Lecture 19: The Vinogradov Mean Value Theorem.

The final topic of the course is a problem in number theory which was spectacularly resolved by Bourgain – Demeter – Guth in 2015 using techniques studied in earlier lectures.

We are interested in the following system of k diophantine equations in $2s$ variables :–

$$\begin{cases} X_1 + \dots + X_s = Y_1 + \dots + Y_s \\ X_1^2 + \dots + X_s^2 = Y_1^2 + \dots + Y_s^2 \\ \quad\vdots \\ X_1^k + \dots + X_s^k = Y_1^k + \dots + Y_s^k \end{cases} \qquad (V)_{s,k}$$

The problem is to count the number of solutions to $(V)_{s,k}$ lying in $[\phi, N]^{2s}$. In particular, let

$$J_{s,k}(N) := \#\text{ sol}^{\text{n}}\text{s to } (V)_{s,k} \text{ with } 1 \leq X_i, Y_i \leq N \text{ for } i = 1, \dots, s.$$

This problem arose in Vinogradov's investigation into the classical Waring problem in the 1930s. We will not cover the background motivation here.

The connection between $(V)_{s,k}$ and Fourier analysis is made via the following elementary observation:

Lemma 1 If

$$f_k(\vec{\alpha}, N) := \sum_{X=1}^{N} e^{2\pi i(\alpha_1 X + \dots + \alpha_k X^k)},$$

then

$$J_{s,k}(N) = \int_{[0,1]^k} |f_k(\vec{\alpha}, N)|^{2s}\, d\vec{\alpha}.$$

It will be useful, in fact, to consider a more general result.

Given $\vec{h} \in \mathbb{Z}^k$, let

$J_{s,k,\vec{h}}(N) := \#$ sol^{n}s to

$$\begin{cases} X_1 + \cdots + X_s - Y_1 - \cdots - Y_s = h_1 \\ X_1^2 + \cdots + X_s^2 - Y_1^2 - \cdots - Y_s^2 = h_2 \\ \vdots \\ X_1^k + \cdots + X_s^k - Y_1^k - \cdots - Y_s^k = h_k \end{cases} \qquad (V)_{s,k,\vec{h}}$$

with $1 \le X_i, Y_i \le N$, $i = 1, \dots, s$.

Thus, $J_{s,k}(N) = J_{s,k,\vec{0}}(N)$.

Lemma 1' :- For f_k as above

$$J_{s,k,\vec{h}}(N) = \int_{[0,1]^k} |f_k(\vec{\alpha}, N)|^{2s} e^{-2\pi i \langle \vec{\alpha}, \vec{h} \rangle} \, d\vec{\alpha}. \qquad (1)$$

Proof :- Note that

$$|f_k(\vec{\alpha}, N)|^{2s} = f_k(\vec{\alpha}, N)^s \, \overline{f_k(\vec{\alpha}, N)^s}$$

where

$$f_k(\vec{\alpha}, N)^s = \sum_{X_1, \dots, X_s = 1}^{N} e^{2\pi i (\alpha_1 \sum_{j=1}^{s} X_j + \cdots + \alpha_k \sum_{j=1}^{s} X_j^k)}$$

so that

$$|f_k(\vec{\alpha}, N)|^{2s} = \sum_{X_1, \dots, X_s = 1}^{N} \sum_{Y_1, \dots, Y_s = 1}^{N} e^{2\pi i (\alpha_1 \sum_{j=1}^{s} (X_j - Y_j) + \cdots + \alpha_k \sum_{j=1}^{s} (X_j^k - Y_j^k))}.$$

Substituting this into the right-hand integral in (1) one obtains

$$\sum_{X_1, \dots, X_s = 1}^{N} \sum_{Y_1, \dots, Y_s = 1}^{N} \prod_{\ell=1}^{k} \int_{[0,1]} e^{2\pi i \alpha_\ell (\sum_{j=1}^{s} X_j^\ell - Y_j^\ell - h_\ell)} \, d\alpha_\ell$$

$$= \sum_{X_1, \dots, X_s = 1}^{N} \sum_{Y_1, \dots, Y_s = 1}^{N} \prod_{\ell=1}^{k} \delta\left(\sum_{j=1}^{s} X_j^\ell - Y_j^\ell - h_\ell\right)$$

by the standard orthogonality relations, where $\delta(X) := \begin{cases} 1 & \text{if } X = 0 \\ 0 & \text{otherwise} \end{cases}$. $\square$

Lower bounds :-

Using Lemma 1' one can prove the following lower bound for $J_{s,k}(N)$.

Lemma 2 :-

$$J_{s,k}(N) \gtrsim N^s + N^{2s - \frac{k(k+1)}{2}}$$

Proof :-

- The bound $J_{s,k}(N) \geq N^s$ is completely trivial. In particular, given any

$$(x_1, \ldots, x_s) \in \{1, \ldots, N\}^s$$

one may form a solution
$(x_1, \ldots, x_s, y_1, \ldots, y_s) \in \{1, \ldots, N\}^{2s}$
by taking $y_j := x_j$, $1 \leq j \leq s$.

- To prove $J_{s,k}(N) \gtrsim N^{2s - \frac{k(k+1)}{2}}$, note that

$$|X_1^\ell + \cdots + X_s^\ell - Y_1^\ell - \cdots - Y_s^\ell| \leq s \cdot N^\ell$$

for $1 \leq X_i, Y_i \leq N$, $1 \leq i \leq s$. Consequently,

$$J_{s,k,\vec{h}}(N) = 0 \quad \text{if} \quad |h_\ell| > s N^\ell$$

and so, by Lemma 1',

$$\sum_{\vec{h} \in \mathbb{Z}^k} J_{s,k,\vec{h}}(N) = \sum_{|h_1| \leq sN} \cdots \sum_{|h_k| \leq sN^k} \int_{[0,1]^k} |f_k(\vec{\alpha}, N)|^{2s} e^{-2\pi i \langle \vec{\alpha}, \vec{h} \rangle} \, d\vec{\alpha}. \quad (2)$$

By the triangle inequality, the right-hand side is bounded by

$$s^k N^{\frac{k(k+1)}{2}} J_{s,k}(N). \quad (3)$$

On the other hand, the left-hand side of (2) is precisely the number of solutions to the system

$$X_1 + \dots + X_s - Y_1 - \dots - Y_s + Z_1 = 0$$
$$\vdots$$
$$X_1^k + \dots + X_s^k - Y_1^k - \dots - Y_s^k + Z_k = 0$$

where $1 \leq X_i, Y_i \leq N$ and there are no constraints on the Z_i. Clearly, for every choice of X_i, Y_i, $i = 1, \dots, s$ there is a unique choice of Z_i, $i = 1, \dots, s$ and so this is equal to N^{2s}. (4).

Comparing (3) and (4),

$$J_{s,k}(N) \gtrsim_s N^{2s - \frac{k(k+1)}{2}}, \text{ as required } \square.$$

Conjecture (Vinogradov mean value theorem)

For all integers $s, k \geq 1$ and all $\varepsilon > 0$,

$$J_{s,k}(N) \lesssim_{s,k,\varepsilon} N^{s+\varepsilon} + N^{2s - \frac{k(k+1)}{2} + \varepsilon}. \quad (5)$$

Despite the unusual naming convention, this statement was a conjecture (not a theorem) until 2015.

- For $k = 1, 2$ the result may be deduced via elementary methods (e.g. divisor bounds).
- For $k = 3$, the conjecture was established by Wooley using a method called 'efficient congruencing'.
- The remaining cases $k \geq 4$ were established by Bourgain - Demeter - Guth using a technique from harmonic analysis called 'decoupling'.

In particular, Bourgain–Demeter–Guth established a significantly stronger 'functional' version of the mean value theorem/conjecture.

Decoupling for the moment curve.

Let $\gamma_0 : \mathbb{R} \to \mathbb{R}^k$ denote the moment curve in $\mathbb{R}^n$, given by

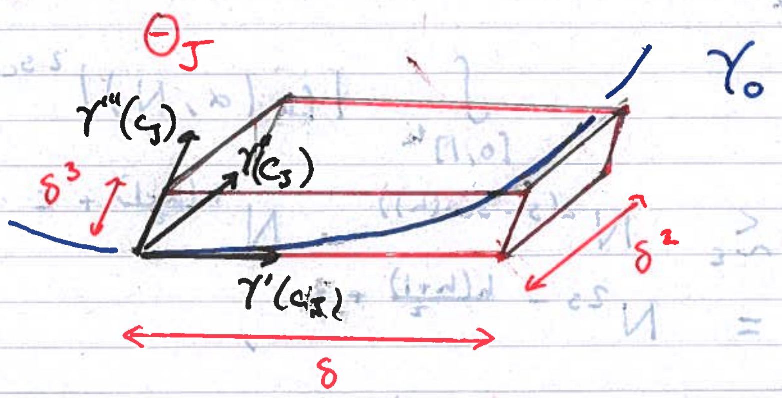
$$\gamma_0(s) := (s, s^2, \dots, s^n).$$

Continuing the theme of the course, we consider functions with Fourier support in some neighbourhood of this curve.

- Here the neighbourhood will be 'anisotropic' in order to match the scaling properties of the curve.

- For $0 < \delta < 1$, let $\mathcal{P}(\delta)$ denote the partition of $[0,1]$ into (roughly equal length) intervals of length $\sim \delta$.

- Given $J \in \mathcal{P}(\delta)$ with centre c_J let Θ_J denote the parallelepiped with
  - centre $\gamma_0(c_J)$
  - dimensions $\delta \times \delta^2 \times \dots \times \delta^n$
  - sides parallel to $\gamma_0'(c_J), \dots, \gamma_0^{(n)}(c_J)$



Theorem 1 (Bourgain – Demeter – Guth) For all $k \in \mathbb{N}$, $2 \le p \le k(k+1)$ and all $\varepsilon > 0$,

$$\Big\| \sum_{J \in \mathcal{D}(\delta)} f_J \Big\|_{L^p(\mathbb{R}^k)} \lesssim_{\varepsilon,k} \delta^{-\varepsilon} \Big(\sum_{J \in \mathcal{D}(\delta)} \| f_J \|^2_{L^p(\mathbb{R}^k)} \Big)^{1/2}$$

whenever $(f_J)_{J \in \mathcal{D}(\delta)}$ satisfies $\operatorname{supp} \hat{f}_J \subseteq \Theta_J$ for all $J \in \mathcal{D}(\delta)$.

Decoupling implies mean value theorem.

Note the two terms $N^{s+\varepsilon}$ and $N^{2s - \frac{k(k+1)}{2} + \varepsilon}$ on the right-hand side of (5) agree if $s = s_{cr}(k)$ is the critical index

$$s_{cr}(k) := \frac{k(k+1)}{2}.$$

Lemma 3:- If (5) holds for some fixed k at the critical index $s_{cr}(k)$, then it holds for k at any index $s \geq 1$.

Proof:- Clearly

$$f_k(\alpha, N) := \sum_{X=1}^{N} e^{2\pi i(\alpha_1 X + \dots + \alpha_k X^k)}$$

satisfies

$$\| f_k(\cdot, N) \|_{L^\infty([0,1]^k)} \le N.$$

Thus, if $s > s_{cr}(k)$, then

$$\int_{[0,1]^k} |f_k(\alpha, N)|^{2s} \, d\alpha \le \| f_k(\cdot, N) \|^{2(s - s_{cr}(k))}_{L^\infty([0,1]^k)} \cdot \int_{[0,1]^k} |f_k(\alpha, N)|^{2 s_{cr}(k)} \, d\alpha$$

$$\lesssim_\varepsilon N^{2(s - s_{cr}(k))} \cdot N^{\frac{k(k+1)}{2} + \varepsilon}$$

$$= N^{2s - \frac{k(k+1)}{2} + \varepsilon},$$

assuming (5) holds at the critical index.

On the other hand, if $1 \leq s < s_{cr}(h)$, then Hölder's inequality yields

$$\int_{[0,1]^h} |f_h(\alpha, N)|^{2s} \, d\alpha \leq \left(\int_{[0,1]^h} |f_h(\alpha, N)|^{2s_{cr}(h)} \, d\alpha \right)^{s/s_{cr}(h)}$$

$$\lesssim_\varepsilon \left(N^{\frac{h(h+1)}{2} + \varepsilon}\right)^{s/s_{cr}(h)} \leq N^{s+\varepsilon}$$

$\square$

Thus, it suffices to show the mean value estimate at the critical exponent:-

$$\int_{[0,1]^h} |f_k(\alpha, N)|^{h(h+1)} \, d\alpha \lesssim_\varepsilon N^{\frac{h(h+1)}{2} + \varepsilon} \quad (6).$$

Proof of (6) assuming Theorem 1:- Let D_N be the diagonal matrix

$$D_N = \begin{pmatrix} N & & & \\ & N^2 & & \\ & & \ddots & \\ & & & N^h \end{pmatrix}$$

so that

$$\gamma_0(Nt) = D_N \gamma_0(t).$$

Thus,

$$f_k(\alpha, N) = \sum_{t=1}^{N} e^{2\pi i \langle \gamma_0(t), \alpha \rangle}$$

$$= \sum_{t=1}^{N} e^{2\pi i \langle \gamma_0(\frac{t}{N}), D_N \alpha \rangle}$$

and so

$$\int_{[0,1]^h} |f_h(\alpha, N)|^{h(h+1)} \, d\alpha =$$

$$= N^{-\frac{k(h+1)}{2}} \int_{\prod_{\ell=1}^{h} [0, N^\ell]} \left| \sum_{t=1}^{N} e^{2\pi i \langle \gamma_0(\frac{t}{N}), x \rangle} \right|^{h(h+1)} dx$$

Now, the function $x_\ell \mapsto e^{2\pi i (\frac{t}{N})^\ell x_\ell}$ is N^ℓ-periodic and we can break $[0, N^h]$ into $N^{k-\ell}$ intervals of length N^ℓ. Thus,

$$\int_{[0,1]^k} |f_k(\alpha,N)|^{k(k+1)}\,d\alpha$$

$$\leq N^{-\frac{k(k+1)}{2}-\frac{k(k-1)}{2}} \int_{[0,N^k]^k} \Big|\sum_{t=1}^{N} e^{2\pi i\langle \gamma_0(\frac{t}{N}),x\rangle}\Big|^{k(k+1)} dx$$

$$\lesssim N^{-k^2} \int_{\mathbb{R}^k} \Big|\sum_{J\in\mathcal{D}(1/N)} g_J\Big|^{k(k+1)} \qquad (7)$$

where :-

- $\mathcal{D}(1/N) := \{ [\frac{t-1}{N},\frac{t}{N}] : t=1,\dots,N\}$
- $g_J(x) := e^{2\pi i\langle\gamma_0(\frac{t}{N}),x\rangle}\,\eta_{N^k}(x)$, $J=[\frac{t-1}{N},\frac{t}{N}]$

for $\eta_{N^k}\in\mathcal{S}(\mathbb{R}^k)$ satisfying :-

- $|\eta_{N^k}(x)|\gtrsim 1$ for $x\in[0,N^k]^k$
- $\|\eta_{N^k}\|_{L^p(\mathbb{R}^k)} \lesssim N^{k^2/p}$
- $\operatorname{supp}\widehat{\eta}_{N^k} \subseteq B(0,N^{-k})$.

Thus, $\widehat{g_J}(\xi) = \widehat{\eta}_{N^k}(\xi-\gamma_0(\frac{t}{N}))$ so that

$\operatorname{supp}\widehat{g_J} \subseteq B(\gamma_0(\frac{t}{N}),N^{-k}) \subseteq \Theta_J$.

Applying Theorem 1,

$$\int_{\mathbb{R}^k} \Big|\sum_{J\in\mathcal{D}(1/N)} g_J\Big|^{k(k+1)} \lesssim_\varepsilon N^{\varepsilon}\left(\#\mathcal{D}(\tfrac{1}{N})^{\frac{1}{2}} N^{k/(k+1)}\right)^{(k+1)k}$$

$$\sim N^{k^2+\frac{k(k+1)}{2}+\varepsilon} \qquad (8).$$

Combining (7) and (8) yields (6) as desired $\square$