Y, D_Y + \sum E_i)$ is said to be *log non-singular*, and if D_Y is a boundary, then one can often use it as an initial model for the LMMP.

(ii) The operation inf in Definition 1.2 is actually the operation min, and it gives either a value $\geqslant -1$ or the value $= -\infty$. The former value can occur, because finite values of dis and tdis can be computed on the divisors of any log resolution (sufficiently non-trivial in the case of dis), and these values are preserved after any subsequent blowups.

(iii) The adjoint $\mathbb{R}$-divisor $K + D$ in the previous definitions can be replaced by a $\mathbb{Q}$-divisor $K + D'$ with the same rational intersection numbers with curves and the same rational discrepancies (see [34], (1.3.2), and [21], 2.12).

(iv) If $\dim X \geqslant 2$, then the condition $\operatorname{tdis}(X, D) \geqslant -1$ is equivalent to the condition $\operatorname{dis}(X, D) \geqslant -1$, and the assumption that $d_i \leqslant 1$ for all i can be omitted in the definition of lc.

Proposition 1.5 ([34], 1.3, [21], 2.17). *Let X be a normal variety and let $D = \sum d_i D_i$ be an $\mathbb{R}$-divisor. Then the following assertions hold.*

(i) *The set of divisors D for which $K + D$ is* lc (*nef, or numerically ample*) *is a convex subset of a direct sum of copies of $\mathbb{R}$. This set belongs to a direct sum of copies of the unit interval $[0, 1]$ if D is a boundary.*

(ii) *The set of divisors D supported by fixed divisors $D_1, \dots, D_s$ for which $K+D$ is* lc *is a finite convex rational polyhedron in $\sum \mathbb{R}D_i$. This polyhedron is a compact subset of $\sum [0, 1]D_i$ if D is a boundary.*

(iii) *If $D' \leqslant D$, $K+D$ is* lc (lt), *and $K+D'$ is an $\mathbb{R}$-Cartier divisor, then $K+D'$ is also* lc (lt, *respectively*), *and moreover $a(E, X, D) \leqslant a(E, X, D')$ for any E.*

(iv) *Suppose that $K + D$ is* lt. *In this case there is an $\varepsilon > 0$ such that $K + D'$ is* lt *for any $\mathbb{R}$-Cartier divisor $K + D'$, where $D' = \sum d'_i D_i$ with $d'_i \leqslant \min\{1, d_i + \varepsilon\}$. In addition, if $K + D$ is* plt, *then so is $K + D'$.*

(v) *If $K + D$ is* plt *and $K + D + D'$ is* lc, *then $K + D + tD'$ is* plt *for any $0 \leqslant t < 1$.*

Let us now consider all kinds of notions of model. We use the standard notion of *log pair* (or *log variety*). A *log pair* is a pair (X, B), where X is a normal variety and $B = \sum b_i D_i$, $b_i \in \mathbb{R}$, $0 \leqslant b_i \leqslant 1$, is an $\mathbb{R}$-*boundary*, or simply a *boundary*, and each factor D_i is a Weil prime divisor on X. A *log canonical divisor* is a divisor of the form $K + B$; it is also called an *adjoint divisor*. We note that the LMMP is the D-MMP, where for D one takes the adjoint divisor $K + B$.

Definition 1.6 ([34], § 1, [21], § 2). (i) A log pair (X, B) with an $\mathbb{R}$-boundary B is called a *log minimal model* if $K+B$ is nef (this assumes that $K+B$ is an $\mathbb{R}$-Cartier divisor and that X is complete) and (X, B) is lt (or the divisor $K+B$ is lt).

(ii) A log pair (X, B) is called a *log canonical model* if $K+B$ is numerically ample and (X, B) is lc. If the pair (X, B) has only cn singularities, it is called a *canonical model*.

(iii) A log pair (X, B) is called a *weakly log canonical model* if $K+B$ is nef and (X, B) is lc. If (X, B) has only trm singularities, then this pair is said to be *terminal* or *minimal*. A *weakly canonical model* has only cn singularities.

(iv) Let (X, B) be a weakly log canonical model. Suppose that the divisor $K+B$ is semi-ample (Conjecture 1.16 below about semi-ampleness claims that this condition always holds, and by 6-1-13 in [17], if B is a $\mathbb{Q}$-boundary, then the condition follows from the abundance conjecture ([17], 6-1-14)). The contraction $I(X, B)\colon X \to U$ in the definition of semi-ampleness of $K+B$ is called an *Iitaka fibration*, and its image U is called a (log canonical) *Iitaka model* of (X, B). Sometimes the image is denoted by X_{can}. The dimension $\dim U$ of the image is equal to the *numerical log Kodaira dimension* of (X, B) ([36], 2.4.4) and is denoted by $\nu(X, B)$. In the case of the $\mathbb{Q}$-boundary B this is the *log Kodaira dimension* $\kappa(X, B)$ of (X, B) (for generalizations, see Remark 1.7(iii) below).

(v) A proper contraction $\varphi\colon X \to S$ of normal varieties is called a *Mori fibration* if the following conditions hold:

a) $\dim S < \dim X$;
b) X has at most $\mathbb{Q}$-factorial terminal singularities;
c) $\rho(X/S) := \rho(X) - \rho(S) = 1$, where $\rho(\ )$ is the Picard number;
d) $-K$ is φ-ample.

The conditions a)–d) taken together mean that $\varphi\colon X \to S$ is an extremal K-contraction of fibred type, and hence an object of the Mori category. In the definition of a Mori log fibration of (X, B), the divisor $-K$ must be replaced in d) by the numerically φ-ample divisor $-(K+B)$, and in b) one must assume that the pair (X, B) satisfies the condition lt rather than trm.

Similarly, for a Fano log fibration one must assume the validity of the condition lc rather than lt, and neither the condition a) nor the $\mathbb{Q}$-factorial property in b) and c) are needed (it is even possible that φ is only a proper morphism rather than a contraction).

The relative versions of these notions can be defined in the standard way.

Remarks 1.7. (i) A (relative) log minimal model is also called a (*relative*) *log terminal model* ([34], English p. 100). The notion of (*relative*) *strictly log minimal*, or (*relative*) *strictly log terminal model*, is also in use if in addition X is $\mathbb{Q}$-factorial and (relatively) projective.

(ii) Of course, if $B = 0$, then the main conjecture of the Mori theory claims that either there is a projective $\mathbb{Q}$-factorial (terminal) minimal model (see Definition 1.9(ii)) or the initial variety is birationally equivalent to a Mori fibration (which we know is proved in dimensions $\leqslant 4$ ([17], [37])). Similar results in the log terminal category have been proved so far only in dimensions $\leqslant 3$.

(iii) Possible generalizations include varieties with *subboundaries*, that is, with $b_i \leqslant 1$, and possibly with negative multiplicities. One can encounter these varieties in auxiliary constructions, for example, in log adjunction ([34], § 3). For models whose singularities are worse than log canonical ones (quasi-log varieties and generalized log varieties), see [1] and [37].

Comment 1.8. The main categories in which we work are the category of canonical log pairs (X,B) and the category of log canonical log pairs together with their subcategories of terminal and log terminal log pairs, respectively. The rational 1-contractions are taken as morphisms, that is, modifications which do not blow up any divisor, or equivalently, the inverse modification does not contract any divisor. Categories of canonical and terminal log pairs (Mori) are well defined with respect to the MMP and LMMP with $B=0$, in the sense that they are closed under elementary transformations, that is, extremal divisorial log contractions and log flips, which are the divisorial contractions in subsection 1.1c) and the D-flips in subsection 1.1d) with $D_i=K+B$, respectively. The same holds if the divisorial log contractions do not contract any component of the boundary, for example, in the category of mobile log pairs if a boundary is composed of generic divisors of linear systems without base components. The following simple example shows that divisorial log contractions need not preserve the canonical property in general. Let $X=\mathbb{F}_n$, where $n\geqslant 1$, be a rational scroll and let $B=s_n$ be its exceptional section. Then the log pair $(\mathbb{F}_n,s_n)$ is canonical, and its unique elementary transformation is a divisorial contraction $(\mathbb{F}_n,s_n)\to(\mathbb{F}_n^*,0)$ onto a cone $\mathbb{F}_n^*$ of degree n which has a non-canonical (non-terminal) singularity at the vertex for $n\geqslant 3$ ($n\geqslant 2$).

This example also shows that it is important to have an appropriate definition of the image of a boundary under a birational transformation (modification) of log pairs. Of course, for a birational transformation $\chi\colon X\dashrightarrow Y$ the notion of birational transform χ_*B of B is standard. However, the notion of *log birational transform* $B_Y^{\log}:=\chi_*B+\sum E_i$, where E_i ranges over the exceptional prime divisors of the inverse transformation $\chi^{-1}\colon Y\dashrightarrow X$, is natural in the log canonical category. Actually, as has been noted more than once (see [21], 2.7, and [36], 1.1.2), there are many ways to define the image of a boundary.

Definition 1.9. Let $\chi\colon(X,B)\dashrightarrow(Y,B_Y^{\log})$ be a modification of complete normal varieties X and Y, where $B_Y^{\log}=B_Y+\sum E_i$ is the log birational transform of the $\mathbb{R}$-boundary $B=\sum b_iD_i$, $B_Y=\chi_*B$ is the birational transform of B, and each E_i is a prime divisor on Y blown up by χ, that is, an exceptional divisor of χ^{-1}. One can assume that $K+B$ is not necessarily lc, and it need not be $\mathbb{R}$-Cartier.

(i) A log pair $(Y,B_Y^{\log})$ is called a *weakly log canonical model* of (X,B) if $(Y,B_Y^{\log})$ is a weakly log canonical model and $a(D_i,Y,B_Y^{\log})\geqslant a(D_i,X,B)=-b_i$ for all χ-exceptional prime divisors D_i of X. A log pair (Y,B_Y) is called a *weakly canonical model* of (X,B) if the log birational transform $B_Y^{\log}$ in the definition is replaced by the birational transform B_Y and the property of being a weak log canonical model is replaced by the corresponding weak canonical property.

(ii) A log pair $(Y,B_Y^{\log})$ is called a *log minimal model* of (X,B) if $(Y,B_Y^{\log})$ is a log minimal model and the inequalities for discrepancies in (i) become strict for any χ-exceptional prime divisors D_i of X. A log pair (Y,B_Y) is called a *minimal model* of (X,B) if the log birational transform $B_Y^{\log}$ in the definition is replaced by the birational transform B_Y and the log minimal model property is replaced by the minimal model property.

(iii) A log pair $(Y, B_Y^{\log})$ is called a *log canonical model* of (X, B) if $(Y, B_Y^{\log})$ is a log canonical model and the inequalities for discrepancies in (i) are satisfied. A *canonical model* (Y, B_Y) of (X, B) can be defined like a weakly canonical model with the weak canonical property replaced by the canonical property.

The relative versions of these notions can be defined in the standard way. Here a modification of log pairs must be replaced by a modification

$$\begin{tikzcd} (X,B) \arrow[rr, dashed, "\chi"] \arrow[dr, "f"'] & & (Y, B_Y^{\log}) \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

of proper morphisms $f\colon X \to Z$ and $g\colon Y \to Z$ or proper relative log pairs $f\colon (X, B) \to Z$ and $g\colon (Y, B_Y^{\log}) \to Z$. For example, a morphism $g\colon (Y, B_Y^{\log}) \to Z$ is called a *relative weakly log canonical model* of $f\colon (X, B) \to Z$ if g is a relative weakly log canonical model and if $a(D_i, Y, B_Y^{\log}) \geqslant a(D_i, X, B) = -b_i$ for any χ-exceptional prime divisors D_i of X.

Remarks 1.10. (i) In Kollár's version of the definition of a log minimal model the inequalities in part (ii) of Definition 1.9 are not strict ([21], (2.15.1)). Under his definition, contractions of reduced components of a boundary are possible if they preserve the log minimal property. For example, if a (-1)-curve D_0 on a non-singular surface X intersects two non-singular curves D_1 and D_2 transversally and simply and if $D_1 \cap D_2 = \varnothing$, then the contraction $\chi\colon X \to Y$ of D_0 gives an ordinary double point of the image $B_Y = \chi_* B$, where $B = \sum D_i$. The model $(Y, B_Y = B_Y^{\log}) \to Y$ of $(X, B) \to Y$ is log minimal in Kollár's sense over a neighbourhood of the point $\chi(D_0) \in Y$ but it is not log minimal in the sense of Definition 1.9(ii).

Similar inequalities hold for minimal models automatically, and these inequalities are always strict, $a(D_i, Y, B_Y) > 0 \geqslant -b_i$, because B is a boundary.

(ii) Actually (as we shall see below), if $g\colon (Y, B_Y^{\log}) \to Z$ is a relative weakly log canonical model of a proper morphism $f\colon (X, B) \to Z$ with an $\mathbb{R}$-Cartier divisor $K + B$, then the inequality $a(F, Y, B_Y^{\log}) \geqslant a(F, X, B)$ holds for all prime b-divisors F of X. The inequality is strict if $(Y, B_Y^{\log}) \to Z$ is a relative log canonical model, and the modification χ is not defined at a generic point of $\mathrm{c}_X F$.

Similar inequalities hold for canonical models of pairs.

(iii) The main problem of the LMMP is to show that a resulting model exists for any proper morphism $f\colon X \to Y$. Satisfactory results have been obtained so far only for $\dim X \leqslant 3$. Some general results, including the cone theorem, non-vanishing theorem, base point free theorem, and contraction theorem have been established in any dimension (see [17], [33], [21], [22]). The existence problems for log flips in dimensions $\geqslant 5$, for log termination in dimensions $\geqslant 4$ (for details, see [37], § 1, and [39]), and for semi-ampleness ([36], 2.6) in dimensions $\geqslant 4$ are still open.

We state now the main conjectures of the LMMP.

Conjecture 1.11 (existence of log flips). *Let (X,B) be an* lt *log pair with $\mathbb{Q}$-factorial object X, and let $\varphi\colon X\to Y$ be an extremal small birational $(K+B)$-contraction. Then it admits a log flip, in the sense that there is a diagram of the form* (1.1) *with the properties* (i)–(iii) *for the divisors $D_i=K+B$, $D_i^+=K_{X_i^+}+B^+$, and $B^+=g_{i*}B$.*

As noted in 1.7(ii) and 1.10(iii), the conjecture has been proved for dimensions $\dim X\leqslant 4$ (see [37], Corollary 1.8 and 1.12, History).

Conjecture 1.12 (termination of log flips). *Under the assumptions of Conjecture* 1.11, *any chain of log flips* (1.2) *for the divisor $K+B$ terminates*:

$$\begin{tikzcd}[column sep=small]
(X,B)=(X_0,B_0) \arrow[r, dashed] \arrow[dr] & (X_0^+,B_0^+)=(X_1,B_1) \arrow[d] \arrow[r, dashed] \arrow[dr] & (X_1^+,B_1^+)=(X_2,B_2) \arrow[d] \arrow[r, dashed] \arrow[dr] & \cdots . \arrow[d] \\
& Y=Y_0 & Y_1 & Y_2
\end{tikzcd} \tag{1.2}$$

This conjecture has been proved for $\dim X\leqslant 3$ (see [33], (2.17), [12], and [36], 5.2).

Conditional Theorem 1.13 (assuming the validity of Conjectures 1.11 and 1.12). *For any log pair (X,B) and, more generally, in the relative situation for a proper morphism $X\to Z$ there is a modification*

$$\begin{tikzcd}
(X,B) \arrow[rr, dashed] \arrow[dr] & & (Y,B_Y^{\log}) \arrow[dl] \\
& Z &
\end{tikzcd},$$

where the pair $(Y,B_Y^{\log})\to Z$ is projective over Z, is $\mathbb{Q}$-factorial, and is either a log minimal model of $(X,B)\to Z$ or has the structure of a Mori log fibration $(Y,B_Y^{\log})\to S$ over Z.

Proof. After a log resolution, one can assume that the log pair (X,B) is as in Conjecture 1.11. Then we can apply the LMMP. For details, see [25], 11-1-4, [17], Introduction, and [36], § 5 (for the generalization to the case of log canonical pairs (X,B) and $\mathbb{R}$-boundaries B).

Conjecture 1.14 (existence of an effective log canonical divisor). *Let $(Y,B_Y)\to Z$ be a weakly log canonical model with a $\mathbb{Q}$-boundary B_Y. Then the relation $|m(K_Y+B_Y)|\neq\varnothing$ holds over Z for some positive integers $m\gg 0$, that is, $\kappa(Y/Z,B_Y)\geqslant 0$.*

This conjecture was proved in [19] for $\dim X\leqslant 3$. In dimensions $\geqslant 4$ the problem remains open.

Conditional Theorem 1.15 (characterization in terms of log Kodaira dimension if Conjectures 1.11, 1.12, and 1.14 are true). *A proper* lc *log pair $(X,B)\to Z$ with a $\mathbb{Q}$-boundary B_Y has a log minimal model if and only if $\kappa(X/Z,B)\geqslant 0$, and it has a Mori log fibration if and only if $\kappa(X/Z,B)=-\infty$.*

At present this theorem has been proved unconditionally only for $\dim X\leqslant 3$ (see [19] and [36], 2.4, 2.7, and also [13] and [21], §§ 11–14).

Conjecture 1.16 (semi-ampleness). *Let $(Y, B_Y) \to Z$ be a weakly log canonical model. Then $K_Y + B_Y$ is semi-ample over Z. For a $\mathbb{Q}$-boundary B_Y this is equivalent to the condition that the linear systems $|m(K_Y + B_Y)|$ are free over Z for some positive integers $m \gg 0$.*

Again as in Conjecture 1.14, this conjecture has been proved at present only for $\dim X \leqslant 3$ (see [19] and [36], 2.7, and also [13] and [21], §§ 11–14).

Remarks 1.17. (i) Conjecture 1.12 is also proved in dimension 4 ([39], Example 9) in the case of cn singularities. Thus, to complete the MMP for $\dim X = 4$, it remains to prove Conjectures 1.14 and 1.16 in this dimension.

(ii) In the case of a $\mathbb{Q}$-boundary B_Y if Conjecture 1.16 holds, then for some positive integers $m \gg 0$ the linear system $|m(K_Y + B_Y)|$ gives an Iitaka fibration $I(Y, B_Y)\colon Y \to Y_{\text{can}}$ onto a normal projective variety Y_{can} which can be characterized by the following property:

$$I(Y, B_Y)(C) = \text{pt.} \iff (K_Y + B_Y)C = 0$$

for each curve $C \subset Y$. By definition, $\kappa(Y, B_Y) = \dim Y_{\text{can}}$ and

$$Y_{\text{can}} \simeq \operatorname{Proj} \bigoplus_{m \geqslant 0} H^0(Y, m(K_Y + B_Y)).$$

The proofs of Theorem 1.13, Theorem 1.15, and Conjecture 1.16 with $B_Y = B_Y^{\log}$ imply that the *log canonical* ring

$$\mathcal{R}(X, B) = \mathcal{R}_X(K + B) = \bigoplus_{m \geqslant 0} H^0(X, m(K + B)) = \bigoplus_{m \geqslant 0} H^0(Y, m(K_Y + B_Y^{\log}))$$

is finitely generated for any log pair (X, B) satisfying the conditions in Theorem 1.15 (if $\kappa(X, B) = -\infty$, then the ring $\mathcal{R}(X, B)$ is the base field k). This ring is an important birational invariant of (X, B).

(iii) If Conjectures 1.11 and 1.12 are true, then a geometric criterion (whose verification is, however, difficult) for the existence of a birational structure of a Mori log fibration on (X, B) is as follows: there is an open non-empty subset $U \subset X$ such that a curve C_x with $(K + B)C_x < 0$ passes through each point $x \in U$.

We discuss below some properties of log models when they exist.

§ 2. Canonical and log canonical models

2.1. Canonical singularities are singularities occurring on canonical models of varieties of general type. Let X be a complete non-singular algebraic variety of general type (that is, of Kodaira dimension $\kappa(X) = \dim X$) and let

$$\mathcal{R}(X) = \mathcal{R}_X(K) = \bigoplus_{m \geqslant 0} H^0(X, mK)$$

be the canonical ring of X with natural multiplication of sections. A canonical model X_{can} of X exists if and only if $\mathcal{R}(X)$ is finitely generated as a

$k(= H^0(X, \mathcal{O}_X))$-algebra, in which case $X_{\mathrm{can}} = \operatorname{Proj}\mathcal{R}(X)$. This implies that a canonical model is unique in its birational class. It is also clear that a model of this kind is a $\mathbb{Q}$-Gorenstein model, or $K_{X_{\mathrm{can}}}$ is a $\mathbb{Q}$-Cartier divisor, that is, $nK_{X_{\mathrm{can}}}$ is a Cartier divisor for some integer $n \neq 0$. As is well known, in dimension 2 the singularities of X_{can} are exactly the rational double points (or *Du Val singularities*, as they are often called).

Canonical singularities in any dimension were defined by Reid [30]. He showed there that these very singularities appear on the canonical models of varieties of general type.

In the log category the ampleness of a log canonical divisor $K + B$ can be generalized to numerical ampleness, which coincides with ordinary ampleness for $\mathbb{Q}$-Cartier divisors. Correspondingly, the canonical property for singularities can be generalized to the log canonical property.

Our nearest goal is to study the uniqueness problem for log canonical and canonical models, in particular, for *mobile log pairs*, that is, log pairs (X, H) where H is a generic divisor in $\sum r_i\mathcal{M}_i$ with $r_i \in [0, 1)$ and $\mathcal{M}_i$ is a linear system without base components for any i; it is also usually assumed that $H \neq 0$, which is equivalent to some linear system $\mathcal{M}_i$ with $r_i > 0$ being mobile (see [4]). We begin with the following statement.

Lemma 2.2 (negativity ([34], 1.1, and [21], 2.19)). *Let $f\colon Y \to X$ be a birational contraction with normal Y and with prime f-exceptional divisors $E_i \subset Y$. Suppose that the relative numerical f-equivalence of $\mathbb{R}$-divisors*

$$\sum c_iE_i \equiv N + G$$

holds, where N is f-nef and G is effective and has no f-exceptional components. Then $c_i \leqslant 0$ for each i, and $c_i < 0$ if N is not numerically trivial on $f^{-1}f(E_i)$ over X.

Proof. This fact is well known (see, for example, [34], English p. 97, [21], 2.19, and [25], 13-1-4). The main idea is to reduce the proof to the surface case and then use the lemma claiming that the intersection form is negative definite on components of contracted curves.

The following assertion is the main result of this section.

Theorem 2.3 (uniqueness of canonical and log canonical models). (i) *A canonical model of a pair* (*in particular*, *of a mobile log pair*) *is unique if it exists.*
(ii) *A log canonical model of any log pair is unique if it exists.*

Proof. The proof is similar for (i) and (ii) and uses the negativity lemma. We note that, by definition, any canonical model of a mobile log pair $(X, \sum r_i\mathcal{M}_i)$ is a canonical model of (X, H) for a generic divisor $H \in \sum r_i\mathcal{M}_i$; the mobile pair is uniquely determined by the generic divisor H. To simplify the notation, we omit the base variety Z over which the pairs are considered; as usual, $Z =$ pt. (a point) in the global case.

Suppose that there are two canonical models $(Y_1, B_1 = B_{Y_1})$ and $(Y_2, B_2 = B_{Y_2})$ of a pair (X, B). Let us consider a Hironaka hut resolving the modification of models $\chi\colon (Y_1, B_1) \dashrightarrow (Y_2, B_2)$,

$$\begin{tikzcd} & (W, B_W) \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ (Y_1, B_1) \arrow[rr, dashed, "\chi"] & & (Y_2, B_2) \end{tikzcd}, \tag{2.1}$$

where (W, B_W) is a simultaneous log resolution of (Y_1, B_1) and (Y_2, B_2), and α and β are the corresponding birational contractions. For an appropriate choice of canonical divisors K_W, K_{Y_1}, and K_{Y_2} one can write

$$K_W + B_W = \alpha^*(K_{Y_1} + B_1) + \sum c_i E_i = \beta^*(K_{Y_2} + B_2) + \sum c_i' E_i', \tag{2.2}$$

where each E_i (E_i') is an exceptional prime divisor of α (of β).

We claim that c_i, $c_i' \geqslant 0$ for all i. By definition, B_1 and B_W are birational transforms of $B = \sum b_i D_i$. Hence, $B_1 = \alpha_* B_W$ and $B_W = (B_1)_W + \sum b_i E_i$, where $(B_1)_W$ is a birational transform of B_1, and

$$b_i = \begin{cases} b_i & \text{if } E_i \text{ is a birational transform of } D_i, \\ 0 & \text{otherwise} \end{cases}$$

(the value 0 must be replaced by 1 in the log category case). Therefore, for any exceptional divisor E_i we have

$$c_i = a(E_i, Y_1, B_{Y_1}) + b_i \geqslant 0$$

by the canonical model property. The same proof works for c_i'.

The formula (2.2) implies the equality

$$\sum (c_i' - c_i) E_i' = \alpha^*(K_{Y_1} + B_1) - \beta^*(K_{Y_2} + B_2) + \sum c_i E_i,$$

where the first sum is taken over the exceptional divisors of β and the second sum over the exceptional divisors of α that are not exceptional for β. In particular, the last sum is effective because $c_i \geqslant 0$. On the other hand, $\alpha^*(K_{Y_1}+B_1) - \beta^*(K_{Y_2}+B_2)$ is β-nef, because $\alpha^*(K_{Y_1}+B_1)$ and $K_{Y_1}+B_1$ are nef and $\beta^*(K_{Y_2}+B_2)$ is numerically β-trivial. Thus, by Lemma 2.2, $c_i \geqslant c_i'$ for each β-exceptional divisor E_i', and since $c_i \geqslant 0$, the same holds for each E_i'. Similarly, $c_i' \geqslant c_i$. Hence, $c_i = c_i'$ everywhere.

Applying (2.2) again, we see that

$$\alpha^*(K_{Y_1} + B_1) = \beta^*(K_{Y_2} + B_2) \tag{2.3}$$

and $\chi\colon (Y_1, B_1) \overset{\sim}{\to} (Y_2, B_2)$ is an isomorphism of the models by the numerical ampleness of $K_{Y_1} + B_1$ and $K_{Y_2} + B_2$. Indeed, if χ has indeterminacy points, then there is a point on Y_1 with image of positive dimension on Y_2. Hence, there is a curve C on W such that $\beta(C) \subset Y_2$ is a curve and $\alpha(C) \subset Y_1$ is an indeterminacy point. However, this is impossible by (2.3), which proves the assertion (i) of the theorem (one can prove (ii) similarly; see the proof of Proposition 2.4 in [36]).

Similarly, Lemma 2.2 implies the following monotonicity (see 1.5 in [34] and § 2 in [21]).

Lemma 2.4. *Let $f\colon (X,B)\to Z$ be a proper relative log pair such that $K+B$ is a $\mathbb{R}$-Cartier divisor, and let $g\colon (Y,B_Y^{\log})\to Z$ be a weakly log canonical model of (X,B). Then*

$$a(E,Y,B_Y^{\log})\geqslant a(E,X,B)$$

for any prime b-divisor E of X (equivalently, of Y). Moreover, if g is a log canonical model of f (in particular, $K_Y+B_Y^{\log}$ is ample over Z) and the modification $\chi\colon X\dashrightarrow Y$ is undefined at a generic point of $\mathrm{c}_X\,E$, then

$$a(E,Y,B_Y^{\log})>a(E,X,B).$$

Proof ([34], (1.5.6), [21], (2.23.3)). The proof depends on the exceptionalness of the divisor E on X.

Case I. If a b-divisor E is not exceptional on X, then the desired inequality follows from the definition of weakly log canonical model (see 1.9(i)). (Here the modification is defined at a generic point of $\mathrm{c}_X\,E=E$.)

Case II. Suppose now that a b-divisor E is exceptional on X. Let us consider the divisorial resolution of E on a variety W, as in the diagram (2.1) with $(Y_1,B_1)=(X,B)$ and $(Y_2,B_2)=(Y,B_Y^{\log})$. Then

$$\alpha^*(K+B)+\sum a(D_i,X,B)D_i=K_W=\beta^*(K_Y+B_Y^{\log})+\sum a(D_i,Y,B_Y^{\log})D_i$$

by the definition of discrepancy. Breaking up the sum over all divisors D_i on W into the sum over exceptional divisors E_i and the sum over non-exceptional divisors D_i on X, we obtain

$$\begin{aligned}&\sum(a(E_i,X,B)-a(E_i,Y,B_Y^{\log}))E_i\\&\qquad=\beta^*(K_Y+B_Y^{\log})-\alpha^*(K+B)+\sum(a(D_i,Y,B_Y^{\log})-a(D_i,X,B))D_i,\end{aligned}$$

in which the divisor $\beta^*(K_Y+B_Y^{\log})-\alpha^*(K+B)$ is nef over X, since $\beta^*(K_Y+B_Y^{\log})$ is nef over Z and $\alpha^*(K+B)$ is numerically trivial over X. On the other hand, the divisor $\sum(a(D_i,Y,B_Y^{\log})-a(D_i,X,B))D_i$ is effective by Case I. Hence, by Lemma 2.2, the divisor $\sum(a(E_i,Y,B_Y^{\log})-a(E_i,X,B))E_i$ is effective, and the desired inequality holds because E can be identified with one of the exceptional divisors E_i.

Suppose in addition that g is a log canonical model of f and that the modification χ is not defined at a generic point of $\mathrm{c}_X\,E$. Then $\beta^*(K_Y+B_Y^{\log})$ is not numerically trivial on $\alpha^{-1}\,\mathrm{c}_X\,E$ by the ampleness of $K_Y+B_Y^{\log}$ over Z. Thus, the desired strict inequality follows from Lemma 2.2 again.

Proposition 2.5. *Let $g\colon (Y,B_Y^{\log})\to Z$ be a weakly log canonical model of a proper morphism $f\colon (X,B)\to Z$ and let $\chi\colon X\dashrightarrow Y$ be the corresponding modification. Then the following assertions hold.*

(i) *The morphism $f\colon (X,B)\to Z$ has neither a Mori log fibration nor a Fano log fibration. The singularities of (X,B) are of no importance here, and it is only* *required that the divisor $K+B$ have the $\mathbb{R}$-Cartier property and that the generic fibre of the Fano log fibration not be* 0*-dimensional.*

(ii) *If (X,B) is lc and the model g is log canonical, then χ^{-1} has no exceptional divisor.*

(iii) *If $g_{\mathrm{can}}\colon H_{\mathrm{can}}\to Z$ is a log canonical Iitaka model of some weakly log canonical model $(Y_1, B_1^{\log}=B_{Y_1}^{\log})\to Z$ of f, then there is a unique morphism $\rho\colon Y\to H_{\mathrm{can}}$ over Z, and $\rho=I(Y/Z,B_Y^{\log})$ is also an Iitaka fibration, so that the Iitaka model is unique as well.*

(iv) *Moreover, a log canonical Iitaka model $H_{\mathrm{can}}\to Z$ exists if and only if $K_Y+B_Y^{\log}$ is semi-ample over Z. In addition, $H_{\mathrm{can}}\to Z$ is a relative log canonical model of f if and only if $K_Y+B_Y^{\log}$ is big over Z.*

(v) *If $K+B$ is an lc $\mathbb{Q}$-Cartier divisor which is big over Z, then a log canonical model, as well as an Iitaka model $g_{\mathrm{can}}\colon H_{\mathrm{can}}\to Z$, exists if and only if the algebra of $f_*\mathcal{O}_X$-modules*

$$\bigoplus_{m\geqslant 0} f_*\mathcal{O}_X(m(K+B))$$

is finitely generated; *if this condition holds*, *then*

$$H_{\mathrm{can}}=\operatorname{Proj}\bigoplus_{m\geqslant 0} f_*\mathcal{O}_X(m(K+B)).$$

The two models are isomorphic if they exist.

Similar statements hold without log as well. Moreover, *the following assertions hold.*

(vi) *Any two minimal models of f are isomorphic in codimension* 1.

(vii) *Any strictly minimal model which is simultaneously a canonical model is isomorphic to any weakly canonical model of f.*

One can replace the pair (X,B) in Lemma 2.4 and in all statements of the proposition by any other log pair $(Y',B_{Y'}^{\log})$ and by $(Y',B_{Y'})$ in the case without log (under the same assumptions). For instance, this simplifies the proof of (iii) below.

Proof. (i) We consider the diagram (2.1) with $(Y_1,B_1)=(X,B)$ and $(Y_2,B_2)=(Y,B_Y^{\log})$. Let $(X,B)\to S\to Z$ be a relative Mori or Fano log fibration with a contraction $\varphi\colon X\to S$ over Z. By definition and by our assumptions, we have the inequality $\dim S<\dim X$, and $-(K+B)$ is numerically ample over S. Let us show that this is impossible.

By the monotonicity lemma (Lemma 2.4) we have

$$\alpha^*(K+B)-\beta^*(K_Y+B_Y^{\log})=\sum(a(D_i,Y,B_Y^{\log})-a(D_i,X,B))D_i\geqslant 0,$$

since $(Y,B_Y^{\log})\to Z$ is a weakly log canonical model. Therefore, this effective divisor non-negatively intersects sufficiently general curves, for instance, those in fibres of $\varphi\circ\alpha$. However, by construction the divisor $\alpha^*(K+B)-\beta^*(K_Y+B_Y^{\log})$ intersects such curves negatively, a contradiction.

(ii) We use the notation and the result of Lemma 2.4. Let E be a χ^{-1}-exceptional divisor. Then $a(E, X, B) < a(E, Y, B_Y^{\log}) = -1$ by the lemma (the last equality holds by the definition of log birational transform). However, this contradicts the lc property of (X, B).

(iii) In general, each of the log pairs $(Y_1, B_1^{\log} = B_{Y_1}^{\log}) \to Z$ and $(Y, B_Y^{\log}) \to Z$ need not be a weakly log canonical model of the other. Indeed, a prime component D_i of B can be non-exceptional on Y and exceptional on Y_1. If $b_i < 1$ in addition, then $B_Y^{\log} \neq (B_1^{\log})_Y^{\log}$. However, each of the log pairs $(Y_1, B_1^{\log}) \to Z$ and $(Y, B_Y^{\log}) \to Z$ is a weakly log canonical model of the other for an appropriate theory of boundary transforms (see the remark on boundary transforms after Comment 1.8), and Lemma 2.4 holds in this more general situation. This generalization of Lemma 2.4 needs a correct generalization of the inequalities in Definition 1.9(ii) to a weakly log canonical model. Thus, as in Lemma 2.4, we always have $a(E, Y_1, B_1^{\log}) = a(E, Y, B_Y^{\log})$ (for details, see the proof of 2.4.2 in [36]). As in the proof of Theorem 2.3, this implies that $\rho = I(Y, B_Y^{\log})\colon Y \dashrightarrow H_{\mathrm{can}}$ is a morphism given by $K_Y + B_Y^{\log}$ (cf. 2.4.3 in [36]).

(iv) If $K_Y + B_Y^{\log}$ is semi-ample over Z, then one can find a contraction

$$I = I(Y/Z, B_Y^{\log})\colon Y \to H_{\mathrm{can}}$$

over Z onto a normal variety H_{can} and an $\mathbb{R}$-divisor H on H_{can} numerically ample over Z such that $I^*H \sim_{\mathbb{R}} K_Y + B_Y^{\log}$ over Z. This is a log canonical Iitaka model. The converse holds by definition. The Iitaka fibration I is birational if and only if $K_Y + B_Y^{\log}$ is big, in which case the model $(H_{\mathrm{can}}, B_{H_{\mathrm{can}}}^{\log}) \to Z$ is log canonical. The birational contraction I is crepant, since $K_Y + B_Y^{\log}$ is numerically trivial over Z, and hence $K_{H_{\mathrm{can}}} + B_{H_{\mathrm{can}}}^{\log}$ is lc. The property of numerical triviality also implies that $K_Y + B_Y^{\log} = I^*(K_{H_{\mathrm{can}}} + B_{H_{\mathrm{can}}}^{\log})$ and that $K_{H_{\mathrm{can}}} + B_{H_{\mathrm{can}}}^{\log} \sim_{\mathbb{R}} H$ is ample. Conversely, a log canonical model is an Iitaka model by (iii) with $Y_1 = H_{\mathrm{can}}$.

(v) The statements concerning log canonical models use a stabilization of the mobile part of pluri-log canonical divisors whenever the condition of finite generation is assumed (cf. the limiting criterion 5.21 below; for details, see the proof of 1.2(II) in [30] for canonical divisors, and 3.18 in [37] for the general case). In addition, if $(Y, B_Y^{\log}) \to Z$ is a log canonical model, then $g_{\mathrm{can}} = g\colon H_{\mathrm{can}} = Y \to Z$ is the Iitaka model of it by (iii)–(iv). This implies the existence of both models if the pluri-log canonical algebra is finitely generated. Suppose now that H_{can} exists (that is, suppose that a log canonical model exists). Then by (iv), $K_Y + B_Y^{\log}$ is a semi-ample $\mathbb{Q}$-divisor. The composition $I \circ \chi\colon X \dashrightarrow Y \to H_{\mathrm{can}}$ induces natural quasi-isomorphisms of graded algebras (see the paragraph before Proposition 5.4 and [37], 4.3) and isomorphisms of their projective spectra,

$$\begin{aligned} H_{\mathrm{can}} &= \operatorname{Proj} \bigoplus_{m \geqslant 0} g_{\mathrm{can}*} \mathcal{O}_{H_{\mathrm{can}}}(mH) \\ &= \operatorname{Proj} \bigoplus_{m \geqslant 0} g_* \mathcal{O}_Y(m(K_Y + B_Y^{\log})) = \operatorname{Proj} \bigoplus_{m \geqslant 0} f_* \mathcal{O}_X(m(K + B)), \end{aligned}$$

where H now stands for a $\mathbb{Q}$-divisor in (iv) and the last isomorphism uses the lc property of (X, B); if the middle algebra is finitely generated here, then so is the last algebra.

(vi) Let $(Y_1, B_1 = B_{Y_1}) \to Z$ and $(Y_2, B_2 = B_{Y_2}) \to Z$ be minimal models of f. Suppose that the modification $\chi\colon Y_1 \to Y_2$ contracts a divisor $E \subset Y_1$. Then the relation $a(E, Y_2, B_2) \leqslant a(E, Y_1, B_1) = -\operatorname{mult}_E B_1 \leqslant 0$ holds by the version of Lemma 2.4 without log, and this contradicts the minimal model property of (Y_2, B_2). Similarly, the inverse modification χ^{-1} does not contract divisors.

(vii) This assertion can be proved in a similar way.

Comment 2.6. The method of proving Theorem 2.3 (which uses the negativity lemma, Lemma 2.2) recalls the method of proving the classical Noether–Fano inequality in the birational theory of Mori fibrations. We can claim that a linear system giving a non-trivial birational map (that is, a map which is not an isomorphism) between Mori fibrations must have a base locus of sufficiently high multiplicity, that is, a *maximal singularity*. Conversely, the condition that a linear system has no maximal singularity can always be interpreted as the canonical property of singularities of an appropriate pair. If the pair is in addition a weakly canonical model, then the birational map given by the linear system is an isomorphism. In this phrasing the Noether–Fano inequality is used as a criterion for termination of the algorithm for factorizing birational maps into links and can be interpreted as the uniqueness of a weakly canonical model in that situation.

Let us take a birational map between two Mori fibrations (the map need not preserve the fibration in general),

$$\mathcal{H} \subset |-\mu K^s + \varphi^* A| \quad \begin{array}{ccc} X & \overset{\chi}{\dashrightarrow} & X' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ S & & S' \end{array} \quad \mathcal{H}' = \left|-\mu' K_{X'} + \varphi'^* A'\right|, \tag{2.4}$$

where the varieties X and S' are projective, $\mathcal{H}'$ is a complete very ample linear system, μ' belongs to $\mathbb{Z}_{>0}$, $A' \in \operatorname{Div}(S')$ is a very ample divisor on S', $\mathcal{H} = \chi_*^{-1}\mathcal{H}'$ is the birational transform of $\mathcal{H}'$ on X, μ belongs to $\mathbb{Q}_{>0}$, K^s is a semicanonical divisor, that is, a canonical divisor up to $\mathbb{Q}$-linear equivalence (if X is non-singular on a generic fibre of φ and the Picard group of the generic fibre is generated by a canonical divisor, then $\mu \in \mathbb{Z}_{>0}$, and K^s can be replaced by K), and $A \in \operatorname{Div}(S) \otimes \mathbb{Q}$ is a $\mathbb{Q}$-divisor. Let us consider two (mobile) log pairs $(X, \frac{1}{\mu}H)$ and $(X', \frac{1}{\mu'}H')$, where $H \in \mathcal{H}$ and $H' = H_{X'} \in \mathcal{H}'$ are generic divisors of the linear systems without base components. By our choice of $\mathcal{H}'$, the pair $(X', \frac{1}{\mu'}H')$ is a minimal model of $(X, \frac{1}{\mu'}H)$ for $\mu' \geqslant 2$. More precisely, for any exceptional prime divisors E of any resolution W (and any closed subset of X') the discrepancies satisfy the relation $a(E, X', H') = a(E, X', 0) > 0$ (and the multiplicity of H' at each generic point of the closed subset is 0) for a sufficiently general divisor H' in a free linear system (the choice of H' depends on a resolution and a closed subset); however, $(X', 0)$ has only terminal singularities. This implies that the pair $(X', \frac{1}{\mu'}H')$ is trm and $H' = H_{X'}$. In addition, $K_{X'} + \frac{1}{\mu'}H' \sim_{\mathbb{Q}} K_{X'} - K_{X'} + \frac{1}{\mu'}\varphi'^* A' = \frac{1}{\mu'}\varphi'^* A'$ is nef on X' since A' is ample on S', and $H' \neq 0$.

For a pair (X, H), its *canonical threshold* is defined as

$$c(X, H) := \max\{t \in \mathbb{Q} \mid K + tH \text{ is cn}\}.$$

We recall that the cn condition means that $\operatorname{dis}(X, tH) \geqslant 0$, that is, the discrepancies $a(E, X, tH)$ are non-negative for each prime b-divisor E of X with $\mathrm{c}_X E$ of codimension $\geqslant 2$. If $H > 0$ is integral, then the threshold $c(X, H)$ is $\leqslant 1$, and its value $c(X, H) < 1$ can be computed explicitly as follows. Let $\alpha\colon W \to X$ be a sufficiently non-trivial log resolution of (X, H) and let the prime divisors E_i be exceptional for α. In this case

$$\begin{aligned} K_W &= \alpha^* K + \sum a_i E_i, \quad a_i > 0, \quad \text{since } X \text{ is trm},\\ H_W &= \alpha^* H - \sum b_i E_i, \quad b_i \geqslant 0, \quad \text{since } H \text{ is effective}. \end{aligned}$$

The canonical property of (X, cH) means that the $a_i - cb_i$ are all non-negative. Hence,

$$c = c(X, H) = \min\left\{\frac{a_i}{b_i}\right\}.$$

The inverse value $\lambda = \frac{1}{c}$ is called the *maximal multiplicity* of the linear system $\mathcal{H}$ on X, and the divisor E_i corresponding to the minimum value is referred to as a *maximal singularity* (the divisor $\mathrm{c}_X E_i$ is also frequently called the maximal singularity). We say that $\mathcal{H}$ *has no maximal singularity on* X if $\lambda \leqslant \mu$, that is, if the pair $(X, \frac{1}{\mu} H)$ is cn.

Theorem 2.7 (Noether–Fano inequality). *In the above notation if* $(X, \frac{1}{\mu} H)$ *is a weakly canonical model* (*in particular, the linear system* $\mathcal{H}$ *on* X *has no maximal singularity*), *then* χ *is an isomorphism.*

If $S = \mathrm{pt.}$ *is a point, then this amounts to asserting that if* χ *is not an isomorphism, then*

$$c(X, H) < \frac{1}{\mu}$$

(*that is,* $\mathcal{H}$ *has a maximal singularity*).

Proof (cf. 1.3 in Chapter II of [8] and [25]). By our construction and by the assumption of the theorem we have $1/\mu \leqslant 1$, and $(X, \frac{1}{\mu} H)$ is a weakly canonical model of the pair $(X', \frac{1}{\mu} H')$ (which is cn and even lt). In particular, the latter pair is not a Mori log fibration (according to (i) of Proposition 2.5 applied to the contraction $X' \to S'$). Hence $1/\mu \geqslant 1/\mu'$, since $1/\mu'$ is the *anticanonical threshold*, that is, the least positive number a such that $K_{X'} + aH'$ is nef over S'. (However, in birational geometry, the inverse value $\mu' = 1/a$, the so-called *quasi-effective threshold*, is used more frequently; in our situation, this is a rational number μ' such that $\mu' K_{X'} + H' \equiv 0$ over S'.) As noted above, $(X', \frac{1}{\mu'} H')$ is a weakly canonical model of the (cn) pair $(X, \frac{1}{\mu'} H)$. Therefore, the inequality $1/\mu' \geqslant 1/\mu$ holds for the same reasons, and thus $1/\mu = 1/\mu'$, or $\mu = \mu'$.

We now show that χ is an isomorphism in codimension 1 (more precisely, a log flop). Since the model $(X', \frac{1}{\mu'}H')$ is minimal, no divisor is contracted by χ (according to the proof of (vi) in Proposition 2.5). By construction, $\varphi' = I(X', \frac{1}{\mu'}H')$ is an Iitaka fibration, and S' is an Iitaka model of $(X', \frac{1}{\mu'}H')$ for a sufficiently ample twist by A'. By the version of (iii) of Proposition 2.5 without log, this implies that the composition $\varphi' \circ \chi \colon X \to S'$ is a morphism (and an Iitaka fibration as well). Since no divisor is contracted by χ, it follows that $\rho(X/S') \leqslant \rho(X'/S') = 1$. Moreover, equality holds here, and thus no divisor is contracted by χ^{-1}. Indeed, $\rho(X/S') \geqslant 1$, since H positively intersects curves in fibres of φ which are also curves over S'. The divisor H is φ-ample because X is projective.

Therefore, $\mu = \mu'$, and $(X, \frac{1}{\mu}H)$ and $(X', \frac{1}{\mu}H')$ are minimal models of each other (cf. (vi) of Proposition 2.5).

We can complete the proof in two ways: we can use the fact that χ is an isomorphism in codimension 1 which transforms a $\varphi' \circ \chi$-ample divisor H into a φ'-ample divisor H', and we can note the trm property of the minimal models $(X, \frac{1}{\mu}H)$ and $(X', \frac{1}{\mu}H')$. Let us choose an ε with $0 < \varepsilon \ll 1$ such that the models $(X', (\frac{1}{\mu}+\varepsilon)H')$ and $(X, (\frac{1}{\mu} + \varepsilon)H)$ are canonical. Then Theorem 2.3(i) yields the desired isomorphism.

Linkage 2.8 (Iskovskikh–Sarkisov–Reid–Corti). *Any birational map between Mori fibrations as in diagram* (2.4) *can* (*conjecturally*) *be factorized into a finite chain of elementary modifications*, *so-called links* (*see* [6], Chapter 13 in [25], and §§ 1, 2 in Chapter II of [8]).

Sketch of the construction. Let us use the notation introduced above. To factorize a modification $\chi \colon X \dashrightarrow X'$, we take a very ample linear system

$$\mathcal{H}' = \left|-\mu' K_{X'} + \varphi'^* A'\right|$$

on X' as in (2.4). For an arbitrary Mori fibration $\varphi \colon X \to S$ and any modification χ of it the given pair $(X', \mathcal{H}')$ determines a *degree* $\deg(\chi, \mathcal{H}') = (\mu, \lambda, e)$, where μ stands for the quasi-effective threshold, that is, a rational number such that $\mu K + \chi_*^{-1} H' = \mu K + H \equiv 0$ over S (in other words, $H \sim_{\mathbb{Q}} -\mu K + \varphi^* A$, where $A \in \operatorname{Div}(S) \otimes \mathbb{Q}$), $\lambda = 1/c$ stands for the maximal multiplicity of $\mathcal{H}$, and e for the number of maximal singularities. The triples (μ, λ, e) are ordered lexicographically, and the untwisting process consists in constructing a chain of links $\Phi_1, \Phi_2, \dots$ such that the degrees are decreasing for $\chi_i = \chi_{i+1} \circ \Phi_{i+1}$, $\chi_0 = \chi$, namely, $\deg(\chi_{i+1}, \mathcal{H}') < \deg(\chi_i, \mathcal{H}')$. The process must terminate at an isomorphism for which the Noether–Fano inequality provides the criterion.

The links can be constructed by the LMMP; to prove the termination of the factorizing algorithm, one needs termination conditions for quasi-effective and canonical thresholds (for details, see [25], Chapter 13, and [8], Chapter II, §§ 1, 2). The existence of factorization in dimension 2 is a classical result due to M. Noether and G. Castelnuovo. In dimension 3 it was proved by Corti [6].

2.9. Geography of linkage and of log models. We say that two $\mathbb{R}$-boundaries B and B' of a variety X are *model* equivalent if the log pairs (X, B) and (X, B') have the same resulting log models, that is, the log minimal models $(Y, B_Y^{\log})$ and

$(Y, {B'_Y}^{\log})$ coincide, or the Mori log fibrations $(Y, B_Y^{\log}) \to S$ and $(Y, {B'_Y}^{\log}) \to S$ have the same first argument (the variety) and the same structural morphisms. By Proposition 2.5(i), more precisely, by its version for more general boundary transforms (cf. the discussion in the proof of Proposition 2.5(iii); for details, see [36], 2.4.1), the following alternative holds:

LMP (log minimal pairs): $(Y, B_Y^{\log}) \to S$ is a log minimal model of (X, B) with the Iitaka fibration if and only if $(Y, {B'_Y}^{\log}) \to S$ is a log minimal model of (X, B') with the same Iitaka fibration, or

MLF (Mori log fibrations): $(Y, B_Y^{\log}) \to S$ is a Mori log fibration if and only if $(Y, {B'_Y}^{\log}) \to S$ is a Mori log fibration.

Take a finite set of prime divisors D_i on X. Then conjecturally the LMMP gives a decomposition of the unit cube of boundaries

$$\sum [0,1] D_i$$

into equivalence classes of boundaries $B = \sum b_i D_i$.

Conjecture 2.10 (geography of log models). *The decomposition is polyhedral. More precisely, each 'country', that is, each equivalence class, is an open polyhedron in the cube. Moreover, all polyhedra are convex, rational, and finite, and the decomposition is locally finite in* $\sum (0,1] D_i$; *for minimal models the value* $b_i = 0$ *can be included. The subpolyhedron of log minimal models is convex, closed, finite, and rational.*

Similar facts are expected in the category with birational transforms of boundaries (*cf. Example* 2.11 *below*). *In this situation it can occur that the models with terminal singularities do not fill the entire cube and form only a subpolyhedron with a coarser equivalence relation.*

The conjecture has been established for 3-fold log minimal models ([36], 6.20).

Example 2.11 (linkage between two models). We consider two Mori fibrations $X \to S$ and $Y \to T$, where X and Y are birationally equivalent, and choose a prime very ample divisor D on X (similar to H' in Linkage 2.8). Analogously, let L_Y on Y correspond to a birational transform L on X. In this case the model equivalence with the birational transform of boundaries gives a decomposition of the square $[0,1]D \oplus [0,1]L$. By construction, the vertices $D = 1D + 0L$ and $L = 0D + 1L$ have minimal[1] models (X, D) and (Y, L_Y), which are also canonical, but the sum $0 = 0D + 0L$ has two Mori (log) fibrations. As in the geography of log models (and even according to this geography), there is a (downwards convex) finite chain of segments (equivalently, a chain of open intervals and points, a *separatrix*) which separates the minimal models from the Fano log fibrations (the sheep from the goats). The points of polygons in the decomposition that do not intersect the edges of the square along a segment correspond to klt pairs $(W, dD_W + lL_W)$, $d, l \in (0,1)$, with $D_W, L_W \neq 0$, so these pairs are cn and even purely trm. In particular, this

[1] Here trm is understood as lt for some resolution; we define similarly the pure trm condition, which is the same as the usual trm condition in dimension $\geqslant 2$, the divisorial trm condition (dtrm), and so on.

holds for the internal points of each segment in the chain. The corresponding Iitaka contractions $(W, dD_W + lL_W) \to U$ are not birational. Otherwise a slight decrease of d or l gives a minimal and even canonical model again after the Iitaka contraction. (Any slight decrease preserves bigness!) Moreover, after a slight decrease of l we can use the LMMP over U to modify the pair $(W, dD_W + lL_W) \to U$ (by using generalized flops) into a pair $(W', dD_{W'} + lL_{W'}) \to U$ which is cn, and $(W', dD_{W'})$ is trm and has a Fano log fibration $(W', dD_{W'}) \to U'$ and a Mori log fibration $W' \to U'$ (over U). This holds for the internal points of each segment. The projection on $[0,1]D$ gives a linkage from $Y \to T$ to $X \to S$. Passage through the ends of segments corresponds to links which can be non-elementary. (One can make them elementary by perturbing D as a b-divisor.) The terminations are hidden in thresholds defining the polyhedral structure.

By 6.20 in [36], this proves the existence of linkages in dimensions $\leqslant 3$.

§ 3. Modifications of log minimal models: flops and the geography of log models

3.1. In the usual Mori theory a minimal model is a normal projective $\mathbb{Q}$-factorial variety X with at worst terminal singularities and a nef canonical divisor K. In the case of surfaces there are no terminal singularities, that is, trm is equivalent to non-singularity and, as is well known, a minimal model is unique in its birational class (if such a model exists). For $\dim X \geqslant 3$ this is no longer so, and there can be many non-isomorphic birationally equivalent models. However, their birational maps can readily be listed, namely, these are isomorphisms in codimension 1, and it is expected that they can be factorized into sequences of elementary modifications, *extremal flops*. Similar expectations apply to the LMMP. Let us recall the definition of a flop.

Definition 3.2. (i) Let X be a normal trm variety, let φ be a small birational contraction of X, and let D be an $\mathbb{R}$-Cartier divisor on X. A small modification χ (over Y) in a commutative diagram

$$\begin{tikzcd} D \text{ on } X \arrow[rr, dashed, "\chi"] \arrow[dr, "\varphi"'] & & X^+ \text{ with } D^+ \arrow[dl, "\varphi^+"] \\ & Y & \end{tikzcd} \tag{3.1}$$

is called a *flop* of X or of the *contraction* φ if K is numerically φ-trivial, φ^+ is also a small birational contraction, and K_{X^+} is numerically φ^+-trivial. A flop is said to be *directed with respect to a divisor* D or a *D-flop* if $-D$ is numerically φ-ample and $D^+ = D_{X^+} = \chi_* D$ is numerically φ^+-ample. Equivalently, χ is a D-flip as in (d) of 1.1 with $\varphi_i = \varphi\colon (X_i, D_i) = (X, D) \to Y_i = Y$. Usually, a flop is also assumed to be *extremal*, that is, the contraction φ is extremal, and the divisor D is assumed to be effective.

(ii) The diagram gives an (ordinary) *log flop* of a pair (X, B) or of a contraction $\varphi\colon (X, B) \to Y$ if an lt or lc log pair (X, B) with a boundary B in (3.1) is given instead of X, and log canonical divisors $K + B$ and $K_{X^+} + B^+$ are taken instead of canonical divisors K and K_{X^+}, respectively, where $B^+ = \chi_* B$. Here a directed log flop with respect to D is defined as in (i) above.

(iii) Let the contraction φ in ii) (which is not assumed to be small and extremal) satisfy only the following conditions:

a) X^+ is normal and the formula (3.1) gives a (generalized) D-flip (that is, $-D$ is numerically φ-ample, φ^+ is a small birational contraction, and $D^+ = \chi_* D$ is numerically φ^+-ample);
b) $K + B$ is numerically φ-trivial and $K_{X^+} + B^+$ is numerically φ^+-trivial.

In this case χ is called a (generalized) *directed log flop* with respect to D.

Remarks 3.3. (i) One can define a more general log flop as a crepant modification by omitting in (iii) the condition that φ^+ is small and assuming instead that φ^+ is crepant (see Definition 4.9(iv)). This determines a unique divisor B^+ which is assumed to be a boundary as above. For a directed flop, a divisor D^+ is determined here (not uniquely) by the equation $\varphi_* D = \varphi^+_* D^+$. Such morphisms can occur in investigations of modifications between weakly log canonical models and restrictions of ordinary log flops (or log flips) to subvarieties.

(ii) By definition (and for effective divisors D), any directed klt log flop and, all the more so, any (terminal) flop are log flips with respect to the divisors $K+B+\varepsilon D$, $0 < \varepsilon \ll 1$, in the items (ii) and (iii) of the definition and with respect to $K + \varepsilon D$ in the item (i). Thus, the existence and termination problem for log flops and flops of this kind is a special case of general conjectures about the existence and termination of log flips. In this general context the existence conjecture was proved in dimensions 3 and 4 and the termination conjecture in dimension 3 and, in the terminal case, in dimension 4 (see the theorem in [34], 5.2 in [36], 1.8 and 1.9 in [37], [17], [12], and Example 9 in [39]). However, the first existence proofs for terminal flops in dimension 3 (and for some flops in dimension 4) were immediate and used the classification of 3-fold terminal singularities (G. N. Tyurina's theorem on the simultaneous resolution of surface Du Val singularities in a one-dimensional family (see 2.2 in [23] and 4.1 in [11])). The termination of such flops can readily be established in dimension 3 by using the above idea of reducing to flips (see 2.17 in [33], [20], and [25]). Here we present a proof illustrating the simplest approach to problems of this type. The general approach is discussed in [39].

Here (and in many other problems of (log) minimal model theory) the main ingredient is the following assertion.

Lemma 3.4 (2.13.3 and 2.15 in [33]). (i) *Let* $(X, D) \dashrightarrow (X^+, D^+)$ *be a log flip of a birational contraction* $\varphi\colon X \to Y$ *into a contraction* $\varphi^+\colon X^+ \to Y$ *as in the diagram* (3.1) (see also the proof). *Then*

$$a(E, X^+, D^+) \geqslant a(E, X, D)$$

for any prime b*-divisor* E *of* X (*or equivalently, of* X^+). *Moreover, the inequality is strict if and only if*

$$\mathrm{c}_X\, E \subset \operatorname{Exc}(\varphi)$$

(*equivalently,* $\mathrm{c}_{X^+}\, E \subset \operatorname{Exc}(\varphi^+)$).

(ii) *Let* (X, B) *be a* 3*-fold* trm *log pair with an* $\mathbb{R}$*-boundary* B. *Then there are finitely many prime* b*-divisors* $E_1, \dots, E_s$ *such that if* $a(E, X, B) < 1$ *for some prime* b*-divisor* E *exceptional on* X, *then either* E *is one of the divisors* E_i,

$i = 1, \dots, s$, or E corresponds to a blowup at a generic point of a curve $\mathrm{c}_X E$ belonging to the non-singular locus of $\operatorname{Supp} B$, $B = \sum b_j D_j$, *and automatically outside the terminal* (*and thus isolated*) *singularities of X. Moreover, in the latter case*

$$a(E, X, B) = 1 - b_j,$$

where j is the index of the component D_j such that $\mathrm{c}_X E \subset D_j$.

Proof. The statement (i) has been well known since [33] and follows immediately from a more general lemma, Lemma 2.4. Thus, one can also assume that the log flip is more general, namely, that $-(K + D)$ is numerically φ-ample and $K_{X^+} + D^+$ is numerically φ^+-ample, without assuming that D is effective nor that φ and φ^+ are extremal.

Let us prove (ii). Let $f\colon V \to X$ be a log resolution of (X, B) on which the irreducible components of $\operatorname{Supp} B_V$ are disjoint. Then $K_V = f^*(K + B) + \sum a_i E_i$, where $a_i > 0$ for all exceptional divisors of f, because X is trm. Any prime b-divisor E exceptional on X and distinct from the E_i can be obtained by subsequent blowups at non-singular centres, starting from V. Thus, if $a(E, X, B) < 1$, then E either coincides with some divisor of the form E_i or corresponds to a blowup at a generic point of a curve $\mathrm{c}_X E$ belonging to the non-singular locus of $\operatorname{Supp} B$.

Theorem 3.5 (termination of terminal flips and flops in dimension 3). (i) *For any initial* trm (*and even for any* cn) *log pair (X, B) with an $\mathbb{R}$-boundary B, any chain of* 3*-fold log flips terminates.*

(ii) *Any chain of* 3*-fold* (*terminal*) *directed D-flops*

$$\begin{tikzcd}[column sep=small]
X = X_0 \arrow[rr, dashed] \arrow[dr, "\varphi=\varphi_0"'] & & X^+ = X_1 \arrow[dl, "\varphi^+=\varphi_0^+"] \arrow[rr, dashed] \arrow[dr, "\varphi_1"'] & & X_1^+ = X_2 \arrow[dl, "\varphi_1^+"] & \dots & X_i \arrow[rr, dashed] \arrow[dr, "\varphi_i"'] & & X_i^+ = X_{i+1} \arrow[dl, "\varphi_i^+"] & \dots \\
& Y = Y_0 & & Y_1 & & & & Y_i & &
\end{tikzcd}$$

terminates provided that D is an effective $\mathbb{R}$-Cartier divisor and X is trm.

Proof (cf. [39], Example 9). For any sufficiently small positive ε the $\mathbb{R}$-divisor $B = \varepsilon D$ is a boundary and (X, B) is trm. In this case any D-flop is a log flip of (X, B) with small birational contractions φ_i and φ_i^+. (The flops are small, that is, all the flops are small modifications; in particular, they do not contract any boundary component, and hence the modifications (X_i, B_{X_i}) are trm, and X_i is trm since B_{X_i} is effective.) Thus, it suffices to establish the termination of log flips. We denote any birational transform of B on any X_i by the same letter B.

Suppose that $B = \sum b_i D_i$. We order the non-zero *values* of the boundary multiplicities, $b_1 > b_2 > \dots > b_s$, and add the values $b_0 = 1$ and $b_{s+1} = 0$.

By Lemma 3.4, since the flipped exceptional locus $\operatorname{Exc}(\varphi_i)$ contains a curve, it follows that $\operatorname{dis}(\varphi_i) < 1$ for the minimal discrepancies given by

$$\operatorname{dis}(\varphi_i) = \min_E \{a(E, X_i, B) \mid \mathrm{c}_{X_i} E \subset \operatorname{Exc}(\varphi_i)\}.$$

Hence, up to finitely many log flips, one can assume that for some integer j with $0 \leqslant j \leqslant s$ the inequality $\operatorname{dis}(\varphi_i) \geqslant 1 - b_j$ holds for all contractions φ_i, and the inequality $\operatorname{dis}(\varphi_i) < 1 - b_{j+1}$ holds for infinitely many contractions φ_i. The termination can be established according to general principles (see [39]), but one needs special tricks, because some general conjectures about the discrepancies are still not proved. In particular, in our case we replace the minimum over all prime b-divisors E with $\mathrm{c}_{X_i} E \subset \operatorname{Exc}(\varphi_i)$ by the minimum over the controlled b-divisors E of this kind; we set $j = s$ and write $\operatorname{dis}(\varphi_i) = 1 - b_s$ for the controlled discrepancy if there are no controlled divisors. By definition, E is said to be *controlled* if $\mathrm{c}_{X_i} E$ is a curve in $\operatorname{Exc}(\varphi_{i-1}^+) \cup \operatorname{Exc}(\varphi_i)$ for infinitely many modifications. We eventually establish the absence of b-divisors E of this kind. On the other hand, controlled divisors E exist by the assumption that the chain of log flips is infinite, and this proves the termination.

Let us show first that

$$a(E, X, B) \geqslant 1 - b_{j+1}$$

for almost all (that is, for all but finitely many) non-controlled b-divisors E of X lying over $\operatorname{Exc}(\varphi_i)$ (or $\operatorname{Exc}(\varphi_i^+)$) for some modification $X_i \dashrightarrow X_{i+1}$. Indeed, otherwise, by Lemma 3.4 there are infinitely many non-controlled b-divisors E of X that are exceptional on X (since the modifications are small) and lie over some exceptional locus $\operatorname{Exc}(\varphi_i)$ (or $\operatorname{Exc}(\varphi_i^+)$), with

$$a(E, X, B) \leqslant 1 - b_l, \qquad l \leqslant j,$$

where the equality $a(E, X, B) = 1 - b_l$ is attained for infinitely many divisors E whose related curve is $\mathrm{c}_X E$. These curves lie on finitely many prime divisors D_j on X with boundary multiplicities b_l. We can also assume that l is minimal for these curves. Then up to finitely many log flips the other flips contract the curves $\mathrm{c}_X E$ on these divisors but do not blow up new curves by Lemma 3.4 and by our choice of the controlled value b_j. However, this is impossible, since the Picard number of such divisors is bounded (cf. Case I below).

Hence, after finitely many log flips, $\mathrm{c}_{X_i} E$ and $\mathrm{c}_{X_{i+1}} E$ are always points for any non-controlled E over $\operatorname{Exc}(\varphi_i)$ and $\operatorname{Exc}(\varphi_i^+)$, and the inequality $a(E, X_i, B) < 1 - b_{j+1}$ holds. Thus, any contraction has a controlled b-divisor E with $\mathrm{c}_{X_i} E \subset \operatorname{Exc}(\varphi_i)$ such that $1 - b_j \leqslant a(E, X_i, B) < 1 - b_{j+1} \leqslant 1$ if a chain of log flips is infinite. Otherwise $j = s$. In the latter case, by Lemma 3.4(i), a blowup over a curve in $\operatorname{Exc}(\varphi_i^+)$ gives a controlled divisor E with $\mathrm{c}_{X_i} E \subset \operatorname{Exc}(\varphi_i)$ such that $a(E, X_i, B) < 1$, and the controlled minimal discrepancy $\operatorname{dis}(\varphi_i) < 1$ is well defined.

Case I. There are infinitely many contractions φ_i with $\operatorname{dis}(\varphi_i) = 1 - b_j$. One can also assume that the minimum for discrepancies is attained on a b-divisor E with a curve $\mathrm{c}_{X_i} E$ for infinitely many contractions φ_i. Otherwise, $\mathrm{c}_{X_i} E = \text{pt.}$ is a point for infinitely many contractions φ_i. By Lemma 3.4(ii), the points $\mathrm{c}_{X_i} E$ for controlled divisors E with $a(E, X_i, B) = 1 - b_j$ form a 0-dimensional subvariety $W_i \subset X_i$. Then by Lemma 3.4(i) and our assumptions, such new points do not appear on the subsequent variety X_{i+1}: the modifications with $\operatorname{dis}(\varphi_i) > 1 - b_j$ do not touch W_i, and at least one point of W_i disappears under the modifications with $\operatorname{dis}(\varphi_i) = 1 - b_j$. This gives the desired family of infinitely many curves $\mathrm{c}_{X_i} E$.

Again by Lemma 3.4(ii), there is a proper subvariety $W_i \subset X_i$ (possibly with non-divisorial components) for which the generic curves of components are of the form $\mathrm{c}_{X_i} E$ for the b-divisors E such that $a(E, X_i, B) = 1 - b_j$. The divisorial components (surfaces) of W_i are exactly the components of B with the multiplicities b_j. It can readily be seen that the usual minimal discrepancy is $\leqslant 1-b_j$ over a generic point of any curve in W_i. Again by Lemma 3.4(i) and our assumptions, the modifications do not blow up any curve in W_i but only contract them if $\operatorname{dis}(\varphi_i)=1-b_j$, and equality is attained in E with a curve $\mathrm{c}_{X_i} E$. Hence, this case is impossible.

Case II. After finitely many log flips we have $\operatorname{dis}(\varphi_i)>1-b_j$ for any φ_i. By Lemma 3.4(ii), values $\operatorname{dis}(\varphi_i)<1-b_{j+1}$ are attained on finitely many controlled divisors E. By the property of being controlled, the numbers $a(E,X_i,B)$ satisfy the ascending chain condition and stabilize. Hence, the value $a=\operatorname{dis}(\varphi_i)>b_j$ also stabilizes, that is, $\operatorname{dis}(\varphi_i)\geqslant a$ for all i, with equality for infinitely many contractions φ_i. The same arguments as in Case I prove termination, that is, the absence of controlled b-divisors E. In this case, W_i is a finite union of curves and points.

3.6. Let us now apply the geography of log models to some problems concerning modifications of minimal (log minimal) models in a birational class. As is well known, for any non-singular surface a minimal model is unique in the corresponding birational class. Moreover, it turns out that a two-dimensional minimal (or terminal) model of a log pair (X,B) is also unique, because such models are isomorphic in codimension 1 (see Proposition 2.5(vi)).

However, this is no longer valid in dimension 3 and in higher dimensions. In general, there are many minimal (log minimal) models in a given birational class (of a log pair or of a morphism of a log pair, respectively). They are related to one another as follows (see Proposition 2.5(vi) above and Proposition 3.10 and Theorem 3.11 below): any two minimal (log minimal) models in the same birational class (akin models of a log pair or a morphism of a log pair, respectively) are isomorphic in codimension 1, and in the strict case can be obtained one from the other by a sequence of flops (log flops). A more global description of the minimal models in the same birational class can be given in terms of geography of log models (see Conjecture 2.10 above).

Definition 3.7. Log minimal models are said to be (*projectively*) *akin* if they have a common (projective) log resolution and are log minimal models of it. One takes the log birational transform as a boundary on the log resolution.

Remark 3.8. The existence of a common (even a projective) log resolution in the last definition always holds for log models having only klt and plt (in particular, only trm) singularities.

The LMMP relation between two log minimal models implies that they are projectively akin. The converse holds modulo the conjectures on the existence and termination of flips. We recall that two log minimal models are *LMMP related* if they can be obtained using the LMMP from the same strictly lt log pair. We also note that log flips preserve both the projectivity of models and the dlt property of singularities, that is, the lc centres are log non-singular ([34], English p. 99). By Lemma 2.4 or, more precisely, by the fact that discrepancies are independent of a weakly log canonical model (see the proof of Proposition 2.5(iii)) and by 3.4(i), these centres are the same on such log minimal models, and thus these two models have a common log resolution.

Example 3.9 (Kulikov's flops). This is an extremal flop and log flop of a log non-singular log pair (X_1,B_1) into (X_2,B_2), where $B_1=B_2=\sum_{i=1}^4 D_i$ in the birational sense, and the flop transforms the curve $C_1 = D_1 \cap D_3 \subset X_1$ into $C_2 = D_2 \cap D_4 \subset X_2$. The log pairs (X_1, B_1) are (X_2, B_2) are not akin, but they are log minimal models of each other.

Proposition 3.10. (i) *Any two log minimal models* (X_1, B_1) *and* (X_2, B_2) *of each other are isomorphic in codimension* 1.

(ii) *Any two akin log minimal models* (X_1, B_1) *and* (X_2, B_2) *are isomorphic in codimension* 1. *Moreover, one can take distinct log resolutions for each of the models in the definition of the akin relation; however, in this case, the two resolutions are assumed to be (log) isomorphic in codimension* 1.

Proof. The assertion (i) follows from the definition, and its proof is simpler than that of Proposition 2.5(vi).

The assertion (ii) can be reduced in essence to (i). Let $(Y_1, B_1^{\log})$ and $(Y_2, B_2^{\log})$ be log resolutions of the pairs (X_1, B_1) and (X_2, B_2) used in the definition of the akin relation, and let these resolutions be isomorphic in codimension 1. Let D be a prime divisor on X_1 which is exceptional on X_2. Then D is a divisor on Y_2. By the definitions of a log terminal resolution and the akin relation we have $a(D, X_2, B_2) > a(D, Y_2, B_2^{\log}) = a(D, Y_1, B_1^{\log}) = a(D, X_1, B_1) = -b$, that is, (X_2, B_2) is a log minimal model of (X_1, B_1). The converse assertion also holds.

Conditional Theorem 3.11 (assuming the validity of the existence and termination conjectures for directed flops in (i) and for log flops in (ii) below).

(i) *Any two projective minimal models in the same birational class are connected by a chain of extremal flops and their inverses.*

(ii) *Any two projectively akin projective log minimal (and minimal) models (of a log pair* (X, B)*) are connected by a chain of extremal log flops and their inverses (see the proof).*

One can assume that the flops and log flops between strictly minimal models (akin strictly log minimal models, respectively) are directed and extremal.

Proof. We shall prove only (ii); (i) can be proved similarly.

Let (X_1, B_1), $B_1 = B_{X_1}^{\log}$, and (X_2, B_2), $B_2 = B_{X_2}^{\log}$, be two projectively akin projective models of a pair (X, B). One can assume that (X, B) is a common projective log resolution. By the projectivity, there is an ample effective $\mathbb{Q}$-divisor D on X_2 such that $(X_2, B_2 + \varepsilon D)$ is a log minimal and simultaneously a log canonical model of $(X, B + \varepsilon D_X)$ for any $0 < \varepsilon \ll 1$. By the akin relation, the model $(X_1, B_1 + \varepsilon D_{X_1})$ is lt for any $0 < \varepsilon \ll 1$ and is akin (up to the nef property) to the former model if X_1 is $\mathbb{Q}$-factorial. The last property can be satisfied after a $\mathbb{Q}$-factorialization (see Corollary 6.7 below) which also preserves the akin relation; after a projective $\mathbb{Q}$-factorialization, a strictly log minimal model can be obtained. However, the model $(X_1, B_1 + \varepsilon D_{X_1})$ is not necessarily a log minimal model of $(X, B + \varepsilon D_X)$, because $K_{X_1} + B_1 + \varepsilon D_{X_1}$ is not necessarily nef. We can apply the LMMP to make it nef. Finally, termination and contraction onto a log canonical model gives the model X_2 according to the uniqueness in Theorem 2.3. The contraction exists for big D. Using the geography of log models, a $\mathbb{Q}$-factorialization can be decomposed into a sequence of inverses of extremal log flops, and the log canonical contraction can be decomposed into extremal log flops ([36], 6.22). We can also assume in addition that $\rho(X^+/Y) \leqslant 1$ (and $\rho(X^+/Y)$ can vanish) for extremal log flops.

By the $\mathbb{Q}$-factorial property in the strict case, the $\mathbb{Q}$-factrialization and log canonical contraction are not needed, and the log flips of $(X_1, B_1 + \varepsilon D_{X_1})$ in the construction are directed extremal log εD_{X_1}-flops of (X_1, B_1).

The existence and termination of directed log flops, which are in fact somewhat more general flops (as in [37], 1.9) in the non-$\mathbb{Q}$-factorial case, is sufficient for the above constructions (see Remark 3.3(ii)).

Conditional Theorem 3.12 (finiteness of the number of minimal models of general type under the existence and termination of directed log flops ([18] and [19]; see also 6.22 in [36] and 12.3 in [25])). *A log pair (X, B) of general type (that is, a pair for which the divisor $K_Y + B_Y$ is big for any cn modification (Y, B_Y) of it) has only finitely many projective minimal models.*

Proof. We follow the proof of 6.22 in [36]. Take the prime components of B and add finitely many prime mobile Weil divisors such that on some resolution $g\colon Y \to X_{\mathrm{can}}$ these divisors (more precisely, their birational transforms), together with the exceptional divisors of g, generate the divisors of Y up to numerical equivalence. (We recall that X_{can} stands for the canonical model of (X, B), which exists for pairs of general type, that is, for pairs such that the divisor $K_Y + B_Y$ is big.) Let us denote the transforms of these prime divisors on X_{can} by D_i and identify the D_i with their birational transforms on the other models. After adding finitely many prime divisors to this family, one can assume that $H \geqslant D_i$ for all i, where H is an ample effective divisor on X_{can} supported by the divisors D_i and passing through the singularities of X_{can}.

We claim that on any minimal model (Y, B_Y) one can form an effective numerically ample divisor $D = \sum d_i D_i + \sum e_i E_i$, where the divisors E_i are exceptional on X_{can}. Indeed, by construction, the divisors D_i and E_i generate the $\mathbb{R}$-Cartier divisors on Y up to numerical equivalence. Thus, one can find a desired D with $d_i, e_i \in \mathbb{R}$. After adding a multiple of g^*H, we obtain an effective divisor D. Hence, $(Y, B_Y + \varepsilon D)$ is a log minimal and also a log canonical model of $(X, B + \varepsilon D_X)$ for any $0 < \varepsilon \ll 1$, and thus a minimal model of this kind is unique. Therefore, we obtain an injection of the minimal models into countries whose boundaries are supported by the divisors D_i and the exceptional divisors on X_{can} (the latter form a finite set, because these are b-divisors for all minimal models). However, there are only finitely many countries in the geography (see [36], 6.20). In general, the geography needs the LMMP for the $\mathbb{R}$-divisors. However, in our situation it suffices to assume the existence and termination of directed log flops (cf. Remark 3.3(ii)).

Remark 3.13 (Batyrev [2]). Another explanation of the finiteness property in Theorem 3.12 is related to a finite polyhedral decomposition of the cone of $\mathbb{R}$-divisors D effective up to the relation $\sim_{\mathbb{R}}$ on a Fano variety with only terminal singularities, into polarization subcones (more precisely, the internal points of the subcones are polarizations) of the projective varieties obtained by the D-MMP starting from the given Fano variety (see [2], 3.4, and [37], 3.33). Here the divisors can be regarded up to numerical, $\mathbb{R}$-linear, and even just identical equivalence, because the Picard group of any Fano variety is finitely generated.

This fact can be generalized to weak Fano klt log fibrations (possibly with 0-dimensional fibres), where the term 'weak' means that the relative ampleness of $-(K + B)$ is replaced by the assumption that this divisor is nef and big.

To prove Theorem 3.12, it suffices to consider another extreme case in which the map $(Y, B_Y) \to X_{\text{can}}$ is a contraction of a (strictly) minimal model onto the canonical model.

Comment 3.14. In general, the question of how many projective minimal models there are in a birational class remains open. However, we also face another question: How should one count the number of such models?

The first way is to consider the number of countries in the geography for the image of an injection of minimal models or the number of polarization cones in the decomposition of the cone of effective divisors. This is exactly the number of minimal models obtained from a sufficiently high non-singular projective blowup by the LMMP. More precisely, any two minimal models X_1 and X_2 in the class are birationally isomorphic by definition. They are identified (that is, treated as equal models) if the modification $X_1 \dashrightarrow X_2$ is an isomorphism, that is, is biregular.

However, other isomorphisms $X_1 \cong X_2$, which need not be induced by birational ones, are also possible. This leads to another method of counting the number of minimal models. Clearly, the number of models counted in the first way is not less than that counted in the second way.

The uniqueness of minimal models in dimension 2 (and thus the finiteness of the set of them) can be established for complete surfaces of general type by using the first approach.

M. Reid gave an example of a 3-fold of non-general type for which the number of minimal models counted in the first way is infinite. However, even in this example (see below) it is unclear whether or not the number of models is infinite for identifications up to arbitrary isomorphism.

Examples 3.15. 1 [31]. Let $f\colon X \to \mathbb{A}^2$ be a family of elliptic curves given by the equation

$$z_1^2 = ((z_2 - a)^2 - x)((z_2 - b)^2 - y), \qquad a \neq b,$$

where x, y are the coordinates of $\mathbb{A}^2$ and z_1, z_2 are the coordinates of fibres. A fibre $f^{-1}p$ is a non-singular elliptic curve if $p \in \{xy \neq 0\}$, a rational curve with an ordinary double point if $p \in \{xy = 0\}$ and $p \neq (0,0)$, and a pair of rational curves with two transversal intersection points if $p = (0,0)$. We regard the family as an analytic germ near the central reduced fibre. Each irreducible component can be flopped step by step in such a way that the result gives a countable set of minimal models (polarization cones) over a germ of $p \in \mathbb{A}^2$.

However, it is clear that all these minimal models are isomorphic over a germ of $p \in \mathbb{A}^2$, and hence there is only one minimal model with respect to the second way of counting.

Of course, this local situation can be globalized by taking an appropriate projective closure (and resolving the singularities if necessary).

2. There can be infinitely many non-projective minimal models (a countable set) even when using the second way of counting. Indeed, let X be a sufficiently general 3-fold quintic. Then there is a smooth rational curve C on X of any positive integer degree d with a normal sheaf $\mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$. Each of these curves C can be contracted, and its flop $X \dashrightarrow X_d$ gives an analytic manifold (or an algebraic space) which is not projective and is not an algebraic variety either. The models X_d are

pairwise non-isomorphic for distinct values of d, because d is the least intersection number of the curve $C_d \subset X$ flopped from C with divisors $D < 0$ on X_d, that is, $DC_d \geqslant d$.

In any case, the following conjecture has been expressed.

Conjecture 3.16 (Kawamata, Matsuki [25], 12-3-6). *The number of projective minimal models in a given birational class is always finite if they are regarded up to arbitrary isomorphism.*

Kawamata showed in [16] that the conjecture holds for 3-fold models X with $\kappa(X) > 0$.

CHAPTER II

3-fold log flips

In this chapter we give a complete proof of the existence of 3-fold Kawamata log terminal (klt) log flips. This proof follows an idea of the second author [37] (see also [10]). The comments below briefly outline other (earlier) approaches to the problem ([26], [11], [34], [21]). In the preparatory section, § 4, a reduction of the initial problem to the existence problem for prelimiting (pl) flips and to a special termination is given. In § 5 the existence problem for pl flips, which is treated as the problem of finite generation for certain function algebras, is reduced to a similar question in dimension 2. This inductive step is one of the main ideas, in [37], of a general approach to the existence problem for n-dimensional log flips. Another main idea in [37], a reduction of the finite generation problem for function algebras to the so-called CCS conjecture, is treated in § 6 by a two-dimensional example in which the CCS conjecture is successfully proved in a more precise form quite easily (see Corollary 6.6 below). This gives the desired result in dimension 3. In § 7 we verify the semistability of 3-fold flips.

§ 4. Reduction to prelimiting flips and special termination

Definition 4.1. Let (X, B) be a klt pair and let $\varphi\colon X \to Y$ be a birational contraction of a normal $\mathbb{Q}$-factorial variety X on a normal variety Y with the following properties:

(i) φ is a small birational contraction, that is, $\operatorname{codim} \operatorname{Exc}(\varphi) \geqslant 2$;
(ii) $-(K + B)$ is numerically φ-ample;
(iii) $\rho(X/Y) = 1$, where $\rho(X/Y) = \rho(X) - \rho(Y)$ is the relative Picard number.
Any such contraction is called a klt birational *log contraction.*

A modification

$$\begin{tikzcd}[column sep=small] (X,B) \arrow[rr, dashed, "\chi"] \arrow[dr, "\varphi"'] & & (X^+,B^+) \arrow[dl, "\varphi^+"] \\ & Y & \end{tikzcd}, \tag{4.1}$$

is called a klt log flip of φ if $\varphi^+\colon X^+ \to Y$ is a birational contraction with a normal $\mathbb{Q}$-factorial variety X^+, (X^+, B^+) is a klt log pair, and the following conditions hold:

(i^+) φ^+ is small, that is, $\operatorname{codim} \operatorname{Exc}(\varphi^+) \geqslant 2$,

(ii$^+$) $K_{X^+} + B^+$ is numerically φ^+-ample,
(iii$^+$) $\rho(X^+/Y) = 1$,

where $B^+ = \chi_* B$ is the birational transform of B and $X \setminus \operatorname{Exc}(\varphi) \xrightarrow{\sim} X^+ \setminus \operatorname{Exc}(\varphi^+)$.

Comment 4.2. The existence problem for klt log flips and the termination problem for chains of them are the main tasks in the LMMP for the category of projective $\mathbb{Q}$-factorial lt varieties. Of course, a solution of these problems in the LMMP with only klt singularities implies a solution of the analogous problems in the Mori category of projective $\mathbb{Q}$-factorial trm varieties as well, namely, it suffices to set $B = 0$ and to note that any log flip does not worsen the singularities and even improves them on the flipping locus (see Lemma 3.4(i), Chapter I). The same holds for non-$\mathbb{Q}$-factorial varieties and/or lt singularities. For the same reasons, in the definition of log flip it suffices to assume that X^+ is normal and that the conditions (i$^+$) and (ii$^+$) hold. In this case the other assumptions follow automatically from the assumptions about (X, B) and φ.

Chronologically, the termination of a chain of flips in the category of 3-fold terminal varieties was proved first ([33], 2.17). The existence of flips in this case was obtained by Mori [26]. The proof used the Kawamata reduction [11] to the case of 3-fold flops (by means of double covers and the elephant conjecture claiming the existence of a good effective divisor in $|-K|$ or in $|-2K|$).

The existence of 3-fold klt log flips was first proved by the second author [34]. To this end, he introduced and used the following notions.

(i) A *special flip* is a log flip of a $\mathbb{Q}$-factorial lt log pair (X, B) with a small birational contraction $f\colon (X, B) \to Y$, where $B = S = \sum S_i \neq 0$ is a reduced Weil divisor, and the divisor $-(K + S)$ and all the divisors $-S_i$ are f-ample.

(ii) A *special termination* is the termination of a chain of log flips

$$(X_1, B_1) \dashrightarrow (X_2, B_2) \dashrightarrow \cdots \dashrightarrow (X_i, B_i) \dashrightarrow \cdots,$$

where (X_i, B_i) are lc log pairs, in the following sense: there is an $i_0 \in \mathbb{N}$ (depending on the chain) such that for any $i \geqslant i_0$ the flipping locus $\operatorname{Exc}(\varphi_i)$ is disjoint from the reduced part $\lfloor B_i \rfloor$ of the boundary (see Definition 4.6 below).

(iii) The *ascending chain condition* (a.c.c.) for the set $\overline{S}^0_d$(local) consisting of the sequences $(b_1, \dots, b_m)$ for which there exist a $\mathbb{Q}$-factorial variety X with $\dim X \leqslant d$, a subset $Z \subset X$, and an $\mathbb{R}$-boundary $B = B_0 + \sum b_i D_i$ with a reduced (and possibly reducible) component $B_0 \neq 0$ such that $\operatorname{Supp} D_i \cap Z \neq \varnothing$ for all i, $Z \subset \operatorname{Supp} B_0$, $K + B_0$ is plt, and $K + B$ is maximally lc near Z, that is, $K + B$ is lc and none of the multiplicities b_i can be increased while preserving the lc property. The *order* is defined as follows:

$$(b_1, \dots, b_m) \leqslant (b'_1, \dots, b'_{m'}) \text{ if } m' < m \text{ or if } m' = m \text{ and } b_i \leqslant b'_i \text{ for all } i.$$

The main result in [34] is the reduction theorem, Theorem 6.4 (see also [21], § 18): if (i)–(iii) hold in dimension 3 (or, more generally, in dimension n), then klt log flips exist in this dimension. It is also proved in 4.1 of [34] (see also 7.1 of [21]) that special termination holds in dimension 3 (as we show in Theorem 4.8 below, this property follows for any dimension n from the LMMP in dimension $n - 1$).

The condition (iii) (a.c.c.) also holds in dimension $\leqslant 3$ (see Chicago Lemma 4.9 in [34] and also 18.19 and 18.25.1 in [21]).

To prove the existence of special flips, the notion of complement was introduced as follows.

(iv) (See [34], § 5, or [21], § 19.) Let X be a normal variety, let $B = \sum b_i D_i$ be a subboundary (that is, $b_i \leqslant 1$ for all i, but negative values are also possible), and let S be the smallest reduced Weil divisor on X such that $\lfloor B - S \rfloor \leqslant 0$. We write $B_0 := B - S$. A divisor $\overline{B} \in |-nK - nS - \lfloor (n+1)B_0 \rfloor|$ is called an *n-complement for* $K+B$ if the divisor $K + B^+$ is lc, where $B^+ := S + \frac{1}{n}(\lfloor (n+1)B_0 \rfloor + \overline{B})$. We say that $K+B$ is *n-complementary* if it has an n-complement. In the special case of (X, B) with a reduced boundary $B = S = \sum S_i$, an n-complement of $K+S$ is a divisor $\overline{B} \in |-nK - nS|$ such that $K + S + B$ is lc, where $B := \overline{B}/n$.

It turns out that the divisor $K+S$ is n-complementary, where $n \in \{1, 2, 3, 4, 6\}$, for a 3-fold special flipping contraction $X \to Y$ with an irreducible boundary S (see [34], 5.12, or [21], 19.6 and 19.8). If the divisor S is reducible, then the existence of a 3-fold special flip can readily be established (see, for example, 6.10 in [34] and 21.2 in [21]). The case of a 1-complimentary divisor $K+S$ with irreducible divisor S can readily be established as well. For $n \geqslant 2$ the problem can be reduced to the case $n = 2$ (see 7.6 in [34] and also §§ 6 and 21 in [21]).

Thus, the following result plays the main role ([34], 8.6–8.8, [21], 22.10, and also [40]): a special flip exists for any 3-fold 2-complementary divisor $K+S$ with irreducible divisor S.

A new existence proof for 3-fold flips and an existence proof for 4-fold klt log flips were obtained in the recent paper [37]. These proofs are based on quite different ideas, namely, they use only the reduction to pl flips (instead of (i); see Definition 4.3 below) and the special termination (ii) (see Definition 4.6 below). To construct pl flips and, in particular, special flips, the apparatus of finitely generated function algebras and induction on the dimension were used instead of n-complements.

In this chapter we give a proof for 3-folds that follows the lines of [37] (see also [10], [7]) and uses these new ideas. We begin with the notion mentioned above.

Definition 4.3. Let (X, B) be a log pair with $\mathbb{Q}$-factorial X and with a $\mathbb{Q}$-boundary B. A birational contraction $\varphi\colon (X, B) \to Y$ is said to be (elementary) *prelimiting* (pl) if the following conditions hold:

(i) φ is small;
(ii) (X, B) is lt;
(iii) there is a reduced part $S = \lfloor B \rfloor = \sum S_i \neq 0$ of the boundary such that each divisor S_i is a prime Weil divisor;
(iv) $\rho(X/Y) = 1$;
(v) $-(K+B)$ is φ-ample;
(vi) all divisors $-S_i$ are φ-ample.

By a *prelimiting* (*pl*) (*elementary*) *flip* we mean a diagram of the form (4.1) satisfying the properties (i$^+$)–(iii$^+$) in Definition 4.1.

Remark 4.4. A more general definition of prelimiting contraction is used in 1.1 of [37], namely, X is not assumed to be $\mathbb{Q}$-factorial, B is an $\mathbb{R}$-boundary, the birational contraction need not be projective and small, and the relative Picard number is arbitrary. The generality involves functorial properties of this general definition under restrictions to subvarieties in the proof by induction. We recall that the notions of lt and of divisorially lt singularities are equivalent for $\mathbb{Q}$-factorial pairs.

The following result reducing the existence of log flips to the finite generation problem for certain relative sheaves of algebras is well known (apparently, since 2.12 in [33] (see also [11], Introduction)).

Proposition 4.5. *Let $\varphi\colon (X,B) \to Y$ be a birational contraction with a $\mathbb{Q}$-boundary B as in Definition* 4.1 *or* 4.2. *Then the corresponding flip exists if and only if the relative $\mathcal{O}_Y$-algebra*

$$\mathcal{R}_{X/Y}(K+B) := \oplus_{m\geqslant 0}\varphi_*\mathcal{O}_X(m(K+B))$$

is finitely generated (*note that a similar statement holds in the most general situation as well* ([37], 3.32)).

Sketch of the proof (cf. [30], 1.6). If a flip $\varphi^+\colon (X^+,B^+) \to Y$ exists, then the $\mathcal{O}_Y$-algebra

$$\begin{aligned}\mathcal{R} = \mathcal{R}_{X/Y}(K+B) &= \bigoplus_{m\geqslant 0}\varphi_*\mathcal{O}_X(m(K+B))\\ &= \bigoplus_{m\geqslant 0}\mathcal{O}_Y(m(\varphi_*(K+B))) = \bigoplus_{m\geqslant 0}\varphi^+_*\mathcal{O}_{X^+}(m(K_{X^+}+B^+))\end{aligned}$$

is finitely generated, because the $\mathbb{Q}$-divisor $K_{X^+}+B^+$ is φ^+-ample. The intermediate equalities follow from the smallness property of both the contractions φ and φ^+.

Conversely, suppose that $\mathcal{R} = \mathcal{R}_{X/Y}(K+B)$ is finitely generated as an $\mathcal{O}_Y$-algebra. The algebra $\mathcal{R}$ is graded. We set

$$X^+ := \operatorname{Proj}\mathcal{R}_{X/Y}(K+B)$$

with a natural morphism $\varphi^+\colon X^+ \to Y$. It is clear that $\mathcal{R}$ is integrally closed (normal) as a graded algebra (see 4.8 in [37]), and thus X^+ is normal and φ^+ is a birational contraction. Let us show that φ^+ is small. Suppose not, that is, let there be a φ^+-exceptional divisor $E^+ \subset X^+$. In this case we have an exact sequence of the form

$$0 \to \mathcal{O}_{X^+} \to \mathcal{O}_{X^+}(E^+) \to \mathcal{K} \to 0$$

with $\mathcal{K} \neq 0$. We argue by contradiction. Let $\mathcal{O}_{X^+}(1)$ be a relative ample sheaf on X^+ over Y corresponding to the construction, that is, let the graded algebra $\mathcal{R}$ be quasi-isomorphic to

$$\mathcal{R}_{X^+/Y}\mathcal{O}_{X^+}(1) = \bigoplus_{m\geqslant 0}\varphi^+_*\mathcal{O}_{X^+}(m).$$

Then $R^1\varphi^+_*\mathcal{O}_{X^+}(m) = 0$ for $m \gg 0$ by Serre's theorem, and the homomorphism $\varphi^{+*}\varphi^+_*(\mathcal{K})\otimes\mathcal{O}_{X^+}(m) \to \mathcal{K}\otimes\mathcal{O}_{X^+}(m)$ is surjective (the image is generated by the global sections). However, the homomorphism α in the exact sequence

$$0 \to \varphi^+_*\mathcal{O}_{X^+}(m) \xrightarrow{\alpha} \varphi^+_*(\mathcal{O}_{X^+}(E^+)\otimes\mathcal{O}_{X^+}(m)) \to \varphi^+_*(\mathcal{K}\otimes\mathcal{O}_{X^+}(m)) \to 0$$

is in fact an isomorphism (this follows from the construction, from the effective and exceptional properties of E^+, and from the equality $\mathcal{R} = \mathcal{R}_{Y/Y}\varphi_*(K+B)$). Hence,

$$\varphi_*^+(\mathcal{K} \otimes \mathcal{O}_{X^+}(m)) = 0 \ \Rightarrow\ \mathcal{K} \otimes \mathcal{O}_{X^+}(m) = 0 \ \Rightarrow\ \mathcal{K} = 0.$$

A contradiction.

Since φ and φ^+ are small, it follows that $\operatorname{Div}_{\mathbb{Q}} X \simeq \operatorname{Div}_{\mathbb{Q}} X^+$. The sheaf $\mathcal{O}_{X^+}(1)$ is invertible by construction. Thus, X^+ is $\mathbb{Q}$-factorial and $\rho(X^+/Y) = 1$. The statement about the singularities follows from Lemma 3.4(i) in Chapter I.

We now reduce the existence problem for klt log flips to the existence problem for pl flips and to special termination. This reduction follows the lines of [34], § 6 and [21], § 18 (see also [10]).

Definition 4.6. Let

$$\begin{tikzcd}
(X_1,B_1) \arrow[rr, dashed] \arrow[dr, "\varphi_1"] && (X_2,B_2) \arrow[rr, dashed] \arrow[dl, "\varphi_1^+"'] \arrow[dr, "\varphi_2"] && \cdots \arrow[rr, dashed] && (X_i,B_i) \arrow[rr, dashed] \arrow[dl, "\varphi_{i-1}^+"'] \arrow[dr, "\varphi_i"] && \cdots \\
& Y_1 && Y_2 && Y_{i-1} && Y_i &
\end{tikzcd} \tag{4.2}$$

be any chain of log flips. We say that *special termination* holds for the chain if there is an i_0 such that the flipping curves are disjoint from the reduced part $\lfloor B_i \rfloor$ of the boundary for all lt pairs (X_i, B_i) with $i \geqslant i_0$.

Theorem 4.7 (reduction to pl flips ([34], 6.4-5, [21], § 18, or [10], 1.7)). *Let* $\varphi\colon (X,B) \to Y$ *be a small birational* klt *log contraction as in Definition* 4.1 *and let* $\dim X = n \geqslant 3$. *Then a* klt *log flip of* φ *exists if*

(i) *there is a flip of any pl contraction in dimension* n;
(ii) *special termination holds for any chain of log flips in the same dimension* n.

Proof (*a construction*). We follow the lines of [34], 6.4-5, and [21], 18.12 (see also [10], 1.7) and apply the so-called log flipping procedure which modifies the small klt contraction $\varphi\colon (X,B) \to Y$ by using the LMMP (and the assumptions (i) and (ii) of the theorem) into a relative strictly log minimal model $\overline{\psi}\colon (\overline{V}, \overline{B} = B_{\overline{V}}^{\log}) \to Y$ (as usual, the term 'strictly' means that $\overline{V}$ is $\mathbb{Q}$-factorial and projective over Y). Let us show that $(\overline{V}, \overline{B}) = (X^+, B^+)$ and that this is the desired log flip. Indeed, by the definition of the log birational transform $B_{\overline{V}}^{\log}$ and by Lemma 2.4, the modification into X^+ is small and $B^+ = B_{X^+}^{\log} = B_{X^+}$. Thus, the birational contraction $\overline{\psi}$ is small by the assertion (i) of Definition 4.1. Then one can readily prove the assertion (ii$^+$) by using the $\mathbb{Q}$-factorial property of X and the assertions (ii) and (iii) of Definition 4.1 (see also Comment 4.2).

The procedure has two phases. We first pass (slightly artificially) from the klt contraction to a relative lt model. Then we remove the non-exceptional reduced part of its boundary.

We begin with the passage to a relative lt model. Let us choose an effective Cartier divisor H on Y and a projective blowup $\sigma\colon V \to X$ with the following properties:

a) $\psi = \varphi \circ \sigma\colon V \to Y$ is an isomorphism outside $\operatorname{Supp} H$;

b) the log pair $(V, B_V^{\log}+H_V)$ has only strictly lt singularities (as above, the term 'strictly' means that V is $\mathbb{Q}$-factorial and projective over Y), where $H_V=\psi_*^{-1}H$ is the birational transform of H;

c) if $\tau\colon Y'\to Y$ is any birational contraction of a $\mathbb{Q}$-factorial and normal variety Y', then the prime components of τ^*H generate the relative numerical $\mathbb{R}$-space $N^1(Y'/Y)$ of divisors.

Only the statements a) and c) need explanations. To satisfy the conditions, it suffices to choose a divisor H such that the support of H contains the singularities of Y, the image $\varphi(\operatorname{Exc}\varphi)$ is a subset of $\operatorname{Supp} H$, and $\operatorname{Supp} H$ contains some prime Weil divisors of Y. Thus, the divisor H is reducible as a rule.

We apply the LMMP with $B=B_V^{\log}+H_V$ to V over Y. Here the needed log flips are pl up to perturbation of the irrational multiplicities of the boundary. Indeed, $B_V^{\log}+H_V$ contains a reduced divisor which intersects a flipping curve C negatively, because $\psi^*HC=H\psi_*C=0$, and by c) there are components intersecting C non-trivially. The log flip remains the same if we remove all reduced components of the boundary that intersect the curve C non-negatively. The other reduced components give the reduced part S required in Definition 4.3, and the remaining part is $B-S$. After a perturbation of $B-S$, one can assume that B is a $\mathbb{Q}$-boundary. By the assumptions (i) and (ii) and by well-known results of the LMMP, after finitely many steps we construct a strictly log minimal model $\overline{\psi}\colon(\overline{V}, B_{\overline{V}}^{\log}+H_{\overline{V}})\to Y$ over Y.

The next problem is to remove $H_{\overline{V}}$ from the boundary $B_{\overline{V}}^{\log}+H_{\overline{V}}$. By construction, $K_{\overline{V}}+B_{\overline{V}}^{\log}+H_{\overline{V}}$ is $\overline{\psi}$-nef. If $K_{\overline{V}}+B_{\overline{V}}^{\log}$ is $\overline{\psi}$-nef, then one can remove $H_{\overline{V}}$, and $\overline{\psi}\colon(\overline{V}, B_{\overline{V}}^{\log})\to Y$ is the desired strictly log minimal model (as at the beginning of the proof). Otherwise there is a curve $C\subset\overline{V}$ over Y such that $H_{\overline{V}}C>0$ and $(K_{\overline{V}}+B_{\overline{V}}^{\log})C<0$. Let ε be the greatest value such that $0\leqslant\varepsilon\leqslant 1$ and $K_{\overline{V}}+B_{\overline{V}}^{\log}+(1-\varepsilon)H_{\overline{V}}$ is $\overline{\psi}$-nef. The case of $\varepsilon=1$ has been studied above. Let $0\leqslant\varepsilon<1$. Then we can increase the number ε according to the following procedure. Let $0<\eta\ll\varepsilon$. If the divisor $K_{\overline{V}}+B_{\overline{V}}^{\log}+(1-\varepsilon-\eta)H_{\overline{V}}$ is not $\overline{\psi}$-nef, then we apply the relative LMMP to $(\overline{V}, B_{\overline{V}}^{\log}+(1-\varepsilon-\eta)H_{\overline{V}})$. Here a log flip (log flop for $B_{\overline{V}_i}^{\log}+(1-\varepsilon)H_{\overline{V}_i}$) at the ith step corresponds to a ray generated by a curve C_i over Y. Moreover,

$$0>(K_{\overline{V}_i}+B_{\overline{V}_i}^{\log}+(1-\varepsilon-\eta)H_{\overline{V}_i})C_i=-\eta H_{\overline{V}_i}C_i\Rightarrow H_{\overline{V}_i}C_i>0.$$

Since $0=\overline{\psi}_i^*HC_i=H_{\overline{V}_i}C_i+\sum\alpha_kE_kC_i, \alpha_k\geqslant 0$, there is an exceptional prime divisor E_k over Y such that $E_kC_i<0$ and $C_i\subset E_k\subset\operatorname{Supp}\left\lfloor B_{\overline{V}_i}^{\log}\right\rfloor$, that is, these flips are also pl.

By our assumptions, these flips exist and terminate. However, there is a subtlety with the termination here, namely, the increments of ε can be very small, and the number ε need not attain the value 1 in finitely many steps. But we can then note that the chain of $(K_{\overline{V}}+B_{\overline{V}}^{\log}+(1-\varepsilon-\eta)H_{\overline{V}})$-flips is also a chain of $(K_{\overline{V}}+B_{\overline{V}}^{\log})$-flips (with $\varepsilon+\eta=1$), since for any flipping curve C_i in the reduction algorithm we have $(K_{\overline{V}_i}+B_{\overline{V}_i}^{\log}+(1-\varepsilon-\eta)H_{\overline{V}_i})C_i<0$ and $H_{\overline{V_i}}C_i>0$. Thus, we obtain the desired log minimal model in finitely many steps.

Let us now discuss the problem of special termination.

Conditional Theorem 4.8 (special termination assuming the LMMP for lt log pairs (X, B) in dimension $\leqslant n-1$). *Consider the chain* (4.2), *where* $\varphi_i\colon X_i \to Y_i$ *is a birational contraction of an extremal ray* R_i *with* $(K_{X_i} + B_i)R_i < 0$, *and let* $\varphi_i^+\colon X_i^+ = X_{i+1} \to Y_i$ *be a log flip of* φ_i. *In this case in dimension* n *the flipping locus* (*and also the flipped locus*) *is disjoint from* $\lfloor B_i \rfloor$ *after finitely many log flips.*

Before passing to the proof, we need some notation to introduce definitions and to comment on them.

Definitions-Remarks 4.9. (i) As usual, let $\mathrm{LCS}(X, B)$ be the union of *log canonical centres*, that is, of the sets $\mathrm{c}_X E$ for prime b-divisors E with $a(E, X, B) = -1$. Since the pair (X, B) is lt, it follows that $\mathrm{LCS}(X, B) = \lfloor B \rfloor$ (we recall that X is assumed to be $\mathbb{Q}$-factorial by the definition of log flip in Conjecture 1.11).

(ii) We say that a curve C on X_i (on $X_{i+1} = X_i^+$) is *flipping* (*flopped*) if $\varphi_i(C) = \mathrm{pt.}$ (if $\varphi_i^+(C) = \mathrm{pt.}$).

(iii) Let S be a log canonical centre of (X, B). Then, as is well known (see, for example, § 3 in [34], § 16 in [21], and also [8]), there is an lt log pair of the form (S, B_S), like the pair (X, B) itself, where $K_S + B_S = (K + B)_{|S}$ by the adjunction formula (in the $\mathbb{Q}$-factorial situation; see Remark 4.4).

(iv) A birational contraction $f\colon (X, B) \to (X', B')$ of log pairs is said to be *crepant* if $K + B = f^*(K_{X'} + B')$, where $f_*B = B'$.

(v) Let $\varphi\colon U \to W$ be a birational contraction. We say that φ is of *type* (D) if $\mathrm{Exc}(\varphi)$ contains a divisor and of *type* (S) if φ is a small contraction. Suppose that two birational contractions are given:

$$U \overset{\varphi}{\to} W \overset{\psi}{\leftarrow} V.$$

For these contractions, the meaning of the notation (DD), (DS), (SS), (SD) is clear.

(vi) Let $B = \sum b_i D_i, 0 \leqslant b_i \leqslant 1$. We introduce the set $\mathbb{B} = \{b_i\}$ of non-negative integers and let

$$S(\mathbb{B}) := \left\{ 1 - \frac{1}{m} + \sum \frac{r_i b_i}{m} \;\middle|\; m \in \mathbb{N},\ r_i \in \mathbb{Z}_{\geqslant 0} \right\}.$$

(vii) Let $S \subset \mathrm{LCS}(X, B)$ be a log canonical centre. We define its *difficulty* by the formula

$$d_{\mathbb{B}}(S, B_S) := \sum_{\alpha \in S(\mathbb{B})} \#\bigl\{E \mid a(E, S, B_S) < -\alpha,\ \mathrm{c}_S E \not\subseteq \lfloor B_S \rfloor\bigr\},$$

where each $\mathrm{c}_S E$ is a centre of a prime b-divisor E of the variety S.

It is clear that the number $d_{\mathbb{B}}(S, B_S)$ is finite, since $(S \setminus \mathrm{Supp}\lfloor B_S \rfloor, B_S)$ is klt, and in this case there are only finitely many prime b-divisors with negative discrepancies (it suffices to consider a log resolution).

Proof of Theorem 4.8. (The same fact can be presented in terms of controlled discrepancies; see [39], Corollary 4.)

Step 1. After finitely many log flips the flipping locus does not contain a log canonical centre. Indeed, the number of log canonical centres is finite. The discrepancies over centres in the flipping locus increase by Lemma 2.4, and thus the number of these centres decreases.

Therefore, for any (X_i, B_i) one can assume that the flipping locus does not contain a log canonical centre. In this case the modifications $\chi_i \colon X_i \dashrightarrow X_{i+1}$ induce modifications $\chi_i|_{S_i} : S_i \dashrightarrow S_{i+1}$ of the log canonical centres. For the latter modifications we have $a(E, S_i, B_{S_i}) \leqslant a(E, S_{i+1}, B_{S_{i+1}})$ (for example, by ([38]; Monotonicity)).

Step 2. Suppose that the modification $\chi_i \colon X_i \dashrightarrow X_{i+1}$ induces a *log isomorphism* for any i, that is, an isomorphism of log pairs on each log canonical centre of dimension $(d-1)$ in LCS. In this case, after finitely many log flips the modification χ_i induces a log isomorphism on all d-dimensional centres as well. Indeed, it is clear that χ_i induces a log isomorphism on all 0-dimensional log canonical centres. It is also clear that if $\chi_i|_{S_i} : (S_i, B_{S_i}) \dashrightarrow (S_{i+1}, B_{S_{i+1}})$ is a log isomorphism, then S_i is disjoint from any flipping curve (see Step 3 below).

The proof below splits into two cases:

a) the pair $(\varphi_i|_{S_i}, \varphi_i^+|_{S_{i+1}})$ is of type (SD) or (DD);

b) the pair $(\varphi_i|_{S_i}, \varphi_i^+|_{S_{i+1}})$ is of type (SS) or (DS).

It follows from the next lemma that the difficulty decreases in the case a), and the termination holds by the LMMP for smaller dimensions in the case b).

Lemma 4.10. *The inequality*

$$d_{\mathbb{B}}(S_i, B_{S_i}) \geqslant d_{\mathbb{B}}(S_{i+1}, B_{S_{i+1}})$$

holds for difficulties for any type. Moreover, if $S_i \to T_i \leftarrow S_i^+ = S_{i+1}$ is of type (SD) *or* (DD), *then the inequality is strict, where T_i is a normalization of $\varphi_i S_i$. For these types there is a prime divisor E on S_{i+1} such that*

$$a(E, S_i, B_{S_i}) < a(E, S_{i+1}, B_{S_{i+1}}) = -\alpha$$

for some $\alpha \in S(\mathbb{B})$. Hence, one can assume that after finitely many log flips the pair $(\varphi_i|_{S_i}, \varphi_i^+|_{S_{i+1}})$ is of type (SS) *or* (DS) *for any i.*

Proof. The inequality for the difficulties follows from the general statement in [38] (Monotonicity) claiming that the discrepancies do not decrease under any log quasi-flips (cf. Lemma 3.4(i)). In addition, in the cases (SD) and (DD) there is a divisor E on S_{i+1} included in $\mathrm{Exc}(\varphi_i^+|_{S_{i+1}})$. Identifying the divisor E with the corresponding b-divisor of S, we obtain the inequality $a(E, S_i, B_{S_i}) < a(E, S_{i+1}, B_{S_{i+1}})$, where $a(E, S_{i+1}, B_{S_{i+1}}) \in S(\mathbb{B})$ by 3.10 and 4.2 in [34]. This implies the strict inequality $d_{\mathbb{B}}(S_i, B_{S_i}) > d_{\mathbb{B}}(S_{i+1}, B_{S_{i+1}})$.

Continuation of the proof of Theorem 4.8. Suppose that each of the consecutive induced modifications $(S_i, B_{S_i}) \dashrightarrow (S_{i+1}, B_{S_{i+1}})$ is of type (SS) or (DS). Here every divisor $-(K_{S_i} + B_{S_i})$ is numerically $\varphi_i|_{S_i}$-ample and every divisor $K_{S_{i+1}} + B_{S_{i+1}}$ is numerically $\varphi_i^+|_{S_{i+1}}$-ample, that is, this is a log flip in the sense of [36], § 5, and [39], Example 2. According to the LMMP (in the version of the second author ([36], § 5)), these flips must terminate. However, here we are using the extremal and strictly lt version in which the log flips are extremal and strictly lt, and termination holds for them (see Conjectures 1.11 and 1.12 above). Thus, we have some more to do. Let $(S_1^0, B_{S_1^0}^{\log}) \to S_1$ be a strictly lt birational contraction, for example, a relative strictly log minimal model of (S_1, B_{S_1}) over S_1. We apply the LMMP to the morphism $(S_1^0, B_{S_1^0}^{\log}) \to T_1$. By our assumptions, after finitely many divisorial contractions and log flips over T_1 we obtain a relative log minimal model $(S_2^0, B_{S_1^0}^{\log}) \to T_1$.

We claim that $(S_2^0, B_{S_1^0}^{\log}) = (S_2^0, B_{S_2^0}^{\log}) \to T_1$ is a relative strictly log minimal model of (S_2, B_{S_2}) over S_2. Note that the projective and $\mathbb{Q}$-factorial conditions of the strict property follow from the LMMP, because the extremal divisorial contractions and flips preserve these conditions. By Proposition 2.5(iii), there is a contraction $S_2^0 \to S_2$, since $(S_2, B_{S_2}^{\log}) \to T_1$ is a relative log canonical model of the log pairs (S_1, B_{S_1}) and (S_2, B_{S_2}) by the smallness of the contraction $S_2 \to T_1$ and by Lemma 3.4(i). We can now apply the LMMP to $(S_2^0, B_{S_2^0}^{\log}) \to T_2$, and so on. The process terminates in finitely many steps by the LMMP, which works under our assumptions.

We still have no information about flipping curves not contained in S_i that intersect the subvariety S_i. This is the next step.

Step 3. By the preceding proof, one can assume that after finitely many log flips the subvariety S_i contains neither flipping nor flipped curves, that is, $S_i \cong S_{i+1}$ for any $i \gg 0$ and the boundaries are preserved. Thus, each pair $(S_i, B_{S_i}) \cong (S_{i+1}, B_{S_{i+1}})$ is a log isomorphism for any $i \gg 0$. Hence, by the monotonicity (see Lemma 3.4(i)) and by the adjunction formula (cf. the proof of 4.1 in [34]), the flipping curves are disjoint from these subvarieties S_i.

Corollary 4.11. *Since the LMMP holds in dimension* 3, *special termination holds in dimensions* $n \leqslant 4$.

§ 5. Reduction of 3-fold pl flips to dimension 2

5.1. In § 4 we proved that the existence problem for klt log flips in dimension 3 can be reduced to the existence of pl flips in the same dimension, as in Definition 4.3. For rational boundaries B both existence problems in any dimension amount to the finite generation of the divisorial algebra

$$\mathcal{R}_{X/Z}(K+B) = \bigoplus_{m \geqslant 0} \varphi_* \mathcal{O}_X(m(K+B)).$$

To clarify the cases when we work in the pl setting, we write $S+B$ instead of B in what follows, where S is the reduced part of the boundary, $B = \sum b_i D_i$, $0 \leqslant b_i < 1$, and each b_i is rational. The problem can readily be reduced to the case in which $K+S+B$ is plt, B is rational, and S is an irreducible surface. Since our objective is a reduction of 3-fold pl flips to dimension 2, after restriction to the surface S we use adjunction: $(K+S+B)|_S = K_S + B_S$, where $B_S = \operatorname{Diff}(B)$ is a boundary on S, the so-called different (see [34], § 3, and [21], § 16). We do not need the explicit form of the different and only note that if $K+S+B$ is plt (which is the case now), then K_S+B_S is klt. The converse also holds (see, for example, [34], 3.3–4, and [21], 17.6). The existence problem for flips is local with respect to the base, and hence we can (and shall) assume that the variety Z is affine. We also recall that X is supposed to be $\mathbb{Q}$-factorial.

If $\varphi\colon X\to Z$ is a pl contraction, then S contains the entire φ-exceptional locus, because $SC<0$ holds by definition for any curve $C\subset \operatorname{Exc}(\varphi)$.

The first step in the proof of finite generation of the algebra $\mathcal{R}_{X/Z}(K+S+B)$ is to determine its restriction $R^0_{S/Z}=\oplus R^0_m$ to S, where

$$R^0_m:=\operatorname{Im}(H^0(X,m(K+S+B))\to H^0(S,m(K_S+B_S))).$$

One can readily see that the algebra $\mathcal{R}_{X/Z}(K+S+B)$ is finitely generated if and only if its restriction $R^0_{S/Z}$ is finitely generated (see Proposition 5.4). However, in general $R^0_{S/Z}$ is not a divisorial algebra (in contrast to $\mathcal{R}_{X/Z}(K+S+B)$), because the above restrictions can fail to be surjective. In this connection we introduced the notion of pseudo-b-divisorial algebra ([37], 4.10). Moreover, since linear systems $|m(K+S+B)|$ and their restrictions to S can have a base locus in general, one must take care about their resolutions. In this connection the language of b-divisors (birational divisors) was introduced. In this language the algebra $R^0_{S/Z}$ (more precisely, its integral closure) is pseudo-b-divisorial (a pbd algebra). The precise definitions are given below.

In general there are no reasons to claim that the pbd algebra $R^0_{S/Z}$ is finitely generated. In [37], § 4, two essential (necessary) conditions are proposed under which this can be proved for the induced birational contraction S/Z (so far, only in dimensions $\leqslant 2$; this is a conjecture in the general case):

a) boundedness (see Definition 5.19 below);
b) log canonical asymptotic (or lca) saturation (see Definition 5.23 below).

Any pbd algebra satisfying the conditions a) and b) for some birational weak Fano log contraction $\varphi\colon X\to Z$ (that is, (X,B) is klt and $-(K+B)$ is φ-nef; the property of being φ-big follows from the birationality of φ) is called an FGA-algebra.[2] The main conjecture claims that each FGA-algebra is finitely generated (see Conjecture 5.25 below).

Any proof of this conjecture, even in the 1-dimensional case, requires some results from the theory of rational approximations (see 4.41 in [37], 2.1 in [3], and 4.10 in [7]). The 2-dimensional case makes essential use of the birational geometry of surfaces. The main statement (see Theorem 6.4 below) claims that any saturated mobile b-divisor $\mathbb{M}$ on a non-singular surface has no base points, that is, it is free. The lca saturation condition is very important as well, and we certainly use the language of function algebras and b-divisors in an essential way. In what follows, we also use the nice survey [7] by Corti.

[2] *Russian Editor's note*: Alessio Corti ("3-fold flips after Shokurov", in *Flips for 3-folds and 4-folds*, a book in preparation; see `http://www.dpmms.cam.ac.uk/ corti/flips.html`) refers to these algebras as Shokurov algebras.

Definition 5.2. Let $\varphi\colon X \to Z$ be a birational contraction onto an affine variety Z with coordinate ring $A = H^0(Z, \mathcal{O}_Z)$. A *function algebra* on X/Z is a graded A-subalgebra of $V \subset k(X)[T]$, that is, $V = \bigoplus_{i\geqslant 0} V_i$, $V_0 = A$, $V_i \subset k(X)$, and $V_iV_j \subset V_{i+j}$, where the multiplication is defined as ordinary multiplication of functions. In addition, each A-module V_i is assumed to be finitely generated over A (is coherent). A function algebra is said to be *bounded* if there is an integral Weil divisor D on X such that $V_i \subset H^0(X, iD)$ for each i. In this case we say that V is *bounded by* D.

A *truncation of the graded algebra* is a graded A-subalgebra $V^{(d)} := \bigoplus_i V_{id}$ for a fixed positive integer $d \neq 0$.

As is well known, a graded function algebra is finitely generated if and only if any truncation of this algebra is finitely generated ([37], 4.6). (However, this assertion fails for more general graded algebras ([37], 4.5).) In what follows, we usually consider function algebras up to truncation or even up to quasi-isomorphism (and often do not mention this explicitly).

Definition 5.3. Let X be a normal variety and let $S \subset X$ be a normal irreducible subvariety of codimension 1. Let $\mathcal{O}_{X,S} \subset k(X)$ be the local ring of regular functions at a generic point of S and let $\mathfrak{m}_{X,S} \subset \mathcal{O}_{X,S}$ be the maximal ideal of this ring. In this case the quotient ring $k(S) = \mathcal{O}_{X,S}/\mathfrak{m}_{X,S}$ is the field of rational functions on S.

A function algebra $V = \bigoplus V_i$ is said to be *regular along* S if the following conditions hold:

(i) $V_i \subset \mathcal{O}_{X,S} \subset k(X)$ for all i;
(ii) $V_1 \not\subset \mathfrak{m}_{X,S}$.

If V is regular along S, then the *restriction* $V^0 = V_{|S} = \operatorname{res}_S V$ is the function algebra $V^0 = \bigoplus V_i^0$, where

$$V_i^0 = \operatorname{Im}(V_i \to k(S)).$$

If V is bounded by a Cartier divisor D and $\operatorname{Supp} D \not\subset S$, then the restriction $V^0 = \operatorname{res}_S V$ is obviously also bounded (for example, by the divisor $D_{|S}$).

Let $(X, S+B)$ be a plt pair and $\varphi\colon X \to Z$ a flipping contraction. As we know, the flip exists if and only if the A-algebra $\mathcal{R}_{X/Z}(K+S+B)$ is finitely generated. This is a function algebra with the natural inclusions $H^0(X, i(K+S+B)) \subset k(X)$ corresponding to the multiple divisors $i(K+S+B)$. Since $\rho(X/Z) = 1$, one can take any φ-negative $\mathbb{Q}$-divisor D instead of $K+S+B$. Then the relation $D \sim_{\mathbb{Q}} r(K+S+B)$ holds for some positive number $r \in \mathbb{Q}$, and thus the divisorial algebras $\mathcal{R}_{X/Z}(K+S+B)$ and $\mathcal{R}_{X/Z}D$ are *quasi*-isomorphic, that is, they have a common truncation up to multiplication by a rational function $s \in k(X)$ (more precisely, by s^i for elements of the algebra of degree i). In the case of a pl contraction we have $S \sim_{\mathbb{Q}} D$ for some divisors D of this kind and, moreover, $S \not\subset \operatorname{Supp} D$ for an appropriate divisor D .

Proposition 5.4 ([37], 3.43, and [7], 3.1.5). *Let $\varphi\colon X \to Z$ be a small contraction, let S be a prime divisor on X, and let D be a Weil $\mathbb{Q}$-divisor such that $D \sim_{\mathbb{Q}} S$ and $S \not\subset \operatorname{Supp} D$. Then a D-flip exists if and only if the restricted algebra $R^0 = \operatorname{res}_S \mathcal{R}_{X/Z} D$ is finitely generated.*

Proof. As above, it suffices to show that the algebra $\mathcal{R}(D) = \mathcal{R}_{X/Z}D$ is finitely generated. Up to quasi-isomorphism, we can assume that $D \sim S$, that is, there is a rational function $t \in k(X)$ such that $S = D + (t)$, where (t) is the divisor of zeros and poles of t for some $t \in \mathcal{R}_1(D)$. By our assumptions, the algebra $\mathcal{R}(D)$ is regular along S. If S is not normal, then the restriction is considered on a normalization of S/Z (in applications S is normal, as follows, for example, from the plt condition). Let us show that the kernel of the restriction $\mathcal{R}(D) \to R^0$ is the principal ideal generated by t. Indeed, suppose that $u \in \mathcal{R}_i(D) \subset k(X)$ is restricted to $0 \in R^0$. This means that $(u) + iD \geqslant 0$, and u vanishes on S, that is, $(u) + iD - S \geqslant 0$. Therefore, we can write $u = tu'$, where $(u') + (i-1)D = (u) + D - S + (i-1)D \geqslant 0$. In other words, $u' \in \mathcal{R}_{i-1}(D)$, and $u \in (t) = \mathcal{R}(D)(t)$. Thus, R^0 is finitely generated if and only if $\mathcal{R}(D)$ is finitely generated.

Let us now turn our attention to b-divisors. We recall that their introduction was connected with a consideration of linear systems with base points together with all possible resolutions of these systems, that is, with chains of proper models over X.

Definition 5.5 [9]. Let X be a normal variety. An *integral* b-*divisor* of X is an element

$$\mathbb{D} \in \lim_{Y \to X} \operatorname{Div} Y$$

of the inverse limit, where the projective limit is taken with respect to proper birational morphisms $f\colon Y \to X$ with homomorphisms $f_*\colon \operatorname{Div} Y \to \operatorname{Div} X$.

The b-divisors with multiplicities in $\mathbb{Q}$ and $\mathbb{R}$ can be defined in a similar way. The group of b-divisors of X is denoted by $\operatorname{Div}(X)$. The direct image homomorphism $f_*\colon \operatorname{Div}(Y) \to \operatorname{Div}(X)$ is an isomorphism, and thus we can identify the b-divisors of X with the b-divisors of Y.

The *trace* of a b-divisor $\mathbb{D}$ on a model $Y \to X$ (or on any proper birational transformation) is the ordinary (Weil) divisor on Y,

$$\mathbb{D}_Y := \sum_{\substack{F \text{ ranges over} \\ \text{the prime divisors on } Y}} d_F F$$

if $\mathbb{D} = \sum d_F F$, where F ranges over the prime b-divisors, that is, over the divisorial valuations of the field $k(X)$ with $\mathrm{c}_X F \neq \varnothing$ (for example, the projective limits for systems of birational transforms of ordinary prime divisors F of X on all models $Y \to X$). We sometimes denote a prime b-divisor identified with a prime divisor $F \in \operatorname{Div} X$ by $\underline{F}$. For a $\mathbb{Q}$-(or $\mathbb{R}$)-Cartier divisor D on X we denote by $\overline{D}$ the *Cartier completion* of D with the trace $\overline{D}_Y = f^*D$ on each model $f\colon Y \to X$.

Examples 5.6. 1. Let $f \neq 0 \in k(X)$ be a rational function. Then the b-divisor of f is defined as

$$\mathbf{div}(f) := \sum \operatorname{mult}_E(f)E,$$

where E ranges over the prime b-divisors of X and $\operatorname{mult}_E(f)$ stands for the multiplicity of f along E (we recall that the divisor E can be identified with a discrete valuation of $k(X)$).

2. Let X be a normal variety. A *canonical* b-*divisor* $\mathbb{K} = \operatorname{div}(\omega)$ is well defined on X, where $\omega \neq 0 \in \Omega_{k(X)}$ is a rational differential form of highest degree on X. Indeed, on a (non-singular) model $f\colon Y \to X$ we have

$$\mathbb{K}_Y = (f^*\omega) = K_Y,$$

and hence $f_*\mathbb{K}_Y = f_*(f^*\omega) = (\omega) = K_X$, that is, this projective limit satisfies the compatibility conditions.

3. Let X be a normal variety and let $D = \sum d_i D_i$ be an $\mathbb{R}$-divisor on X such that $K + D$ is an $\mathbb{R}$-Cartier divisor. In this case a *discrepancy* b-*divisor* $\mathbb{A} = \mathbb{A}(X, D) = \mathbb{K} - \overline{(K + D)}$ is well defined for the pair (X, D) with the traces $\mathbb{A}_Y$ on the models $f\colon Y \to X$, where $\mathbb{A}_Y$ is also given by the formula

$$K_Y = f^*(K + B) + \mathbb{A}_Y.$$

For b-divisors, as well as for ordinary divisors, one can define sheaves of $\mathcal{O}_X$-modules, which need not be coherent in general.

Definition 5.7. Let $\mathbb{D} = \sum d_F F$ be a b-divisor of X (X can be affine). In this case a special sheaf $\mathcal{O}_X(\mathbb{D})$ of $\mathcal{O}_X$-modules on X is well defined. Namely, for each open subset $U \subset X$ we let

$$\Gamma(U, \mathcal{O}_X(\mathbb{D})) = \{f \in k(X) \mid \operatorname{mult}_F f + d_F \geqslant 0 \\ \text{for the prime b-divisors } F \text{ of } X\}.$$

By definition, it is clear that $\mathcal{O}_X(\mathbb{D})$ is a subsheaf of the constant sheaf $k(X)$ (and even of the sheaf $\mathcal{O}_X(\mathbb{D}_X)$). We note that $H^0(X, \mathbb{D}) := H^0(X, \mathcal{O}_X(\mathbb{D})) \subset H^0(X, \mathbb{D}_X)$, that is, the b-divisor $\mathbb{D}$ determines a linear (sub)system of the complete linear system of the divisor $\mathbb{D}_X$. Sometimes this determination is given by base conditions which were the starting point when introducing b-divisors.

A sheaf $\mathcal{O}_X(\mathbb{D})$ can be coherent even if the b-divisor $\mathbb{D}$ is defined by infinitely many non-trivial base conditions. For instance, this can happen if there is a finite divisor $D' = \mathbb{D}_Y$ on some model $f\colon Y \to X$ such that $\mathcal{O}_X(\mathbb{D}) = f_*\mathcal{O}_Y(D')$ (stabilization).

The crucial point in the construction of 3-fold pl flips is to find a single model $f\colon Y \to X$ such that the condition $\mathcal{O}_X(\mathbb{D}_i) = f_*\mathcal{O}_Y(\mathbb{D}_{iY})$ holds for a special infinite sequence of b-divisors $\mathbb{D}_i$ of X.

The following lemma gives natural conditions ensuring the coherence of the sheaf $\mathcal{O}_X(\mathbb{D})$ in some cases. We use this result below.

Lemma 5.8 ([37], the proof of 4.46, and [7], 3.2.7). *Let X be a non-singular variety and let $D = \sum d_i D_i$ be an $\mathbb{R}$-divisor whose support $\sum D_i$ is a divisor with* (*simple*) *normal crossings. Let $\mathbb{A} = \mathbb{A}(X, D)$ be the discrepancy b-divisor, let $f\colon Y \to X$ be a normal blowup, and let*

$$\lceil \mathbb{A}_Y \rceil = f^* \lceil \mathbb{A}_X \rceil + \sum e_i E_i,$$

where each divisor E_i is exceptional. Then $e_i \geqslant 0$ for all i.

The first corollary of this result is as follows.

Corollary 5.9 ([37], 4.47, [7], 3.2.8). *Let X be a normal variety and let $D = \sum d_i D_i$ be an $\mathbb{R}$-divisor for which a discrepancy b-divisor $\mathbb{A} = \mathbb{A}(X, D)$ is well defined. Then $\mathcal{O}_X(\lceil \mathbb{A} \rceil)$ is a coherent sheaf.*

We note that the divisor D in the corollary is an $\mathbb{R}$-divisor (under the assumption that $K + D$ is an $\mathbb{R}$-Cartier divisor), that is, no assumptions about the singularities of (X, D) are used.

To prove the corollary, it suffices to choose a log resolution and then use the previous lemma.

The following definition generalizes some notions natural for ordinary divisors to the case of b-divisors.

Definition 5.10. A b-divisor $\mathbb{D} \in \operatorname{Div}(X)$ is called a b-*Cartier* b-divisor (a b-*free*, b-*nef*, b-*semi-ample* b-divisor, and so on) if there is a model $Y \to X$ such that $D = \mathbb{D}_Y$ is a Cartier divisor (a free, nef, semi-ample divisor, and so on) and $\mathbb{D} = \overline{D}$. One can define similar notions for $\mathbb{D} \in \operatorname{Div}(X) \otimes \mathbb{Q}$ or $\operatorname{Div}(X) \otimes \mathbb{R}$.

We now consider the very important notion of saturation for divisors and b-divisors. This notion was originally introduced in [37], 4.33.

Definition 5.11. (i) The *mobile part* of an (effective) integral divisor D on X is the divisor

$$\operatorname{Mov} D = D - F,$$

where $F = \operatorname{Fix} |D|$ is the fixed part of the complete linear system $|D|$.

(ii) Let D be a divisor as above, and let C be an $\mathbb{R}$-divisor on X such that $\lceil D + C \rceil$ is effective up to linear equivalence, that is, $|\lceil D + C \rceil| \neq \varnothing$. We say that D is C-*saturated* if $\operatorname{Mov} \lceil D + C \rceil \leqslant D$. A divisor D is always C-saturated when $|\lceil D + C \rceil| = \varnothing$.

(iii) An (effective) b-divisor $\mathbb{D}$ of X is said to be $\mathbb{C}$-*saturated* if $\mathbb{D}_Y$ is $\mathbb{C}_Y$-saturated on any *sufficiently high* model $Y \to X$. This means that there is a concrete model $Y \to X$ on which saturation holds and remains valid for any higher model $Y' \to Y \to X$. If $\mathbb{C} = \mathbb{A}(X, B)$ is the discrepancy b-divisor of (X, B), then we say that $\mathbb{D}$ is *log canonically* (lc) *saturated*, more precisely, log canonically saturated over (X, B).

(iv) An (effective) b-divisor $\mathbb{D}$ of X is said to be *exceptionally saturated* if it is $\mathbb{E}$-saturated for any $\mathbb{R}$-b-divisor $\mathbb{E}$ of X that is exceptional on X.

Example 5.12. The Cartier completion $\overline{D}$ of an integral $\mathbb{Q}$-Cartier Weil divisor D is exceptionally saturated. This follows from the well-known fact that

$$f_* \mathcal{O}_Y\Bigl(\Bigl\lceil f^* D + \sum e_i E_i \Bigr\rceil\Bigr) = f_* \mathcal{O}_Y(\overline{D}) = \mathcal{O}_X(D)$$

for all models $f \colon Y \to X$, where all divisors E_i are exceptional and $e_i \geqslant 0$ for all i.

To prove a statement on the saturation of a b-divisor, one must first find a concrete model $Y \to X$ on which saturation holds. The following lemma indicates conditions under which the saturation on a model $Y \to X$ can be extended to higher models $Y' \to Y \to X$. This lemma readily follows from Lemma 5.8.

Lemma 5.13 ([37], 6.36, [7], 3.3.8). *Let (X, D) be a pair with an $\mathbb{R}$-divisor D on X such that $K + D$ is an $\mathbb{R}$-Cartier divisor. Let $\mathbb{D}$ be a b-divisor of X and let $Y \to X$ be a model such that the following conditions hold*:

(i) *Y is non-singular, and $\mathbb{D}_Y + \mathbb{A}_Y$ is a divisor with (simple) normal crossings*;
(ii) $\mathbb{D} = \overline{\mathbb{D}_Y}$.

Then $\mathbb{A}$-saturation, where $\mathbb{A} = \mathbb{A}(X, D)$, holds on Y if and only if it holds for any higher model $Y' \to Y \to X$, that is,

$$\operatorname{Mov}\lceil \mathbb{D}_Y + \mathbb{A}_Y \rceil \leqslant \mathbb{D}_Y \iff \operatorname{Mov}\lceil \mathbb{D}_{Y'} + \mathbb{A}_{Y'} \rceil \leqslant \mathbb{D}_{Y'}.$$

The following obvious statement is useful in our situation when taking the restriction to S.

Lemma 5.14. *Let $(X, S + B)$ be a* plt *log pair with the discrepancy b-divisor $\mathbb{A} = \mathbb{A}(X, S + B)$. If a b-divisor $\mathbb{D}$ is exceptionally saturated, then it is $(\mathbb{A} + \underline{S})$-saturated.*

Proof. The positive part of the b-divisor $\mathbb{A} + \underline{S}$ is exceptional on X.

We now return to our problem of finite generation for $\mathcal{R}_{X/Z}(K + S + B) = \bigoplus_{m \geqslant 0} H^0(X, m(K + S + B))$ and note that the divisors $m(K + S + B)$ can be replaced by their mobile parts $\operatorname{Mov}(m(K + S + B))$, because this does not change the A-submodules $H^0(X, m(K+S+B))$ of functions and does not affect the problem of finite generation. However, after the fixed parts are removed, the algebra can fail to be divisorial in general. Thus, taking into account the base conditions necessarily leads to pbd algebras. To this end, one must isolate the mobile part of a b-divisor and assign a function algebra to a sequence of mobile parts. Let us pass to this construction.

Definition 5.15. (i) An integral b-divisor $\mathbb{D}$ is said to be *mobile* if $\mathbb{D}$ is b-free.

(ii) A b-*mobile part* of an integral $\mathbb{Q}$-Cartier (effective) divisor D on X with respect to an A-submodule $L \subset H^0(X, D)$ ($D \in L$ up to linear equivalence) is a b-divisor $\operatorname{Mov}_L D$ with the trace $(\operatorname{Mov}_L D)_Y = f^*D - \operatorname{Fix} f^* |L|$ on each model $f\colon Y \to X$.

(For a $\mathbb{Q}$-factorial divisor X the b-mobile part is well defined for any integral divisor D on X.) The mobile part $\operatorname{Mov}_L D$ is a mobile b-divisor (this amounts to the existence of a resolution of the base locus of a linear system which is not necessarily complete).

The b-mobile part can be characterized by the formula (see [37], 4.15, [10], 2.5, [7], 3.4.4)

$$\operatorname{mult}_E \operatorname{Mov}_L D = -\min_{\varphi \in L} \operatorname{mult}_E \varphi$$

for all prime b-divisors E of X, where $\operatorname{mult}_E(-)$ stands for the multiplicity along the b-divisor E. The right-hand side seems to be independent of D. However, this is not true, since the inclusion $L \subset k(X)$ depends on D (by definition, $H^0(X, D) = \{\varphi \in k(X) \mid (\varphi) + D \geqslant 0\}$).

In the case of complete linear systems (for $L = H^0(X, D)$) we simply write $\operatorname{Mov} D$ for the b-mobile part of D. It is characterized by the formula

$$(\operatorname{Mov} D)_Y = \operatorname{Mov} \lfloor f^*D \rfloor$$

for all modules $f\colon Y \to X$.

Lemma 5.16. *The b-mobile part* $\operatorname{Mov} D$ *of an integral* $\mathbb{Q}$*-Cartier Weil divisor* D *on* X *is exceptionally saturated.*

Proof. This follows from the characterization $(\operatorname{Mov} D)_Y = \operatorname{Mov} \lfloor f^*D \rfloor$ on the models $Y \to X$.

Of course, an arbitrary mobile b-divisor $\mathbb{M}$ need not be exceptionally saturated (as can be seen by simple examples, for instance, for a blowup of a point in the plane).

We now proceed to a discussion of the following important result: under certain conditions, exceptional saturation over X implies lc saturation over S. In general it is unclear how to define a restriction of an arbitrary b-divisor $\mathbb{D}$ to S (even if S is not contained in the support of $\mathbb{D}_X$). However, this can be done successfully for any mobile b-divisor.

Definition 5.17 ([37], 7.1, [7], 3.5.1). Let $\mathbb{M}$ be a mobile b-divisor of a variety X and let $S \subset X$ be a prime divisor not contained in the support of $\mathbb{M}_X$. We define the *mobile restriction* $\mathbb{M}^0 = \operatorname{res}_S \mathbb{M}$ to S (more precisely, to a normalization of S if S is not normal as a subvariety) as follows. We choose a model $Y \to X$ such that the trace $\mathbb{M}_Y$ is free and $\mathbb{M} = \overline{\mathbb{M}_Y}$. Let S' be the birational transform of S (or its normalization if S is not normal) on Y. Then we set

$$\operatorname{res}_S \mathbb{M} = \overline{\mathbb{M}_{H}\big|_{S'}}.$$

In other words, $\operatorname{res}_S \mathbb{M}$ is the Cartier completion of the ordinary restriction $\mathbb{M}_{H}\big|_{S'}$ for Cartier divisors. Since $\operatorname{Div} S' = \operatorname{Div} S$, we obtain a mobile b-divisor $\mathbb{M}^0 = \operatorname{res}_S \mathbb{M}$ of S. One can readily see that the mobile restriction is independent of the choice of $Y \to X$.

We note that mobile restriction is *additive*, that is, it satisfies the condition

$$(\mathbb{M}_1 + \mathbb{M}_2)^0 = \mathbb{M}_1^0 + \mathbb{M}_2^0.$$

Lemma 5.18 ([37], 4.50, [7], 3.5.4). *Let* $\varphi\colon (X, S+B) \to Z$ *be a weak Fano* plt *birational contraction onto* Z, *that is, the divisor* $K+S+B$ *is* plt *and* $-(K+S+B)$ *is* φ*-nef* $(-(K+S+B)$ *is* φ*-big since* $X \to Z$ *is birational*$)$. *Let* $\mathbb{M}$ *be a mobile b-divisor of* X. *Suppose that* S *is not contained in the support of* $\mathbb{M}_X$.

Let $\mathbb{A} = \mathbb{A}(X, S+B)$ *be a discrepancy b-divisor as usual. Suppose that* $\mathbb{M}$ *is* $(\mathbb{A} + \underline{S})$*-saturated* (*by Lemma* 5.3, *this holds if* $\mathbb{M}$ *is exceptionally saturated*). *In this case the mobile restriction* $\mathbb{M}^0 = \operatorname{res}_S \mathbb{M}$ *is* lc *saturated over* (S, B_S), *that is,* $\mathbb{A}(S, B_S)$*-saturated.*

Proof. Let $f\colon Y \to X$ be a log resolution of $(X, S+B)$, let F_i be the f-exceptional prime divisors, and let D'_j and S' be the birational transforms of the prime components of B and S, respectively. One can choose the model $Y \to X$ to be high enough so that the following conditions hold:

a) $\mathbb{M}_Y$ is free and $\mathbb{M} = \overline{\mathbb{M}_Y}$;
b) $\mathbb{A}'$-saturation for $\mathbb{M}$ holds on Y, that is,

$$\operatorname{Mov} \lceil \mathbb{M}_Y + \mathbb{A}'_Y \rceil \leqslant \mathbb{M}_Y,$$

where $\mathbb{A}' = \mathbb{A} + \underline{S}$.

We write

$$K_Y = f^*(K+S+B) - S' - \sum b_j D'_j + \sum a_i F_i.$$

Then

$$\mathbb{A}_Y = -S' - \sum b_j D'_j + \sum a_i F_i$$

by the definition of discrepancies. We claim that the restriction $\mathbb{M}^0 = \operatorname{res}_S \mathbb{M}$ is lc saturated. To this end, we first verify the saturation on $S' \to S$. Let $\mathbb{A}^0 = \mathbb{A}(S, B_S)$. By the adjunction formula one has $\mathbb{A}^0_{S'} = \mathbb{A}'_{Y}{}_{|S'}$. On the other hand,

$$H^1(Y, \lceil \mathbb{M}_Y + \mathbb{A}_Y \rceil) = H^1(Y, K_Y + \lceil -f^*(K+S+B) + \mathbb{M}_Y \rceil) = 0$$

by the Kawamata–Viehweg vanishing theorem, because $-(K+S+B)$ is nef and big over Z by our assumptions. Therefore, the natural restriction

$$H^0(Y, \lceil \mathbb{M}_Y + \mathbb{A}'_Y \rceil) \to H^0\left(S', \left\lceil \mathbb{M}_{H_{|S'}} + \mathbb{A}^0_{S'} \right\rceil\right)$$

is surjective. Since $\operatorname{Mov} \lceil \mathbb{M}_Y + \mathbb{A}'_Y \rceil \leqslant \mathbb{M}_Y$, it follows from the surjectivity that $\operatorname{Mov} \left\lceil \mathbb{M}_{H_{|S'}} + \mathbb{A}^0_{S'} \right\rceil$ contains nothing but the restriction $\operatorname{Mov} \mathbb{M}_{H_{|S'}}$. This proves saturation on $S' \to S$.

Saturation on any higher model $S'' \to S' \to S$ follows from Lemma 5.13.

We now consider some properties of pbd algebras regarded as function algebras associated with an (upper) semi-additive system (sequence) $\mathbb{M}_* = \{\mathbb{M}_i\}$, $0 < i \in \mathbb{N}$, of integral mobile b-divisors $\mathbb{M}_i$, $i \in \mathbb{N}$, of a variety X. *Semi-additivity* means that $\mathbb{M}_1 \geqslant 0$ and $\mathbb{M}_i + \mathbb{M}_j \leqslant \mathbb{M}_{i+j}$ for any $i, j \in \mathbb{N}$. To such a system we can assign its (upper) *convex characteristic* system of $\mathbb{Q}$-b-divisors $\mathbb{D}_* = \{\mathbb{D}_i\}$, $0 < i \in \mathbb{N}$,

$$\mathbb{D}_i = \frac{1}{i}\mathbb{M}_i.$$

Convexity means that $\mathbb{D}_1 \geqslant 0$ and $i\mathbb{D}_i + j\mathbb{D}_j \leqslant (i+j)\mathbb{D}_{i+j}$ for any $i, j \in \mathbb{N}$. We say that a system $\mathbb{D}_*$ is *bounded* (above) if there is a b-divisor $\mathbb{D}$ such that $\mathbb{D}_i \leqslant \mathbb{D}$ for all $i \in N$.

A convex system increases in the (multiplicative arithmetic) sense, that is, $\mathbb{D}_i \leqslant \mathbb{D}_j$ if i divides j. If $\mathbb{D}_*$ is bounded, then the convexity implies the existence of the following limit which determines an $\mathbb{R}$-b-divisor:

$$\mathbb{D} = \lim_{i\to\infty} \mathbb{D}_i = \sup \mathbb{D}_i \in \operatorname{Div} X \otimes \mathbb{R}.$$

If a semi-additive system of (mobile) b-divisors $\mathbb{M} = \{\mathbb{M}_i\}$ is given, then after passing to the limit it is more convenient to work with the convex characteristic system $\mathbb{D}_* = \{\mathbb{D}_i\}$, where $i\mathbb{D}_i = \mathbb{M}_i$.

Definition 5.19. By a pbd *algebra* we mean a function algebra $R = R(X, \mathbb{D}_*)$ associated with a convex sequence $\mathbb{D}_*$ of $\mathbb{R}$-b-divisors. By definition,

$$R_i = H^0(X, i\mathbb{D}_i) = H^0(X, \mathcal{O}_X(i\mathbb{D}_i)) \text{ for all } i \geqslant 1, \text{ and } R_0 = A = H^0(Z, \mathcal{O}_Z)$$

on an affine variety Z (as above). The definition of the sheaves $\mathcal{O}_X(i\mathbb{D}_i)$ implies that $R_i \subset k(X)$, and the convexity of $\mathbb{D}_*$ implies that $R_iR_j \subset R_{i+j}$, where the product is defined by the multiplication of functions. We also assume that the sequence $\{\mathbb{D}_i\}$ is such that the finite generation property (the coherence property) holds over A: each A-module R_i is finitely generated. Thus, a pbd algebra is a function algebra, and $\mathbb{D}_*$ is called a *characteristic system* of it if each divisor $i\mathbb{D}_i$ is mobile. The algebra is bounded if and only if its characteristic sequence is bounded.

A pbd algebra can be defined similarly for a semi-additive sequence $\mathbb{M}_*$ of mobile b-divisors by setting $R_i = H^0(X, \mathbb{M}_i)$ and $R_0 = A$. In this case the sequence $\mathbb{M}_*$ is called the *mobile sequence* of the pbd algebra. The prefix 'pseudo' is added to distinguish pbd algebras from b-divisorial algebras of the form

$$\bigoplus_{m\geqslant 0} H^0(X, m\mathbb{D})$$

associated with a single b-divisor $\mathbb{D}$.

We have $R(X, \mathbb{D}_*) = R(Z, \mathbb{D}_*)$ for any birational contraction $X \to Z$ onto a normal affine variety Z, and thus we can sometimes work with Z instead of X, that is, assume that X is affine (but possibly non-$\mathbb{Q}$-factorial).

Lemma 5.20 ([37], 4.15, [7], 3.6.7). *Let $X \to Z$ be a birational contraction onto an affine variety Z and let $R = R(X, \mathbb{D}_*)$ be a* pbd *algebra on X/Z. Then there is a semi-additive sequence $\mathbb{M}_*$ of mobile b-divisors of X such that $R = R(X, \mathbb{M}_*)$. In other words, any* pbd *algebra is associated with a mobile sequence, and hence with a characteristic sequence.*

Proof. Let us consider Z instead of X. Then we can suppose that $Z = X$ is affine. Each $V_i = H^0(X, i\mathbb{D}_i) \subset H^0(X, i\mathbb{D}_{iX})$ is an A-submodule of finite type. We take the b-mobile part of $i\mathbb{D}_{iX}$ with respect to V_i,

$$\mathbb{M}_i = \operatorname{Mov}_{V_i} i\mathbb{D}_{iX}.$$

It is clear that $H^0(X, \mathbb{M}_i) = H^0(X, i\mathbb{D}_i)$.

Lemma 5.21 (limiting criterion). *Let $X \to Z$ be a birational contraction onto an affine variety Z. A* pbd *algebra $R = R(X, \mathbb{D}_*)$ (given by its characteristic sequence $\mathbb{D}_*$) is finitely generated if and only if there is a positive integer m such that $\mathbb{D}_{mi} = \mathbb{D}_m$ for any i.*

By the previous lemma, each pbd algebra can be defined by a characteristic sequence of b-divisors $\mathbb{D}_i = \mathbb{M}_i/i$.

Proof. Suppose that $\mathbb{D}_{mi} = \mathbb{D}_i$ for all i. We claim that R is finitely generated. Taking a truncation, we can assume that $m = 1$. Then

$$R = R(X, \mathbb{M}_*) = \bigoplus_{i\geqslant 0} H^0(X, i\mathbb{M})$$

is an ordinary b-divisorial algebra associated with a mobile b-divisor $\mathbb{M} = \mathbb{D}_1$. Let $Y \to X$ be a model on which $\mathbb{M}_Y$ is free and $\mathbb{M} = \overline{\mathbb{M}_Y}$. In this case $R = R(Y, \mathbb{M}_Y)$ is a divisorial algebra associated with the free divisor $\mathbb{M}_Y$. As is well known, such algebras are finitely generated. Conversely, suppose that R is finitely generated. One can assume that, up to truncation, R is generated by functions of degree 1 corresponding to the mobile b-divisor $\mathbb{M}_1 = \mathbb{D}_1$. Then

$$\operatorname{Proj} R = \operatorname{Proj} \bigoplus_{i \geqslant 0} H^0(X, i\mathbb{M}_1),$$

that is, the mobile system of R is b-divisorial. This implies that $\mathbb{M}_{mi} = i\mathbb{M}_m$ for all i and for any positive integer m. Passing to the characteristic sequence, we obtain the desired result (see [37], 4.16 and 4.28).

Lemma 5.22 ([37], 4.15, [7], 3.8.2). *Let $V = \oplus_{i \geqslant 0} V_i$ be a function algebra on X/Z. Then there is a* pbd *algebra $V \subset R = R(X, \mathbb{M}_*)$ that is integrally closed in $k(X)/Z$ and such that*

a) *V is bounded $\Longleftrightarrow$ R is bounded;*
b) *V is finitely generated $\Longleftrightarrow$ R is finitely generated.*

Proof. Working with Z instead of X and identifying Z with X, one can assume that X is affine. By the definition of a function algebra, $V_i \subset k(X)$ is an A-submodule of finite type (a fractional ideal) for each i, where $A = H^0(X, \mathcal{O}_X)$ is the coordinate ring of X. It is clear that $V_i \subset H^0(X, \mathbb{M}_i)$ for a mobile b-divisor $\mathbb{M}_i$ (cf. Lemma 5.20) given by the formula

$$\operatorname{mult}_E \mathbb{M}_i = -\min_{\varphi \in V_i} \operatorname{mult}_E \varphi$$

for any prime b-divisor E. Since $V_i V_j \subset V_{i+j}$, the system $\mathbb{M}_* = \{\mathbb{M}_i\}$ is semi-additive. We set $R = R(X, \mathbb{M}_*)$.

If the algebra V is bounded by D, then R is bounded by the same divisor. Obviously, the converse holds as well.

Let us verify the assertion b). Finite generation implies boundedness. Thus, if one of the algebras is finitely generated, then both the algebras are bounded. Multiplying by an appropriate rational function, one can assume that any V_i satisfies the condition $V_i \subset A \subset k(X)$ as an ideal of A. In this case the family $R_i \subset k(X)$ consists of the functions φ satisfying an equation of integral closure

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0,$$

where $a_j \in V_i^j$ (the superscript stands for a power of the ideal V_i). Hence, the algebra R is integral over V. The algebras V and R have the same field of fractions, $k(X)$. Since the integral closure is of finite type, R is of finite type as a V-module.

If the algebra V is finitely generated, then R is also finitely generated, because R is a module of finite type over V. Conversely, suppose that R is finitely generated. One can assume that R is generated by R_1 up to truncation. For every i the truncation $R^{(i)} \subset R$ is integral over the finitely generated algebra $\oplus_{j\geqslant 0} V_i^j$. This gives finite surjective morphisms

$$Y = \operatorname{Proj} R = \operatorname{Proj} R^{(i)} \to Y_i = \operatorname{Proj} \oplus_{j\geqslant 0} V_i^j.$$

These morphisms have natural decompositions $Y \to Y_{mi} \to Y_i$ determined by the inclusion $V_i^{mj} \subset V_{mi}^j$. For some i_0 the relation $Y_{mi_0} = Y_{i_0}$ holds for any $m \geqslant 1$. One can again assume that $i_0 = 1$ up to truncation. Then the relation $Y_m = Y_1$ means that the equality $V_m^j = V_1^{jm}$ holds for some $j \gg 1$ and for any m. As above, we assume that each set $V_m \subset A$ is an ideal. In this case it is clear that the A-module V_m is integral over the A-module V_1^m, and therefore the algebra V is integral over $\oplus_{m\geqslant 0} V_1^m$. Thus, V is of finite type over the last algebra. Hence, V is also finitely generated.

Finally, we discuss the most important point of the theory, that is, the conjecture on the finite generation of certain pbd algebras. We note immediately an obvious necessary condition, namely, the pbd-algebra R in question must be *bounded*.

However, the most subtle and essential condition is *asymptotic saturation*, originally introduced in [37], 4.33.

Definition 5.23. A convex system $\mathbb{D}_* = \{\mathbb{D}_i\}$ of (effective) b-divisors is said to be *asymptotically* $\mathbb{C}$-*saturated* if

$$\operatorname{Mov} \lceil j\mathbb{D}_{iY} + \mathbb{C}_Y \rceil \leqslant j\mathbb{D}_{jY} \tag{5.1}$$

for all positive integers i, j (up to truncation) on any sufficiently high model $Y \to X$. This means that (after a truncation) for every pair (i, j) there is a model $Y_{(i,j)} \to X$ such that the inequality (5.1) holds on any model $Y \to Y_{(i,j)} \to X$.

If $\mathbb{C} = \mathbb{A} = \mathbb{A}(X, B)$ is the discrepancy b-divisor of (X, B), then we say that the sequence is *log canonically asymptotically* (lca) *saturated* over (X, B). Correspondingly, a pbd algebra R is said to be lca *saturated* if the characteristic sequence of R is lca saturated.

To use asymptotic saturation in practice, one must find a model Y independent of i, j on which the asymptotic saturation holds uniformly. The existence of a model of this kind is rather non-trivial; this is the so-called CCS conjecture ([37], 6.14), and at present we can construct such a model only in the case of surfaces (see the next section, § 6, and, in particular, Corollary 6.6).

The meaning of asymptotic saturation is that the inequality (5.1) becomes stronger as $i \to \infty$. In the case of uniform asymptotic saturation, it is customary to use the following corollary to saturation: if $\mathbb{D} = \lim_{i\to\infty} \mathbb{D}_i = \sup\{\mathbb{D}_i\}$ and if saturation holds on Y for any i, j, then

$$\operatorname{Mov} \lceil j\mathbb{D}_Y + \mathbb{C}_Y \rceil \leqslant j\mathbb{D}_{jY} \leqslant j\mathbb{D}_Y \quad \text{for any} \quad j.$$

In some situations this enables us to prove that the limiting b-divisor is rational (see the next section, § 6, and, in particular, Corollary 6.12).

Definition 5.24. Let (X, B) be a klt pair and let $X \to Z$ be a weak Fano birational contraction of the pair onto an affine variety Z. A bounded lca saturated pbd algebra on X/Z is called an FGA-*algebra*.

Conjecture 5.25 (on finite generation). *Any* FGA-*algebra is finitely generated.*

§ 6. Any FGA-algebra in dimension 2 is finitely generated

6.1. In this section we show that the FGA conjecture, Conjecture 5.25, implies the existence of pl flips, and at the end of the section we establish the conjecture for surfaces.

Let $\varphi\colon (X, S+B) \to Y$ be a pl contraction; in particular, let $K+S+B$ be plt and φ-anti-ample. We choose a φ-anti-ample divisor M on X whose support does not contain S. A pl flip exists if the algebra

$$\mathcal{R} = \mathcal{R}_{X/Z} M = \bigoplus_{i \geqslant 0} H^0(X, iM)$$

is finitely generated, or equivalently, if an M-flip exists. We restrict the algebra to S, $R^0 = \operatorname{res}_S \mathcal{R}$. By definition, $R^0 = \bigoplus R_i^0$, where $R_i^0 = \operatorname{Im}(H^0(X, iM) \to H^0(X, iM_{|S}))$. In Proposition 5.4 we showed that an M-flip exists if R^0 is finitely generated. By Lemma 5.22, the integral closure of R^0 in $k(S)/Z$ is a pbd algebra R_S, and R^0 is finitely generated if and only if R_S is finitely generated. It follows from the proof of Lemma 5.22 that $R_S = R(S, \mathbb{M}_*^0)$, where

$$\mathbb{M}_i^0 = \operatorname{res}_S \operatorname{Mov} iM$$

is a mobile restriction of the b-mobile part of iM, as in Definition 5.15(ii). Since the divisorial algebra $\mathcal{R}$ is bounded, so is the pbd algebra R_S (again by Lemma 5.22).

Lemma 6.2 ([37], 4.50.1, [7], 4.1.1). *R_S is an* FGA-*algebra.*

Proof. One can readily see that $(S, B_S) \to Z$ is also a weak Fano birational contraction. Thus, we must show that the algebra R_S is lca saturated. The lca saturation is invariant under integral closure, since the mobile part of corresponding linear systems does not change upon taking the integral closure, and only the dimension of these systems can change. By Lemma 5.16, every b-divisor $\operatorname{Mov} iM$ is exceptionally saturated. Therefore, Lemma 5.18 is applicable. This implies that any mobile restriction $\mathbb{M}_i^0$ is $\mathbb{A}(S, B_S)$-saturated. To prove the lca saturation, we return to the proof of Lemma 5.18 and make the necessary changes.

To simplify our notation, we write $\mathbb{M}_i = \operatorname{Mov} iM$. By the construction, and since M is a Cartier divisor, we have

$$\mathbb{M}_{iY} = \operatorname{Mov} f^* iM$$

on all models $f\colon Y \to X$. It follows from what was said above that $\mathbb{M}_i^0 = \operatorname{res}_S \mathbb{M}_i$ (the mobile restriction).

To verify the lca saturation, we take a pair of indices (i, j). Let $f\colon Y \to X$ be a log resolution of $(X, S+B+\mathbb{D}_{iX})$, let F_m be f-exceptional prime divisors, and let D_l' and S' be birational transforms of the prime components of B and S, respectively. Suppose that $Y \to X$ is a sufficiently high model, in the sense that

a) $\mathbb{M}_{iY}$ is free and $\mathbb{M}_i = \overline{\mathbb{M}_{iY}}$;
b) the same holds for $\mathbb{M}_j$.

In general the model $Y \to X$ depends on i, j. We write

$$K_Y = f^*(K + S + B) - S' - \sum b_l D'_l + \sum a_m F_m$$

on Y, where $\mathbb{A}_Y = -S' - \sum b_l D'_l + \sum a_m F_m$ by definition. We claim that the pbd algebra $R_S = R(S, \mathbb{M}^0_*)$ is lca saturated for the chosen indices i, j. This is a property of the characteristic sequence, and hence we consider the $\mathbb{Q}$-b-divisors $\mathbb{D}_i = \frac{1}{i}\mathbb{M}_i$. Let us first prove that the saturation property holds for $j\mathbb{D}_i$ and $j\mathbb{D}_j$ on the model $S' \to S$. For simplicity, we set $\mathbb{A}^0 = \mathbb{A}(S, B_S)$. One must compare $\operatorname{Mov}\left\lceil j\mathbb{D}_{iY}|_{S'} + \mathbb{A}^0_{S'}\right\rceil$ with $j\mathbb{D}_{jY}|_{S'}$. We note that $\mathbb{A}^0_{S'} = \mathbb{A}'_Y|_{S'}$ by the adjunction formula, where $\mathbb{A}' = \mathbb{A} + \underline{S}$.

By the Kawamata–Viehweg vanishing theorem we have

$$H^1(Y, \lceil (j\mathbb{D}_i + \mathbb{A})_Y \rceil) = 0,$$

because $-(K + S + B)$ is nef and big over Z. Therefore, the restriction

$$H^0(Y, \lceil (j\mathbb{D}_i + \mathbb{A}')_Y \rceil) \to H^0(S', \lceil (j\mathbb{D}^0_i + \mathbb{A}^0)_{S'} \rceil),$$

where $\mathbb{D}^0_i = \frac{1}{i}\mathbb{M}^0_i = \operatorname{res}_S \mathbb{D}_i$, is surjective. The operation of taking the integral part commutes with the restriction operation by our choice of the log resolution. To prove the claim, it suffices to show that

$$\operatorname{Mov}\lceil (j\mathbb{D}_i + \mathbb{A}')_Y \rceil \leqslant j\mathbb{D}_{jY} = \mathbb{M}_{jY}.$$

We noted that $\mathbb{M}_{iY} = \operatorname{Mov} f^* iM$. Therefore, the desired assertion is equivalent to the inequality

$$\operatorname{Mov}\left\lceil \frac{j}{i}\operatorname{Mov} f^* iM + \mathbb{A}'_Y \right\rceil \leqslant \operatorname{Mov} f^* jM.$$

This is simple, because the divisor

$$f_* \operatorname{Mov}\left\lceil \frac{j}{i}\operatorname{Mov} f^* iM + \mathbb{A}'_Y \right\rceil = jM = f_* \operatorname{Mov} f^* jM$$

is integral and the positive part of $\mathbb{A}'_Y$ is exceptional. Hence, the inequality in question follows from the exceptional saturation of $\operatorname{Mov} jM$ by Lemma 5.16.

By Lemma 5.13, asymptotic saturation for i, j holds now on any model $S'' \to S' \to S$. This completes the proof.

We proceed to work with surfaces. Let (X, B) be a surface klt pair, let $X \to Z$ be a weak Fano birational contraction, and let $R = R(X, \mathbb{D}_*)$ be an FGA-algebra with characteristic sequence $\mathbb{D}_* = \{\mathbb{D}_i\}$. We want to prove that R is finitely generated. The first real difficulty is to find a model $Y \to X$ on which all asymptotic saturations

$$\operatorname{Mov}\lceil j\mathbb{D}_{iY} + \mathbb{A}_Y \rceil \leqslant j\mathbb{D}_{jY}$$

hold uniformly with respect to i, j. Below we show that for this model one can take the so-called crepant terminal model $\psi\colon (X', B') \to (X, B)$, that is, (X', B') is a trm pair and $K_{X'} + B' = \psi^*(K + B)$.

Definition 6.3. Let $\mathbb{D}$ be a b-divisor of X. We say that $\mathbb{D}$ *descends* to X if $\mathbb{D} = \overline{\mathbb{D}_X}$.

The case of surfaces is very special, because mobile divisors on a (non-singular) surface are nef. This is a crucial point in the proof of the following main result of the present section.

Theorem 6.4. *Let (X, B) be a surface* trm *pair. This means that X is a non-singular surface and B, $0 \leqslant B = \sum b_i D_i$, is an effective $\mathbb{R}$-divisor on X such that* $\operatorname{mult}_x B < 1$ *at every (closed) point $x \in X$. Let $X \to Z$ be a weak Fano birational contraction onto an affine surface Z. If $\mathbb{M}$ is a mobile $\mathbb{A}$-saturated b-divisor of X, where $\mathbb{A} = \mathbb{A}(X, B)$, then $\mathbb{M}$ descends to X.*

Proof. Let $f\colon Y \to X$ be a sufficiently high log resolution of (X, B) such that $\mathbb{M} = \overline{\mathbb{M}_Y}$, and therefore the linear system

$$|\mathbb{M}_Y| = |\mathbb{M}| = \{\operatorname{div}(\varphi) + \mathbb{M} \mid 0 \neq \varphi \in H^0(X, \mathbb{M})\}$$

is free.

We claim that the divisor $E = \lceil \mathbb{A}_Y \rceil$ is integral and f-exceptional and that the following assertions hold:

(i) each (effective) f-exceptional divisor is supported by E (with positive multiplicities), that is, $\operatorname{Supp} E$ consists of all the exceptional prime divisors;

(ii) $H^1(Y, E) = 0$.

To prove these statements, we write

$$K_Y = f^*(K + B) - B_Y + \sum a_i E_i,$$

where $\mathbb{A}_Y = -B_Y + \sum a_i E_i$ is the discrepancy and all exceptional multiplicities satisfy the inequality $a_i > 0$. In particular, the assertion (i) holds for $E = \lceil \mathbb{A}_Y \rceil$. We note that the $\mathbb{R}$-divisor

$$-f^*(K + B) = -K_Y + \mathbb{A}_Y$$

is nef and big over Z, and hence $H^1(Y, \lceil \mathbb{A}_Y \rceil) = H^1(Y, E) = 0$ by the Kawamata–Viehweg vanishing theorem. This proves (ii).

By our assumptions, $\mathbb{M}$ is $\mathbb{A}$-saturated. This means that

$$E = \operatorname{Fix}(\mathbb{M}_Y + E).$$

Let $M_Y \in |\mathbb{M}_Y|$ be a generic element. We can assume that M_Y is a curve, because $\mathbb{M} = 0 = \overline{0_X}$ otherwise. In this case, by the vanishing result (ii), the restriction

$$H^0(Y, \mathbb{M}_Y + E) \to H^0(M_Y, (\mathbb{M}_Y + E)_{|M_Y})$$

is surjective. Therefore,

$$E_{|M_Y} = \operatorname{Fix}((\mathbb{M}_Y + E)_{|M_Y}).$$

However, M_Y is an affine curve. Thus, any complete linear system on M_Y is free. Since the restriction is surjective, the system $|\mathbb{M}_Y + E|$ is free on Y near M_Y. Hence, $\operatorname{Supp} E \cap M_Y = \varnothing$. Since the support of E contains all exceptional divisors, the curve M_Y is disjoint from any exceptional curve, that is, $\mathbb{M}_Y = f^*\mathbb{M}_X$ and $\mathbb{M} = \overline{\mathbb{M}_X}$.

We recall the following well-known result on the existence of a crepant terminal blowup.

Conditional theorem, Theorem 6.5 (assuming the LMMP in the dimension $\dim X$ ([36], 3.1)). *Let* (X, B) *be a* klt *pair. Then there is a projective* ($\mathbb{Q}$-*factorial*) *blowup* $\psi\colon X' \to X$ *such that*

(i) (X', B') *is a* trm *pair*;

(ii) $K_{X'} + B' = \psi^*(K + B)$, *that is*, (X', B') *is crepant over* (X, B).

Such a pair (X', B') *is called a crepant terminal model of* (X, B) (see Definition 1.6(iii) and cf. Definition 1.9(ii)).

For $\dim X \leqslant 4$ *we can drop the LMMP assumption. In the case of surfaces a crepant terminal model exists and is unique.*

Before proving the theorem, we derive the following corollary.

Corollary 6.6. *Let* (X, B) *be a* klt *log surface, let* $X \to Z$ *be a weak Fano birational contraction, and let* $R = R(X, \mathbb{M}_*)$ *be an* FGA-*algebra with a mobile sequence* $\mathbb{M}_* = \{\mathbb{M}_i\}$. *If* $(X', B') \to (X, B)$ *is a crepant terminal model, then* $X' \to Z$ *is also a weak Fano birational contraction and each b-divisor* $\mathbb{M}_i$ *descends to* X'. *Let* $G = \sum G_l$ *be a divisor on* X' *containing the supports of all the* $\mathbb{M}_{iX'}$ *and let* $X'' \to X'$ *be a log resolution for* $(X', B' + G)$. *Then asymptotic* $\mathbb{A}$-*saturation holds uniformly on the models* $Y \to X''$ *blown up over* X'', *that is, on all these models*

$$\operatorname{Mov} \lceil j\mathbb{D}_{iY} + \mathbb{A}_Y \rceil \leqslant j\mathbb{D}_{jY}.$$

Proof. This follows from Lemma 5.13. We also note that asymptotic $\mathbb{A}$-saturation for $i = j$ implies $\mathbb{A}$-saturation for the b-divisor $\mathbb{M}_i = i\mathbb{D}_i$.

Corollary 6.7. *Under the assumptions of Theorem* 6.5, *a variety* (*a space*, *a scheme*) X *has a projective* $\mathbb{Q}$-*factorialization* (*locally over a compact*, *complete subset of* X), *that is*, *a small blowup* $X' \to X$ (*analytic*, *formal*) *such that* X' *is* $\mathbb{Q}$-*factorial* (*locally over a compact*, *complete subset of* X).

For $\dim X \leqslant 4$ *we can drop the LMMP assumption and weaken the* klt *property to the* lt *property of* (X, B) *if there is a projective resolution in the definition of the* lt *property.*

Sketch of the proofs of Theorem 6.5 *and Corollary* 6.7. The results are well known for $\dim X \leqslant 3$, and the construction in their proof possibly goes back to [11]. Let $Y \to X$ be a relative (analytic, formal) projective log resolution of (X, B) (locally over a compact, complete subset of X) with the exceptional prime divisors E_i including all prime b-divisors with discrepancies $\leqslant 0$. Let (X', B') be a log pair obtained by the LMMP applied to $(Y, B_Y + \sum b_i E_i)$ over X (locally over a compact, complete subset of X, respectively), where the summation ranges over the exceptional divisors E_i with $b_i = -a(E_i, X, B) \geqslant 0$ only. In this case the following remarks imply the assertion of the theorem:

a) (X', B') is a strictly minimal model and, at the same time, a log minimal model of $(Y, B_Y + \sum b_i E_i)$ which contracts neither the components of B_Y nor the exceptional prime divisors E_i with discrepancies $a(E_i, X, B) \leqslant 0$;

b) (X', B') is a crepant blowup of (X, B) of precisely the exceptional prime divisors whose discrepancies are $\leqslant 0$;

c) (X', B') is a strictly terminal pair.

The assertion a) follows from the inequality of Lemma 2.4 (up to a decrease of the boundary multiplicities) or from the divisorial version (together with the flipping version) in the monotonicity assertion of 3.4(i), since (X, B) is a log canonical model of $(Y, B_Y + \sum b_i E_i)$ and (X', B'). For the same reasons we obtain the assertion b) (cf. [34], 1.5.7). By construction, the assertion c) also holds. For log surfaces, the construction amounts to blowdowns on a log resolution of all (-1)-curves with positive discrepancies.

By 1.8 in [37] and by Example 9 in [39], one can omit the LMMP for $\dim X \leqslant 4$, because one can assume that the initial model $(Y, B_Y + \sum b_i E_i)$ is trm, as well as the subsequent ones.

We note that if (X, B) is a trm pair, then the construction gives a small blowup, which is the desired $\mathbb{Q}$-factorialization. To obtain a similar $\mathbb{Q}$-factorialization for klt singularities, we must take $(Y, B_Y^{\log})$ as an initial pair, that is, replace the exceptional multiplicities b_i by 1. Then by enlarging the boundary we can manage with only the special termination of Theorem 4.8 instead of the general termination (see the proof of 3.1 in [36] and the reduction in Theorem 4.7), and the special termination holds for $\dim X \leqslant 4$ by Corollary 4.11.

Let us now prove Conjecture 5.25 for surfaces. As we already know, this implies the existence of 3-fold flips.

Theorem 6.8. *Let (X, B) be a surface* klt *pair and let $\varphi\colon X \to Z$ be a weak Fano birational contraction of it onto an affine surface Z (that is, $\varphi_* \mathcal{O}_X = \mathcal{O}_Z$ and $-(K + B)$ is nef over Z; the property of being φ-big follows from the birationality of φ). In this case every* FGA-*algebra on X/Z is finitely generated.*

The rest of the section is devoted to the proof of this theorem. Several assertions hold in all dimensions. We note the places where we use the assumption that X is a surface.

Proof. Step 1. Notation and preliminary remarks. Let $R = R(X, \mathbb{D}_*)$ be an FGA-algebra with characteristic sequence $\mathbb{D}_* = \{\mathbb{D}_i\}$ and mobile sequence $\mathbb{M}_* = \{\mathbb{M}_i\}$, $\mathbb{M}_i = i\mathbb{D}_i$. Let $f\colon (X', B') \to (X, B)$ be a crepant terminal model of (X, B). Since $-(K_{X'} + B') = -f^*(K + B)$, the contraction $X' \to Z$ is still a weak Fano birational contraction. It is clear that $R = R(X', \mathbb{D}_*)$ is also an FGA-algebra, but on X'/Z rather than on X. Therefore, passing to the crepant terminal model, we can assume that (X, B) is a trm pair.

The first requirement for an FGA-algebra is boundedness. The algebra R is bounded; in particular, there is a reduced divisor $G = \sum G_j$ on X such that $\operatorname{Supp} \mathbb{D}_{iX} \subset \operatorname{Supp} G$ for all i. In addition, it follows from the boundedness that the system $\mathbb{D}_*$ has a limit

$$\mathbb{D} = \lim_{i \to \infty} \mathbb{D}_i \in (\operatorname{Div} X) \otimes \mathbb{R}$$

as a b-divisor, possibly with real multiplicities, and $\operatorname{Supp} \mathbb{D}_X \subset G$. Our objective is to show that $\mathbb{D}$ is a b-divisor with rational multiplicities only, and that $\mathbb{D} = \mathbb{D}_m$ for some $m \gg 1$ and thus for any l greater than and divisible by m.

Step 2. Semi-ampleness of $\mathbb{D}_X$. Here we assume that X is a surface.

Lemma 6.9. *The divisor $\mathbb{D}_X$ is semi-ample.*

Proof. The b-divisors $\mathbb{M}_i$ are mobile and their traces on the surface X are nef over Z. Therefore, the limit $\mathbb{D}_X = \lim_{i\to\infty} \mathbb{D}_{iX}$ is also nef over Z. The Kleiman–Mori cone on X/Z is polyhedral, because $X \to Z$ is a weak Fano birational contraction. The dual cone of the nef $\mathbb{R}$-divisors is generated by the semi-ample divisors that are supporting for the (contracting extremal) faces of the Kleiman–Mori cone. Therefore, all nef divisors on X are semi-ample.

Step 3. Diophantine approximation. We work with an integral lattice $N^1_{\mathbb{Z}} = \oplus\mathbb{Z}G_j \subset \operatorname{Div} X$ and the vector spaces $N^1_{\mathbb{Q}} = N^1_{\mathbb{Z}} \otimes \mathbb{Q}$ and $N^1_{\mathbb{R}} = N^1_{\mathbb{Z}} \otimes \mathbb{R}$. Since $\mathbb{D}_X$ is semi-ample, we can choose finitely many free divisors $P_j \in N^1_{\mathbb{Z}}$ forming a basis of $N^1_{\mathbb{Q}}$ and such that $\mathbb{D}_X$ belongs to the cone

$$\mathcal{P} = \sum \mathbb{R}_+ P_j \subset \sum \mathbb{R}G_j \subset N^1_{\mathbb{R}}$$

generated by the P_j, where $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r \geqslant 0\}$. The following statement holds and can readily be proved in the spirit of Diophantine approximations.

Lemma 6.10. *For every $\varepsilon > 0$ there exist a positive integer m and a divisor $M \in N^1_{\mathbb{Z}}$ such that*

(i) *the linear system $|M|$ is free*;
(ii) *$\|m\mathbb{D}_X - M\| < \varepsilon$, where the norm is defined as the maximal absolute value of the coordinates in the basis $\{G_j\}$ of $N^1_{\mathbb{R}}$*;
(iii) *if the divisor $\mathbb{D}_X$ is not rational, then $m\mathbb{D}_X - M$ is not effective.*

Proof. The proof follows immediately from Kronecker's theorem ([3], 1.3). In the case of a rational trace $\mathbb{D}_X$ we take an integer $m > 0$ such that $M = m\mathbb{D}_X$ is free. Such a number m exists by Lemma 6.9.

If $\mathbb{D}_X$ is irrational, then we work with the integral lattice $N_{\mathbb{Z}}$ generated by $\{P_k\}$, $k = 1, \dots, l$, and with the vector spaces $N^1_{\mathbb{Q}}$ and $\mathbb{N}^1_{\mathbb{R}}$ with this basis. Thus, we identify these objects with $\mathbb{Z}^l$, $\mathbb{Q}^l$, and $\mathbb{R}^l$, respectively. We write $\mathbb{D}_X$ as a vector, $\mathbb{D}_X = \mathbf{d} = (d_1, \dots, d_l) \in \mathbb{R}^l = \mathbb{N}^1_{\mathbb{R}}$, where all the components d_i are non-negative real numbers. Since the vector $\mathbf{d}$ is not rational, it belongs to a minimal vector subspace of $\mathbb{R}^l$ over $\mathbb{Q}$ of dimension $\geqslant 1$ that is contained in the subspace of the quadrant corresponding to positive coordinates of $\mathbf{d}$ and in which the multiples of $\mathbf{d}$ have arbitrarily close approximations by nodes of the lattice $\mathbb{N}_Z$. Therefore, by 1.3 in [3], for every $\varepsilon > 0$ one can find a positive integer multiple $m\mathbf{d}$ of $\mathbf{d}$ and an integral vector $\mathbf{m} = (m_1, \dots, m_l) \in N_Z$ such that

1) $m_i \geqslant 0$ for all i;
2) $\|m\mathbf{d} - \mathbf{m}\| < \varepsilon$;
3) the divisor $m\mathbf{d} - \mathbf{m}$ is not effective (approximation from above).

The last property can be obtained by the strict convexity of the set of effective divisors, namely, this set contains no lines.

We set $M = \sum m_i P_i \in N_{\mathbb{Z}} = \mathbb{Z}^l$. Then the linear system $|M|$ is free by 1).

Step 4. Non-vanishing for linear systems. Let G be the (effective) divisor in Step 1. We choose a very small number $\gamma > 0$ such that the log pair $(X, B + \gamma G)$ is still klt.

If $\mathbb{A} = \mathbb{A}(X, B)$ is the discrepancy b-divisor, then the klt property for $(X, B + \gamma G)$ means the inequality $\operatorname{mult}_E(\mathbb{A} - \gamma\overline{G}) > -1$ for any prime b-divisor E, and this is equivalent to the effectiveness

$$\lceil \mathbb{A} - \gamma\overline{G} \rceil \geqslant 0.$$

Lemma 6.11 (non-vanishing). *Take the divisor M as in Lemma* 6.10. *If $\varepsilon < \gamma$, then the inequality*

$$\operatorname{Mov} \lceil m\mathbb{D}_Y + \mathbb{A}_Y \rceil \geqslant (\overline{M})_Y = f^*M$$

holds on each model $f\colon Y \to X$.

Proof. We write $F = m\mathbb{D}_X - M$ and set

$$m\mathbb{D}_Y + \mathbb{A}_Y = f^*(m\mathbb{D}_X) + \mathbb{A}_Y = f^*M + f^*F + \mathbb{A}_Y \geqslant f^*M - \varepsilon f^*G + \mathbb{A}_Y.$$

The first equality holds by the descent in Corollary 6.6; this descent commutes with the limits. Passing to the upper integral parts and then taking their mobile parts, we obtain the desired inequality

$$\operatorname{Mov} \lceil m\mathbb{D}_Y + \mathbb{A}_Y \rceil \geqslant \operatorname{Mov}(f^*M + \lceil \mathbb{A}_Y - \varepsilon f^*G \rceil) \geqslant \operatorname{Mov} f^*M = f^*M.$$

Step 5. Rationality of $\mathbb{D}$. Here, as at the previous step, we make essential use of the condition that the dimension is 2.

Lemma 6.12. *The b-divisor $\mathbb{D}$ is rational.*

Proof. We recall that a log pair (X, B) is assumed to be terminal. Corollary 6.6 claims that all the divisors $\mathbb{D}_i$ descend to X, and hence so does the divisor $\mathbb{D}$. Thus, to establish the rationality of the b-divisor $\mathbb{D}$, it suffices to prove that the trace $\mathbb{D}_X$ is rational. Suppose the contrary, that is, let $\mathbb{D}_X$ be irrational. Let $Y \to X$ be an arbitrary blowup of some log resolution of $(X, B + G)$. Corollary 6.6 also claims that asymptotic $\mathbb{A}$-saturation holds (uniformly) on Y; in particular, for any $i \gg 0$ we have

$$\operatorname{Mov} \lceil m\mathbb{D}_Y + \mathbb{A}_Y \rceil = \operatorname{Mov} \lceil m\mathbb{D}_{iY} + \mathbb{A}_Y \rceil \leqslant m\mathbb{D}_Y.$$

However, by the non-vanishing in Lemma 6.11,

$$\operatorname{Mov} \lceil m\mathbb{D}_Y + \mathbb{A}_Y \rceil \geqslant (\overline{M})_Y = f^*M.$$

Together with the previous inequality, this implies that

$$m\mathbb{D}_Y \geqslant (\overline{M})_Y.$$

Therefore, $m\mathbb{D}_X - M \geqslant 0$, which contradicts Lemma 6.10(iii).

Step 6. Stabilization.

Lemma 6.13. *The characteristic system* $\mathbb{D}_* = \{\mathbb{D}_i\}$ *stabilizes, that is,* $\mathbb{D} = \mathbb{D}_m$ *for some* $m \gg 0$.

Proof. Let $m \gg 0$ be some integer such that the multiple $m\mathbb{D}_X$ is free. The existence of m follows from Lemmas 6.12 and 6.9. By the non-vanishing Lemma 6.11 with $M = m\mathbb{D}_X$ we have

$$\operatorname{Mov} \lceil m\mathbb{D}_Y + \mathbb{A}_Y \rceil \geqslant (\overline{M})_Y = f^*M = m\mathbb{D}_Y.$$

Using a more precise saturation

$$\operatorname{Mov} \lceil m\mathbb{D}_Y + \mathbb{A}_Y \rceil = \operatorname{Mov} \lceil m\mathbb{D}_{iY} + \mathbb{A}_Y \rceil \leqslant m\mathbb{D}_{mY}$$

than the above one and applying the inequality $\mathbb{D}_m \leqslant \mathbb{D}$, we obtain the relation

$$\mathbb{D}_m = \overline{M}/m = \mathbb{D}$$

instead of a contradiction.

Thus, the FGA conjecture, Conjecture 5.25 on finite generation, is proved in dimension 2. This is immediate by the limiting criterion (Lemma 5.21).

Hence, we obtain the following assertion.

Corollary 6.14. klt *log flips*, lt *log flips* (*as in Conjecture* 1.11), *and even* lc *log flips exist for* 3*-folds.*

Proof. This follows immediately from Theorem 4.7, Corollary 4.11, Proposition 5.4, Lemma 6.2, and Theorem 6.8. The proof ([36], 6.13) of the existence of lc log flips (see the definition in [36], § 5) needs the 3-fold LMMP and the semi-ampleness in dimension 2.

Corollary 6.15. *Directed flops and* klt *directed log flops exist for* 3*-folds.*

Proof. See Remark 3.3(ii).

§ 7. Semistable 3-fold flips

7.1. In § 6 we proved the existence of 3-fold klt log flips. This clearly implies the existence of classical 3-fold terminal (trm) flips [26]: it suffices to take the trivial boundary $B = 0$ and consider varieties having only terminal singularities. There is a special case of a trm flip, namely, a semistable 3-fold flip, which arises when studying the (relative) MMP for semistable degenerations, in which case there is an additional structure, that is, a projective (or proper) morphism $f\colon X \to \Delta$, where $\Delta = \operatorname{Spec} \mathcal{O}$ stands for the spectrum of a local ring $\mathcal{O}$. It is of importance to study the 3-fold singularities arising here and to coordinate 3-fold flips (which exist and terminate, as we already know) with the morphism f and the singularities, in particular, to show that these flips are semistable (see Theorem 7.4 below).

Kulikov [24] was possibly the first to investigate birational transformations (flips) in a similar situation. In contrast to the MMP, he used non-projective transformations even when assuming that the initial model is projective. What is worse, his flips usually go out of the category of algebraic varieties. The main objective of the present section is Corollary 7.16 below, whose proof explains the relationship between the Mori and Kulikov models (see [35], § 5, and also Comment 7.17). The proof uses the existence of flips (cf. another application of flips in [38]) and the classification of the singularities thus arising (see [35], and Remark 7.6 below). Thus, we shall show how to obtain a concrete description of singularities from general results like the existence results for flips.

Definition 7.2. A *semistable singularity* is a singularity of a relative log pair $(X, S) \to \Delta = \operatorname{Spec} \mathcal{O}$, where f is a proper morphism onto the spectrum of a local ring $\mathcal{O}$ and S is a reduced divisor such that

(i) the variety X is $\mathbb{Q}$-factorial;
(ii) (X, S) is lt;
(iii) $S \sim 0/\Delta$;
(iv) there is a *semistable resolution*, that is, a log resolution $g\colon Y \to X$ of (X, S) that is an isomorphism outside S with reduced divisor g^*S; the resolution is assumed to be only locally projective over Δ.

We note that the conditions (ii)–(iv) imply that X has only terminal singularities. Indeed, by the lt property and (iv), this must be verified at the isolated points on S over which lie exceptional b-divisors E with $a(E, X, S) > -1$. However, S is a Cartier divisor by (iii). Hence, we have $a(E, X, 0) > 0$ for these b-divisors E, and the point $\mathrm{c}_X E$ is terminal. We also note that, by (iii), there is a regular function t on X over Δ with the divisor $(t) = S$ of zeros. (The function t is unique up to multiplication by an invertible function.) In this section, the singularities are usually regarded up to analytic (or formal) equivalence. The main scheme X is assumed to be a 3-fold. We present a typical example.

Examples 7.3. In the following examples we take the identity for the morphism $X \to \Delta$. Let A_0^d be a localization at the point $0 = (0, \dots, 0)$ of the d-dimensional affine space with the coordinates $x_1, \dots, x_d$.

1. *Triple point.*

$$X = \mathbb{A}_0^3, \quad S = (x_1x_2x_3), \quad \text{and} \quad t = x_1x_2x_3,$$

that is, the boundary $S = S_1 + S_2 + S_3$ has three prime components $S_i = (x_i)$, $i = 1, 2, 3$, intersecting at 0. The point 0 is non-singular (in the log sense).

2. *Quotient singularity.* For two coprime positive integers $a \leqslant r$ we set

$$X = \frac{1}{r}(a, -a, 1), \quad S = (x_1x_2) \subset X, \quad \text{and} \quad t = x_1x_2,$$

where $\frac{1}{r}(a, -a, 1)$ stands for the quotient of the scheme $\mathbb{A}_0^3$ by the action of the cyclic group $\mathbb{Z}_r$ with the weights $(a, -a, 1)$, namely, the generator $1 \in \mathbb{Z}_r$ acts according to the rule

$$(x_1, x_2, x_3) \mapsto (\zeta^a x_1, \zeta^{-a} x_2, \zeta x_3),$$

where $\zeta = \sqrt[r]{1}$ is a primitive rth root of unity. The function $t = x_1x_2$ is invariant under the action and is well defined on X. Moreover, the boundary $S = S_1 + S_2$ consists of two prime $\mathbb{Q}$-Cartier Weil surfaces S_1 and S_2, which are the quotients of the planes $x_1 = 0$ and $x_2 = 0$, respectively. The index of the point 0 (that is, the index of the canonical divisor K at this point) is equal to r. The quotient is semistable; it is singular exactly when $r > 1$. It is better to describe both the corresponding resolution and the singularity itself in terms of the toric category as a chain of weighted blowups ([35], 1.2.3, [32], 5.7), as in the next example.

3. *Moderate singularity of type* (r, a). The quotient of the hypersurface singularity

$$X = (x_1x_2 = x_3^r + x_4^n) \subset \frac{1}{r}(a, -a, 1, 0) \quad \text{and} \quad S = (x_4) \subset X,$$

where $\frac{1}{r}(a, -a, 1, 0)$ stands for the quotient of $\mathbb{A}_0^4$ by the action of the cyclic group $\mathbb{Z}_r$ with the weights $(a, -a, 1, 0)$. The function $t = x_4$ is invariant under this action and well defined on X. In addition, the boundary S is a prime Cartier divisor. The index of 0 is equal to r. The quotient is semistable, and it is singular if and only if $r > 1$. For $n = 1$ the singularity is as in Example 2 with another boundary $S = (x_1x_2 - x_3^r = 0)$. A semistable resolution can be obtained by a chain of weighted blowups. Indeed, the weighted blowup with the weights $\frac{1}{r}(a, r-a, 1, r)$ (see, for example, [34], Appendix, and [8], Chapter II, 3.6) has only semistable singularities, and there are at most three singularities:

a) a moderate singularity that is locally analytically (or formally) isomorphic to $X = (x_1x_2 = x_3^r + x_4^{n-1}) \subset \frac{1}{r}(a, -a, 1, 0)$ if and only if $n > 2$;
b) two quotient singularities that are locally analytically (or formally) isomorphic to

$$\frac{1}{a}(r \bmod a, -(r \bmod a), 1) \quad \text{and} \quad \frac{1}{r-a}(r \bmod r-a, -(r \bmod r-a), 1),$$

where $x \bmod y$ stands for the remainder $0 \leqslant z < y$ upon dividing x by y, and these are singularities if and only if $a > 1$ and $r - a > 1$, respectively.

Induction on n gives a resolution of the singularity in a). Similarly, induction on the index r gives a resolution of the singularities in b). (For details, see [35], § 4, and [32], 5.7.)

Theorem 7.4 (semistability of flips). *Extremal divisorial K-contractions and extremal flips are semistable for any 3-fold, that is, they preserve (the class of) semistable singularities.*

We give the proof below together with the proof of Proposition 7.9.

Corollary 7.5 (Mori's model). *Any 3-fold scheme projective over Δ that has only semistable singularities admits a resulting model having only semistable singularities. In addition, a relative minimal model of this kind exists if and only if the Kodaira dimension of the fibre of X over a generic point of Δ is $\geqslant 0$.*

Proof. This follows immediately from the theorem by using the MMP and by the termination in Theorem 3.5. Here the schemes can be replaced by varieties over a neighbourhood of a point with the given localization Δ of the base.

Remark 7.6. In Theorem 7.4 the existence result for flips can be established immediately. Historically, the existence of flips of this type was established first ([41],

[11], [35]). Coverings enable us to prove the existence of more general flips ([34], 2.6–9), and even in the positive characteristic [15]. Corti gave a more abstract proof [5] (see also Remark 7.14(ii) below), which uses the reduction to prelimiting, limiting, and special flips of § 6 in [34] (see also § 4 above) and reduces the existence of semistable flips to the case of special semistable flips. The latter existence result can readily be established according to the existence of 1-complements in the sense of the second author (see [34], § 5, [21], § 19). New constructions and generalizations of semistable flips are being published [29] up to the present time. All these constructions and proofs make essential use of the classification of lc singularities in dimension 2.

The statement of Theorem 7.4 about divisorial contractions is immediate by definition. However, the flips require a more invariant approach (independent of a resolution).

Definition 7.7. A normal algebraic variety (or a scheme) is said to be *analytically* (or *formally*) *$\mathbb{Q}$-factorial* if this property holds locally analytically (formally) near each point, that is, if each point is *analytically* (*formally*) *$\mathbb{Q}$-factorial*.

Lemma 7.8. *Any* klt 3*-fold* (*space or scheme*) X *and even any* 3*-fold log pair has a $\mathbb{Q}$-factorialization* (*an analytic or formal $\mathbb{Q}$-factorialization*), *that is*, *there is a small proper* (*possibly non-projective*) *morphism* $Y \to X$ (*analytic*, *formal over a neighbourhood of a compact*, *complete subset of* X) *such that each point of* Y *is $\mathbb{Q}$-factorial* (*analytically*, *formally $\mathbb{Q}$-factorial*, *respectively*).

Proof. We note that the non-$\mathbb{Q}$-factorial points (in any sense indicated above) are closed and isolated in dimension 3, and the number of these points is finite (for some analytic, formal neighbourhood of a compact, complete set in X; this assertion can fail for the entire analytic space X in general). Hence, it suffices to construct a $\mathbb{Q}$-factorialization over the corresponding neighbourhood. To this end, one can use Corollaries 6.7 and 6.14. (By Remark 0.9, this can be applied to the analytic and formal categories.) Since the birational contraction $Y \to X$ is small and the surface klt singularities are $\mathbb{Q}$-factorial in any sense indicated above, the $\mathbb{Q}$-factorial property at the closed points of the contracted curves is equivalent to the usual $\mathbb{Q}$-factorial property in a neighbourhood of the curves, that is, locally over X (in the corresponding topology).

Proposition 7.9 ([35], 1.3, 4.1). *A germ of any point of a* 3*-fold relative log pair* $(X, S) \to \Delta$ *having only semistable singularities is analytically* (*or formally*) *isomorphic to one of the following germs*:

(i) *a triple point of Example* 7.3.1;
(ii) *a quotient singularity of Example* 7.3.2;
(iii) *one of the singularities of type* (r, a), *where the boundary* S *of every germ of this kind is prime*, *and its analytic* (*or formal*) *$\mathbb{Q}$-factorialization has only singularities that are analytically* (*or formally*) *isomorphic to moderate singularities of type* (r, a) *in Example* 7.3.3 (*for details*, *see the next comment*).

Comment 7.10 ([35], 1.3.6, 4.6). The type of a singularity in the assertion (iii) of the theorem is defined by the type of the singularity of S. The singularity is of type $(r,a)=(1,1)$, or equivalently, of index $r=1$, if and only if the singularity on S is a Du Val singularity. In this case a $\mathbb{Q}$-factorialization is a small resolution of the singularity, and all types of Du Val singularities are possible.

In the remaining cases of (iii) (that is, for the indices $r>1$) the singularity of S is a cyclic quotient singularity of type $\frac{1}{r^2}(a,r-a)$ ([14], 3.1). In this case the exceptional curves of a $\mathbb{Q}$-factorialization $Y\to X$ lie on S_Y (the birational transform of S on the $\mathbb{Q}$-factorialization); they are (-1)-curves on a minimal resolution of the surface S_Y and form a chain on S_Y. (For details, see the proof of Corollary 4.6 in [35].)

Definition 7.11. A pair, a variety, or a scheme X is said to be *stably* (*analytically* or *formally*) $\mathbb{Q}$-*factorial* along a Cartier divisor S if every finite cover along S is again $\mathbb{Q}$-factorial (analytically or formally). Here a finite cover means a finite morphism of normal varieties induced by an algebraic function $g(t)$, where $t=0$ is an equation of S, that is, $S=(t)$, and the term 'induced' means that the cover (a variety or a scheme) is a normalization of the fibre product $X\times_{\mathbb{A}^1}Z$ with respect to the morphism $X\to\mathbb{A}^1, x\mapsto t(x)$, and a finite cover $Z\to\mathbb{A}^1$ corresponding to the extension $k(\mathbb{A}^1)(g)$ of the field of rational functions. In other words, the cover is a normalization of X in the extension $k(X)(g)$.

Examples 7.12. 1. Let (X,S) be an analytic germ of a log non-singular point 0 in a 3-fold with a prime boundary S, that is, both the variety X and the irreducible surface S are non-singular at 0. In this case the point is stably analytically and formally $\mathbb{Q}$-factorial. It suffices to consider formal covers with one branch, that is, covers corresponding to the algebraic functions $g(t)=\sqrt[d]{t}$. The corresponding covers are log non-singular, and hence $\mathbb{Q}$-factorial in all senses.

2. Let us now place an ordinary double singularity at the origin 0 on S, assuming that X itself is non-singular, that is, X is formally equivalent to $x^2+y^2+z^2+t=0$; here $S=(t)$. The function $g=\sqrt{t}$ gives a cover with a singularity which is equivalent to the ordinary double singularity $x^2+y^2+z^2+t^2=0$ and has a small semistable resolution. Hence, the pair (X,S) is not stably $\mathbb{Q}$-factorial. In addition, (X,S) is not semistable since it has no semistable resolution.

The main technical result of this section is in the same spirit.

Theorem 7.13. *The following statements about a* 3-*fold analytic* (*or formal*) *germ of a log pair* (X,S) *at a point with a prime boundary* S *are equivalent*:

(i) (X,S) *is a moderate singularity*;

(ii) (X,S) *is a semistable singularity which is analytically* (*or formally*) $\mathbb{Q}$-*factorial*, *and the semistable resolution need not be projective here*;

(iii) (X,S) *is stably* $\mathbb{Q}$-*factorial*, *where* (X,S) *is a* plt *log pair and* S *is a Cartier divisor.*

The singular property of the point in the theorem is not essential by Example 7.12.1.

Proof of Theorem 7.13. The assertion (ii) follows from (i) by Example 7.3.3. In particular, the $\mathbb{Q}$-factorial property follows from the extremal property of weighted blowups. (Moreover, the exceptional divisors are dual to the curves in the exceptional locus.)

The assertion (ii) implies (iii) according to the following arguments used in the proof of 5.7 in [5]. Let $X' \to X$ be a finite cyclic cover of degree d along S and let $g\colon Y \to X$ be a semistable resolution. We take the fibre product $Y' = Y \times_X X'$. Let $D' \in \operatorname{Div} X'$ be a Weil divisor. We must show that D' is a $\mathbb{Q}$-Cartier divisor.

The following observations imply the statement.

a) The germ Y' is normal. In particular, the natural birational contraction $g'\colon Y' \to X'$ corresponds to the Stein factorization of $Y' \to X$. In addition, Y' has only toroidal simplicial singularities (and hence analytically (formally) $\mathbb{Q}$-factorial singularities).

b) The cyclic group μ_d acts on Y' and X' in such a way that $Y'/\mu_d = Y$ and $X'/\mu_d = X$. The birational contraction g' is μ_d-equivariant, and, the most important point, μ_d acts trivially on the boundary $Y'^{\mu_d} = S' = \pi_*^{-1} S_Y^{\log}$, where $\pi\colon Y' \to Y$ stands for the quotient morphism and $S_Y^{\log} = S_Y + \sum E_i$ for the log birational transform of the boundary, and the E_i are prime g-exceptional divisors and can be identified with the divisors $\pi_*^{-1} E_i$ on Y'.

Indeed,

$$\sum_{\zeta \in \mu_d} \zeta {g'_*}^{-1} D' = \pi^* \pi_* {g'_*}^{-1} D'.$$

Further, since Y is analytically $\mathbb{Q}$-factorial, the equivalence $\pi_* {g'_*}^{-1} D' + \sum e_i E_i \sim_{\mathbb{Q}} 0$ holds over X for some $e_i \in \mathbb{Q}$, and therefore

$$\sum_{\zeta \in \mu_d} \zeta {g'_*}^{-1} D' + \sum d e_i E_i = \pi^* \pi_* \Big({g'_*}^{-1} D' + \sum e_i E_i \Big) \sim_{\mathbb{Q}} 0$$

over X'. Hence,

$${g'_*}^{-1} D' + \sum e_i E_i \equiv 0$$

over X', because the intersection number with any curve $C \subset S'$ over X' is well defined by a), and $\zeta {g'_*}^{-1} D' C = \zeta {g'_*}^{-1} D' \zeta C = {g'_*}^{-1} D' C$ by b). The germ X' has only rational singularities (as X has). Thus, the numerical equivalence $\equiv$ over X' can be replaced by the equivalence $\sim_{\mathbb{Q}}$, that is, D' is a $\mathbb{Q}$-Cartier divisor.

The assertion (i) follows from (iii) (in essence, by the semistable reduction theorem). According to our assumptions, the singularity is a hypersurface quotient singularity of the type

$$(x_1 x_2 = F(x_3^r, x_4 = t)) \subset \frac{1}{r}(a, -a, 1, 0),$$

where $F(x_3^r, 0) \neq 0$. By the stable $\mathbb{Q}$-factorial property, after an analytic or formal change of coordinates we obtain $F(x_3^r, x_4) = x_3^r + x_4^n$ (cf. [14], where moderate singularities were originally introduced).

Remarks 7.14. (i) The meaning of moderate singularities is in their discrete nature. Along with the type (r, a), each of these singularities has another natural parameter, n. Moreover, a general deformation of a moderate point of type (r, a) corresponds to the hypersurface $x_1 x_2 = x_3^r + x_4$ with $n = 1$, and thus this is a quotient singularity.

(ii) The stable $\mathbb{Q}$-factorial property of moderate singularities is an innovation introduced by Corti in [5], where a generalization of Theorem 7.4 to a broader class of singularities is also given (cf. Corti's somewhat free definition of semistable singularities in [5], 2.5; in the Russian terminology, these singularities are said to be q-semistable [29]).

(iii) One of the possible answers to Corti's question posed at the beginning of § 5 in [5] may be a higher-dimensional generalization of Theorem 7.13 on the equivalence between analytic (formal) stable $\mathbb{Q}$-factoriality and ordinary $\mathbb{Q}$-factoriality for singularities having a semistable resolution. However, one can expect an explicit description only for rather mild singularities, for example, if the discrepancies over a point of this kind are $> d-3$ in dimension d.

Proof of Theorem 7.4 *and of Proposition* 7.9. As noted above, in order to prove Theorem 7.4, it suffices to show the semistable property of extremal flips. In other words, one must prove that the flips preserve this property of singularities. It should be noted that flips do preserve the properties (i)–(iii) of semistable singularities (according to the general properties of extremal log flips). The property (iv), the existence of a semistable resolution, is more complicated. However, this property is immediate in the case of analytically (or formally) $\mathbb{Q}$-factorial singularities by Theorem 7.13. Indeed, extremal flips preserve both the analytic (formal) $\mathbb{Q}$-factorial property and its stable version; in particular, flips preserve moderate singularities. The general semistable flips can be reduced to extremal flips by using an analytic (or formal) $\mathbb{Q}$-factorialization (see Lemma 7.8). Moreover, the well-known singularities of Proposition 7.9(i–ii) are analytically (formally) stably $\mathbb{Q}$-factorial.

Of course, in turn, the $\mathbb{Q}$-factorialization needs the semistable property of flips to produce semistable singularities. Thus, to complete the proof of both the theorem and Proposition 7.9(iii), we must use induction on the minimal number of exceptional prime divisors needed for a semistable resolution. In this case, at each induction step we first establish Proposition 7.9(iii) and then the theorem.

A posteriori, a semistable resolution (as in Definition 7.2(iv)) admits the usual algebraic projectivity over X (and over Δ) if X is an algebraic variety or scheme (projective over Δ) (see [35], 4.11), and therefore a resolution of this kind is itself algebraic.

Remark 7.15 (cf. Remark 7.14(ii).). The meaning of semistable singularities is related to their generic property (for details, see 4.7 in [35]). According to Proposition 7.9(iii) and Remark 7.14(i), for a fixed singularity on S a general deformation of a semistable point of type (r, a) can have an analytic (formal) $\mathbb{Q}$-factorialization only with the same number of exceptional curves and with the quotient singularities of the same type; in particular, a $\mathbb{Q}$-factorialization of this kind gives a small resolution for the index $r = 1$.

This can possibly explain why any quotient singularity in the McKey conjecture has a small resolution and can lead to a generalization of this result. Indeed, the quotient singularities must have no deformation for a fixed singularity on S, and thus they must be generic. Although the statement is probably true, it needs explanation. In this case a small resolution can be obtained as an analytic (formal) $\mathbb{Q}$-factorialization.

A possible generalization in dimension 3 is as follows: an analytic (formal) $\mathbb{Q}$-factorialization of a terminal quotient singularity of index r has only quotient singularities of either the same index or an index dividing r. Most likely, the terminal property is not sufficient for higher-dimensional generalizations, but one must assume that the singularities are rather mild, for example, as in Remark 7.14(iii).

Corollary 7.16 (Kulikov's model). *Any* 3*-fold scheme which is projective over* Δ *and has only semistable singularities has a resulting analytic* (*formal*) *model whose singularities are only the semistable singularities of Examples* 7.3.1–3, *with* $r \geqslant 2$ *in Example* 7.3.3. *This gives a relative minimal model if and only if the Kodaira dimension of the fibre of* X *over a generic point of* Δ *is* $\geqslant 0$.

Moreover, *the resulting model has the triple points of Example* 7.3.1 (*the singular quotients of Example* 7.3.2) *if and only if the initial scheme has triple points* (*singular quotients*).

Proof. According to Proposition 7.9(iii) a Kulikov model is obtained by an analytic (or formal) $\mathbb{Q}$-factorialization from a model of Corollary 7.5. For $r = 1$, this gives a small log resolution.

The statement on triple points (singular quotients) requires the connectedness of the locus of log canonical singularities ([34], 5.7).

Comment 7.17. The proof of Corollary 7.16 shows that a Kulikov model can be constructed from a Mori model. However, a Kulikov model can probably be constructed directly by working in a non-projective category even in a more general situation, and not just for semistable degenerations of K3 surfaces. Here the main difficulty is an analytic (formal) construction of extremal birational contractions, especially of small ones. Thus, the projectivity in the definition of semistable singularities (even local analytic) is perhaps not at all necessary.

On the other hand, when working in the algebraic Mori category, that is, in the projective MMP, we preserve the semistability (see the comparison theorem in 4.11 of [35] and the subsequent results in § 4 of [35]). The same assertion possibly holds for the algebraic non-projective category and for algebraic spaces.

Corollary 7.18. *Let* $f\colon X \to \Delta$ *be a projective semistable degeneration of non-singular surfaces with numerically trivial canonical divisor*, *that is*, $\Delta = \operatorname{Spec} \mathcal{O}$ *is the spectrum of a* (*local*) *discrete valuation ring* $\mathcal{O}$, *and the log pair* (X, B) *with boundary* $B = f^{-1}0$ (*the central fibre*) *is semistable log non-singular. Assume also that the surface* X_η *has numerically trivial divisor* K_{X_η} *for a generic point* $\eta \in \Delta$, *or equivalently*, $mK_{X_\eta} \sim 0$ *for some positive integer* m. *Then this degeneration has a semistable analytic* (*formal*) *model over* Δ *which is semistable minimal and has only the semistable singularities of Examples* 7.3.1–3, *with* $r \geqslant 2$ *in Example* 7.3.3. *In addition*, r *divides* m.

Moreover, *the resulting model has the triple points of Example* 7.3.1 (*the singular quotients of Example* 7.3.2) *if and only if the initial model has triple points* (*singular quotients*).

Proof. This follows immediately from Corollary 7.16. We also note that the Gorenstein index of the minimal model divides m, since the scheme $f^*0 = f^{-1}0$ (the central fibre) is reduced. Hence, r divides m.

Remark 7.19. By the classification of surfaces in Corollary 7.18, one can assume that $m = 1, 2, 3, 4$, or 6. More precisely, $m = 1$ except for the Enriques surfaces with $m = 2$ and the hyperelliptic surfaces with $m = 2, 3, 4$, or 6.

Examples 7.20. 1. (Semistable degenerations of K3 surfaces.) If X_η is a K3 surface, then $m = 1$, and any minimal model in Corollary 7.18 is log non-singular (and can fail to be projective). This is the main result of the paper [24] of Kulikov. By construction, such a model is an analytic (formal) $\mathbb{Q}$-factorialization of a Mori projective model in Corollary 7.5.

2. (Semistable degenerations of Enriques surfaces.) For such degenerations we have $m = 2$. Therefore, the singularities are those of Examples 7.3.2–3 with $r = 2$ and $a = 1$. The same holds for degenerations of hyperelliptic surfaces with $m = 2$. In addition, if an initial model has no triple points, then the Kulikov model has only the singularities of Example 7.3.3 with $r = 2$ and $a = 1$ that have a resolution in the form of a flower pot ([28], 3.3.1). Degenerations with singularities of this kind really do exist, which indicates an error in Kulikov's method for Enriques surfaces [24] (possibly because the method was numerical).

Bibliography

[1] F. Ambro, "Quasi-log varieties", *Tr. Mat. Inst. Steklov.* **240** (2003), 220–239; English transl., *Proc. Steklov Inst. Math.* **240** (2003), 214–233.

[2] V. V. Batyrev, "The cone of effective divisors on threefolds", *Contemp. Math.* **131,** Part 3 (1992), 337–352.

[3] A. A. Borisov and V. V. Shokurov, "Directional rational approximations with some applications to algebraic geometry", *Tr. Mat. Inst. Steklov.* **240** (2003), 73–81; English transl., *Proc. Steklov Inst. Math.* **240** (2003), 66–74.

[4] I. A. Cheltsov, "On a smooth quintic 4-fold", *Mat. Sb.* **191**:9 (2000), 139–162; English transl., *Sb. Math.* **191** (2000), 1399–1419.

[5] A. Corti, "Semistable 3-fold flips", alg-geom/9505035, 1995.

[6] A. Corti, "Factoring birational maps of threefolds after Sarkisov", *J. Algebraic Geom.* **4** (1995), 223–254.

[7] A. Corti, "3-fold flips after Shokurov", preprint, Univ. of Chicago, 2003.

[8] V. A. Iskovskikh, "Birational rigidity of Fano hypersurfaces in the framework of Mori theory", *Uspekhi Mat. Nauk* **56**:2 (2001), 3–86; English transl., *Russian Math. Surveys* **56** (2001), 207–291.

[9] V. A. Iskovskikh, "b-divisors and functional algebras according to Shokurov", *Tr. Mat. Inst. Steklov.* **240** (2003), 8–20; English transl., *Proc. Steklov Inst. Math.* **240** (2003), 4–15.

[10] V. A. Iskovskikh, "On Shokurov's paper 'Prelimiting flips'", *Tr. Mat. Inst. Steklov.* **240** (2003), 21–42; English transl., *Proc. Steklov Inst. Math.* **240** (2003), 16–36.

[11] Y. Kawamata, "Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces", *Ann. of Math.* (2) **127** (1988), 93–163.

[12] Y. Kawamata, "Termination of log flips for algebraic 3-folds", *Internat. J. Math.* **3** (1992), 653–659.

[13] Y. Kawamata, "Abundance theorem for minimal threefolds", *Invent. Math.* **108** (1992), 229–246.

[14] Y. Kawamata, "Moderate degenerations of algebraic surfaces", *Lecture Notes in Math.* **1507** (1992), 113–132.

[15] Y. Kawamata, "Semistable minimal models of threefolds in positive or mixed characteristic", *J. Algebraic Geom.* **3** (1994), 463–491.

[16] Y. Kawamata, "On the cone of divisors of Calabi–Yao fiber spaces", *Internat. J. Math.* **8** (1997), 665–687.

[17] Y. Kawamata, K. Matsuda, and K. Matsuki, "Introduction to the minimal model problem", Algebraic Geometry (Sendai, 1985) (T. Oda, ed.), (Adv. Stud. Pure Math., vol. 10) North-Holland, Amsterdam 1987, pp. 283–360.

[18] Y. Kawamata and K. Matsuki, "The number of the minimal models for a 3-fold of general type is finite", *Math. Ann.* **276** (1987), 595–598.

[19] S. Keel, K. Matsuki, and J. McKernan, "Log abundance theorem for threefolds", *Duke Math. J.* **75** (1994), 99–119; Corrections in: *Duke Math. J.* **122** (2004), 625–630.

[20] J. Kollár, "Flops", *Nagoya Math. J.* **113** (1989), 15–36.

[21] J. Kollár (ed.), *Flips and abundance for algebraic threefolds*, Summer Seminar at the University of Utah (Salt Lake City, 1991), (Astérisque, no. 211) Soc. Math. France, Paris 1992.

[22] J. Kollár and S. Mori, with the collaboration of C. H. Clemens and A. Corti, *Birational geometry of algebraic varieties*, (Cambridge Tracts in Math., vol. 134) Cambridge Univ. Press, Cambridge 1998.

[23] J. Kollár and N. I. Shepherd-Barron, "Threefolds and deformations of surface singularities", *Invent. Math.* **91** (1988), 299–338.

[24] V. S. Kulikov, "Degeneration of K3 surfaces and Enriques surfaces", *Izv. Akad. Nauk SSSR Ser. Mat.* **41** (1977), 1008–1042; English transl., *Math. USSR Izv.* **11** (1977), 957–989.

[25] K. Matsuki, *Introduction to the Mori program*, Springer-Verlag, New York 2002.

[26] S. Mori, "Flip theorem and the existence of minimal models for 3-folds", *J. Amer. Math. Soc.* **1** (1988), 117–253.

[27] N. Nakayama, "The lower semicontinuity of the plurigenera of complex varieties", *Internat. J. Math.* **8** (1997), 551–590.

[28] U. Persson, "On degenerations of algebraic surfaces", *Mem. Amer. Math. Soc.* **189** (1977).

[29] Yu. G. Prokhorov, "On semi-stable Mori contractions", *Izv. Ross. Akad. Nauk Ser. Mat.* **68**:2 (2004), 147–158; English transl., *Russian Acad. Sci. Izv. Math.* **68** (2004), 365–374.

[30] M. Reid, "Canonical 3-folds", Journeés de Géométrie Algébrique d'Angers (Angers, 1979) (A. Beauville, ed.), Sijthoff and Noordhoff, Alphen aan den Rijn 1980, pp. 273–310.

[31] M. Reid, "Minimal models of canonical 3-folds", Algebraic Varieties and Analytic Varieties (Tokyo, 1981), (Adv. Stud. Pure Math., vol. 1) North-Holland, Amsterdam 1983, pp. 131–180.

[32] M. Reid, "Young person's guide to canonical singularities", *Proc. Sympos. Pure Math.* **46** (1987), 345–414.

[33] V. V. Shokurov, "The nonvanishing theorem", *Izv. Akad. Nauk SSSR Ser. Mat.* **49** (1985), 635–651; English transl., *Math. USSR Izv.* **26** (1986), 591–604.

[34] V. V. Shokurov, "3-fold log flips", *Izv. Ross. Akad. Nauk Ser. Mat.* **56** (1992), 105–203; English transl., *Russian Acad. Sci. Izv. Math.* **40** (1993), 95–202.

[35] V. V. Shokurov, "Semistable 3-fold flips", *Izv. Ross. Akad. Nauk Ser. Mat.* **57**:2 (1993), 165–222; English transl., *Russian Acad. Sci. Izv. Math.* **42** (1994), 371–425.

[36] V. V. Shokurov, "3-fold log models", *J. Math. Sci.* **81** (1996), 2667–2699.

[37] V. V. Shokurov, "Prelimiting flips", *Tr. Mat. Inst. Steklov.* **240** (2003), 82–219; English transl., *Proc. Steklov Inst. Math.* **240** (2003), 75–213.

[38] V. V. Shokurov, "Letters of a bi-rationalist. IV. Geometry of log flips", Algebraic geometry (M. C. Beltrametti et al., eds.), de Gruyter, Berlin 2002, pp. 313–328.

[39] V. V. Shokurov, "Letters of a bi-rationalist, V. Minimal log discrepancies and termination of log flips", *Tr. Mat. Inst. Steklov.* **246** (2004), 328–351; English transl., *Proc. Steklov Inst. Math.* **246** (2004), 315–336.

[40] H. Takagi, "3-fold log flips according to V. V. Shokurov", preprint, 1999.

[41] S. Tsunoda, "Degenerations of surfaces", *Internat. J. Math.* **8** (1997), 755–764.

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