Harmonic Analysis Working Group :

Bourgain's circular maximal theorem

Lecture 3

Henceforth we will focus on the proof of Bourgain's circular maximal theorem, following the argument in Bourgain's original papers.

Recap of earlier reductions

By the reductions of the previous lecture and restricted strong-type interpolation, it suffices to show the following:-

Proposition 1. For all $1 < q < 2$ there exists some exponent $\varepsilon(q') > 0$ such that

$$\Big\| \sum_{C \in \mathcal{C}} \tilde{\chi}_{C^\delta} \Big\|_{L^q(\mathbb{R}^2)} \lesssim_q \delta^{2/q - 1 + \varepsilon(q')} [\#\mathcal{C}]^{1/q} \qquad (1)$$

holds whenever $0 < \delta \ll 1$ and $\mathcal{C}$ is a collection of unit scale circles in the plane with δ-separated centres.

Here a "unit scale circle" is a circle $C = C(x, r)$ in $\mathbb{R}^2$ with centre $x \in [-1/10, 1/10]^2$ and radius $r \in [1, 2]$.

From Lecture 2, we already have

$$\Big\| \sum_{C \in \mathcal{C}} \tilde{\chi}_{C^\delta} \Big\|_{L^2(\mathbb{R}^2)} \lesssim [\#\mathcal{C}]^{1/2} \qquad (2)$$

and it is not difficult to show that

$$\Big\| \sum_{C \in \mathcal{C}} \tilde{\chi}_{C^\delta} \Big\|_{L^1(\mathbb{R}^2)} \leq \sum_{C \in \mathcal{C}} \| \tilde{\chi}_{C^\delta} \|_{L^1(\mathbb{R}^2)} \lesssim \delta \cdot \#\mathcal{C} \qquad (3)$$

Interpolating (2) and (3) via Hölder's inequality,

$$\Big\| \sum_{C \in \mathcal{C}} \tilde{\chi}_{C^\delta} \Big\|_{L^q(\mathbb{R}^2)} \lesssim \delta^{2/q - 1} [\#\mathcal{C}]^{1/q} \quad \text{for } 1 \leq q \leq 2$$

which just misses (1) by an $\varepsilon(q')$-power of δ. Thus, our task is simply to obtain an ε-improvement over the estimates established in the previous lecture.

Overall strategy : Interpolation.

The strategy used to prove Proposition 1 is to partition $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ such that, for some fixed $\varepsilon > 0$, we have

$$\Big\| \sum_{C \in \mathcal{C}_2} \tilde{\chi}_{C^\delta} \Big\|_{L^2(\mathbb{R}^2)} \lesssim \delta^{\varepsilon} [\#\mathcal{C}]^{1/2} \qquad (2')$$

and

$$\Big\| \sum_{C \in \mathcal{C}_1} \tilde{\chi}_{C^\delta} \Big\|_{L^1(\mathbb{R}^2)} \lesssim \delta^{1+\varepsilon} \#\mathcal{C} \qquad (3')$$

Once we have (2') and (3'), we can apply the triangle inequality to bound

$$\Big\| \sum_{C \in \mathcal{C}} \tilde{\chi}_{C^\delta} \Big\|_{L^q(\mathbb{R}^2)} \leq \Big\| \sum_{C \in \mathcal{C}_1} \tilde{\chi}_{C^\delta} \Big\|_{L^q(\mathbb{R}^2)} + \Big\| \sum_{C \in \mathcal{C}_2} \tilde{\chi}_{C^\delta} \Big\|_{L^q(\mathbb{R}^2)}$$

for any fixed $1 < q < 2$. We now write $\frac{1}{q} = \frac{1-\theta}{1} + \frac{\theta}{2}$ for some $0 < \theta < 1$ and apply Hölder's inequality to further bound

$$\Big\| \sum_{C \in \mathcal{C}} \tilde{\chi}_{C^\delta} \Big\|_{L^q(\mathbb{R}^2)} \leq \Big\| \sum_{C \in \mathcal{C}_1} \tilde{\chi}_{C^\delta} \Big\|^{1-\theta}_{L^1(\mathbb{R}^2)} \Big\| \sum_{C \in \mathcal{C}_1} \tilde{\chi}_{C^\delta} \Big\|^{\theta}_{L^2(\mathbb{R}^2)}$$
$$+ \Big\| \sum_{C \in \mathcal{C}_2} \tilde{\chi}_{C^\delta} \Big\|^{1-\theta}_{L^1(\mathbb{R}^2)} \Big\| \sum_{C \in \mathcal{C}_2} \tilde{\chi}_{C^\delta} \Big\|^{\theta}_{L^2(\mathbb{R}^2)}.$$

To estimate the contribution from $\mathcal{C}_1$ we use (3') and (2). To estimate the contribution from $\mathcal{C}_2$ we use (2') and (3). Altogether, this gives a bound of

$$\Big\| \sum_{C \in \mathcal{C}} \tilde{\chi}_{C^\delta} \Big\|_{L^q(\mathbb{R}^2)} \lesssim \left(\delta^{(1+\varepsilon)(1-\theta)} + \delta^{(1-\theta) + \varepsilon\theta} \right) [\#\mathcal{C}]^{1/q}$$
$$= \delta^{2/q - 1 + \varepsilon(q')} [\#\mathcal{C}]^{1/q}$$

where $\varepsilon(q') := \varepsilon \cdot \min\{2/q - 1,\ 2 - 2/q\} > 0$ for $1 < q < 2$. $\square$

The transverse case

To motivate the decomposition $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ we consider some special situations where it is possible to show ( ) fairly directly.

Recall the weak orthogonality estimate from the previous lecture, which was the key ingredient in the proof of (2).

Lemma (Weak orthogonality): For all pairs of unit scale circles C_1, C_2 and all $N \in \mathbb{N}$,

$$|\langle \tilde{\chi}_{C_1^\delta}, \tilde{\chi}_{C_2^\delta} \rangle| \lesssim_N \delta \cdot (1 + \delta^{-1} \mathrm{dist}(C_1, C_2))^{-1/2} (1 + \delta^{-1} \Delta(C_1, C_2))^{-N}.$$

Here, for $C_1 = C(x_1, r_1)$, $C_2 = C(x_2, r_2)$,

- $\mathrm{dist}(C_1, C_2) := |x_1 - x_2|$ is the distance between the centres
- $\Delta(C_1, C_2) := ||x_1 - x_2| - |r_1 - r_2||$ measures the (interior) tangency between C_1 and C_2.

From this, we esepect a significant gain over (2) if there are few tangent pairs of circles $(C_1, C_2) \in \mathcal{C} \times \mathcal{C}$. We refer to this as the "transverse case". This observation is made precise by the following lemma.

Lemma 2 (Transverse case $\Rightarrow L^2$ improvement) Suppose $\mathcal{C}$ is a collection of unit scale circles with δ-separated centres which satisfies the additional hypothesis

$$\text{(Trans:)} \quad \#\{C_2 \in \mathcal{C} : \Delta(C_1, C_2) \leq \delta^{1-\eta}\} \leq \delta^{2/3 + 4\varepsilon} \delta^{-2} \tag{4}$$

for some fixed $0 < \varepsilon, \eta < 1$. Then

$$\Big\| \sum_{C \in \mathcal{C}} \tilde{\chi}_{C^\delta} \Big\|_{L^2(\mathbb{R}^2)} \lesssim_{\eta,\varepsilon} \delta^{\varepsilon} [\#\mathcal{C}]^{1/2}.$$

Remarks: • The $\tilde{\chi}_{C^\delta}$ are frequency localised a scale $2^j = \delta^{-1}$, and, by the uncertainty principle, the support of $\check{\chi}_{C^\delta}$ is essentially C^δ. Thus, δ is the smallest scale at which it makes sense to analyse the problem on the physical side. In particular, the condition

$$\Delta(C_1, C_2) \leq \delta$$

can be read as C_1^δ and C_2^δ are "maximally tangent".

• The condition $\Delta(C_1, C_2) \leq \delta^{1-\eta}$ therefore means C_1^δ and C_2^δ are tangent up to a relatively small tolerance factor of $\delta^{-\eta}$.

• Trivially,

$$\#\{C_2 \in \mathcal{C} : \Delta(C_1, C_2) \leq \delta^{1-\eta}\} \leq \#\mathcal{C} \lesssim \delta^{-2}$$

and so the hypothesis (4) represents a gain over the trivial bound by a factor of $\delta^{2/3+4\varepsilon}$.

Proof: We write

$$\Big\| \sum_{C \in \mathcal{C}} \check{\chi}_{C^\delta} \Big\|_{L^2(\mathbb{R}^2)}^2 = \sum_{C_1, C_2 \in \mathcal{C}} \langle \check{\chi}_{C_1^\delta}, \check{\chi}_{C_2^\delta} \rangle$$

$$= \sum_{C_1 \in \mathcal{C}} \sum_{\substack{C_2 \in \mathcal{C} \\ \Delta(C_1, C_2) \leq \delta^{1-\eta}}} \langle \check{\chi}_{C_1^\delta}, \check{\chi}_{C_2^\delta} \rangle + \sum_{C_2 \in \mathcal{C}} \sum_{\substack{C_1 \in \mathcal{C} \\ \Delta(C_1, C_2) > \delta^{1-\eta}}} \langle \check{\chi}_{C_1^\delta}, \check{\chi}_{C_2^\delta} \rangle.$$

$$=: \Sigma_{\text{tang}} + \Sigma_{\text{trans}}. \tag{5}$$

We first deal with the transverse contribution Σ_{trans}. If $\Delta(C_1, C_2) > \delta^{1-\eta}$, then the weak orthogonality lemma implies

$$|\langle \check{\chi}_{C_1^\delta}, \check{\chi}_{C_2^\delta} \rangle| \lesssim_{N,\eta} \delta\, (1 + \delta^{-1}\Delta(C_1, C_2))^{-\lceil N/\eta \rceil}$$
$$\lesssim \delta^N.$$

Since $\#\mathcal{C} \lesssim \delta^{-2}$, we have

$$\Sigma_{\text{trans}} \lesssim_\eta \delta^{100} \# \mathcal{C} \qquad (6).$$

We now turn to the tangential contribution Σ_{tang}. Again by the orthogonality lemma,

$$|\langle \tilde{\chi}_{C_1^*}, \tilde{\chi}_{C_2^*} \rangle| \lesssim \delta\,(1 + \delta^{-1}\operatorname{dist}(C_1, C_2))^{-1/2}$$

and so, by Hölder's inequality,

$$\Sigma_{\text{tang}} \lesssim \delta \cdot \sum_{C_1 \in \mathcal{C}} \sum_{\substack{C_2 \in \mathcal{C} \\ \Delta(C_1, C_2) \leq \delta^{1-\eta}}} (1 + \delta^{-1}\operatorname{dist}(C_1, C_2))^{-1/2}$$

$$\lesssim \delta \sum_{C_1 \in \mathcal{C}} \#\{C_2 \in \mathcal{C} : \Delta(C_1, C_2) \leq \delta^{1-\eta}\}^{3/4} \Big(\sum_{C_2 \in \mathcal{C}} (1 + \delta^{-1}\operatorname{dist}(C_1, C_2))^{-2} \Big)^{1/4}.$$

By a familiar argument (dyadically decomposing according to $\operatorname{dist}(C_1, C_2)$),

$$\sum_{C_2 \in \mathcal{C}} (1 + \delta^{-1}\operatorname{dist}(C_1, C_2))^{-2} \lesssim \log_2 \delta^{-1}$$

whilst, by hypothesis,

$$\#\{C_2 \in \mathcal{C} : \Delta(C_1, C_2) \leq \delta^{1-\eta}\} \leq \delta^{-\frac{4}{3}(1-3\varepsilon)}$$

for all $C_1 \in \mathcal{C}$. Combining these observations,

$$\Sigma_{\text{tang}} \lesssim \delta \cdot \sum_{C_1 \in \mathcal{C}} \delta^{-(1-3\varepsilon)} (\log_2 \delta^{-1})^{1/4}$$

$$\lesssim_\varepsilon \delta^{2\varepsilon} \# \mathcal{C}. \qquad (7).$$

Plugging the bounds (6) and (7) into the right-hand side of (5) yields

$$\Big\| \sum_{C \in \mathcal{C}} \check{\chi}_{C^*} \Big\|_{L^2(\mathbb{R}^2)}^2 \lesssim_{\varepsilon,\eta} \delta^{2\varepsilon} \# \mathcal{C},$$

as required. $\square$

A first glimpse of the tangent case.

We now consider the situation where the hypothesis (4) of Lemma 2 fails. In particular, we suppose $\mathcal{C}$ is a collection of unit scale circles with δ-separated centres and that there exists some $C_0 \in \mathcal{C}$ such that

$$\#\{C \in \mathcal{C} : \Delta(C_0, C) \leq \delta^{1-?}\}$$

is large. (In Kakeya parlance, this is a "bush" configuration).

To simplify matters we assume

- $\Delta(C_0, C) \leq \beta$ for all $C \in \mathcal{C}$ (8)

where $C_0 \in \mathcal{C}$ is some fixed circle and $\delta \leq \beta \ll 1$ is a fixed parameter. We will further assume

- $e/2 \leq \mathrm{dist}(C_0, C) \leq e$ for all $C \in \mathcal{C}$
- Either $r \geq r_0$ for all $C(x,r) \in \mathcal{C}$ or $r \leq r_0$ for all $C(x,r) \in \mathcal{C}$

where $\delta < e \ll 1$ is another fixed parameter. We will always be able to reduce to this situation by dyadic pigeonholing.

All the circles $C \in \mathcal{C}$ are "almost tangent" to the fixed circle C_0. But what we are really interested in is the total number of almost-tangent pairs $(C_1, C_2) \in \mathcal{C} \times \mathcal{C}$. Trivially,

$$\#\{(C_1, C_2) \in \mathcal{C} \times \mathcal{C} : \Delta(C_1, C_2) \leq \beta\} \leq [\#\mathcal{C}]^2.$$

Furthermore, this bound can be attained if all the circles are arranged in a 'clam shell' configuration, so that all the annuli are pairwise tangent along a common arc of C_0.

It turns out that the clam shell configuration is, in some sense, the only situation in which there are many tangent pairs $(C_1, C_2) \in \mathcal{C}$, under the hypothesis (8). A key step towards the proof of Proposition 1 will be to and prove a precise statement of this characterisation.

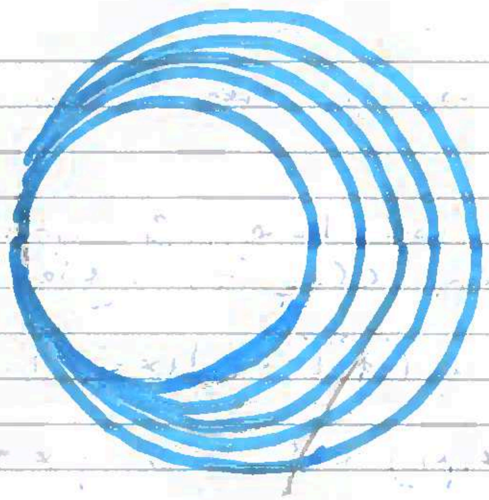

Fig 1: The Clam shell configuration.

Some geometric preliminaries.

In order to analyse the tangent case, we will need a number of elementary observations about intersecting pairs and triples of circles. We begin with some notation.

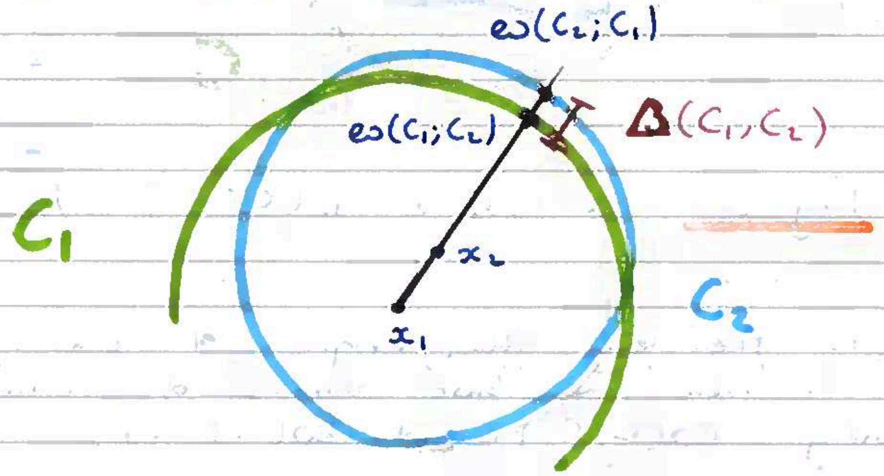

Fig 2: The point $\omega(C_1;C_2) \in C_1$ of "almost tangency".

Given a pair of circles $C_1 = C(x_1, r_1)$, $C_2 = C(x_2, r_2)$ we let

$$\omega(C_1;C_2) := x_1 - r_1 \,\mathrm{sgn}(r_1 - r_2)\, \frac{x_1 - x_2}{|x_1 - x_2|}.$$

If C_1, C_2 are interior tangent (ie $\Delta(C_1,C_2) = 0$), then $\omega(C_1,C_2)$ is precisely the location on C_1 where the tangency occurs. More generally, for a pair of intersecting circles C_1, C_2 with $\Delta(C_1, C_2) \leq \beta$, the point $\omega(C_1;C_2)$ is the location of the point of "almost tangency" on C_1, centred in the midpoint of the arc between the two points of intersection between C_1 and C_2.

Now suppose we have 3 circles $C = C(x, r)$, $C_1 = C(x_1, r_1)$ and $C(x_2, r_2)$. We define

$$\measuredangle(C; C_1, C_2) := \measuredangle\left(\mathrm{sgn}(r - r_1)(x - x_1),\ \mathrm{sgn}(r - r_2)(x - x_2)\right)$$

(assuming $x \notin \{x_1, x_2\}$ and $r \notin \{r_1, r_2\}$ so this is well-defined).

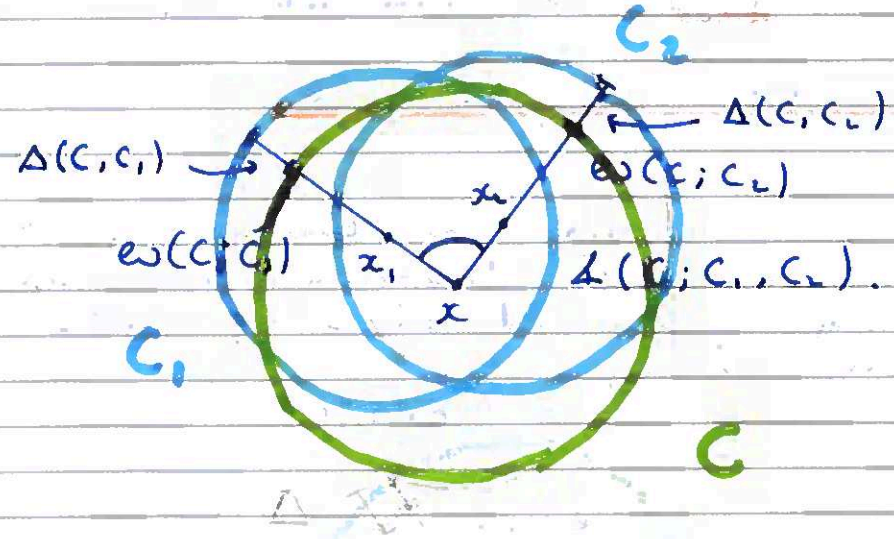

Fig 3: The angle $\measuredangle(C; C_1, C_2)$

This is precisely the angle of the arc of C between the two points of almost tangency $\omega(C; C_1)$ and $\omega(C; C_2)$.

Lemma 3 (Uniqueness of the Clamshell configuration)
Let $\delta \leq \beta, e, \varkappa \ll 1$ and suppose $C_0 = C(x_0, r_0)$, $C_1 = C(x_1, r_1)$, $C_2 = C(x_2, r_2)$ are unit scale circles such that

i) $\Delta(C_0, C_1), \Delta(C_0, C_2) \leq \beta$
ii) $e/2 \leq \mathrm{dist}(C_0, C_1), \mathrm{dist}(C_0, C_2) \leq e$ ← We can probably drop this.
iii) Either $r_1, r_2 \geq r_0$ or $r_1, r_2 \leq r_0$.

Further suppose

$$\Delta(C_1, C_2) + \beta \leq \varkappa \cdot \mathrm{dist}(C_1, C_2). \quad (9)$$

a) If $r_1, r_2 \geq r_0$, then

$$\measuredangle\,(C_{max}\,;\,C_0, C_{min}) \lesssim \kappa^{1/2} \qquad (10a)$$

b) If $r_1, r_2 \leq r_0$, then

$$\measuredangle\,(C_{min}\,;\,C_0, C_{max}) \lesssim \kappa^{1/2} \qquad (10b)$$

where $C_{max} \in \{C_1, C_2\}$ is any circle of maximal radius and $C_{min} \in \{C_1, C_2\}$ is any circle of minimal radius.

To understand the content of this lemma, first note i), ii) and iii) correspond precisely to the hypotheses of the tangent case introduced in the previous subsection. In addition to this, we assume (9), which tells us $\Delta(C_1, C_2)$ is small and therefore $(C_1, C_2) \in \mathcal{C} \times \mathcal{C}$ forms an almost-tangent pair. Finally, the conclusion in (10a) and (10b) tells us that the point of "tangency" between C_1 and C_2 must lie on an arc of some C_ℓ close to the point of "tangency" between C_ℓ and C_0. In particular, the point of tangency between C_1 and C_2 is constrained to lie in a certain direction relative to the centre of one of the circles. This lemma, therefore, is a rigorous interpretation of the claim that under hypotheses i), ii) and iii), the only way for (C_1, C_2) to form a tangent pair is if $\{C_0, C_1, C_2\}$ lie in a clam shell configuration.

The proof of Lemma 3 relies, unsurprisingly, on simple trigonometry. Recall, the law of cosines (see Figure 4) implies for any triple of vectors $x, y, z \in \mathbb{R}^2$ we have

$$|z-y|^2 = |y-x|^2 + |z-x|^2 - 2|y-x||z-x| \cos\alpha \qquad (11)$$

where α is the angle between $y-x$ and $z-x$.

Rearranging the identity (11), we obtain a "quantitative" version of the triangle inequality.

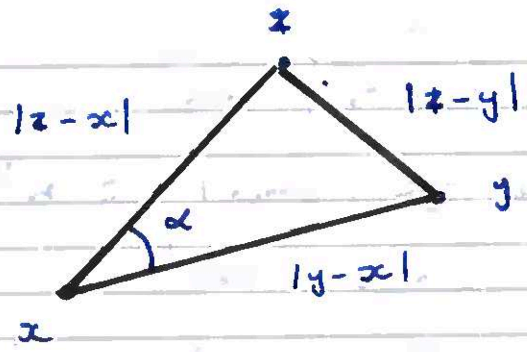

Fig 4: The law of cosines:

$$|z-y|^2 = |z-x|^2 + |y-x|^2 - 2|z-x||y-x| \cos\alpha .$$

Lemma 4: (Quantitative triangle inequality). If $x, y, z \in \mathbb{R}^2$ and $\alpha := \measuredangle(y-x, z-x)$, then

$$|z-y| + |y-x| - |z-x| \gtrsim |y-x|\alpha^2 . \qquad (12)$$

Furthermore, if $|y-x| \leq |z-x|, |z-y|$, then

$$|z-y| + |y-x| - |z-x| \sim |y-x|\alpha^2 .$$

By the standard triangle inequality, $|z-x| \leq |z-y| + |y-x|$ and so

$$|z-y| + |y-x| - |z-x| \geq 0 \qquad (13).$$

Thus, (12) gives a quantified improvement over (13) in the case where α is large.

Proof:- Recalling (11) and completing the square, we have

$$\begin{aligned} |z-y|^2 &= |z-x|^2 + |y-x|^2 - 2|z-x||y-x| \\ &\qquad + 2|z-x||y-x|(1-\cos\alpha) \\ &= (|z-x| - |y-x|)^2 + 2|z-x||y-x|(1-\cos\alpha). \end{aligned}$$

Rearranging and factorising,

$$\begin{aligned} 2|z-x||y-x|(1-\cos\alpha) &= |z-y|^2 - (|z-x| - |y-x|)^2 \\ &= (|z-y| - |z-x| + |y-x|) \cdot \\ &\qquad (|z-y| + |z-x| - |y-x|). \end{aligned}$$

On the one hand, $1-\cos\alpha \sim \alpha^2$. On the other hand, by the triangle inequality

$|z-y| \le |z-x| + |y-x|$ and so
$$|z-y| + |z-x| - |y-x| \le 2|z-x|.$$
Combining these observations,

$$\alpha^2 |z-x||y-x| \lesssim (|z-y| + |y-x| - |z-x|)|z-x|$$

and, assuming $|z-x| \neq 0$ (or else the situation is trivial), this gives (12).

For the equality case we simply note
$$|z-y| + |z-x| - |y-x| \geq |z-x|$$
if $|x-y| \le |z-y|$. $\square$.

Proof (of Lemma 3):- a) Suppose $r_1, r_2 \geq r_0$ and, without loss of generality, $r_1 = \max\{r_1, r_2\}$.

Let $\theta := \measuredangle(C_1; C_0, C_2)$ so that $\theta := \angle(x_2 - x_1, x_0 - x_1)$.
By the quantitative triangle inequality (12) we have

$$|x_1 - x_2|\theta^2 \lesssim |x_2 - x_0| - |x_1 - x_0| + |x_2 - x_1|. \quad (14)$$

By hypothesis i),

$$\Delta(C_2, C_0) = ||x_2 - x_0| - (r_2 - r_0)| \le \beta$$
$$\Delta(C_1, C_0) = ||x_1 - x_0| - (r_1 - r_0)| \le \beta$$

and so
$$|x_2 - x_0| \le r_2 - r_0 + \beta, \quad (15a)$$
$$|x_1 - x_0| \geq r_1 - r_0 - \beta. \quad (15b)$$

Combining (14), (15a) and (15b),

$$\begin{aligned}|x_1 - x_2|\theta^2 &\lesssim r_2 - r_0 - (r_1 - r_0) + |x_2 - x_1| + 2\beta \\ &= |x_2 - x_1| - (r_1 - r_2) + 2\beta \\ &= \Delta(C_1, C_2) + 2\beta.\end{aligned}$$

Consequently, $$\theta^2 \lesssim \frac{\Delta(C_1, C_2) + \beta}{\mathrm{dist}(C_1, C_2)}.$$

Finally, applying the hypothesis (9), we deduce $\theta^2 \lesssim \kappa$, as required.

b) Suppose $r_1, r_2 \le r_0$ and, without loss of generality, that $r_1 = \min\{r_1, r_2\}$. The argument now proceeds identically to that for part a). In place of (15a) and (15b) we now have

$$|x_2 - x_0| \le r_0 - r_2 + \beta,$$
$$|x_1 - x_0| \ge r_0 - r_1 - \beta,$$

which again gives

$$\begin{aligned} |x_1 - x_2|\,\theta^2 &\lesssim r_0 - r_2 - (r_0 - r_1) + |x_1 - x_2| + 2\beta \\ &= |x_1 - x_2| - (r_2 - r_1) + 2\beta \\ &= \Delta(C_1, C_2) + 2\beta \end{aligned}$$

and the desired result follows. ◻.

Where are we heading?

- By Lemma 2, we are led to consider the tangent case, in which all the circles in $\mathcal{C}$ are almost tangent to a fixed circle C_0.
- Again by Lemma 2, we can expect to be in a favourable situation unless $\mathcal{C} \times \mathcal{C}$ contains many pairs (C_1, C_2) of almost tangent circles.
- By Lemma 3, the only way $\mathcal{C} \times \mathcal{C}$ can contain many pairs of almost tangent circles is if the circles in $\mathcal{C}$ are arranged in a clam shell-like configuration.

Thus, the sum total of this lecture is to, at least on a heuristic level, reduce our attention to the special case where $\mathcal{C}$ is a clam shell configuration.

So far, we have focused on obtaining an improvement over the L^2 bound (2) via Lemma 2. In order to deal with the tangencies in the clam shell configuration, we can no longer wholly rely on L^2 methods, but must also work with L^1 bounds. In particular, we will obtain an improvement over the L^1 bound (1) in this case.

We sketch the remaining ingredients of the proof, which will be discussed in detail in the next lecture.

The first observation is that for a clam shell configuration $\mathcal{C}$, the sets $C_1^\delta \cap C_2^\delta$ must be highly localised in space.

Lemma 5 : Under the hypotheses of Lemma 3, there exists an absolute constant $A_1 \geq 1$ such that for $\sigma :=: \sigma(\mathcal{C}, \varkappa) := A_1 \ell \varkappa$ we have

$$C_1^\delta \cap C_2^\delta \subseteq C_0^\sigma .$$

This lemma follows as a corollary of Lemma 3 together with some simple geometric observations about circle intersections. We discuss the details in the next lecture.

Recall from the uncertainty principle that each $\tilde{\chi}_{C^\delta}$ is essentially supported on C^δ. In particular, (since also the $\tilde{\chi}_{C^\delta}$ are L^∞-normalised), we should have

$$|\langle \tilde{\chi}_{C_1^\delta}, \tilde{\chi}_{C_2^\delta} \rangle_{L^2(\mathbb{R}^2 \setminus C_0^\sigma)}| \lesssim |C_1^\delta \cap C_2^\delta \cap \mathbb{R}^2 \setminus C_0^\sigma| \text{ for all } C_1, C_2 \in \mathcal{C}.$$

In view of Lemma 5, if $\mathcal{C}$ is a clam shell configuration, then we should expect

$$\Big\| \sum_{C \in \mathcal{C}} \tilde{\chi}_{C^\delta} \Big\|_{L^2(\mathbb{R}^2 \setminus C_0^\sigma)}$$

to satisfy favourable bounds. Indeed, squaring and expanding in terms of the inner product, we have

$$\sum_{(C_1, C_2) \text{ tangent}} \langle \tilde{\chi}_{C_1^\delta}, \tilde{\chi}_{C_2^\delta} \rangle_{L^2(\mathbb{R}^2 \setminus C_0^\sigma)} + \sum_{(C_1, C_2) \text{ transverse}} \langle \tilde{\chi}_{C_1^\delta}, \tilde{\chi}_{C_2^\delta} \rangle_{L^2(\mathbb{R}^2 \setminus C_0^\sigma)}$$

where the sum over tangent pairs (C_1, C_2) is a sum over (C_1, C_2) for which $\Delta(C_1, C_2)$ is small. Provided the parameters are correctly chosen, the tangent contribution should be negligible by Lemma 5. On the other hand, we expect the transverse contribution to satisfy favourable bounds as in Lemma 2.

Remark:- In dealing with the transverse case, we can no longer directly apply the weak orthogonality lemma to bound the $\langle \tilde{\chi}_{C_1^\delta}, \tilde{\chi}_{C_2^\delta} \rangle_{L^2(\mathbb{R}^2 \setminus C_0^\sigma)}$ because of the restriction to integration over $\mathbb{R}^2 \setminus C_0^\sigma$ in the inner product. To circumvent this issue, we will instead bound [1]

$$|\langle \tilde{\chi}_{C_1^\delta}, \tilde{\chi}_{C_2^\delta} \rangle_{L^2(\mathbb{R}^2 \setminus C_0^\sigma)}| \lesssim |C_1^\delta \cap C_2^\delta|$$

and use geometric arguments to show $|C_1^\delta \cap C_2^\delta|$ is small in the transverse case.

It then remains to find a favorable bound for $\sum_{C \in \mathcal{C}} \tilde{\chi}_{C^\delta}$ over the complementary set C_0^σ. Here we use the simple L^1 bound

$$\Big\| \sum_{C \in \mathcal{C}} \tilde{\chi}_{C^\delta} \Big\|_{L^1(C_0^\sigma)} \leq \sum_{C \in \mathcal{C}} \| \tilde{\chi}_{C^\delta} \|_{L^1(C_0^\sigma)}$$

$$\lesssim \sum_{C \in \mathcal{C}} |C^\delta \cap C_0^\sigma|$$

A simple geometric argument will show (again working with C satisfying the hypotheses in Lemma 3) that

$$|C^\delta \cap C_0^\sigma| \lesssim \varkappa^{1/2} \delta ;$$

in particular the intersection $C^\delta \cap C_0^\sigma$ occurs along an arc of C of length $\varkappa^{1/2}$. Thus, we have

$$\Big\| \sum_{C \in \mathcal{C}} \tilde{\chi}_{C^\delta} \Big\|_{L^1(C_0^\sigma)} \lesssim \varkappa^{1/2} \delta \, \# \mathcal{C},$$

which represents a (crucial) gain of $\varkappa^{1/2}$ over the trivial L^1 bound (3).

Altogether, these observations can be used to deal with the close-shell-like configuration and complete the proof.

[1] This is somewhat schematic. The rigorous argument will also exploit the oscillation between the functions $\tilde{\chi}_{C_1^\delta}$ and $\tilde{\chi}_{C_2^\delta}$.