MODERN DEVELOPMENTS IN FOURIER ANALYSIS

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Syllabus: Recently, there have been a number of remarkable developments in euclidean harmonic analysis, related to the *Fourier restriction conjecture*. Broadly speaking, one is interested in studying functions f whose Fourier transform is supported in a neighbourhood of a submanifold of $\mathbb{R}^n$, such as a paraboloid or a cone or a sphere. Such situations arise naturally in PDE, as well as in harmonic analysis and analytic number theory.

One of the goals of this course is to understand the so-called *decoupling inequalities* of Wolff [4] and Bourgain–Demeter [2]. The idea is that whilst f may be difficult to analyse, it can be broken up as a sum of pieces f_θ which are much easier to understand (in particular, the pieces f_θ are localised in frequency to small regions where the submanifold is essentially flat). The key question is then to understand how the various f_θ interact with one another. In decoupling theory this is achieved via a norm inequality of the form

$$\|f\|_{L^p(\mathbb{R}^n)} \lessapprox \Big(\sum_\theta \|f_\theta\|^2_{L^p(\mathbb{R}^n)} \Big)^{1/2}. \tag{1}$$

The key feature of (1) is that an ℓ^2 expression appears on the right-hand side, rather than the trivial ℓ^1 expression given by the triangle inequality; this crucially takes into account complex destructive interference patterns between the different f_θ.

Decoupling theory has had a profound impact on a wide range of (ostensibly) distinct areas of mathematical analysis. A large portion of the course will investigate applications.

Possible topics include:

- Fourier analysis philosophy and uncertainty principle heuristics.
- Multilinear harmonic analysis: the Bennett–Carbery–Tao theorem via induction-on-scale [1].
- The Bourgain–Guth method for estimating oscillatory integral operators [3].
- Proof of the ℓ^2-decoupling theorem of Bourgain–Demeter [2].
- Relation to incidence geometry.
- Applications of decoupling to PDE: Strichartz estimates on the torus, spectral theory, local smoothing for the wave equation.
- Applications of decoupling to harmonic analysis: Bochner–Riesz means, Fourier restriction, L^p-Sobolev and maximal bounds for generalised Radon transforms.
- Applications of decoupling to analytic number theory: diophantine equations, the proof of the Vinogradov mean value theorem, Weyl sum bounds, the Lindelöf hypothesis.
- Variable coefficient extensions and analysis on manifolds.

Relevant indicated interests: Harmonic analysis, the ℓ^2 decoupling theorem, pseudo-differential operators, dispersive PDEs, wave equations, spectral theory, operator theory, functional analysis, geometric measure theory.

Schedule: The class meets Tuesdays and Fridays 9:15 am – 10:15 am from 6th January to 13th March 2020. The lectures will be supplemented with additional contact hours and meetings for discussion between students.

Textbooks: The topic of this course is a very recent development and no textbooks are available.

A comprehensive bibliography will be made available to the students on the course webpage.

Assessment: Whilst the initial lectures will be given by the instructor, the students will be asked to choose a topic to present later as the course progresses. A list of topics will be prepared and made available at the outset of the course, with the option for students to suggest their own (relevant) topic if they so wish. The syllabus is particularly conducive to this approach: we hope to explore various applications of the decoupling theory and each application should fill 1 - 2 sessions. Support for the students will be provided through office hours. The students will also be asked to prepare a latex write up of the topic they present. Ideally, these reports will be compiled at the end of the course and will be made publicly available and are likely to prove a valuable reference for the wider mathematical community.

Prerequisites: A modest background in functional analysis and measure theory: L^p spaces, interpolation of operators, Hölder and Minkowski inequalities, etc. Elementary theory of the Fourier transform: Schwartz functions, Fourier inversion, Plancherel's theorem. The course will aim to develop some basic understanding of the Fourier transform at a heuristic level and should be accessible to students working in pure analysis in a broad sense.

References

[1] J. Bennett, A. Carbery, and T. Tao. On the multilinear restriction and Kakeya conjectures. *Acta Math.*, 196(2):261–302, 2006.

[2] J. Bourgain and C. Demeter. The proof of the l^2 decoupling conjecture. *Ann. of Math. (2)*, 182(1):351–389, 2015.

[3] J. Bourgain and L. Guth. Bounds on oscillatory integral operators based on multilinear estimates. *Geom. Funct. Anal.*, 21(6):1239–1295, 2011.

[4] T. Wolff. Local smoothing type estimates on L^p for large p. *Geom. Funct. Anal.*, 10(5):1237–1288, 2000.