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Rational Cherednik algebras

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Abstract. We survey a number of results about rational Cherednik algebra representation theory and its connection to symplectic singularities and their resolutions.

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1. Introduction

This paper explores some rational Cherednik algebra representation theory and its interaction with constructions in algebraic geometry with a symplectic flavour. Although the rational Cherednik algebras were constructed as degenerations of Cherednik's double affine Hecke algebra and so have many links with the theory developed there, see [18], it turns out that a connection with the theory of symplectic resolutions, and particularly Hilbert schemes, has played a particularly important role. Such a connection was already foreseen at the birth of the algebras, and over the last decade the subject has developed significantly in this direction. There have been constructions of symplectic resolutions via moduli spaces of representations and localisation theorems from the categories of representations to sheaves on quantisations of the resolutions. Since symplectic resolutions turn up remarkably often in representation theory this in turn has led to the study of the geometry and algebra of such resolutions in general. Here the Cherednik algebras are key examples helping to form the subject. The goal of this brief survey is to present a little of this.

We completely omit lots of interesting aspects of Cherednik algebras, including realisations as Hecke algebras for double loop groups, as equivariant K-groups of affine flag manifolds, as Hall algebras of elliptic curves and equivariant K-theory of the Hilbert scheme. There are, however, a number of surveys on rational Cherednik algebras where many more details can be found, [30], [25], [62], [42], [71], [28].

The structure of the article is as follows. We begin in Section 2 by defining rational Cherednik algebras. In the third section we discuss symplectic singularities, representation theory at $t = 0$, and the existence of symplectic resolutions of orbit singularities. In Section 4 we explain the KZ functor, induction and restriction functors, and results on supports of representations. In the final section we present a number of different approaches to localisation of the rational Cherednik algebras of type A to the Hilbert scheme of points on the plane.

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2. Definitions

Rational Cherednik algebras are defined for any finite complex reflection group W.

Definition 1. A *complex reflection group* W is a group acting on a finite dimensional complex vector space $\mathfrak{h}$ that is generated by complex reflections: non-trivial elements that fix a complex hyperplane in $\mathfrak{h}$ pointwise. We say W is *irreducible* if $\mathfrak{h}$ is an irreducible representation of W.

Such groups, which include the finite Coxeter groups, play a major role in Lie theory and invariant theory, as well as appearing in many other fields. The irreducible complex reflection groups were classified in [65]: one infinite family appears, labelled $G(d,e,n)$ where d,e,n are positive integers such that e divides d (the Weyl groups of type A_{n-1}, B_n and D_n are $G(1,1,n), G(2,1,n)$ and $G(2,2,n)$ respectively); there are 34 exceptional cases.

Given a complex reflection group W, let $\mathcal{S}$ denote its set of complex reflections, and for $s \in \mathcal{S}$ let $\alpha_s \in \mathfrak{h}^*$ have kernel the hyperplane fixed by s. We set

$$\mathbf{k} = \mathbb{C}[\mathbf{t}, \mathbf{c}_s : s \in \mathcal{S}, \, \mathbf{c}_s = \mathbf{c}_{s'} \text{ if } s \text{ and } s' \text{ are conjugate in } W].$$

Definition 2 (Etingof-Ginzburg, [27]). The *rational Cherednik algebra* $\mathsf{H}_{\mathbf{k}}(W)$ is the $\mathbf{k}$-subalgebra of $\mathrm{End}_{\mathbf{k}}(\mathbf{k}[\mathfrak{h}])$ generated by the following operators:

- the action of $w \in W$
- multiplication by each $p \in \mathfrak{h}^* \subset \mathbf{k}[\mathfrak{h}]$
- for each $y \in \mathfrak{h}$, $T_y := \mathbf{t}\partial_y + \sum_{s\in\mathcal{S}} \mathbf{c}_s \alpha_s(y)\alpha_s^{-1}(s-1)$, where ∂_y is the $\mathbf{k}$-linear derivative on $\mathbf{k}[\mathfrak{h}]$ in the direction of y.

The operators T_y are called Dunkl operators (these were introduced by Dunkl for Coxeter groups [22]; for complex reflection groups see [24]). Remarkably, the Dunkl operators commute with one another – the subalgebra of $\mathsf{H}_{\mathbf{k}}(W)$ they generate is isomorphic to $\mathbf{k}[\mathfrak{h}^*]$. This is part of the following "PBW theorem".

Theorem 2.1 ([27]). *There is a $\mathbf{k}$-module isomorphism*

$$\mathsf{H}_{\mathbf{k}}(W) \xrightarrow{\sim} \mathbf{k}[\mathfrak{h}] \otimes_{\mathbf{k}} \mathbf{k}[W] \otimes_{\mathbf{k}} \mathbf{k}[\mathfrak{h}^*]$$

where each tensorand is a subalgebra of $\mathsf{H}_{\mathbf{k}}(W)$.

Specialisation $\mathbf{k} \longrightarrow \mathbb{C}$ to parameters $t \in \mathbb{C}$ and $c \in \mathbb{C}[\mathcal{S}]^{\mathrm{ad}W}$ leads to the rational Cherednik algebra $\mathsf{H}_{t,c}(W)$, a $\mathbb{C}$-algebra. The PBW theorem says that the $\mathsf{H}_{t,c}(W)$ are deformations of $\mathsf{H}_{0,0}(W) \cong \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*] \rtimes W$, the coordinate ring of the quotient stack $[(\mathfrak{h} \times \mathfrak{h}^*)/W]$.

Definition 3. Let $e = |W|^{-1} \sum_{w\in W} w \in \mathbb{C}W$, the trivial idempotent. The *spherical Cherednik algebra* $\mathsf{U}_{\mathbf{k}}(W)$ is the $\mathbf{k}$-algebra $e\mathsf{H}_{\mathbf{k}}(W)e$.

Specialisation this time leads to the family of $\mathbb{C}$-algebras $\mathsf{U}_{t,c}(W)$. These are deformations of $\mathsf{U}_{0,0}(W) = e(\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*] \rtimes W)e \cong \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^W$, the coordinate ring of the orbit space $(\mathfrak{h} \times \mathfrak{h}^*)/W$.

If $\lambda \in \mathbb{C}^*$ then $\mathsf{H}_{t,c}(W) \cong \mathsf{H}_{\lambda t,\lambda c}(W)$ and $\mathsf{U}_{t,c}(W) \cong \mathsf{U}_{\lambda t,\lambda c}(W)$ so we can assume that either $t = 0$ or $t = 1$. There is now a dichotomy: $\mathsf{U}_{0,c}(W)$ is commutative, but $\mathsf{U}_{1,c}(W)$ has a trivial centre; similarly, $\mathsf{H}_{0,c}(W)$ is a finite module over its centre, but the centre of $\mathsf{H}_{1,c}(W)$ is trivial. See [27] and [16].

Remarks 1. If $W = \mathbb{Z}_2$, the cyclic group of order 2, then $U_{1,c}(W) \cong U(\mathfrak{sl}_2)/(\Omega - \lambda(c))$ where Ω is the Casimir and $\lambda(c)$ a weight depending quadratically on c. More generally, for $W = \mathbb{Z}_d = G(d, 1, 1)$ the spherical algebras were studied in the context of generalisations of the above Lie theoretic quotient and also as (commutative and noncommutative) deformations of the kleinian singularity of type A_{d-1}. For these W the algebras $H_{t,c}(W)$ were then introduced by Crawley-Boevey and Holland in [19] where they also studied the other kleinian singularities.

3. Resolutions and deformations

The varieties $(\mathfrak{h} \times \mathfrak{h}^*)/W$ appearing above have symplectic singularities, a class of examples with rich algebraic, geometric and representation theoretic properties.

Definition 4. (Beauville, [1]) Let X be a normal affine variety over $\mathbb{C}$ that admits a symplectic 2-form ω on its smooth locus $\mathsf{sm}(X)$. We say that X has *symplectic singularities* if for any resolution of singularities $\pi : \widetilde{X} \to X$ the 2-form induced on $\pi^{-1}(\mathsf{sm}(X))$ extends to a regular 2-form on $\widetilde{X}$. If, in addition, there is a contracting $\mathbb{C}^*$-action on X with unique fixed point and such that $\lambda \cdot \omega = \lambda^n \omega$ for some positive integer n and for all $\lambda \in \mathbb{C}^*$, then we say that X has *contracting symplectic singularities.*

The paper [1] shows that $(\mathfrak{h}\times\mathfrak{h}^*)/W$ has contracting symplectic singularities: its smooth locus is the set of orbits of cardinality $|W|$ and the symplectic form on them is inherited from the natural W-equivariant symplectic form on $\mathfrak{h} \times \mathfrak{h}^*$; dilation on the vector space $\mathfrak{h} \times \mathfrak{h}^*$ produces the $\mathbb{C}^*$-action. There are many other examples of contracting symplectic singularities in representation theory: $\mathcal{N}(\mathfrak{g})$, the nullcone of reductive Lie algebra $\mathfrak{g}$; the normalisation of the closure of a nilpotent orbit in $\mathcal{N}(\mathfrak{g})$; Slodowy's transverse slices to nilpotent orbits in $\mathcal{N}(\mathfrak{g})$; hypertoric varieties; affine Nakajima quiver varieties.

A systematic study of symplectic singularities in [48] shows they have a canonical stratification by *finitely* many symplectic leaves.

Definition 5. Suppose X has symplectic singularities. A resolution $\pi : \widetilde{X} \longrightarrow X$ is called a *symplectic resolution* if the extension of the 2-form to $\widetilde{X}$ is non-degenerate.

We have that $\pi : \widetilde{X} \longrightarrow X$ is a symplectic resolution if and only if it is a crepant resolution, see [33]. Thus, since the canonical bundle of $\widetilde{X}$ is obviously trivial in this case, the bounded derived category of coherent sheaves on $\widetilde{X}$ are of significant interest in algebraic geometry, see [49] for important results in this direction. Moreover, the Springer resolution $\pi : T^*(G/B) \longrightarrow \mathcal{N}(\mathfrak{g})$, resolutions of kleinian singularities, and many Nakajima quiver varieties are symplectic resolutions, so the notion pervades geometric representation theory.

If X has symplectic singularities then ω defines a Poisson bracket on $\mathcal{O}_X$. A Poisson deformation of X is simultaneously a deformation of the variety X *and* its bracket. There is a satisfying theory of such Poisson deformations: building on work of Ginzburg-Kaledin, [38], and using the minimal model programme, Namikawa proved

Theorem 3.1 ([59]). *Let X have contracting symplectic singularities. The following are equivalent:*

1. *X has a smooth Poisson deformation,*
2. *X has a symplectic projective resolution.*

The Grothendieck-Springer resolution illustrates this theorem:

$$\begin{array}{ccccc} T^*(G/B) & \xrightarrow{\pi} & \mathcal{N}(\mathfrak{g}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ G\times_B \mathfrak{b} & \longrightarrow & \mathfrak{g} & \xrightarrow{\delta} & \mathfrak{g}/\!/G \end{array}$$

Here $T^*(G/B)$ is a symplectic resolution of $\mathcal{N}(\mathfrak{g})$, whilst the generic fibre of δ is G/T, a Poisson smoothing of $\mathcal{N}(\mathfrak{g})$. This also illustrates that the resolution deforms as well, a general fact for symplectic resolutions of contracting symplectic singularities.

The Grothendieck-Springer resolution is the source of a lot of remarkable representation theory; it is hoped that there is an equally rich picture around other symplectic singularities. Rational Cherednik algebras have proved very useful in understanding this: they are related to $(\mathfrak{h}\times\mathfrak{h}^*)/W$ in the way that the enveloping algebra of $\mathfrak{g}$ is related to $\mathcal{N}(\mathfrak{g})$, but there are several new phenomena which lead to many interesting and sometimes surprising developments.

Recall that the spherical algebra $\mathsf{U}_{0,c}(W)$ is commutative for all choices of c. In fact $\mathsf{U}_{0,c}(W)\cong Z(\mathsf{H}_{0,c}(W))$, the centre of $\mathsf{H}_{0,c}(W)$, [27]. Let $\mathsf{X}_c(W)=\operatorname{Spec}(\mathsf{U}_{0,c})$. These varieties are Poisson deformations of $\mathsf{X}_0(W)=(\mathfrak{h}\times\mathfrak{h}^*)/W$, the Poisson structure on $\mathsf{U}_{0,c}(W)$ being inherited from the commutator on the flat family $\mathbb{C}[\mathbf{t}]\longrightarrow\mathsf{U}_{\mathbf{t},c}(W)$: $\{F|_{\mathbf{t}=0},G|_{\mathbf{t}=0}\}=(t^{-1}[F,G])|_{\mathbf{t}=0}$ for $F,G\in\mathsf{U}_{\mathbf{t},c}(W)$. Thus the rational Cherednik algebras provide a family of Poisson deformations over $\mathbb{C}[\mathcal{S}]^{\mathrm{ad}W}$ as well as a coherent sheaf $\mathcal{R}_c(W)$ on $\mathsf{X}_c(W)$, corresponding to the $\mathsf{U}_{0,c}(W)$-module $e\mathsf{H}_{0,c}(W)$, whose endomorphism ring is $\mathsf{H}_{0,c}(W)$.

If L is an irreducible representation of $\mathsf{H}_{0,c}(W)$, then $Z(\mathsf{H}_{0,c}(W))$ acts by scalars on it, and we have a surjective map

$$\chi_c : \mathsf{Irrep}(\mathsf{H}_{0,c}(W)) \longrightarrow \mathsf{X}_c(W).$$

This is finite-to-one and from general principles of noncommutative algebra, we can use χ to study the singularities of $\mathsf{X}_c(W)$.

The prototype of such a principle is the theorem that the "Azumaya locus equals the smooth locus". Since $\mathsf{H}_{0,c}(W)$ is a finite module over its centre, there is an upper bound on the complex dimension of an irreducible $\mathsf{H}_{0,c}(W)$-representation; the Azumaya locus is by definition the set of maximal dimensional irreducible representations. It transpires that χ is one-to-one precisely on this locus, and that its image is the smooth locus of $\mathsf{X}_c(W)$, [27]. Over this locus, $\mathcal{R}_c(W)$ is actually a vector bundle of rank $|W|$, the maximal dimension of an irreducible, and we then deduce that over this locus $\mathsf{H}_{0,c}(W)$ is a matrix ring over $\mathcal{O}_{\mathsf{sm}(\mathsf{X}_c(W))}$.

Each $\mathsf{X}_c(W)$ has symplectic singularities and so is stratified by finitely many symplectic leaves. Thanks to [16] the irreducible representation theory of $\mathsf{H}_{0,c}(W)$ is constant along each leaf; elegant work of Losev, [53], and of Bellamy, [5], reduces the problem of studying a general leaf to a leaf of dimension 0, i.e. a point.

Remarks 2. There are general theorems on algebras that are finite modules over their centres that imply the Azumaya result mentioned here, [51], [15], [67]. Common to all these results is that the Azumaya locus should be relatively large (e.g. of codimension two) in the spectrum of the centre. Symplectic-like structures usually ensure this, since symplectic leaves are always even dimensional. One sees this in many Lie theoretic examples: the result holds for enveloping algebras of reductive Lie algebras in positive characteristic because of the symplectic structure on coadjoint orbits; it fails for affine Hecke algebras because there is no non-degenerate enough Poisson structure on their centre. Similarly, passing from an arbitrary leaf to a point by considering transverse slices is a normal tactic. For instance, Premet's work on Lie algebras in positive characteristic, [61], shows that along each coadjoint orbit the representation theory is equivalent to that of the associated finite W-algebra, which is attached to the transverse slice of the orbit, and in which the orbit shrinks to a point.

There is an embedding of $R := \mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W$ into $\mathsf{U}_{0,c}(W)$, and hence a (finite) morphism $\Upsilon_c : \mathsf{X}_c(W) \to \mathfrak{h}/W \times \mathfrak{h}^*/W$. If a point $x \in X_c$ is a symplectic leaf, then it must belong to the fibre $\Upsilon_c^{-1}(0)$. By studying this fibre and applying Theorem 3.1 one can prove the following.

Theorem 3.2 ([39], [38], [3]). *For some (and hence for generic) $c \in \mathbb{C}[\mathcal{S}]^{ad\ W}$ the variety $\mathsf{X}_c(W)$ is smooth if and only if $W = G(d,1,n)$ or $W = G_4$. It follows that $(\mathfrak{h} \times \mathfrak{h}^*)/W$ admits a symplectic projective resolution if and only if W is one of these groups.*

For $W = G(d,1,n)$ we obtain a symplectic resolution as follows, [72]. Let $Y = \mathbb{C}^2/\mathbb{Z}_d$ be the kleinian singularity of type A_{d-1} and let $\widetilde{Y}$ be its minimal resolution. Then

$$\pi : \mathsf{Hilb}^n(\widetilde{Y}) \to \mathsf{Sym}^n(\widetilde{Y}) \to \mathsf{Sym}^n(Y) = (\mathfrak{h} \times \mathfrak{h}^*)/W \tag{1}$$

is a symplectic projective resolution. This is a quiver variety; variation of GIT gives several other resolutions.

The group $W = G_4$ is an exceptional complex reflection group in the list of [65]. Two symplectic resolutions of $(\mathfrak{h} \times \mathfrak{h}^*)/W$, a four dimensional variety, are given in [52]. It remains to see whether these can be adequately described by some quiver variety construction.

The reduction of Losev and Bellamy shows that it is crucial to understand $\Upsilon_c^{-1}(0)$ and the corresponding representations of $\mathsf{H}_{0,c}(W)$. The points in $\Upsilon_c^{-1}(0)$ are equivalent to blocks in the *restricted rational Cherednik algebra* $\mathsf{H}_{0,c}(W) \otimes_R \mathbb{C}$. The irreducible representations of this algebra are labelled by the irreducible representations of W. It follows that the fibres of χ_c above $\Upsilon_c^{-1}(0)$ induce a partition of $\mathsf{Irrep}(W)$ which depends crucially on the parameter $c \in \mathbb{C}[\mathcal{S}]^{\mathrm{ad}W}$. It is conjectured, [44] and [54], that this partition essentially agrees with the decomposition of the cyclotomic Hecke algebra of W (specialised according to the choice of c) into blocks – these are called Rouquier families. Furthermore the dimension of the scheme theoretic fibre of $\Upsilon_c^{-1}(0)$ at this point should be the dimension of the corresponding Hecke algebra block. The first claim of this conjecture is confirmed for $W = G(d, e, n)$, [44] and [4], and the second claim holds whenever the given point of $\Upsilon_c^{-1}(0)$ is smooth in $\mathsf{X}_c(W)$. There is, however, no conceptual understanding of why this should be so; in particular in the Weyl group case, this suggests a link between the singularities of the spaces $\mathsf{X}_c(W)$ and Kazhdan-Lusztig theory.

4. Representations and Hecke algebras

The algebra $\mathsf{H}_{1,c}(W)$ is sensitive to the choice of parameter $c \in \mathbb{C}[\mathcal{S}]^{\mathrm{ad}W}$: for most choices $\mathsf{H}_{1,c}(W)$ is simple; for infinitely many values of c, however, there are finite dimensional representations, and hence two-sided ideals of finite codimension. Thus we need a robust category of representations to study. Motivated by Theorem 2.1 we have the following definition, [24].

Definition 6. $\mathcal{O}_c(W)$ is the full subcategory of finitely generated $\mathsf{H}_{1,c}(W)$-modules that are locally nilpotent for the action of $\mathfrak{h} \subset \mathbb{C}[\mathfrak{h}^*] \subset \mathsf{H}_{1,c}(W)$.

This an analogue of the BGG category $\mathcal{O}$ for semisimple Lie algebras. There are related versions of $\mathcal{O}_c(W)$ where $\mathfrak{h}$ acts by non-zero eigenvalues, but [10] shows that such categories are equivalent to $\mathcal{O}_c(W')$ for some subgroup W' of W.

There is an isomorphism $\mathsf{H}_{1,0}(W) \cong D(\mathfrak{h}) \rtimes W$, the ring of W-equivariant differential operators on $\mathfrak{h}$. Hence $\mathcal{O}_0(W)$ corresponds to W-equivariant holonomic $\mathcal{D}(\mathfrak{h})$-modules whose support equals $\mathfrak{h}$, in other words to finite rank W-equivariant vector bundles on $\mathfrak{h}$ with trivial connection. This category is equivalent to the category of finite dimensional $\mathbb{C}[W]$-modules: $V \in \mathbb{C}[W]\text{-mod} \mapsto \Delta_0(V) := H_{1,0}(W) \otimes_{\mathbb{C}[\mathfrak{h}] \rtimes W} V \cong \mathbb{C}[\mathfrak{h}] \otimes V$.

In general, we can define standard modules $\Delta_c(V) \in \mathcal{O}_c(W)$, but they may no longer be the only objects in the category. If $V \in \mathsf{Irrep}(W)$ then $\Delta_c(V)$ does, however, have a unique irreducible quotient, $L_c(V)$, and $\mathcal{O}_c(W)$ becomes a highest weight category with these standard and irreducible objects. It is an important open problem to determine the composition multiplicities $[\Delta_c(V) : L_c(V')]$ for $V, V' \in \mathsf{Irrep}(W)$.

Definition 2 shows that $\mathsf{H}_{1,c}(W)[\alpha_s^{-1} : s \in \mathcal{S}] \cong D(\mathfrak{h}_{\mathrm{reg}}) \rtimes W$ where $\mathfrak{h}_{\mathrm{reg}} = \{z \in \mathfrak{h} : \alpha_s(z) \neq 0 \text{ for all } s \in \mathcal{S}\}$, the subset of $\mathfrak{h}$ on which W acts freely. Hence, on restricting to $\mathfrak{h}_{\mathrm{reg}}$, we may pass from $\mathcal{O}_c(W)$ to a category of W-equivariant bundles on $\mathfrak{h}_{\mathrm{reg}}$ with flat connections, which in turn corresponds to some category of representations of the fundamental group $\pi_1(\mathfrak{h}_{\mathrm{reg}}/W)$, a generalised Artin braid group. These representations satisfy certain Hecke-type relations.

Theorem 4.1 ([37]). *There is an exact and essentially surjective functor*

$$\mathsf{KZ}_c : \mathcal{O}_c(W) \longrightarrow \mathcal{H}_q(W)\text{-mod}$$

where $\mathcal{H}_q(W)$ denotes the (topological) Hecke algebra of W at parameter $q = \exp(2\pi i c)$ (see [14] for a definition).

This functor has many good properties. In particular it generally restricts to an equivalence on $\mathcal{O}_c(W)^{\Delta}$, the subcategory of objects that have a filtration by standard objects. Remarkably, in [63], Rouquier shows that the data of such a functor on a highest weight category together with a compatible partial order on its simple objects determines the highest weight category up to equivalence.

For $W = S_n$, there is a Schur functor $S_q(n)\text{-mod} \longrightarrow \mathcal{H}_q(S_n)\text{-mod}$ from the q-Schur algebra, $S_q(n)$, which has analogous properties to KZ_c, see for instance [21]. Thus Rouquier's result above implies that there is an equivalence of categories between $\mathcal{O}_c(S_n)$ and $S_q(n)$-mod which sends standard modules to Weyl modules (or dual Weyl modules if c is a negative number). In particular, since the decomposition numbers are known for the q-Schur algebra, [69], we can describe the composition multiplicities $[\Delta_c(V) : L_c(V')]$ in this case in terms of parabolic Kazhdan-Lusztig polynomials of type $\widehat{A}$.

For $W = G(d, 1, n)$ and for $c \in \mathbb{C}[\mathcal{S}]^{\mathrm{ad}W}$ in a certain cone, one can show similarly that $\mathcal{O}_c(W)$ is Morita equivalent to a cyclotomic q-Schur algebra. A conjecture of Yvonne, [73], describes $[\Delta_c(V) : L_c(V')]$ in terms of a canonical basis of a level d Fock space, introduced in [68]. This conjecture is generalised to more general $c \in \mathbb{C}[\mathcal{S}]^{\mathrm{ad}W}$ in [63].

Remarks 3. There is another approach to the decomposition numbers of $\mathcal{O}_c(S_n)$ by Suzuki, [66]. Using conformal coinvariants, he constructs a functor from the Kazhdan-Lusztig category of modules for the affine Lie algebra of type $\widehat{A}$ at negative level to $\mathcal{O}_c(S_n)$. This produces an appropriate equivalence which again yields the above decomposition numbers. This functor is generalised to the $G(d, 1, n)$ case in [70] using conformal coinvariants twisted by a cyclic group action, but the corresponding decomposition numbers do not yet follow.

KZ_c is not generally a category equivalence since the passage from $\mathfrak{h}$ to $\mathfrak{h}_{\mathrm{reg}}$ kills any object of $\mathcal{O}_c(W)$ supported on $\mathfrak{h} \setminus \mathfrak{h}_{\mathrm{reg}}$, the union of reflecting hyperplanes of reflections in W. The support of an irreducible object is always a W-orbit of an intersection of reflecting hyperplanes, [35], so has, up to conjugacy, a parabolic subgroup W' attached to it by taking the stabiliser of a generic point in the intersection of these hyperplanes. Despite there usually being no non-trivial homomorphism from $\mathsf{H}_{1,c}(W')$ to $\mathsf{H}_{1,c}(W)$, Bezrukavnikov-Etingof have proved the following theorem by completing the rational Cherednik algebras at a point in the intersection of the relevant hyperplanes.

Theorem 4.2 ([10]). *Let $x \in \mathfrak{h}$ with stabiliser W_x. There are induction and restriction functors*

$$\mathcal{O}_c(W) \underset{\mathsf{Ind}_x}{\overset{\mathsf{Res}_x}{\rightleftarrows}} \mathcal{O}_c(W_x)$$

Up to isomorphism, these functors are independent of the choice of $x \in \mathfrak{h}_{reg}^{W_x} := \{z \in \mathfrak{h} : W_z = W_x\}$.

The isomorphism of functors is not canonical, and so the functor Res_x has monodromy on $\mathfrak{h}_{\mathrm{reg}}^{W_x}$. If $x \in \mathfrak{h}_{\mathrm{reg}}$ so that $W_x = 1$, the monodromy of the functor $\mathsf{Res}_x : \mathcal{O}_c(W) \longrightarrow \mathcal{O}_c(W_x) = \mathbb{C}$-mod recovers KZ_c. These functors are crucial to understanding $\mathcal{O}_c(W)$ and restriction to non-generic points preserves information killed by KZ_c. In [64], Shan has refined these functors to produce a crystal structure on the irreducible objects in $\mathcal{O}_c(G(d,1,n))$-modules (where n varies); this crystal is isomorphic to the one attached to the canonical basis of the level d Fock space above.

In studying induction and restriction it is important to know the support of representations. Etingof uses the Macdonald-Mehta integral for Weyl groups in [26] to give a beautiful description of the support of $L_c(\mathsf{triv})$, generalising the work of [70] which describes when $L_c(\mathsf{triv})$ is finite dimensional, i.e. is supported at $0 \in \mathfrak{h}$. In the case c is a positive constant function, his result states that $x \in \mathfrak{h}$ is in the support of $L_c(\mathsf{triv})$ if and only if $P_W/P_{W_x}(e^{2\pi i c}) \neq 0$, where $P_W(q) = \sum_{w\in W} q^{\ell(w)}$ is the Poincaré polynomial of W.

The induction and restriction functors help to determine the set of aspherical values of W, [10].

Definition 7. The parameter $c \in \mathbb{C}[\mathcal{S}]^{\mathrm{ad}W}$ is an *aspherical value* of W if $eL_c(V) = 0$ for some $V \in \mathsf{Irrep}(W)$; such an $L_c(V)$ is called an *aspherical representation.* We let $\Sigma(W)$ denote the set of aspherical values of W.

It can be shown that $c \notin \Sigma(W)$ if and only if the functor $\mathsf{H}_{1,c}(W)$-mod $\longrightarrow \mathsf{U}_{1,c}(W)$-mod, $M \mapsto eM$ is an equivalence. Thus for $c \notin \Sigma(W)$, $\mathsf{U}_{1,c}(W)$ inherits many favourable properties from $\mathsf{H}_{1,c}(W)$.

Using the restriction functors, one can show that $\Sigma(W)$ is the union of the $\Sigma(W')$ for proper parabolic subgroups $W' < W$ and of the set of finite dimensional aspherical representations of $\mathsf{H}_{1,c}(W)$. For $W = S_n$, this observation allows an inductive determination of the aspherical values, [10]. Remarkably, Bezrukavnikov and Etingof note that the number of aspherical representations matches phenomena in the $(\mathbb{C}^*)^2$-equivariant small quantum cohomology of $\mathsf{Hilb}^n(\mathbb{C}^2)$. Namely, multiplication in the quantum cohomology ring can be encoded by the so-called quantum differential equation which defines a flat connection on $\mathbb{C}$ for the trivial bundle associated with $H^*(\mathsf{Hilb}^n(\mathbb{C}^2), \mathbb{C})$, and this connection has regular singularities at $q = -\exp(2\pi i c)$ for $c \in \Sigma(S_n)$, [60]. Furthermore, the rank of the residue of the connection at each of these points equals the number of aspherical representations! For $W = G(d, 1, n)$, the set $\Sigma(W)$ has been calculated by Dunkl and Griffeth, [23]; the quantum differential equation for $\mathsf{Hilb}^n(\widetilde{Y})$ of (1) has been described by Maulik and Oblomkov, [55]. A matching of data is again expected.

These surprising coincidences are part of a large programme involving several people which aims to study the quantum cohomology, and particularly the quantum differential equation, of symplectic resolutions of contracting symplectic singularities, [13]. Amongst other things, intriguing connections with geometric representation theory and with derived categories of symplectic resolutions are predicted, and representations of rational Cherednik algebras have an important role.

5. Reduction and localisation

The spherical subalgebras $\mathsf{U}_{1,c}(W)$ share many properties with the quotients of enveloping algebras of reductive Lie algebras $U_\lambda(\mathfrak{g})$ at a central character λ. They are filtered with associated graded ring being the coordinate ring of a contracting symplectic singularity: $(\mathfrak{h} \times \mathfrak{h}^*)/W$ in the Cherednik case; $\mathcal{N}(\mathfrak{g})$ in the Lie case. This already produces a lot of structure including noetherianity, the Auslander-Gorenstein property, and a bound on the number of finite dimensional irreducible representations, [29]. Furthermore, it is only at very special values of the parameter where global dimension is infinite: at the aspherical values in the Cherednik case; at values such as $-\rho$ in the Lie case.

In the Lie case, a direct connection between $U_\lambda(\mathfrak{g})$ and the Springer resolution $\pi : T^*(G/B) \longrightarrow \mathcal{N}(\mathfrak{g})$ is made by the localisation theorem of Beilinson-Bernstein, [2]: this produces an equivalence between $U_\lambda(\mathfrak{g})$-modules and twisted $D_{G/B}$-modules. Combined with the Riemann-Hilbert correspondence, this relates BGG category $\mathcal{O}(\mathfrak{g})$ with perverse sheaves on G/B, and hence with Kazhdan-Lusztig theory for the Hecke algebra of the Weyl group of $\mathfrak{g}$.

We would like to produce an analogue of this for $\mathsf{U}_{1,c}(W)$ whenever there is a symplectic resolution $\pi : \widetilde{X} \longrightarrow (\mathfrak{h} \times \mathfrak{h}^*)/W$. This has been carried out for $W = S_n$ with $\widetilde{X} = \mathsf{Hilb}^n(\mathbb{C}^2)$, first in [45] algebraically, then in [34] and [50] using differential operators, then microlocal differential operators. (See [11] for similar results in positive characteristic.) Although these contructions are at their heart similar, and all have admitted various generalisations, the approaches in [34] and [50] connect directly to the mainstream of geometric representation theory. An interesting point is that, unlike $T^*(G/B)$, $\mathsf{Hilb}^n(\mathbb{C}^2)$ is not the cotangent bundle of a variety. This leads to a new point of view on localisation theorems which should be applicable to any symplectic resolution of a contracting symplectic singularity.

The first approach to quantising the Hilbert scheme follows Haiman's work on the $n!$ theorem, [47]. Here $\mathsf{Hilb}^n(\mathbb{C}^2)$ is constructed as the blow-up of $\mathsf{Sym}^n(\mathbb{C}^2)$ along the big diagonal, that is at the ideal $(\mathbb{C}[\mathfrak{h}\times\mathfrak{h}^*]^{\mathsf{sign}})^2$ where $\mathbb{C}[\mathfrak{h}\times\mathfrak{h}^*]^{\mathsf{sign}}$ denotes the polynomials that transform according to the sign representation under the S_n action. Thus $\mathrm{Coh}\,\mathsf{Hilb}^n(\mathbb{C}^2)$ is equivalent to a category of graded modules for the associated Rees ring. The first part of the following theorem asserts that there is a noncommutative version of this category.

Theorem 5.1 ([45]). *Assume that $c \not< 0$. There exists a category $\mathbb{X}_c$ of coherent sheaves on a noncommutative variety such that*

1. *$\mathbb{X}_c$ is a deformation of* $\mathrm{Coh}\,\mathsf{Hilb}^n(\mathbb{C}^2)$*,*
2. *There is an equivalence* $\mathsf{U}_{1,c}(S_n)\text{-mod} \xrightarrow{\sim} \mathbb{X}_c$.

The category $\mathbb{X}_c$ is a category of graded modules over an algebra which deforms the above Rees ring, replacing $\mathbb{C}[\mathfrak{h}\times\mathfrak{h}^*]^{S_n}$ with $\mathsf{U}_{1,c}(S_n)$ and $\mathbb{C}[\mathfrak{h}\times\mathfrak{h}^*]^{\mathsf{sign}}$ with $e\mathsf{H}_{1,c}(S_n)e_-$ where $e_- \in \mathbb{C}[S_n]$ is the idempotent corresponding to the sign representation. By an important result of Heckman-Opdam, see [9], $e\mathsf{H}_{1,c}(S_n)e_-$ is a $(\mathsf{U}_{1,c}(S_n), \mathsf{U}_{1,c+1}(S_n))$-bimodule and one can show it induces an equivalence $U_{1,c}(S_n)\text{-mod} \xrightarrow{\sim} U_{1,c+1}(S_n)\text{-mod}$ whenever c and $c+1$ are not aspherical values. Thus the glueing data in the category $\mathbb{X}_c$ produces Morita equivalences, giving the second claim.

The advantage of this construction is that one can apply Haiman's work directly. This leads in [46] to the calculation of the characteristic cycle of any object from $\mathcal{O}_c(S_n)$, i.e. the support cycles in $\mathsf{Hilb}^n(\mathbb{C}^2)$ of the degeneration of the corresponding objects in $\mathbb{X}_c$; one can also show that the image in $\mathbb{X}_c$ of the $\mathsf{U}_{1,c}(S_n)$-module $e\mathsf{H}_{1,c}(S_n)$ is a deformation of the Procesi bundle $\mathcal{P}$ on $\mathsf{Hilb}^n(\mathbb{C}^2)$. In fact, since c is not aspherical $e\mathsf{H}_{1,c}(S_n)$ induces an equivalence between $\mathsf{U}_{1,c}(S_n)\text{-mod}$ and $\mathsf{H}_{1,c}(S_n)\text{-mod}$ and is thus a projective $\mathsf{U}_{1,c}(S_n)$-module carrying the regular representation of S_n. These properties are analogous to crucial properties of $\mathcal{P}$: it is an enduring hope that the representation theory of $\mathsf{H}_{1,c}(S_n)$ may be used to give a new proof of the $n!$ theorem.

Remarks 4. A similar algebraic analysis is carried out for kleinian singularities, [12] and [57], and for Cherednik algebras with $W = G(d,1,n)$, [41], but in this general case the geometry of the associated varieties generalising $\mathsf{Hilb}^n(\mathbb{C}^2)$ is not yet completely understood. There is also a localisation theorem for Harish-Chandra bimodules of finite W-algebras in this spirit, [36].

$\mathsf{Hilb}^n(\mathbb{C}^2)$ can be realised as a quiver variety, [58]. Let V be an n-dimensional vector space, and let $GL(V)$ act naturally on $Y = \mathsf{End}(V)\times V$. Set $X = T^*Y$ and let $\mu_X : X \to \mathfrak{gl}(V)^*$ be the moment map. Nakajima proved that the hamiltonian reduction $\mu_X^{-1}(0)//GL(V)$ is isomorphic to $\mathsf{Sym}^n(\mathbb{C}^2)$, and that there is an open set $X^s \subset X$ of "stable" representations on which $GL(V)$ acts freely such that $\mu_X^{-1}(0)^s/GL(V)$ is isomorphic to $\mathsf{Hilb}^n(\mathbb{C}^2)$ where $\mu_X^{-1}(0)^s := \mu_X^{-1}(0)\cap X^s$.

Differentiating the action of $GL(V)$ on Y produces a homomorphism $\tau_X : U(\mathfrak{gl}(V)) \longrightarrow D(Y)$, a noncommutative analogue of μ_X. If $\nu : \mathfrak{gl}(V) \longrightarrow \mathbb{C}$ is a character, let I_ν be the left ideal of $U(\mathfrak{gl}(V))$ generated by $A + \nu(A)$ for all $A \in \mathfrak{gl}(V)$ and let $(D_Y, GL(V))_\nu$-mod denote the category of $GL(V)$-equivariant D_Y-modules whose derived action of $\mathfrak{gl}(V)$ equals the action defined through $\tau_X + \nu$.

Theorem 5.2 ([34]). *Given a character $\nu : \mathfrak{gl}(V) \longrightarrow \mathbb{C}$, there is a parameter $c_\nu \in \mathbb{C}$ such that*

1. $(D(Y)/D(Y)\tau_X(I_\nu))^{GL(V)} \cong \mathsf{U}_{1,c_\nu}(S_n)$.
2. *There is a functor $\mathbb{H} : (D_Y, GL(V))_\nu$-mod $\longrightarrow \mathsf{U}_{1,c_\nu}(S_n)$-mod defined by $\mathbb{H}(M) = M^{GL(V)}$ which is exact and essentially surjective.*

The first part of this theorem quantises the quiver theoretic description of $\mathsf{Sym}^n(\mathbb{C}^2)$; the second part allows one to study $\mathsf{U}_{1,c}(S_n)$-modules via D-modules on Y.

To realise the Hilbert scheme instead, we must pass to the stable locus X^s. But D_Y-modules are local on the base Y rather than on $X = T^*Y$, and X^s is an open set defined on X. Thus we are led to a microlocal point of view, considering sheaves of algebras on X rather than on Y. There is a standard quantisation of the symplectic manifold $T^*\mathbb{C}^n$ via the Moyal product, producing a sheaf of $\mathbb{C}[[h]]$-algebras. Denote by $\mathcal{W}(T^*\mathbb{C}^n)$ the sheaf we get from this by inverting h. It is a sheaf of $\mathbb{C}((h))$-algebras.

Definition 8 ([50]). A *quantised differential operator algebra* on a smooth symplectic variety X is a sheaf of $\mathbb{C}((h))$-algebras, $\mathcal{W}_X$, such that for each $x \in X$ there is a neighbourhood U of x and a symplectic morphism $\phi : U \longrightarrow T^*\mathbb{C}^n$ such that $\mathcal{W}_X|_U \cong \phi^*\mathcal{W}(T^*\mathbb{C}^n)$.

Going back to our specific case let $U = X^s$, a symplectic manifold with a proper and free symplectic $GL(V)$-action and orbit map $p : \mu_X^{-1}(0)^s \longrightarrow \mu_X^{-1}(0)^s/GL(V) \cong \mathsf{Hilb}^n(\mathbb{C}^2)$. There is a noncommutative moment map: $\tau_U : \mathfrak{gl}(V) \longrightarrow \mathcal{W}_U$. Kashiwara and Rouquier, [50], show that

$$\mathcal{W}_{\mathsf{Hilb},\nu} := p_*\mathcal{E}nd_{\mathcal{W}}(\mathcal{W}_U/\mathcal{W}_U\tau_U(I_\nu))^{GL(V)}$$

is a quantised differential operator algebra on $\mathsf{Hilb}^n(\mathbb{C}^2)$ and that there is an equivalence of categories

$$(\mathcal{W}_{X^s}, GL(V))_\nu\text{-mod} \longrightarrow \mathcal{W}_{\mathsf{Hilb},\nu}\text{-mod}$$

for appropriate categories of $\mathcal{W}$-modules.

The categories above are $\mathbb{C}((h))$-linear and thus cannot be $D(Y)$-modules or $\mathsf{U}_{1,c}(S_n)$-modules. To remedy this, extend the good $\mathbb{C}^*$-actions that arise from the contracting action on $\mathsf{Sym}^n(\mathbb{C}^2)$ to the quantised differential operator algebras by letting h be an eigenvector of appropriate weight. Then categories of $\mathbb{C}^*$-equivariant $\mathcal{W}$-modules are equivalent, under taking fixed points, to $\mathbb{C}$-linear categories: for instance $(\mathcal{W}(T^*\mathbb{C}^n), \mathbb{C}^*)$-mod $\xrightarrow{\sim} D(\mathbb{C}^n)$-mod for appropriate $\mathbb{C}^*$-actions. This produces an equivalence

$$(\mathcal{W}_{X^s}, GL(V) \times \mathbb{C}^*)_\nu\text{-mod} \longrightarrow (\mathcal{W}_{\mathsf{Hilb},\nu}, \mathbb{C}^*)\text{-mod},$$

the quantisation of the quiver theoretic description of $\mathsf{Hilb}^n(\mathbb{C}^2)$. Kashiwara and Rouquier then prove the following elegant Beilinson-Bernstein style theorem.

Theorem 5.3 ([50]). *For a character $\nu : \mathfrak{gl}(V) \longrightarrow \mathbb{C}$ such that $c_\nu \geq 0$, the global sections functor induces an equivalence*

$$(\mathcal{W}_{\mathsf{Hilb},\nu}, \mathbb{C}^*)\text{-mod} \longrightarrow \mathsf{U}_{1,c_\nu}(S_n)\text{-mod}.$$

With the approaches of [34] and of [50] one can begin a D-module or microlocal study of the representation theory of $\mathsf{U}_{1,c}(S_n)$ or $\mathsf{H}_{1,c}(S_n)$. This has been carried out (in a slightly different context) in [31] and [32]. Recently, McGerty, [56], gives a new construction for $W = S_n$ of the KZ-functor, new versions of induction and restriction functors, and recovers the characteristic cycle computations of objects in $\mathcal{O}_c(S_n)$, all via microlocal fundamental groups and classical D-module theory from geometric representation theory.

Remarks 5. The above analysis should apply to other symplectic resolutions of contracting symplectic singularities that are realised by hamiltonian reduction. For finite W-algebras see [20] and for hypertoric varieties see [6] and the works of Braden, Licata, Proudfoot and Webster. For general quiver varieties one of the most intriguing aspects is to discover the algebras appearing as global sections, replacing the spherical Cherednik algebras in the Hilbert scheme case. It is still challenging to find the correct tools and concepts to unlock the properties of the categories of $\mathcal{W}$-modules.

Remarks 6. Back in the world of rational Cherednik algebras, it seems that the case $W = G(d, 1, n)$ will be understood via D-modules or microlocalisation. But, with the exception of G_4, all other complex reflection groups have no corresponding symplectic resolution; how to study these examples geometrically is unclear at the moment. That these cases have wider significance is clear from applications to integrable systems, D-module theory and the representation theory of complex reflection groups, see e.g. [8] and [7], and applications to algebraic combinatorics, see e.g. [40] and [43].

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