Discrete Geometry
The (Refined) Polynomial Method

Michael Tang

13th January 2021

Outline

Recall

Recall we have

$$g_j(p, \gamma, \alpha, \lambda) = \frac{1}{\binom{\lambda+k_j}{k_j}} |\mathcal{B}_{p,\gamma}(\alpha, \lambda)|,$$

where

$$\mathcal{B}_{p,\gamma}(\alpha, \lambda) := \cup_{r \in \mathbb{N}} \mathcal{B}_p^r$$

were the sets of carefully chosen basis elements.

I find it useful to think of $g_j = O(1)$, and $|\mathcal{B}| = O(\lambda^{k_j})$.

Properties We Need

Lemma (Uniform Boundedness)

Let $\lambda \in \mathbb{N}$. $\exists p, q$ *are such that* $\alpha_p < \alpha_q - \lambda$, *then* $g(p, \gamma, \alpha, \lambda) = 0$ *for all* p *and* γ.

Lemma (Monotonicity)

If $\alpha^{(1)}$ *and* $\alpha^{(2)}$ *are such that* $\exists p \in J \cap \gamma$ *so that* $\alpha_p^{(1)} - \alpha_{p'}^{(1)} \leq \alpha_p^{(2)} - \alpha_{p'}^{(2)}$ *for all* $p' \in \mathcal{P}$, *then*

$$|\mathcal{B}_p(\alpha^{(1)})| \leq |\mathcal{B}_p(\alpha^{(2)})|.$$

Lemma (Lipschitz Continuity)

Fix p, γ, λ. *Then for every* $p \in J$

$$||\mathcal{B}_p(\alpha^{(1)})| - |\mathcal{B}_p(\alpha^{(2)})|| \lesssim \lambda^{\dim \gamma - 1} \sum_{p' \in J} |(\alpha_p^{(1)} - \alpha_{p'}^{(1)}) - (\alpha_p^{(2)} - \alpha_{p'}^{(2)})|.$$

Elementary Work

Suppress j. Fix a k-plane γ, and let $\mathcal{P} = J \cap \gamma$. We may restrict our attention to $\mathcal{P}$.

Let $v = (v_p)_p : \mathcal{P} \to \mathbb{Z}_{\geq 0}$.[1] Define

$$\mathbb{T}(v, \lambda) := \{f \in \mathbb{F}_\lambda[x_1, \ldots, x_k] : f \text{ vanishes to order } \geq v_p \quad \forall p \in \mathcal{P}\} .$$

Let

$$b_p(v, \lambda) := \operatorname{codim}_{\mathbb{T}(v,\lambda)} \mathbb{T}(v+e_p, \lambda) := \dim \mathbb{T}(v, \lambda) - \dim(\mathbb{T}(v+e_p, \lambda).$$

This describes in how many ways we can increase the order of vanishing by 1 at p.

Examples:

- $b_{p_1,\gamma}((1,2),5) = 3 - 2 = 1, \qquad (k = 1, \mathcal{P} = \{p_1, p_2\})$.
- $b_{p,\gamma}(1,2) = 5 - 3 = 2 \qquad (k = 2, \mathcal{P} = \{p\})$.

[1] Think of this is a vector of order of vanishing.

Elementary Work

Lemma (Prelim. Uniform Boundedness)

If $v_p > \lambda$ *for some* $p \in \mathcal{P}$ *them* $\dim \mathbb{T}(v, \lambda) = 0$.

Proof.

Suggestions? A polynomial of degree at most λ cannot vanish to order greater than λ at any point, so $\mathbb{T}(v, \lambda)$ contains only the zero polynomial. □

Elementary Work

Lemma (Prelim. Monotonicity)

Let $p \in \mathcal{P}$. Suppose $v^{(1)}, v^{(2)} \in \mathbb{Z}_{\geq 0}^{\mathcal{P}}$ satisfy $v^{(1)} \geq v^{(2)}$, with equality at p. Then $b_p(v^{(1)}, \lambda) \leq b_p(v^{(2)}, \lambda)$.

Facts:

- Rank of a linear map $= \text{codim}_{\text{domain}}\text{kernel}$.
- Let $U, W \leq V$ be subspaces of V. Then
  - $\text{codim}_U(W \cap U) \leq \text{codim}_V W$.
- So restriction of a linear map to a subspace decreases the rank.

Proof.

- $\mathbb{T}(v + e_p, \lambda)$ is the kernel of the vector valued map $\mathcal{D}$ sending $f \in \mathbb{T}(v, \lambda)$ to all its v_p-th order derivatives at p. Thus $b_p(v, \lambda)$ is the rank of this map.
- So for $i = 1, 2$, $b_p(v^{(i)}, \lambda)$ is the rank of these maps $\mathcal{D}_{(i)}$. Since $v^{(1)} \geq v^{(2)}$, $\mathbb{T}(v^{(1)}, \lambda) \leq \mathbb{T}(v^{(2)}, \lambda)$. So $\mathcal{D}_{(1)}$ is the restriction of $\mathcal{D}_{(2)}$ to $\mathbb{T}(v^{(1)}, \lambda)$, and the rank of a linear map decreases.

□

Elementary Work

Lemma (Prelim. Continuity)

Let $p, q \in \mathcal{P}$. Suppose $v^{(i)}$ is an increasing sequence in $\mathbb{Z}_{\geq 0}^{\mathcal{P}}$, doing so strictly at p. Then

$$0 \leq \sum_{r \in N} b_p(v^{(r)}, \lambda) - \sum_{r \in \mathbb{N}} b_p(v^{(r)} + e_q, \lambda) \leq \operatorname{codim}_{\mathbb{T}(0,\lambda)} \mathbb{T}(0, \lambda - 1).$$

Note:

$$\operatorname{codim}_{\mathbb{T}(0,\lambda)} \mathbb{T}(0, \lambda - 1) = \binom{k + \lambda - 1}{k - 1} = O(\lambda^{k-1}).$$

(1/4).

Firstly, for all r,

$$b_p(v^{(r)}, \lambda) \geq b_p(v^{(r)} + e_q, \lambda),$$

establishing the lower bound. Fix r and consider $b_p(v^{(r)}, \lambda) - b_p(v^{(r)} + e_q, \lambda)$. We will next show that this is at most

$$b_p(v^{(r)}, \lambda) - b_p(v^{(r)}, \lambda - 1),$$

by showing that

$$0 \leq b_p(v^{(r)} + e_q, \lambda) - b_p(v^{(r)}, \lambda - 1).$$

Elementary Work

Proof.

(2/4) Let f be an arbitrary linear factor which vanishes at q but no other point in $\mathcal{P}$. Then

$$\begin{aligned} b_p(v^{(r)} + e_q, \lambda) &= \operatorname{codim}_{\mathbb{T}(v^{(r)}+e_q,\lambda)} \mathbb{T}(v^{(r)} + e_q + e_q, \lambda) \\ &\geq \operatorname{codim}_{f\cdot\mathbb{T}(v^{(r)},\lambda-1)} f \cdot \mathbb{T}(v^{(r)} + e_p, \lambda - 1) \\ &= \operatorname{codim}_{\mathbb{T}(v^{(r)},\lambda-1)} \mathbb{T}(v^{(r)} + e_p, \lambda - 1) \\ &= b_p(v^{(r)}, \lambda - 1). \end{aligned}$$

Elementary Work

Proof.

(3/4) Start with the quantity from (1/3):

$$
\begin{aligned}
& b_p(v^{(r)}, \lambda) - b_p(v^{(r)}, \lambda - 1) \\
&= \operatorname{codim}_{\mathbb{T}(v^{(r)}, \lambda)} \mathbb{T}(v^{(r)} + e_p, \lambda) - \operatorname{codim}_{\mathbb{T}(v^{(r)}, \lambda - 1)} \mathbb{T}(v^{(r)} + e_p, \lambda - 1) \\
&= \Big(\dim \mathbb{T}(v^{(r)}, \lambda) - \dim \mathbb{T}(v^{(r)} + e_p, \lambda)\Big) \\
&\qquad - \Big(\dim \mathbb{T}(v^{(r)}, \lambda - 1) - \dim \mathbb{T}(v^{(r)} + e_p, \lambda - 1)\Big) \\
&= \Big(\dim \mathbb{T}(v^{(r)}, \lambda) - \dim \mathbb{T}(v^{(r)}, \lambda - 1)\Big) \\
&\qquad - \Big(\dim \mathbb{T}(v^{(r)} + e_p, \lambda) - \dim \mathbb{T}(v^{(r)} + e_p, \lambda - 1)\Big) \\
&= \operatorname{codim}_{\dim \mathbb{T}(v^{(r)}, \lambda)} \mathbb{T}(v^{(r)}, \lambda - 1) - \operatorname{codim}_{\mathbb{T}(v^{(r)} + e_p, \lambda)} \mathbb{T}(v^{(r)} + e_p, \lambda - 1) \\
&\leq \operatorname{codim}_{\dim \mathbb{T}(v^{(r)}, \lambda)} \mathbb{T}(v^{(r)}, \lambda - 1) - \operatorname{codim}_{\mathbb{T}(v^{(r+1)}, \lambda)} \mathbb{T}(v^{(r+1)}, \lambda - 1),
\end{aligned}
$$

where the last inequality follows because of the subspace inequality, and that $v^{(r)}$ is strictly increasing at p.

Proof.

(4/4) We now sum over all r:

$$\begin{aligned}
&\sum_{r\in\mathbb{N}} b_p(v^{(r)},\lambda) - b_p(v^{(r)},\lambda-1) \\
&\leq \sum_{r\in\mathbb{N}} \operatorname{codim}_{\dim \mathbb{T}(v^{(r)},\lambda)} \mathbb{T}(v^{(r)},\lambda-1) - \operatorname{codim}_{\mathbb{T}(v^{(r+1)},\lambda)} \mathbb{T}(v^{(r+1)},\lambda-1) \\
&= \dim \mathbb{T}(v^{(0)},\lambda) - \dim \mathbb{T}(v^{(0)},\lambda-1) \\
&= \operatorname{codim}_{\mathbb{T}(v^{(0)},\lambda)} \mathbb{T}(v^{(0)},\lambda-1) \\
&\leq \operatorname{codim}_{\mathbb{T}(0,\lambda)} \mathbb{T}(0,\lambda-1),
\end{aligned}$$

where we have once again used the subspace inequality. □

Choose v

All the Lemmas we want happen at a particular p. Given a handicap $\alpha \in \mathbb{Z}$, it turns out that we can define a $v = v^{(p,r)}$ so that $b_p^\gamma(v, \lambda) = |\mathcal{B}^r_{p,\gamma}(\alpha, \lambda)|$. For each $(p, r) \in J \times \mathbb{N}$, define

$$v_{p'}^{(p,r)}(\alpha) := \begin{cases} \max\{r - (\alpha_p - \alpha_{p'}) + 1, 0\} & p' < p \\ \max\{r - (\alpha_p - \alpha_{p'}), 0\} & \text{o/w.} \end{cases}$$

For each p', this is the least r' so that $(p, r) \preceq (p', r')$. To see this:

- If $p' < p$, then equality can't occur, so p' needs r' just larger than r to appear after (p, r).
- If $p' = p$ then the same order (accounting for handicaps) will do.

More Useful Description

- The vector $v = (v_q^{(p,r)})_q$ collects the number of times each q has been counted + 1 until state (p, r) in the priority order.
- The idea is then to check that given hypothesis on α, or sequences $\alpha^{(r)}$, that the associated $v^{(p,r)}$ satisfy the hypothesis of our preliminary Lemmas.

Polynomial Spaces

Define $\mathcal{B}^r_{p,\gamma}(\alpha,\lambda)$ and $\mathcal{B}^r_{p,\gamma}(\alpha,\lambda)$ as previously, as collections of well chosen dual basis elements. Then

$$\operatorname{span}\left(\mathcal{B}^{r'}_{p'} : (p',r') \prec (p,r)\right) = \operatorname{span}\left(\mathbb{B}^{r'}_{p'} : (p',r') \prec (p,r)\right).$$

By definition, if a polynomial f lies in the kernel of this space of operators, then it vanishes to order v. Adding $\mathcal{B}^r_p$ to this span increases the order of vanishing at p by 1. Hence

$$|\mathcal{B}^r_p| = \operatorname{codim}_{\mathbb{T}(v,\lambda)} \mathbb{T}(v+e_p,\lambda) = b_p(v,\lambda).$$

Uniform Boundedness.

By hypothesis, p, q satisfy $\alpha_p < \alpha_q - \lambda$. For each $r \in \mathbb{N}$, let $v = v^{(p,r)}(\alpha)$. Then

$$v_q \geq r - (\alpha_p - \alpha_q) > r + \lambda > \lambda.$$

Apply preliminary Uniformity Lemma to get $\dim \mathbb{T}(v, \lambda) = 0$. Hence $b_p(v, \lambda) = \operatorname{codim}_{\mathbb{T}(v,\lambda)} \mathbb{T}(v + e_p, \lambda) = 0$. □

Monotonicity

Monotonicity.

Fix $r \in \mathbb{N}$. Let p be such that $\alpha_p^{(1)} - \alpha_{p'}^{(1)} \leq \alpha_p^{(2)} - \alpha_{p'}^{(2)}$ for all $p' \in \mathcal{P}$.[a] Recall

$$v_{p'}^{(p,r)}(\alpha) := \begin{cases} \max\{r - (\alpha_p - \alpha_{p'}) + 1, 0\} & p' < p \\ \max\{r - (\alpha_p - \alpha_{p'}), 0\} & \text{o/w.} \end{cases}$$

Let $v^{(i)} := v^{(p,r)}(\alpha^{(i)})$. Then automatically the hypothesis of Preliminary Monotonicity Lemma. The conclusion follows. □

[a] Since α is not "rooted", this just says α increasing at p.

Lipschitz Continuity

Lipschitz Continuity.

Let $\alpha = \alpha^{(1)}$ and consider the vector $(\alpha_p - \alpha_{p'})_{p'}$. By incrementally increasing each entry, one at a time, we can increase this to $(\alpha_p^{(2)} - \alpha_{p'}^{(2)})_{p'}$ in precisely $|(\alpha_p^{(1)} - \alpha_{p'}^{(1)}) - (\alpha_p^{(2)} - \alpha_{p'}^{(2)})|$ moves. So, it suffices to check that for consecutive iterates,

$$0 \leq |\mathcal{B}_p(\alpha, \lambda)| - |\mathcal{B}_p(\alpha + e_q, \lambda)| = O(\lambda^{k-1}).$$

Recall that $b_p^r(v, \lambda) = |\mathcal{B}_p^r(v, \lambda)|$, and the sets $\mathcal{B}$ are disjoint. The lower bound hence follows from Preliminary Continuity Lemma. It remains to check the upper bound.

Continued...

It suffices to consider

$$\sum_{r\in\mathbb{N}} b_p(v(\alpha),\lambda) - \sum_{r\in\mathbb{N}} b_p(v(\alpha+e_q),\lambda).$$

This is almost ready for Preliminary Continuity Lemma, but need to pass e_q outisde the bracket.

Lipschitz Continuity

Continued...

Recall that $v_{p'}^{(p,r)}(\alpha)$ is one more than the number of times p' has been counted until just before (p,r). So, there is an r_0 so that if $r < r_0$, then $v(\alpha + e_q) = v(\alpha)$, and if $r \geq r_0$, then $v(\alpha + e_q) = v(\alpha) + e_q$. Easier to see this with an example:

$$\begin{aligned} (0,1,3,-1,0) &\leftrightarrow c|c|bc|abce|acbde|abcde\cdots \\ (0,1,3,0,0) &\leftrightarrow c|c|bc|abcde|acbde|abcde\cdots. \end{aligned}$$

We now restrict sum to $r \geq r_0$ because initial terms have positive contribution, then bound using Prelim Continuity Lemma. □

Properties

We have now deduced that the map $\alpha \mapsto (b_p(\alpha, \lambda)$ is

- bounded,
- monotonically increasing,
- Lipschitz Continuous.

Remains to:

- Establish Vanishing Lemma, and,
- check there is a good handicap.